Triple-α reactions at low energy

Y. Suzuki (Niigata, Riken)

Physics motivation of triple-alpha reactions
Reaction rate and strength function
Continuum of charged three particles near threshold
  Difficulties and discrepancies
Preliminary result with adiabatic hyperspherical method

Synthesis of ^{12}C element (the fourth abundant element)

Hydrogen burning creates ^{4}He

$4p \to {}^{4}He + 2e^{+} + 2\nu$

Because of mass gap at A =5 and 8, ^{12}C synthesis is blocked

A way out is Helium burning by Triple-α process (Salpeter, 1952)

First step: resonance formation $\alpha + \alpha \leftrightarrow {}^{8}Be(0^{+})$

Property of ^{8}Be resonance ($Q = -92$ keV, $\Gamma = 6$ eV)

$\tau \approx 10^{-16}$s >> Transit time $\approx 10^{-19}$ s for $E = |Q|$

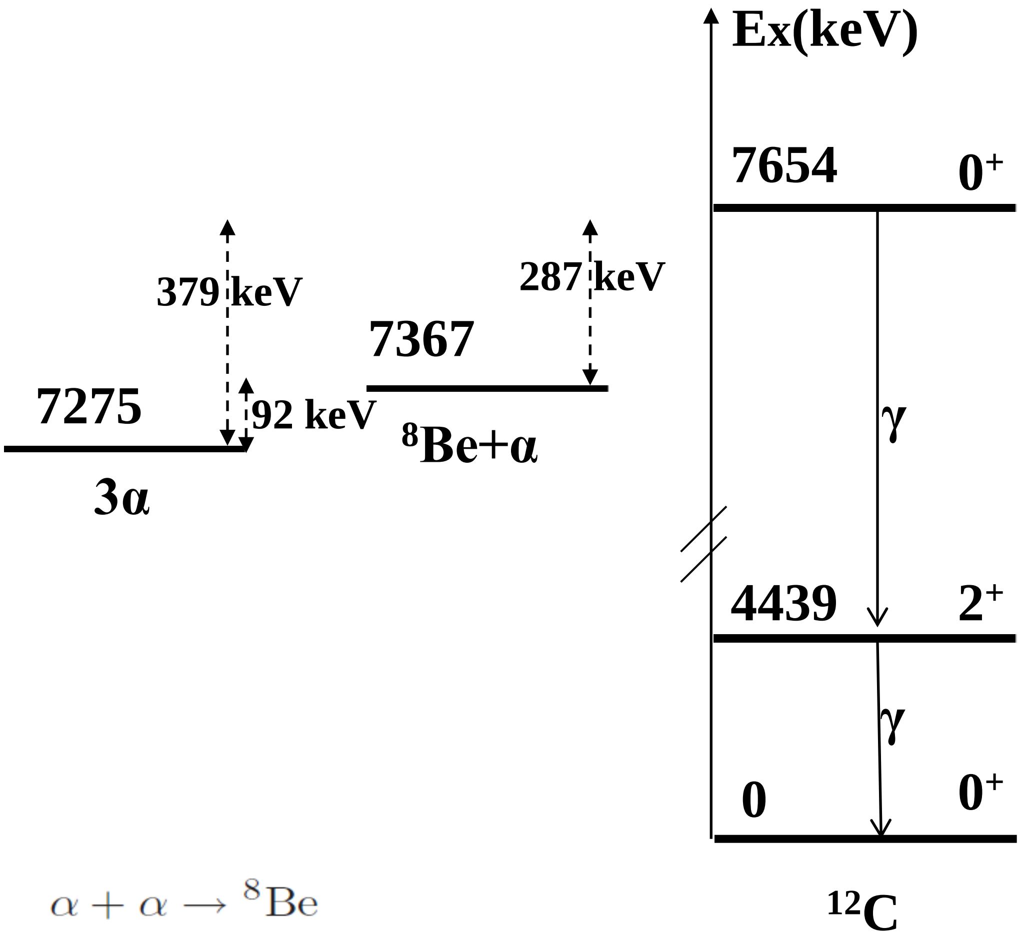
Second step: radiative capture $^{8}Be(\alpha, \gamma)^{12}C$

Hoyle's prediction (1953)

To explain the abundance, the second step should proceed via S-wave resonant state near $\alpha+{}^{8}$Be threshold

The Hoyle state confirmed experimentally (Cook et al., 1957)

$\Gamma \approx 8$ eV, $\Gamma_{\gamma} = 3.7$ meV



Hoyle state

Efimov, PLB33(1970)

$$\alpha + \alpha \rightarrow {}^{8}\mathrm{Be}$$

$$\alpha + {}^{8}\mathrm{Be} \rightarrow {}^{12}\mathrm{C}(2_1^+) + \gamma$$

Measurements at low energies are impossible
Theoretical inputs are needed

Standard rate (NACRE)

C.Angulo et al., NPA 656(1999)

Sequential process:

$\alpha\alpha$ forms ^{8}Be resonance

α^8Be forms the Hoyle resonance and decays by γ emission

Use of experimental data on E_r Γ_α Γ_γ

Use of Breit-Wigner resonance formula and energy-dependent width

$$N_A^2 \langle \sigma v \rangle^{\alpha\alpha\alpha} = 3 N_A \left(\frac{8\pi\hbar}{\mu_{\alpha\alpha}^2} \right) \left(\frac{\mu_{\alpha\alpha}}{2\pi k_B T} \right)^{3/2} \times \int_0^\infty \frac{\sigma_{\alpha\alpha}(E)}{\Gamma_\alpha(^8\mathrm{Be}, E)} \exp(-E/k_B T) N_A \langle \sigma v \rangle^{\alpha^8\mathrm{Be}} E \, dE$$

$$N_A \langle \sigma v \rangle^{\alpha^8\mathrm{Be}} = N_A \frac{8\pi}{\mu_{\alpha^8\mathrm{Be}}^2} \left(\frac{\mu_{\alpha^8\mathrm{Be}}}{2\pi k_B T} \right)^{3/2} \int_0^\infty \sigma_{\alpha^8\mathrm{Be}}(E'; E) \exp(-E'/k_B T) \; E' \, dE'$$

cross section for $^8\mathrm{Be}(\alpha, \gamma)^{12}\mathrm{C}$

What we need to calculate

Energy averaged reaction rate for $a+b+c \rightarrow A+\gamma$

$$< abc >= \int R_{abc}(E)\psi(E)dE$$

$$\psi(E) = \frac{1}{2(k_BT)^3}E^2\exp(-\frac{E}{k_BT})$$

Energy distribution for three identical particles with mass m

$R_{abc}(E)$ related to photoabsorption cross section for $A+\gamma \rightarrow a+b+c$

