Contacts for Identical Bosons near Unitarity

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Phys. Rev. Lett. 112, 110402

INT, April 9 2014, University of Washington

Outline

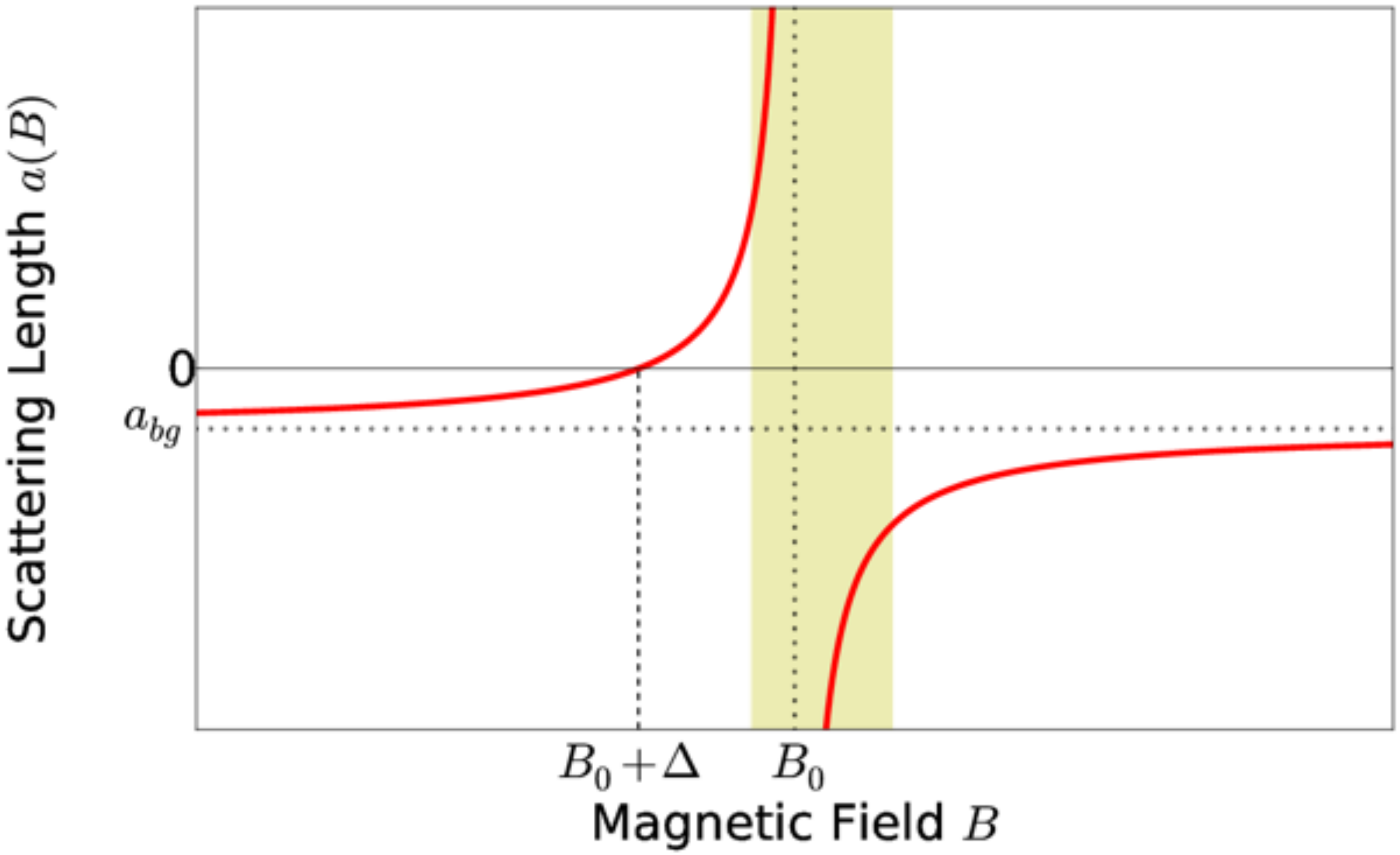
- Introduction
- Universal relations from Effective Field Theory
- JILA experiment
- Determining the contacts
- Other probes of the contacts
- Conclusions

