INT Program INT-14-1
Universality in few-body systems: Theoretical challenges and new directions
April 4, 2014

Borromean States of Three Identical Particles in Two Spatial Dimensions

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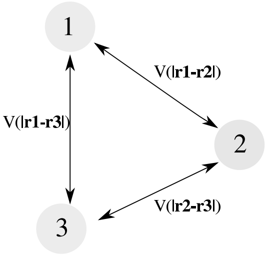
arXiv:1312.6535

Outline

Hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{3}\frac{\partial^2}{\partial \mathbf{r_i}^2} + g\sum_{i<j} V(|\mathbf{r_i} - \mathbf{r_j}|), \qquad \mathbf{r_i} = (r_{xi}, r_{yi})$$

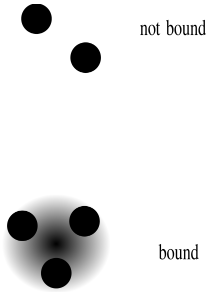


- bosons: $\Psi(\mathbf{r_i}, \mathbf{r_j}) = \Psi(\mathbf{r_j}, \mathbf{r_i})$
- spinless fermions: $\Psi(\mathbf{r_i}, \mathbf{r_j}) = -\Psi(\mathbf{r_j}, \mathbf{r_i})$

Potential, $gV(r)$

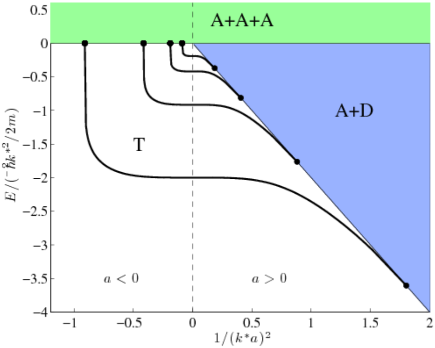
- $V(r)$ is a bounded function of 'short' range ($\int Vr^n \mathrm{d}r < \infty, \forall n$).
- $g > 0$ can be tuned. If $g = g_2$ – no two-body bound states.

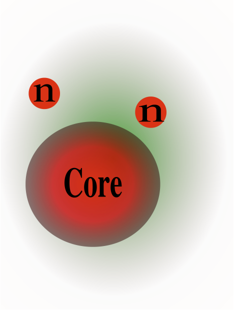
Our basic idea is to consider three-body systems at $g = g_2$.
If the three body system at $g = g_2$ has a bound state it is called Borromean.



Motivation from 3D

$a \to \infty$ infinitely many bound states.
Tune g to eliminate two-body bound states - get infinetely many three-body Borromean states.





Motivation and Aims

Theoretical understanding

- occurence conditions for Borromean states in 2D
- mechanism for occurence of Borromean states, also in 1D, 2D, 3D
- understanding of Super Efimov states using real space

Motivation and Aims

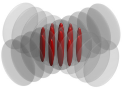
Numerical analysis

- produce a set of numerical examples
- establish reference points for future numerical computations
- test different numerical techniques

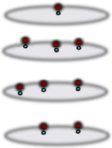
Motivation ans Aims

Talk by Sergei Moroz on Tuesday 1st of April

Experimental relevance



from E. Haller et al., *Science* **325**, 1224



- few-body recombination loss
- matter of trimers, that might be stabilized for spinless fermions

What should be taken into account

- quasi 2D
- few-body calculations for loss
- many-body physics

Two Dimensions (Two bosons)

Consider dimensionless equation for two body bound states in attractive square well potential

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\Psi(r) + gV(r)\Psi(r) = -B\Psi(r), B > 0$$

$$V = \begin{cases} -1; & 0 < r < 1 \\ 0; & r > 1 \end{cases} \quad \Psi = \begin{cases} \mathrm{C}J_0(\sqrt{g+B}r); & 0 < r < 1 \\ \mathrm{C}_1 K_0(\sqrt{B}r); & r > 1 \end{cases}$$

Condition for binding energy

$$\sqrt{g+B}\frac{J_1(\sqrt{g+B})}{J_0(\sqrt{g+B})} = \sqrt{B}\frac{K_1(\sqrt{B})}{K_0(\sqrt{B})}$$

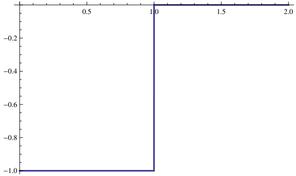
For $g \to 0$

$$\frac{g}{2} \simeq \frac{1}{-\ln\sqrt{B}} \to B \simeq \exp(-4/g)$$

If $B \sim 1/(a_{2D})^2 \to a_{2D} \sim \exp(2/g)$

Two Dimensions (no Efimov effect for bosons)

For a purely attractive potential: $g_2 = 0$.
This observation leads to absence of three-body states at g_2.