Detailed balance $$R_{abc}(E) = F(abc, A)\left(\frac{E_\gamma}{E}\right)^2\sigma_\gamma(E_\gamma)$$

$1440\sqrt{3}\pi\hbar^3/m_\alpha^3c^2$ for 3α case

$E_\gamma = E + |B|$

|B|=2.836 MeV

Photoabsorption cross section and strength function

$$\sigma_\gamma(E_\gamma) = \frac{(2\pi)^3(\lambda+1)}{\lambda((2\lambda+1)!!)^2}\left(\frac{E_\gamma}{\hbar c}\right)^{2\lambda-1}S_{E\lambda}(E_\gamma)$$

$$S_{E\lambda}(E_\gamma) = \mathcal{S}_{\mu f}|\langle\Psi_f|\mathcal{M}_{E\lambda\mu}|\Psi_i\rangle|^2\delta(E_f - E_i - E_\gamma)$$

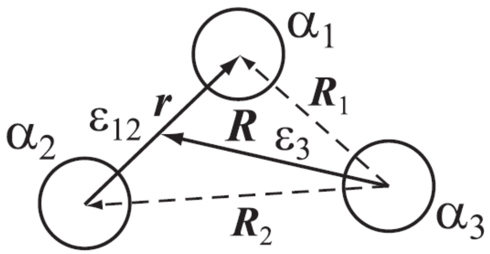
Transition matrix elements for bound to 3-body continuum
Strength function

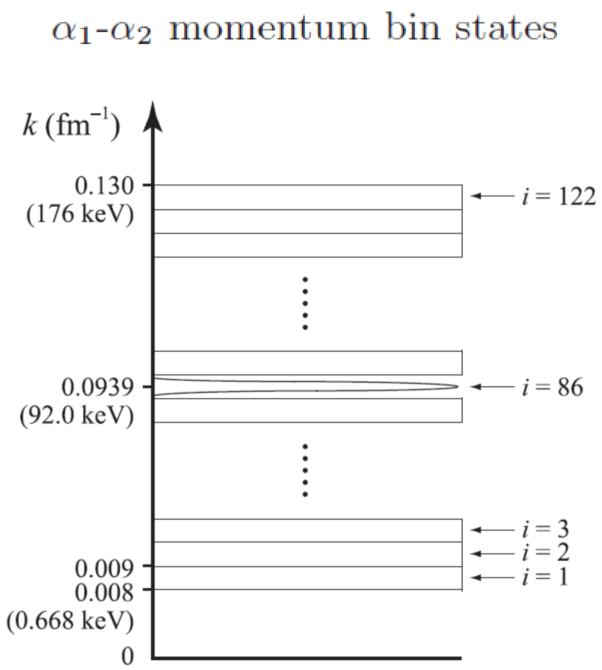
Challenge to low-energy reaction rate

Both nonresonant and resonant process at low energies

Ogata et al. Prog. Theor. Phys. 122 (2009)

CDCC (continuum-discretized coupled-channels) **calculation**





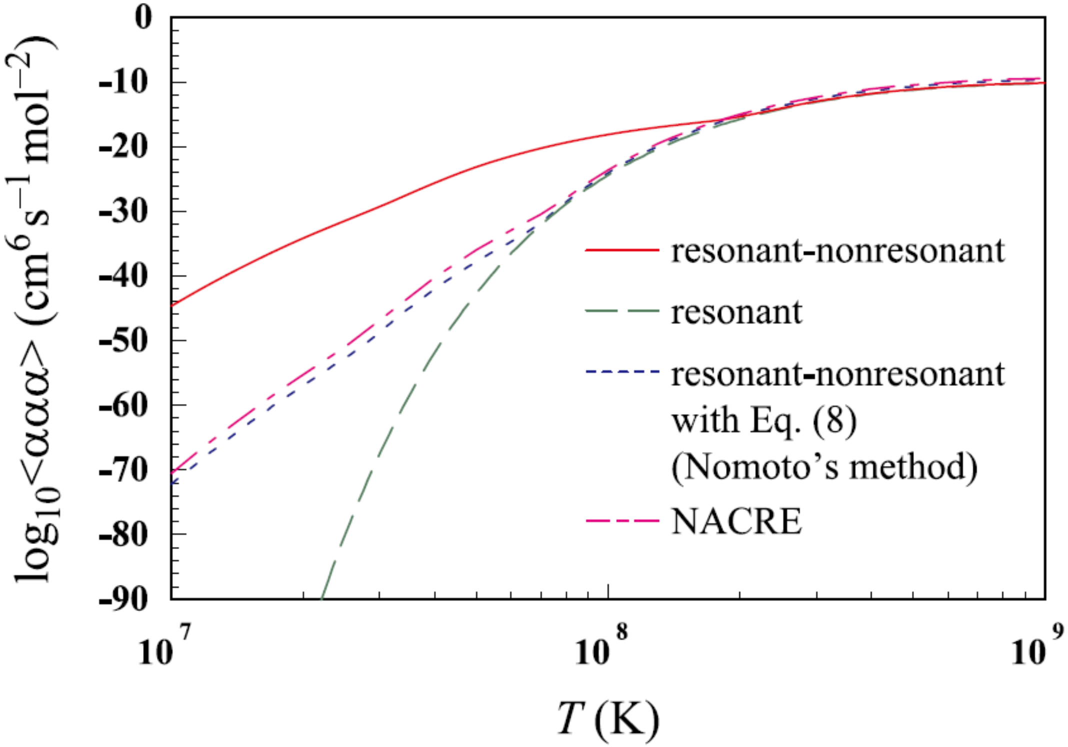
$$\Psi^{0+}_{\hat{k}_{i_0},E}(r,R) = \sqrt{\frac{2}{\pi}}\frac{1}{32\pi^2}\frac{1}{\hat{k}_{i_0}\hat{K}_{i_0}}\sum_{i=1}^{i_{\max}}\frac{\hat{u}_i(r)}{r}\frac{\hat{\chi}_i^{(i_0)}(R)}{R}$$

$$\hat{u}_i(r) = C_i\int_{k_i}^{k_{i+1}} f_i(k)u(k,r)dk$$

Weight function $f_i(k)$:

constant (for nonresonant) or Breit-Wigner (for resonant)

$$[T_R + V_{ii}(R) - (E - \hat{\epsilon}_{12,i})]\,\hat{\chi}_i^{(i_0)}(R) = -\sum_{i'\neq i} V_{ii'}(R)\,\hat{\chi}_{i'}^{(i_0)}(R)$$