Introduction:

- Unitarity
- Universal relations for 2-component Fermi gas
- Universal relations for a Bose gas

Introduction:
Unitarity

- Largest cross sections allowed by quantum mechanics
- $a \rightarrow \infty$: unitarity
- Large, finite a: universal regime
- S-wave scattering length, a, can be controlled with Feshbach resonance.
- At unitarity, T and n are only remaining scales



Introduction:
Universal relations: Fermi gas with 2 spin states

- Few-body physics controls aspects of many-body physics
- Tan's Contact, C, for fermions with 2 spin states:
  - Gives the dependence of the energy on the scattering length:

$$\frac{dE}{da} = \frac{C}{4\pi a^2}$$

  - Gives the large-momentum tail of the momentum distribution:

$$k^4 n(k) \rightarrow C$$

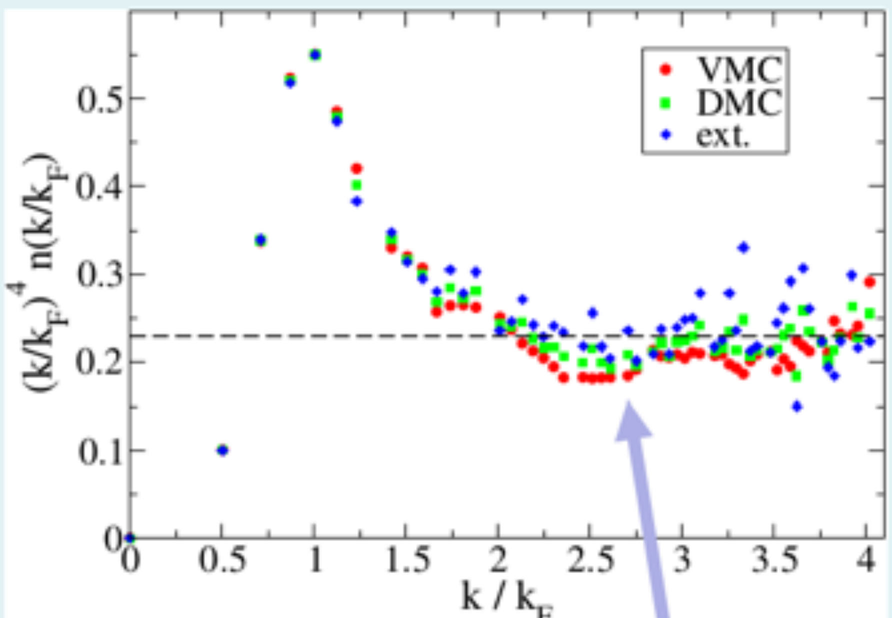
  - There are many more relations involving the contact!
    - Virial theorem, equation of state, etc. Tan, Ann. Phys. **323,** (2008)
- Note: the Fermi gas is stable at unitarity.

Introduction:
Universal relations: Fermi gas with 2 spin states

The tail of the momentum distribution is determined by the contact:

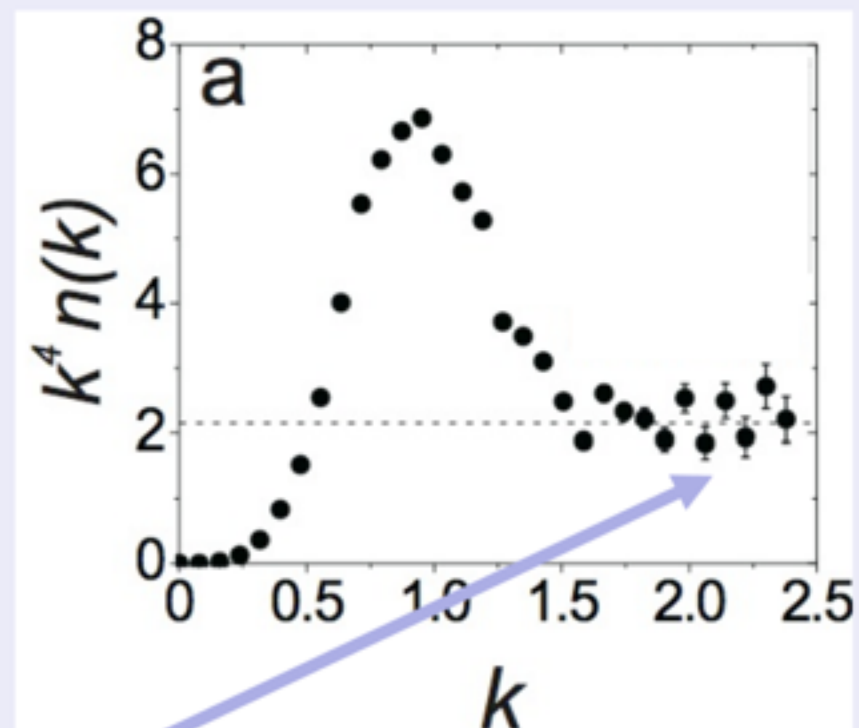
$$k^4 n(k) \rightarrow C$$

Quantum Monte Carlo



Gandolfi, Schmidt, Carlson [PRA 2011]

Experiment



JILA group [PRL 2010]

scaled by Fermi momentum

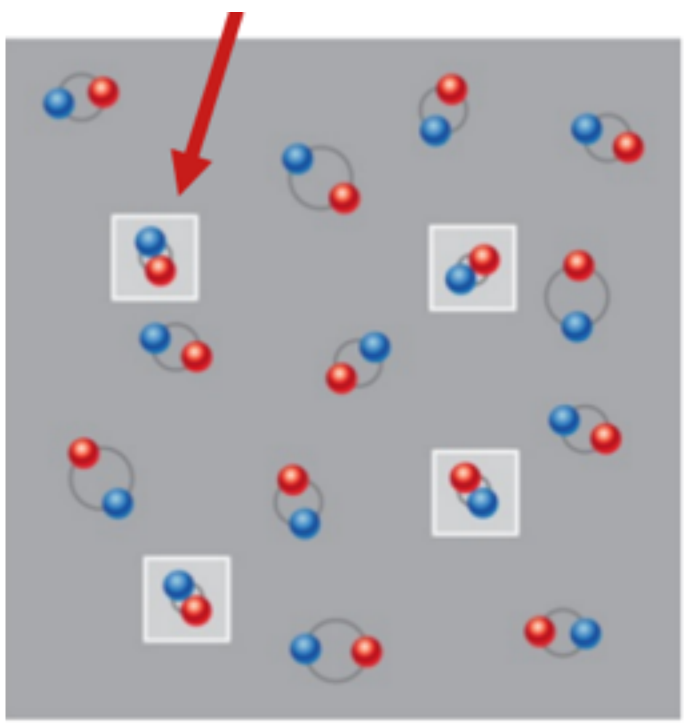
$k_F = (3\pi^2 \langle n \rangle)^{1/3}$

Plateau (1/k^4 tail) above 2 k$_F$!

Introduction:
Universal relations: Fermi gas with 2 spin states

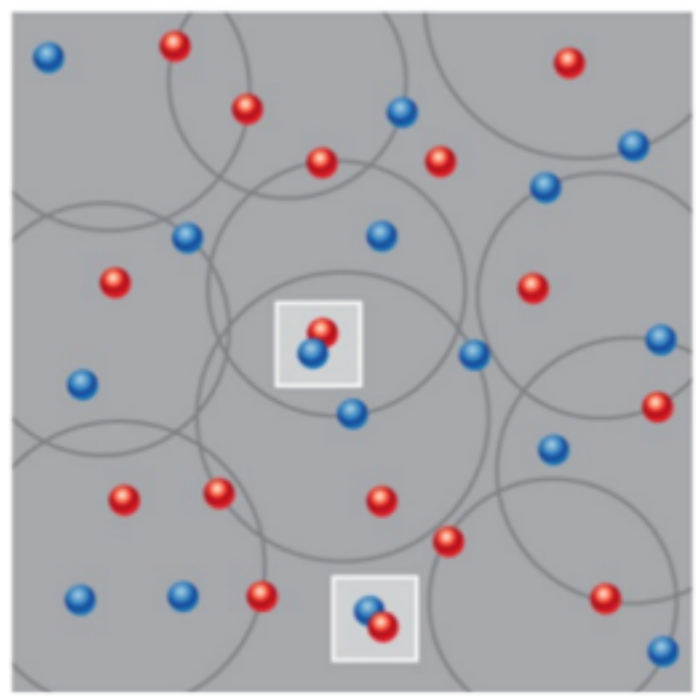
Tan's Contact:

- An extensive, thermodynamic variable conjugate to $1/a$
- Measures how likely it is for two atoms to be close together
- For homogeneous Fermi gas at zero temperature:



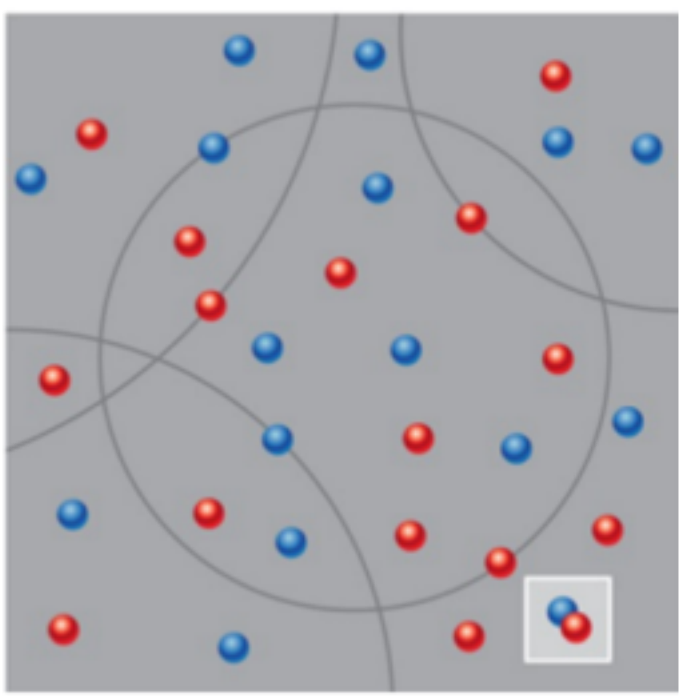
BEC limit (a>0)

$$8\pi/a \times n/2$$



unitary limit (a→ $\pm\infty$)

$$10.51(3)\, n^{4/3}$$



BCS limit (a<0)

$$4\pi^2 a^2 n^2$$

Introduction:
Universal relations: Bose gas

- Two- and three-body contacts:

$$C_2 = \int d^3r\, \mathcal{C}_2 \qquad\qquad C_3 = \int d^3r\, \mathcal{C}_3$$

- The large-momentum tail becomes:

$$k^4 n(k) \to C_2 + \frac{A \sin(2s_0 \ln(k/\kappa_*) + \phi)}{k} C_3$$

- A, s_0, and ϕ are known universal numbers:

  $s_0 = 1.00624$ [1]

  $A = 89.2626$ [2]