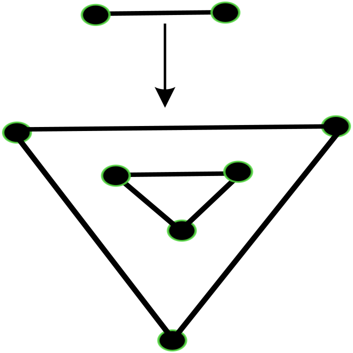


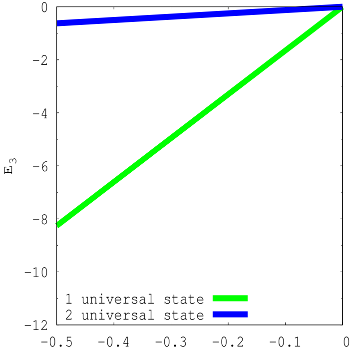
This means that one would have universal three-body states that vanish together with two-body bound states

L. W. Bruch, J. A. Tjon *PRA* **19** 425 (1979)
S. K. Adhikari et al. *PRA* **37**, 3666 (1988)
E. Nielsen, D. V. Fedorov, and A. S. Jensen *PRA* **56** 3287 (1997)
H.-W. Hammer and D.T. Son *PRL* **93** 250408 (2004)
D. Blume *PRB* **72** 094510 (2005)
O. I. Kartavtsev and A. V. Malykh *PRA* **74** 042506 (2006)
F. F. Bellotti et al. *J. Phys. B: At. Mol. Opt. Phys.* **44** 205302 (2011)
...

Universal properties of three bosons in 2D

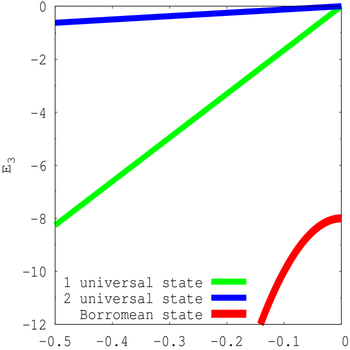
For $E_2 \to -0$ ($a_{2D} \to \infty$) - two universal states with $E_3/E_2 \simeq 16.52, 1.27$

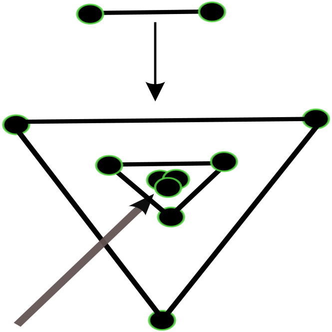




L. W. Bruch, J. A. Tjon *PRA* **19** 425 (1979)

Borromean bosonic systems in 2D



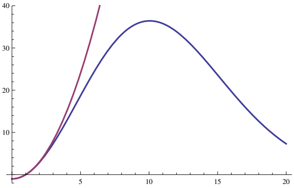


Occurence of Borromean Systems

We consider following potential

$$gV(r) = g\frac{m\omega^2}{2}\left(r^2 - l\frac{\hbar}{m\omega}\right)f(r)$$

for small r: $f(r) = 1$;
for large r: $f(r) \sim exp(-r)$



Approximating potential as an oscillator we get
$g_2 = 8/l^2$
$E_3(g_2) \simeq -2.20\hbar\omega/l$
$g_3 = 16/(3l^2)$ – three-body body threshold
$g_3/g_2 = 2/3$ (the smallest possible: J.-M. Richard and S. Fleck, *PRL* **73** 1464, (1994))
Borromean window $g_3 < g < g_2$

Fermionic systems in 2D

Two spinless fermions

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\Psi + \frac{1}{r^2}\Psi(r) + gV(r)\Psi(r) = E_2\Psi(r)$$

- For $E_2 = 0$: $\Psi_0(r \to \infty) \simeq (r - a_p^2/r)$, resonant state if $a_p \to \infty$.
- To bind two spinless fermions potential should be of finite depth.