10^{26} times NACRE rate at T=0.01 GK corresponding to 0.86 keV

Incompatible with observation:

Dotter, Paxton, Astron.Astrophys. 507 (2009)

(I) Hyperspherical harmonics & R-matrix method (HHR)

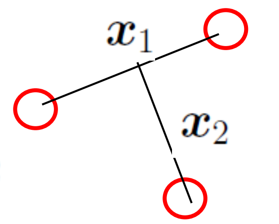
Nguyen et al. PRC87 (2013)

$$\Psi^{LM} = \rho^{-5/2} \sum_{K l_x l_y} \chi_{K l_x l_y}(\rho)\, \varphi_K^{l_x l_y}(\theta)\, [Y_{l_x} \otimes Y_{l_y}]_{LM}$$

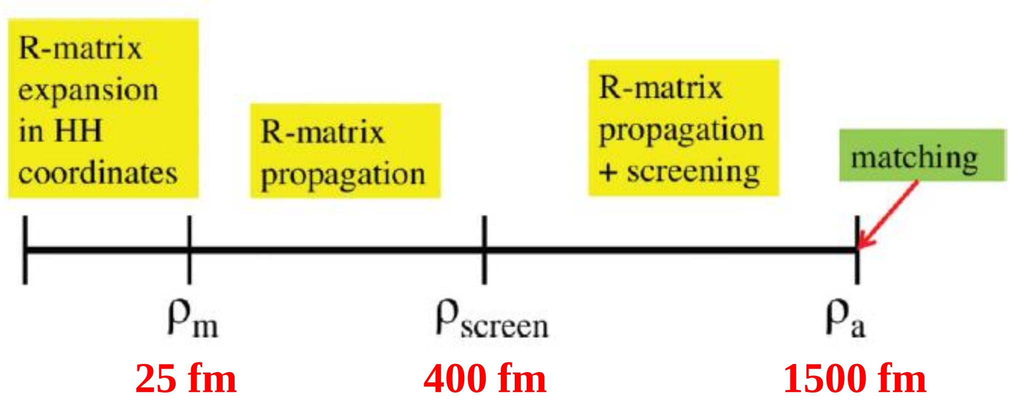
$\varphi_K^{l_x l_y}(\theta)$: hyperspherical harmonics

$$\rho^2 = x_1^2 + x_2^2$$

$$\theta = \tan^{-1} x_1/x_2$$



$$\left(\frac{\hbar^2}{2m}\left[\frac{d^2}{d\rho^2} - \frac{\Delta(\Delta+1)}{\rho^2}\right] + E\right) \chi_\gamma(\rho) = \sum_{\gamma'} V_{\gamma\gamma'}(\rho)\, \chi_{\gamma'}(\rho) \qquad \gamma = \{K, l_x, l_y\} \qquad \Delta = K + 3/2$$

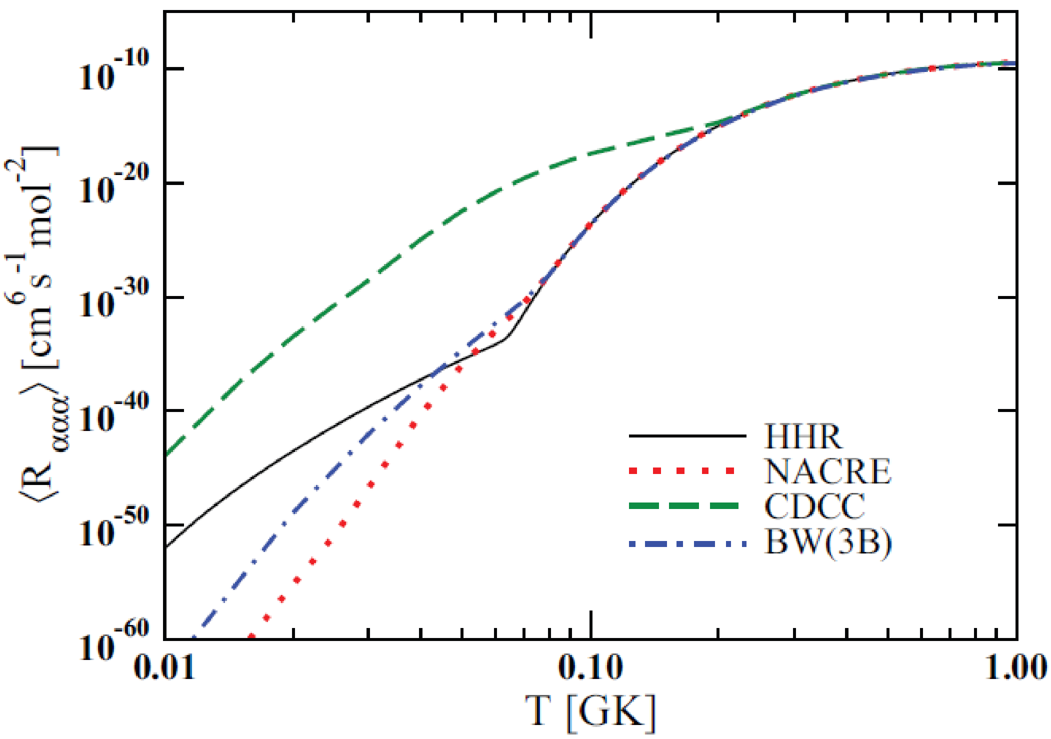


off diagonal $V_{\gamma\gamma'}(\rho)$ screened by $\{1 + \exp[(\rho - \rho_{\rm screen})/a_{\rm screen}]\}^{-1}$

$$\chi^{\rm HHR}_{\gamma\gamma'}(\rho) \overset{\rho\to\infty}{\longrightarrow} H^-_\Delta(\eta_\gamma, \kappa\rho)\delta_{\gamma\gamma'} - H^+_\Delta(\eta_\gamma, \kappa\rho) S_{\gamma\gamma'}$$

Justifiable?