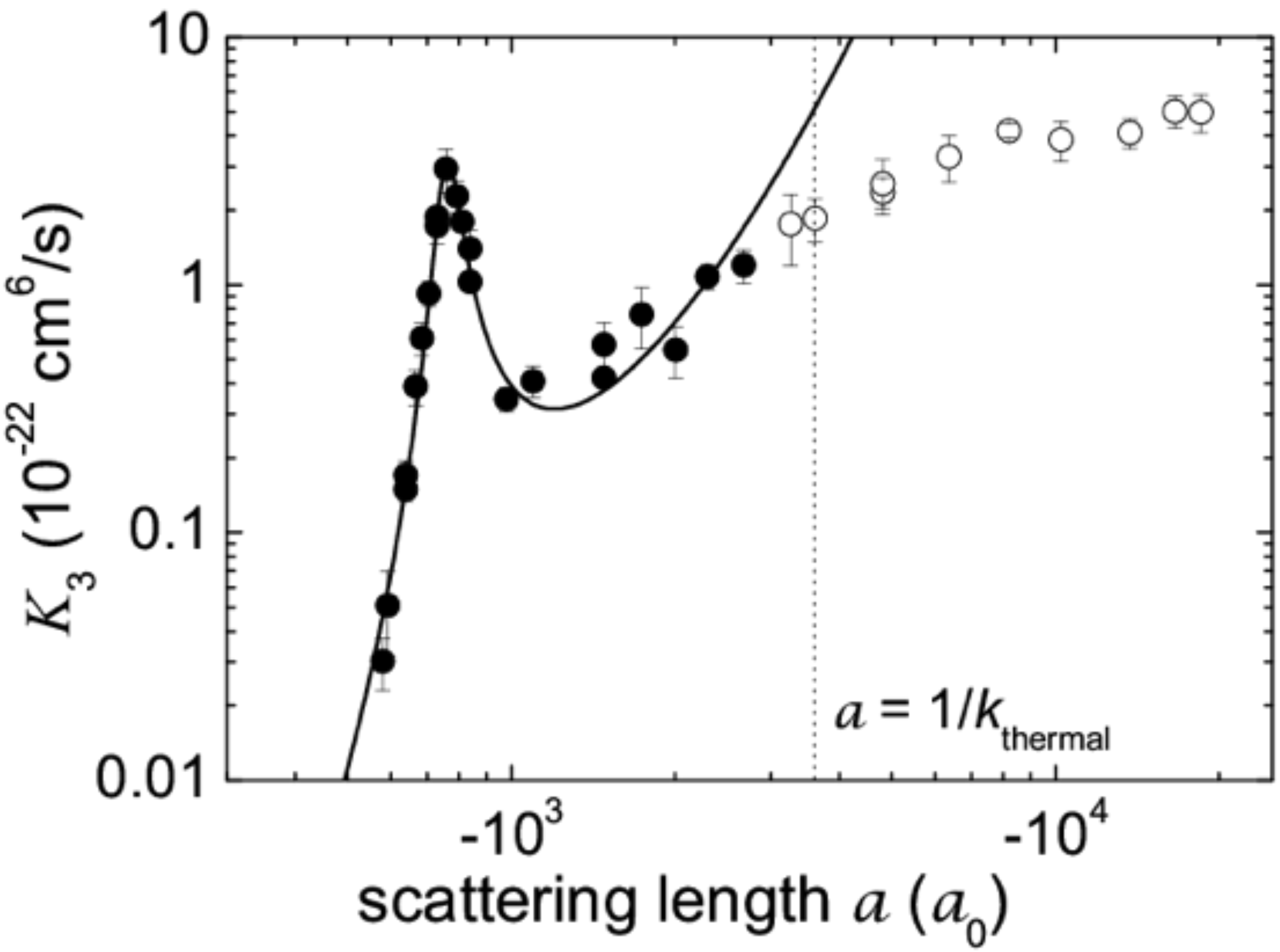
  $\phi = -1.33813$ [2]

- κ_* can be determined experimentally

[1] Efimov Phys. Lett. **33B** 563 (1970)
[2] Braaten, Kang, and Platter PRL **106** 153005 (2011)

Introduction:
Determining κ_* from experiment

Measure Efimov resonance in the three body loss rate at scattering length a_-



- For $^{85}\mathrm{Rb}$, $a_- = -759\ a_0$ [1]
- κ_* is universally related to a_- [2]:

$$a_-\kappa_* = -1.50763$$

- For $^{85}\mathrm{Rb}$, $\kappa_* = 1/(503\ a_0)$
- Note: simple extrapolation of the loss rate to unitarity suggests that the Bose gas is unstable at unitarity.

[1] JILA, PRL **108,** 145305 (2012)
[2] Gogolin, Mora, and Egger, PRL **100,** 140404 (2008)

Introduction:
Universal relations: Bose gas

- At unitarity with T=0, n is the only dimensionful scale in the system (assuming log-periodic dependence on κ_* is weak).
- By dimensional analysis,

$$\mathcal{C}_2 = \alpha n^{4/3} \implies C_2 = \alpha N \langle n^{1/3} \rangle$$
$$\mathcal{C}_3 = \beta n^{5/3} \implies C_3 = \beta N \langle n^{2/3} \rangle$$

- α and β are universal numbers

Effective Field Theory:

- Interaction hamiltonian
- Two-body scattering amplitude
- Two- and three-body contacts
- Tail of the Momentum Distribution