Three particles at $a_p \to \infty$ – Super Efimov states with angular momentum ± 1 and energy $E_3^{(n)} \propto \exp(-2e^{3\pi n/4+\theta})$

Talk by Sergej Moroz on Tuesday 1st of April

Yusuke Nishida, Sergej Moroz, Dam Thanh Son *Phys. Rev. Lett.* **110**, 235301 (2013)

Hyperspherical Formalism (Coordinate transformation)

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{3}\frac{\partial^2}{\partial \mathbf{r_i}^2} + g\sum_{i<j} V(|\mathbf{r_i} - \mathbf{r_j}|), \qquad \mathbf{r_i} = (r_{xi}, r_{yi})$$

$$(r_{x1}, r_{y1}, r_{x2}, r_{y2}, r_{x3}, r_{y3}) \rightarrow (R_x, R_y, X_x, X_y, Y_x, Y_y)$$

$$\mathbf{X} = (\mathbf{r_1} - \mathbf{r_2})/\sqrt{2}, \mathbf{Y} = \sqrt{1/6}(\mathbf{r_1} + \mathbf{r_2} - \mathbf{2r_3})$$

$$\mathbf{R} = \sqrt{1/3}(\mathbf{r_1} + \mathbf{r_2} + \mathbf{r_3})$$

$$(R_x, R_y, X_x, X_y, Y_x, Y_y) \rightarrow (R_x, R_y, \rho, \alpha, \phi_X, \phi_Y)$$

$$\rho = \sqrt{X_x^2 + X_y^2 + Y_x^2 + Y_y^2}$$

$$\rho\sin(\alpha) = \sqrt{X_x^2 + X_y^2}, \rho\cos(\alpha) = \sqrt{Y_x^2 + Y_y^2}$$

Hyperspherical Formalism

$$H = T_R - \frac{\hbar^2}{2m}\left(\frac{1}{\rho^{3/2}}\frac{\partial^2}{\partial\rho^2}\rho^{3/2} - \frac{3}{4\rho^2}\right) + \frac{\hbar^2}{2m\rho^2}\Lambda^2 + \sum V$$

where

$$\Lambda^2 = -\frac{\partial^2}{\partial\alpha^2} - 2\cot(2\alpha)\frac{\partial}{\partial\alpha} - \frac{1}{\sin^2(\alpha)}\frac{\partial^2}{\partial\phi_X^2} - \frac{1}{\cos^2(\alpha)}\frac{\partial^2}{\partial\phi_Y^2}$$

first we solve part with angles for given ρ:

$$(\Lambda^2 - \lambda(\rho))\Psi = \left(\sum \frac{2m\rho^2}{\hbar^2}V\right)\Psi(\rho;\alpha,\phi_X,\phi_Y)$$

using Faddeev decomposition

$$(\Lambda^2 - \lambda(\rho))\psi = \frac{2m\rho^2}{\hbar^2}V(\sqrt{2}\sin(\alpha)\rho)\Psi(\rho;\alpha,\phi_X,\phi_Y)$$

Hyperspherical Formalism

After obtaining eigenstates for angular part we decompose the total wave function in this basis set

$$\Phi(\rho, \alpha, \phi_X, \phi_Y) = \frac{1}{\rho^{3/2}} \sum_{\{\lambda\}} f_{\lambda_n}(\rho) \Psi_n(\rho; \alpha, \phi_X, \phi_Y)$$

where f_{λ_n} should solve the set of equations

$$\left(-\frac{\partial^2}{\partial \rho^2} + \frac{\lambda_n + 3/4}{\rho^2} - \frac{2mE}{\hbar^2}\right) f_{\lambda_n} = \frac{2m}{\hbar^2} \sum_{n'} \left(2P_{nn'} \frac{\partial}{\partial \rho} f_{\lambda_{n'}} + Q_{nn'} f_{\lambda_{n'}}\right)$$

$$P_{nn'} = \langle \Psi_n | \frac{\partial}{\partial \rho} | \Psi_{n'} \rangle$$

$$Q_{nn'} = \langle \Psi_n | \frac{\partial^2}{\partial \rho^2} | \Psi_{n'} \rangle$$