$K_{max} \approx 26$

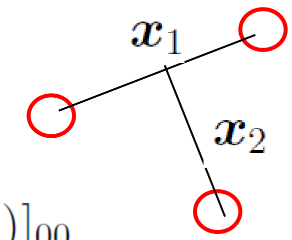


T ≥ 0.07 GK ~ NACRE rate

T ≈ 0.02 GK ~ 10^{12} NACRE rate

Questions: (1) Convergence of hyperspherical harmonics expansion

Slow convergence in K of the wave functions at large ρ



$$\rho^2 = x_1^2 + x_2^2$$

$$\Phi(s_1, s_2, \boldsymbol{x}_1, \boldsymbol{x}_2) = \sum_K f_K(s_1, s_2, \rho)\varphi_K^{l_x l_y}(\theta)[Y_0(\hat{\boldsymbol{x}}_1)Y_0(\hat{\boldsymbol{x}}_2)]_{00}$$

$$||f_K(s_1, s_2)||^2 = \int_0^\infty d\rho\, \rho^5 [f_K(s_1, s_2, \rho)]^2$$ probability of K component

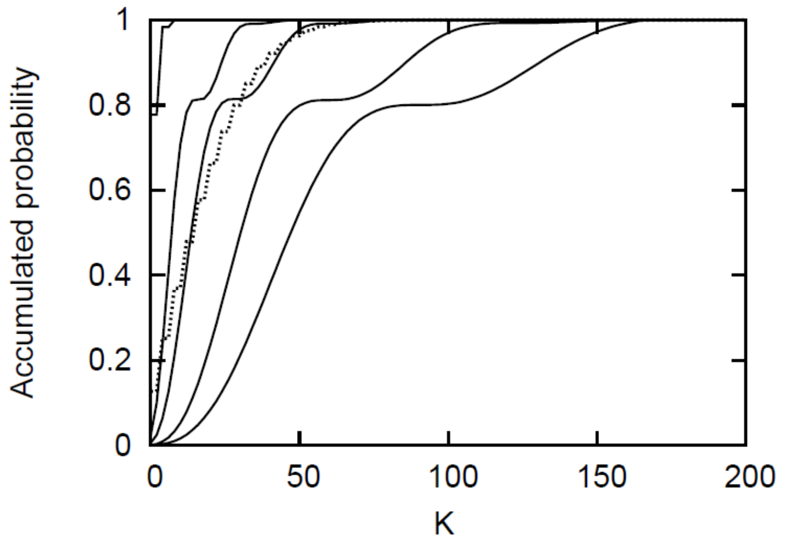
Example: product of Gauss packets

$$\Phi(s_1, s_2, \boldsymbol{x}_1, \boldsymbol{x}_2) = \mathcal{N}(s_1, s_2) g(s_1, x_1) g(s_2, x_2) [Y_0(\hat{\boldsymbol{x}}_1)Y_0(\hat{\boldsymbol{x}}_2)]_{00}$$

$$g(s_1, x_1) = \frac{1}{\beta s_1 x_1}\left[\exp(-\tfrac{\beta}{2}(x_1 - s_1)^2) - \exp(-\tfrac{\beta}{2}(x_1 + s_1)^2)\right]$$

$$\gamma_1 = \sqrt{\beta} s_1 \qquad \gamma_2 = \sqrt{\beta} s_2$$

$$\sum_{K'=0}^{K} ||f_{K'}(s_1, s_2)||^2$$



(γ_1, γ_2)	$\langle \rho^2 \rangle^{1/2}$
(3.5, 3.5)	≈ 3
(3.5, 21)	15
(3.5, 35)	25
(3.5, 70)	50
(3.5, 105)	70 fm
(25, 25)	

(2) Asymptotics of charged three particles

Coupling potential in HH basis

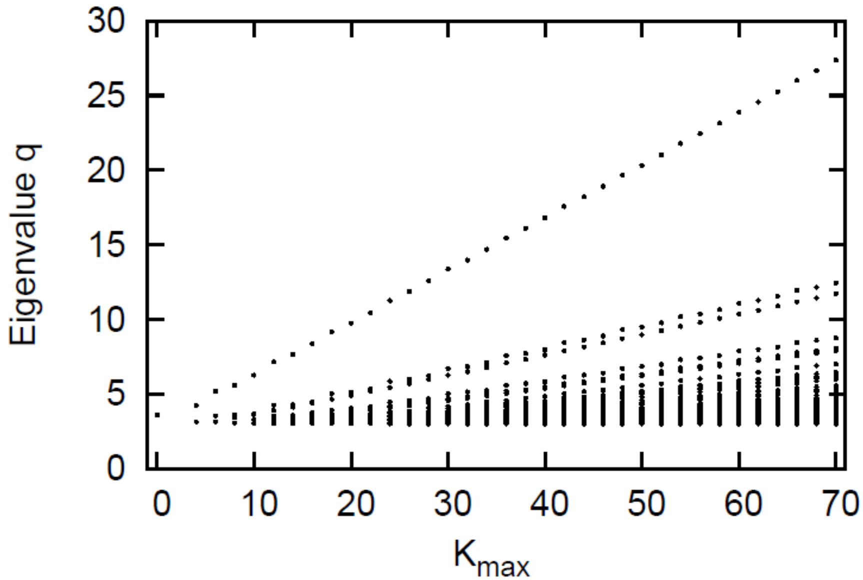
$$V_{\gamma\gamma'}(\rho) \to \frac{\Lambda^2_{\gamma\gamma'}}{\rho^2} + \frac{C_{\gamma\gamma'}}{\rho} \text{ for large } \rho$$