Effective Field Theory:
2- and 3-body interactions

The large scattering length zero-range limit can be described by a quantum field theory with

$$\mathcal{H}_{\rm int} = \frac{g_2}{4m}\psi^\dagger\psi^\dagger\psi\psi + \frac{g_3}{36m}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi$$

In terms of the S-wave scattering length:

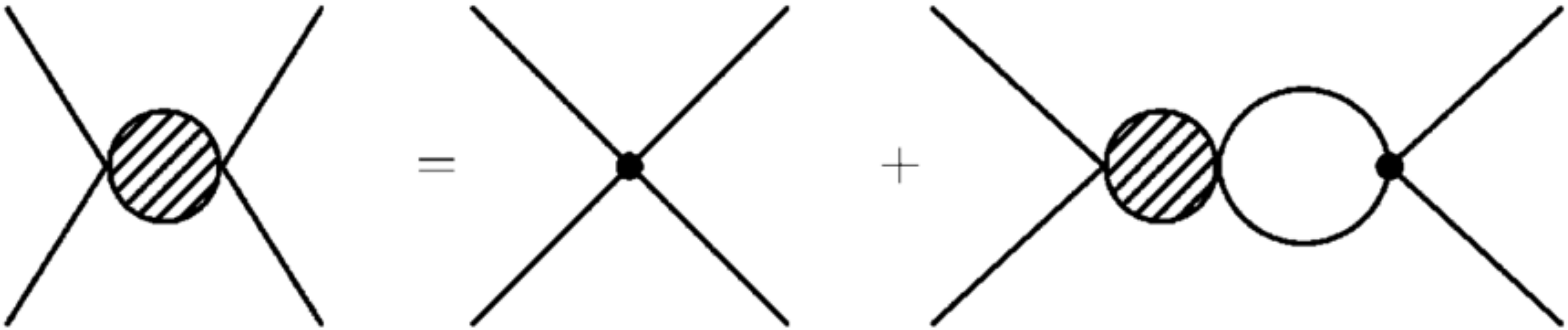
$$g_2 = 8\pi/(1/a - 2\Lambda/\pi)$$

$$g_3 = -\frac{9g_2^2 H}{\Lambda^2}$$

- Λ is an explicit momentum cutoff
- H is a log-periodic function of the cutoff

Effective Field Theory:
2-body scattering amplitude

The 2-body scattering amplitude can be derived by solving the integral equation:



$$\mathcal{A}(E) = \frac{8\pi/m}{-1/a + \sqrt{-mE - i\epsilon}}$$

Effective Field Theory:
The Contacts

Use the adiabatic relations to define the 2-body contact operator (assume there is no 3-body interaction):

$$\left(a\frac{\partial\mathcal{H}}{\partial a}\right)_{\kappa_*} = \frac{1}{8\pi a m}\tfrac{1}{4}g_2^2\psi^\dagger\psi^\dagger\psi\psi$$

$$\Longrightarrow \quad \left(a\frac{\partial E}{\partial a}\right)_{\kappa_*} = \frac{1}{8\pi a m}\int \mathrm{d}^3 R\langle\tfrac{1}{4}g_2^2\psi^\dagger\psi^\dagger\psi\psi\rangle = \frac{C_2}{8\pi a m}$$

Similarly, for 3-body contact, consider

$$\left(\kappa_*\frac{\partial\mathcal{H}}{\partial\kappa_*}\right)_a$$

Effective Field Theory:
The Contacts

The contact densities are

$$\mathcal{C}_2 = \frac{g_2^2}{4}\langle\psi^\dagger\psi^\dagger\psi\psi\rangle - \frac{g_2^3}{2}\frac{H(\log(\Lambda))}{\Lambda}\langle\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi\rangle$$

$$\mathcal{C}_3 = -\frac{g_2^2}{8}\frac{H'(\log(\Lambda))}{\Lambda^2}\langle\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi\rangle$$

Where:

- Λ is an explicit momentum cutoff
- H is a log-periodic function of the cutoff
- H' is the derivative of H with respect to $\log(\Lambda)$

Note:

- The expectation values have cutoff dependence
- The contact densities are independent of the cutoff
- The contact densities are state-dependent