Hyperspherical Formalism

Hyperspherical formalism is particulaly useful to investigate universality that arises from long-distance behavior of λ. For example in 3D (D. Fedorov and A. Jensen *PRL* **71** 4103):

$$\left(-\frac{\partial^2}{\partial\rho^2} + \frac{\lambda_n + 15/4}{\rho^2} - \frac{2mE}{\hbar^2}\right) f_{\lambda_n} = \frac{2m}{\hbar^2}\sum_{n'}\left(2P_{nn'}\frac{\partial}{\partial\rho}f_{\lambda_{n'}} + Q_{nn'}f_{\lambda_{n'}}\right)$$

For $a \to \infty$ Q, P decay faster than $1/r^2$ and using only the lowest λ we recover Efimov scenario

$$\left(-\frac{\partial^2}{\partial\rho^2} + \frac{-1.01 - 1/4}{\rho^2} - \frac{2mE}{\hbar^2}\right) f_{\lambda_0} = 0$$

In 2D lowest adiabatic curve is larger than $-1/(4\rho^2)$ at two-body threshold (E. Nielsen, D. V. Fedorov, and A. S. Jensen *PRA* **56** 3287 (1997)) - no Efimov effect.

Restrictions on potentials for Borromean binding

- Two-body problem with $\int V(r) r \mathrm{d}r \leq 0$ always has a bound state
- Lowest adiabatic potential in hypershperical formalism has $3/(4\rho^2)$ repulsive core

- Potentials should have positive net volume, $\int V(r) r \mathrm{d}r > 0$
- Potentials should have finite depth, $g_2 \int V(r) r \mathrm{d}r \not\to 0$

Numerical Search for Borromean states

Correlated Gaussians approach

- $E[f] = \frac{\langle f|H|f\rangle}{\langle f|f\rangle}$
- $f = \sum c_i \exp\left[-a^2(\mathbf{X}-\mathbf{S_x})^2 - 2c(\mathbf{X}-\mathbf{S_x})(\mathbf{Y}-\mathbf{S_y}) - b^2(\mathbf{Y}-\mathbf{S_y})^2\right]$

The potential is the sum of three Gaussians

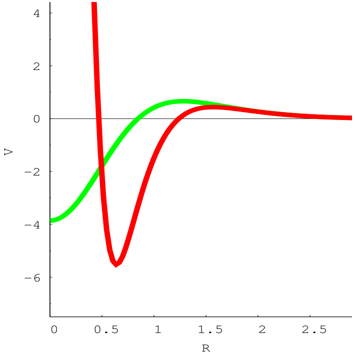
$$\frac{mr_0^2}{\hbar^2}V(r) = b_1\exp(-r^2/r_0^2) + b_2\exp(-a_1 r^2/r_0^2) + b_3\exp(-a_2 r^2/r_0^2)$$

tuned to resonant two-body state.

Borromean Binding (results)

Numerical search: no Borromean states for potentials without barrier*.

Possible (handwaving) explanation: two-body state is bound with smaller overall depth of the attractive part, which leads to small attractive part in the lowest adiabatic potential for three particles.



*Also for known numerical calculations in the literature

Properties of Borromean states

Similar to 3D

- deep potentials with an outer barrier and without a core - similar to harmonic oscillator, described above
  1. large Borromean window ($g_3/g_2 \sim 2/3$)
  2. binding energy and extend at g_2 is of natural order
- potentials with a large core and a barrier
  1. small Borromean window ($g_3/g_2 \rightarrow 1$)
  2. small binding energy and large extend at g_2

The three-body r.m.s radius diverges at g_3.

Unanswered Questions

- Can we have Borromean states in a potential without a barrier?
- Is it possible to construct a theory with one additional parameter produced by overall features of the potential that will tell if there is a three-body bound state at g_2?