Λ^2 is diagonal and takes $K(K+4)$ in HH but C is not

They do not commute

$$V_C = \sum_{j>i=1}^{3} \frac{4e^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} = \frac{4e^2}{\rho} q(\Omega_x)$$

$$q(\Omega)F_q(\Omega) = qF_q(\Omega)$$



In principle we need to diagonalize $\Lambda^2 + \rho C$

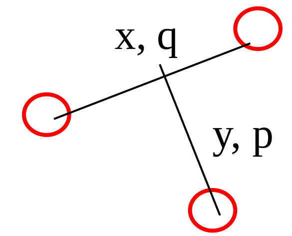
Macek, JPB 1 (1968)

Hard (or impossible) to determine the asymptotic Coulomb wave functions in ρ

(II) Faddeev method

Ishikawa, PRC87 (2013)

Jacobi coordinate



Transition amplitude for triple-α reaction

$$F^{(\mathrm{B})}(q, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \langle \Psi_b | H_\gamma | \boldsymbol{q}, \boldsymbol{p} \rangle^{(+)}$$

$$E = \frac{\hbar^2}{m_\alpha} q^2 + \frac{3\hbar^2}{4m_\alpha} p^2$$

Asymptotic form of the wave function for the inverse process

$$|\Psi\rangle = \frac{1}{E + \imath\epsilon - H_{3\alpha}} H_\gamma |\Psi_b\rangle$$

$$\Psi(\boldsymbol{x}, \boldsymbol{y}) \underset{\substack{x \to \infty \\ y/x \text{ fixed}}}{\longrightarrow} \frac{e^{\imath(K_0 + \mathcal{O}(R^{-1}))R}}{R^{5/2}} F^{(\mathrm{B})*}(q, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$$

$$K_0 = \sqrt{\frac{m_\alpha}{\hbar^2} E}$$

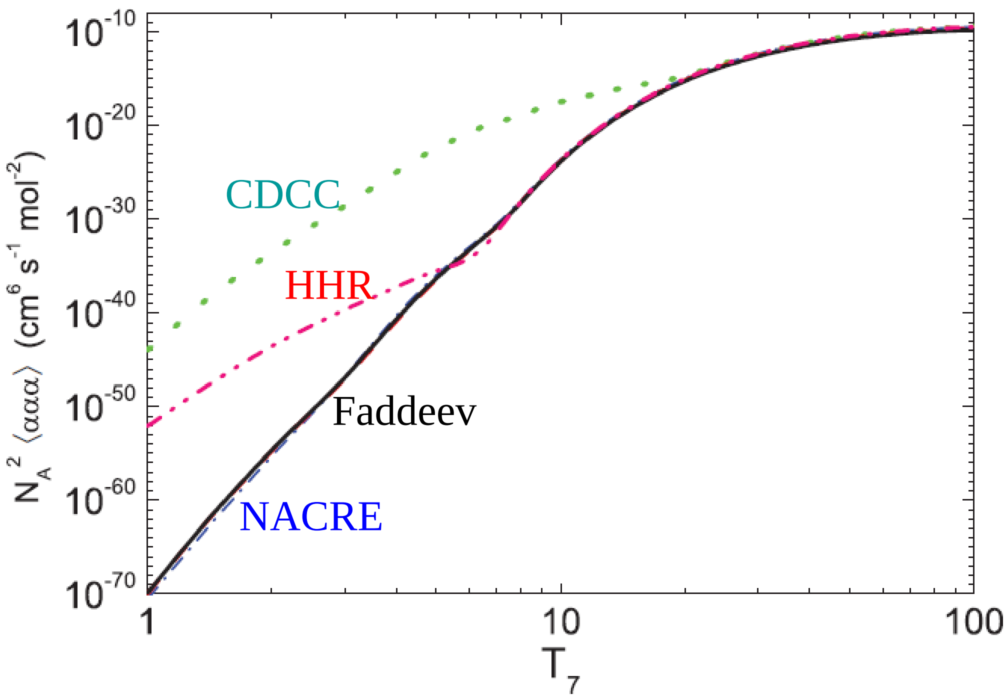
Faddeev method to solve 3-body equation

$$(E - H_{3\alpha}) |\Psi\rangle = H_\gamma |\Psi_b\rangle$$

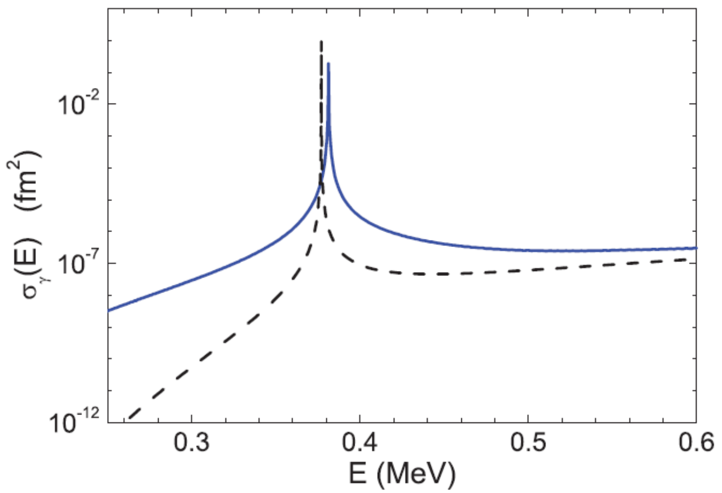
Strength function

$$\int_0^{K_0} dq\, q^2 p |F^{(\mathrm{B})}(q, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})|^2$$

Components Ψ_0 satisfying $(E - H_{3\alpha})\Psi_0 = 0$ are excluded from Ψ?