Super Efimov States (Hyperspherical approach)

$$(\Lambda^2 - \lambda(\rho))\psi = \frac{2m\rho^2}{\hbar^2} V(\sqrt{2}\sin(\alpha)\rho)\Psi(\rho;\alpha,\phi_X,\phi_Y)$$

$$\Lambda^2 = -\frac{\partial^2}{\partial\alpha^2} - 2\cot(2\alpha)\frac{\partial}{\partial\alpha} - \frac{1}{\sin^2(\alpha)}\frac{\partial^2}{\partial\phi_X^2} - \frac{1}{\cos^2(\alpha)}\frac{\partial^2}{\partial\phi_Y^2}$$

First notice that the Hamiltonian commutes with the operator of total angular momentum $\hat{L} = i\frac{\partial}{\partial\phi_X} + i\frac{\partial}{\partial\phi_Y}$ with eigenstates $exp(-iM(\phi_X + \phi_Y)),\ M = ..., -1, 0, 1, ...$

$$\psi_M(\rho;\alpha,\phi_X,\phi_Y) = \sum_{l=-\infty}^{\infty} e^{-il\phi_X} e^{-i(M-l)\phi_Y} \phi_{Ml}(\rho;\alpha)$$

for spinless fermions l - odd.
for Super Efimov states $M = \pm 1$

Super Efimov States (Hyperspherical approach)

We are interested in large distance behavior of $\lambda(\rho)$. For the lowest potential we need the smallest $l = \pm 1$.

$$\left(-\frac{\partial^2}{\partial\alpha^2} - 2\cot(2\alpha)\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha} + \frac{2gm\rho^2}{\hbar^2} V - \lambda^1 \right)\phi_{11} = -\frac{2gm\rho^2}{\hbar^2} V(\sqrt{2}\rho\sin(\alpha))(R_{111} + R_{11-1}) \, ,$$

$$\left(-\frac{\partial^2}{\partial\alpha^2} - 2\cot(2\alpha)\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha} + \frac{4}{\cos^2\alpha} + \frac{2gm\rho^2}{\hbar^2} V - \lambda^1 \right)\phi_{1-1} = -\frac{2gm\rho^2}{\hbar^2} V(\sqrt{2}\rho\sin(\alpha))(R_{1-11} + R_{1-1-1}) \, .$$

Chao Gao and Zhenhua Yu arxiv1401.0965

Super Efimov States (lowest adiabatic potential at large distance)

$$\lambda_0^1 = -1 - \frac{Y}{\ln(\rho/r_0)} - \frac{16}{9\ln^2(\rho/r_0)} + o\left(\frac{1}{\ln^2(\rho/r_0)}\right), \; Y > 0$$

$$\left(-\frac{\partial^2}{\partial\rho^2} + \frac{\lambda_n^1 + 3/4}{\rho^2} - \frac{2mE}{\hbar^2}\right) f_{\lambda_n^1} = \frac{2m}{\hbar^2}\sum_{n'}\left(2P_{nn'}\frac{\partial}{\partial\rho} f_{\lambda_{n'}^1} + Q_{nn'} f_{\lambda_{n'}^1}\right)$$

first we neglect couplings, as in 3D

$$\left(-\frac{\partial^2}{\partial\rho^2} - \frac{1}{4\rho^2} - \frac{Y}{\rho^2\ln(\rho/r_0)} - \frac{16}{9\rho^2\ln^2(\rho/r_0)} - \frac{2mE}{\hbar^2}\right) f_{\lambda_0^1} = 0$$

For $E = 0$ and neglecting terms $\sim 1/(\ln^2(\rho/r_0))$ the solution is $f_{\lambda_0^1} = \sqrt{\rho\ln(\rho/r_0)}\left[AJ_1(2\sqrt{Y\ln(\rho/r_0)}) + AY_1(2\sqrt{Y\ln(\rho/r_0)})\right]$, $E_n \sim \exp(-(\pi n)^2/(2Y))$,

Super Efimov States (couplings)

For previous slide to be valid Q, P should be small at large distance. However, $Q_{00} = -\frac{Y}{\rho^2 \ln(\rho/r_0)} + \frac{?}{\rho^2 \ln^2(\rho/r_0)}$.

Prospective

$$\left(-\frac{\partial^2}{\partial \rho^2} - \frac{1}{4\rho^2} - \frac{s_0 + 1/4}{\rho^2 \ln^2(\rho/r_0)} - \frac{2mE}{\hbar^2}\right) f_{\lambda_0^1} = 0$$

with $s_0 = 16/9 + ? - 1/4$.