10 times NACRE rate at T=0.01 GK



Width of the Hoyle state estimated from the Breit-Wigner fit to σ_γ

	$\Gamma_{3\alpha}$ (eV)
Faddeev	9.1
CDCC	126

(III) Adiabatic hyperspherical method

Under way with H. Suno (Riken at Kobe)

Esry et al. PRA54 (1996)

Hyperradius R Hyperangles $\Omega = (\theta, \phi, \text{Euler angles})$

Smith-Whitten

$$\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial R^2} + \frac{\Lambda^2}{2\mu R^2} + \frac{15\hbar^2}{8\mu R^2} + V(R,\theta,\varphi)\right]\psi = E\psi$$

Channel functions and adiabatic potentials

Stecher, Greene, PRA80 (2009)

$$\left[\frac{\Lambda^2}{2\mu R^2} + \frac{15\hbar^2}{8\mu R^2} + V(R,\theta,\varphi)\right]\Phi_\nu(R;\Omega) = U_\nu(R)\Phi_\nu(R;\Omega)$$

With use of $\psi_n(R,\Omega) = \sum_{\nu=0}^{\infty} F_{\nu n}(R)\Phi_\nu(R;\Omega)$, a coupled equation for F is given

Upper and lower bounds to the ground state energy

Starace, Webster, PRA19 (1979)

Complex absorbing potential (CAP) to localize continuum

$$U_\nu(R) \to U_\nu(R) - iW(R)$$

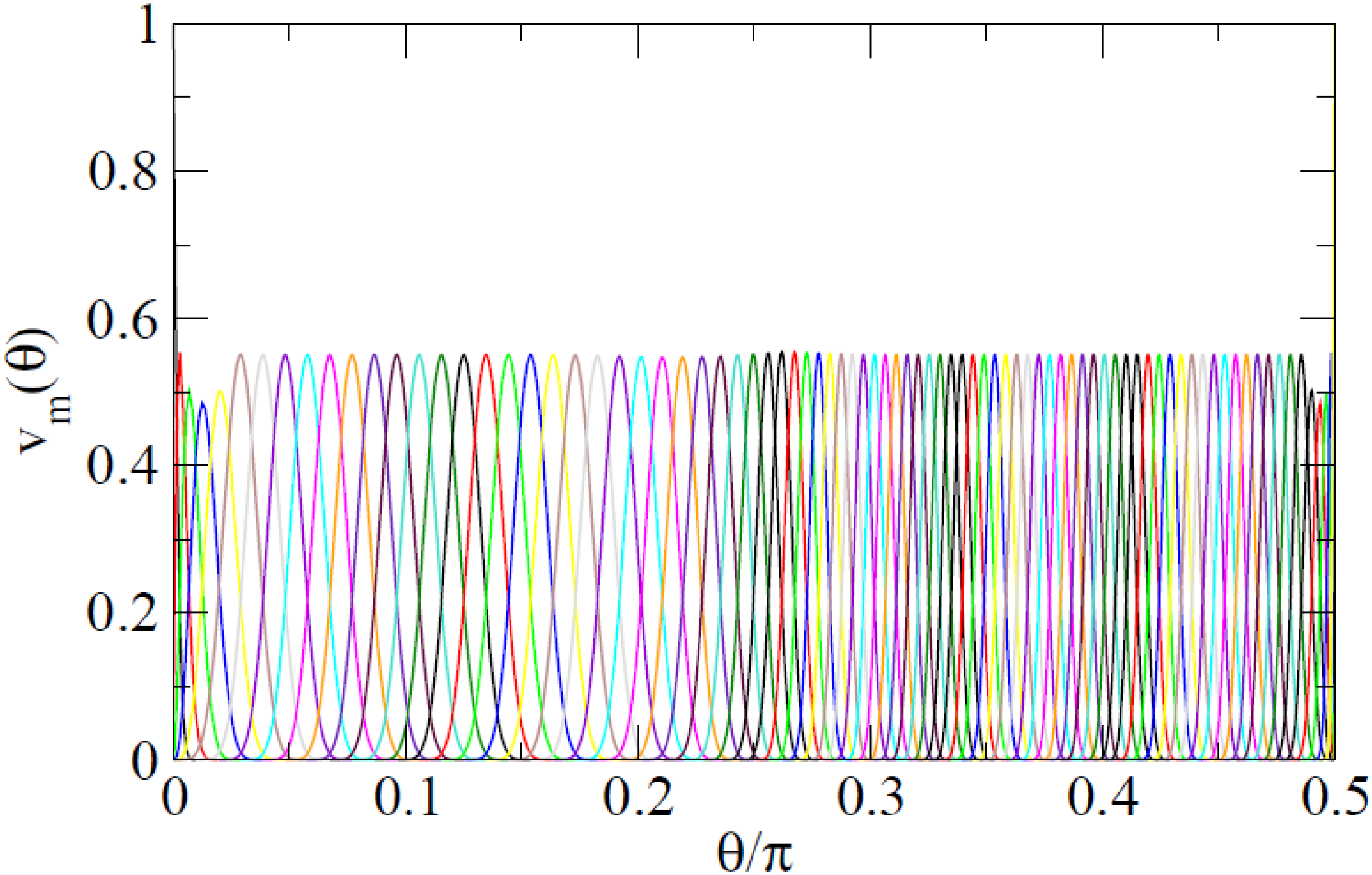
$$E_n = E_n^r - i\Gamma_n/2$$

Transmission-free CAP
Manolopoulos, JCP117(2002)