The equation has zero-energy solution

$f_{\lambda_0^1} = \sqrt{\rho \ln(\rho/r_0)} \cos(\sqrt{s} \ln(\ln(\rho/r_0)) + \delta)$

Conclusions

We recover Super Efimov if $? = 1/4$ and all other P, Q are small.

Other Angular Momenta

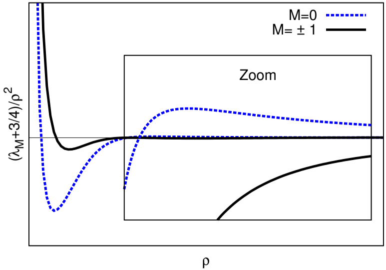
Previous slides considered $M = \pm 1$.

Other angular momenta?

- at $\rho \to \infty$ angular momenta with $M \neq \pm 1$ produce positive adiabatic potentials for $E_2 \to 0$: this leads to finite amount of bound states with angular momenta other than ± 1.
- states with $|M| > 1$ will generally have potential curves higher than $M = 1$.
- for $\rho \to 0$ the lowest adiabatic curve is $M = 0$.

Zero Angular Momentum, $M = 0$

Schematic plot of lowest adiabatic potentials ($M = \pm 1$, $M = 0$)



Let us take the oscillator potential considered before. We get that the deeply bound ground state has $M = 0$ at the two-body threshold. Again the largest Borromean window is reached: $g_3/g_2 = 2/3$.

Numerical findings

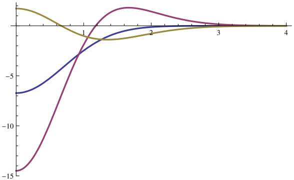
We again use variational approach with correlated Gaussians, using sums of Gaussians as potentials

$$V_1(r) = -g\frac{\hbar^2}{2mb^2}\exp(-r^2/b^2)$$

$$V_2(r) = -g\frac{\hbar^2}{2mb^2}(-\exp(-r^2/b^2) + 0.5\exp(-0.5r^2/b^2))$$

$$V_3(r) = -g\frac{\hbar^2}{2mb^2}(\exp(-r^2/b^2) - 0.8\exp(-0.5r^2/b^2))$$

Not the best choice



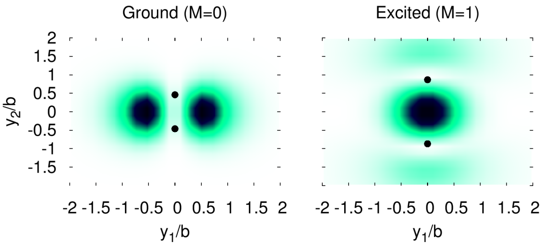
Numerical findings

V	g_2^{cr}	g_3^{cr}/g_2^{cr}	$E_3(g_2^{cr})$	$E_3^*(g_2^{cr})$	$\langle\rho^2\rangle_{gr}$	$\langle\rho^2\rangle_{exc}$
$V_1(r)$	6.72	0.72	−1.50	−0.18	1.65	6.0
$V_2(r)$	28.98	0.68	−5.55	−0.47	0.56	1.13
$V_3(r)$	8.63	0.72	−0.439	−0.045	5.9	22.7

Table : Ground and excited states have angular momentum 0 and ± 1. Lengths and energies are in units of b and $\hbar^2/(mb^2)$, respectively.

Numerical findings

Density distribution for the third particle, when two other particles are placed in the most favorable configuration (black dots).



Summary (Bosons in 2D)

- Borromean states can only occur for potentials with *substantial attractive part and positive net volume*.
- Numerical search did not yield Borromean states for potentials without barrier.
- For potentials with barrier properties of Borromean states are similar to 3D Borromean ground state.

Summary (Spinless fermions in 2D)

- Borromean states always exist at the two-body threshold (follows from the existence of the Super Efimov states).
- To establish the Super Efimov scenario in the hyperspherical formalism more work is needed.
- The ground state can be with $M = 0$.

Outlook

- Occurence conditions for Borromean states for three bosons
- Super Efimov states in hyperspherical formalism. Is it possible to construct different scheme of dividing Hamiltonian that will produce $\sim \frac{1}{1 \ln^2(\rho/r_0)}$ as leading order.
- Investigate lowest state of three spinless fermions, is it always $M = 0$?
- Quasi 2D