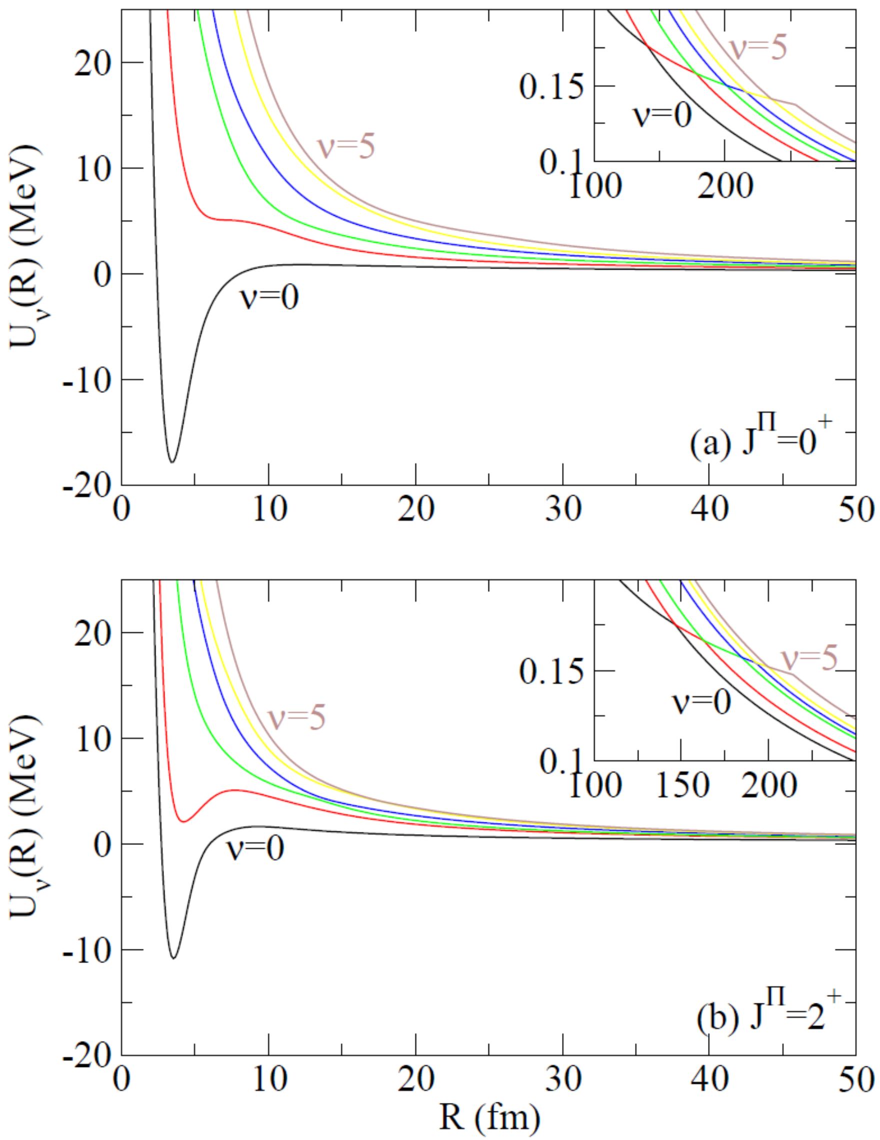
Symmetry of 3α $\theta \in [0, \pi/2]$ $\varphi \in [0, \pi/3]$

Basis splines $N_\theta = 80\,\mathrm{mesh}$ $N_\varphi = 40\,\mathrm{mesh}$

$R_{\mathrm{max}} \approx 10,000\,\mathrm{fm}$



αα short-ranged nuclear and Coulomb potentials + 3α hyperscalar potential

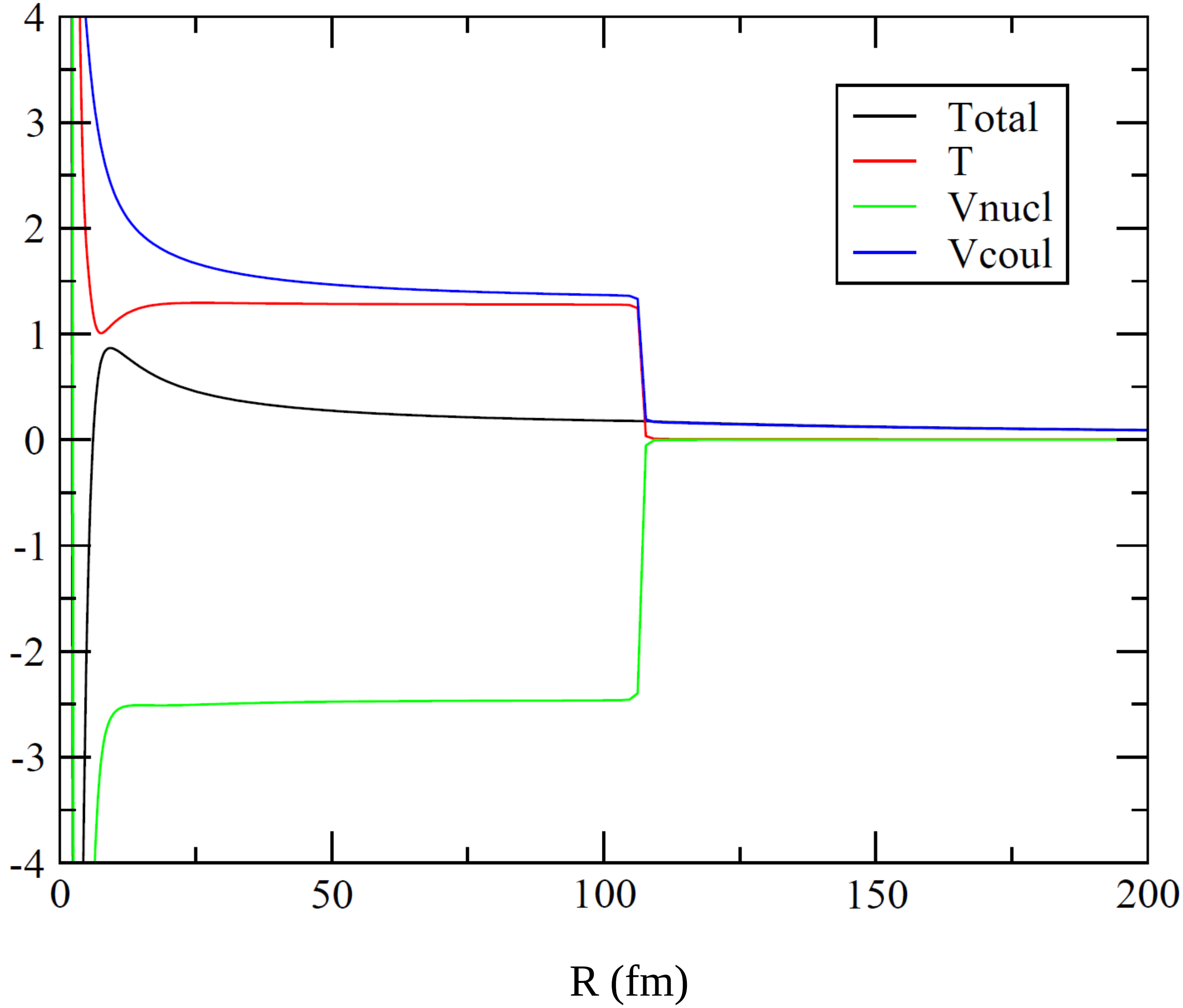


Avoided crossings with ^{8}Be+α occur at R > 100 fm

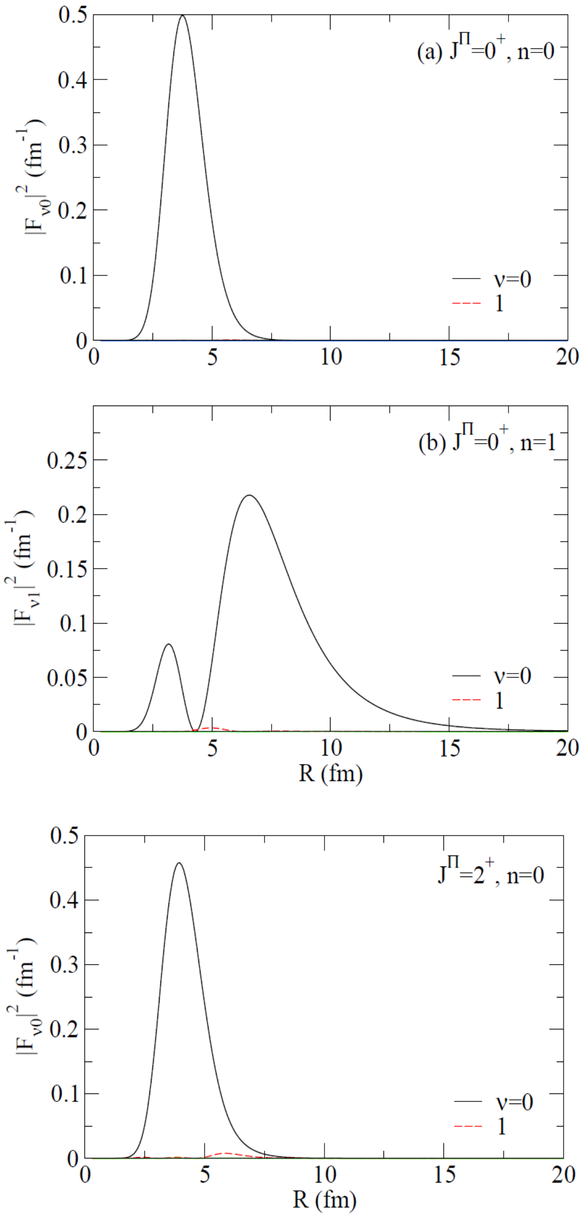
3α threshold

$$R^2 = \sqrt{3}\sum_{i=1}^{3}(\boldsymbol{r}_i - \boldsymbol{R}_{\rm cm})^2$$

Contributions to $U_{\nu=0}(R)$



Hyperradial components



	$J^\Pi = 0^+$		$J^\Pi = 2^+$
	$n = 0$	$n = 1$	$n = 0$
BO	-9.415	$0.255 - i1.3 \times 10^{-8}$	-3.096
Adiabatic	-9.201	$0.486 - i2.1 \times 10^{-5}$	-2.521
$\nu_{\max} = 1$	-9.293	$0.383 - i1.3 \times 10^{-5}$	-2.861
$\nu_{\max} = 2$	-9.299	$0.371 - i7.7 \times 10^{-6}$	-2.876
$\nu_{\max} = 3$	-9.301	$0.368 - i6.9 \times 10^{-6}$	-2.884
$\nu_{\max} = 4$	-9.301	$0.367 - i6.6 \times 10^{-6}$	-2.888
$\nu_{\max} = 5$		$0.367 - i6.5 \times 10^{-6}$	-2.890
$\nu_{\max} = 6$		$0.367 - i6.5 \times 10^{-6}$	-2.890

Hoyle state

$E_r = 0.38\,\mathrm{MeV},\ \Gamma_r = 8.3 \pm 1\,\mathrm{eV}$

Cf. Asymptotics of U(R)
Fedorov, Jensen PLB389 (1996)
^{8}Be+α approximation to U(R)
Fedotov et al. PRC70 (2004)
Complex scaling method
Rodriguez et al. EPJA31 (2007)

Calculation of E2 strength function

Continuum sum replaced by Green's function

$$S(E) = \frac{1}{5}\sum_M \sum_n |<\Psi_n|W_{2\,-M}|\Phi_{2M}>|^2 \delta(E_n - E)$$

$$= -\frac{1}{5}\mathrm{Im} < [W_2\Phi_2]_{00}|\mathcal{G}(E+i\epsilon)|[W_2\Phi_2]_{00} >$$

$$\mathcal{G}(E+i\epsilon) = \frac{1}{E+i\epsilon - H} \qquad [W_2\Phi_2]_{00} = \sum_M < 2M\ 2\ -M|00 > W_{2M}\Phi_{2\,-M}$$

$$\Psi = \mathcal{G}(E+i\epsilon)[W_2\Phi_2]_{00}$$

Driven equation of motion for **Ψ** $\quad (H-E)\Psi = -[W_2\Phi_2]_{00}$

$$\Psi = \sum_\nu \rho^{-\Lambda} f_\nu(\rho)\Phi_\nu(\rho,\Omega) \qquad [W_2\Phi_2]_{00} = \sum_\nu \rho^{-\Lambda} w_\nu(\rho)\Phi_\nu(\rho,\Omega)$$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{d\rho^2} + U_\nu(\rho) - E\right] f_\nu(\rho) - \frac{\hbar^2}{2m}\sum_{\nu'}\left[2P_{\nu\nu'}(\rho)\frac{d}{d\rho} + Q_{\nu\nu'}(\rho)\right] f_{\nu'}(\rho) = -w_\nu(\rho)$$

$$S(E) = -\frac{1}{5}\mathrm{Im}\sum_\nu \int_0^\infty w_\nu^*(\rho) f_\nu(\rho)d\rho$$

Summary

- challenge to the triple-alpha reaction at very low energies
- some controversial results in theory
- adiabatic hyperspherical approach, complex absorbing potential
  - description of both resonant and nonresonant continua
  - calculation of U(R) at large distances
  - narrow width of Hoyle resonance

- calculate the reaction rate soon