Effective Field Theory:
Tail of the Momentum Distribution

The momentum distribution for the bosons is

$$n(\mathbf{k}) = \int \mathrm{d}^3 R \int \mathrm{d}^3 r e^{-i\mathbf{k}\cdot\mathbf{r}} \langle \psi^\dagger(\mathbf{R} + \tfrac{1}{2}\mathbf{r}) \psi(\mathbf{R} - \tfrac{1}{2}\mathbf{r}) \rangle$$

We can use the OPE to expand $\psi^\dagger(\mathbf{R} + \frac{1}{2}\mathbf{r}) \psi(\mathbf{R} - \frac{1}{2}\mathbf{r})$:

$$\psi^\dagger(\mathbf{R} + \tfrac{1}{2}\mathbf{r}) \psi(\mathbf{R} - \tfrac{1}{2}\mathbf{r}) = \sum_n c_n(\mathbf{r}) \mathcal{O}_n(\mathbf{R})$$

- Some of the coefficients are non-analytic at $\mathbf{r}=0$.
- These can give power law tails in $n(\mathbf{k})$.

Aside: Operator Product Expansion

Express product of local operators at nearby points as a series of local operators:

$$\mathcal{O}_A(\mathbf{R} + \tfrac{1}{2}\mathbf{r})\ \mathcal{O}_B(\mathbf{R} - \tfrac{1}{2}\mathbf{r}) = \sum_n c_n(\mathbf{r})\mathcal{O}_n(\mathbf{R})$$

- Expansion coefficients of lowest dimension operators can be determined by matching expectation values in few-body states
- The coefficients are the same for any state of the system
- In many cases, the series can be truncated after only a few terms
- Construct operators $\mathcal{O}_n(\mathbf{R})$ from fields and gradients of fields
- The set of operators includes those from Taylor expanding the left hand side and operators resulting from quantum fluctuations

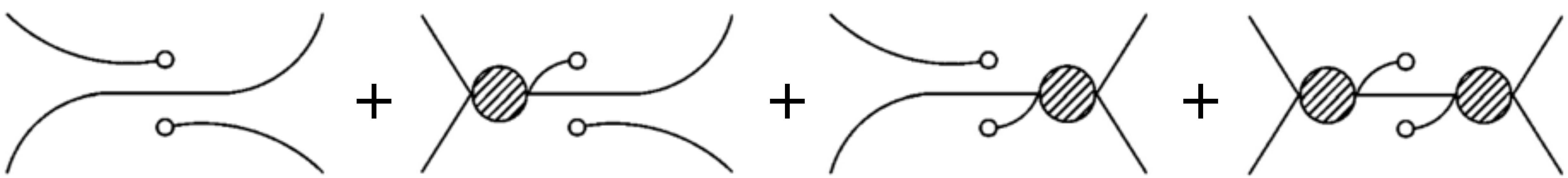
Effective Field Theory:
Tail of the Momentum Distribution

Assume our system is homogeneous:

$$\langle \psi^\dagger(\tfrac{1}{2}\mathbf{r})\psi(-\tfrac{1}{2}\mathbf{r})\rangle = \sum_n c_n(\mathbf{r})\langle \mathcal{O}_n(0)\rangle$$

Calculate LHS in 2-body scattering state:

$$\langle \psi^\dagger(\tfrac{1}{2}\mathbf{r})\psi(-\tfrac{1}{2}\mathbf{r})\rangle =$$



$$= \frac{im^2\mathcal{A}^2(p^2/m)}{8\pi p}e^{ipr}$$

Effective Field Theory:
Tail of the Momentum Distribution

Now consider RHS of

$$\langle \psi^\dagger(\tfrac{1}{2}\mathbf{r})\psi(-\tfrac{1}{2}\mathbf{r})\rangle = \sum_n c_n(\mathbf{r})\langle \mathcal{O}_n(0)\rangle$$

Calculate expectation values of relevant local operators in 2-body scattering states:

$$\langle \psi^\dagger \psi(0)\rangle = \frac{im^2 \mathcal{A}^2(p^2/m)}{8\pi p}$$

This matches the r^0 term from the LHS. So, the expansion coefficient for $\langle \psi^\dagger \psi(0)\rangle$ is just 1. This term does not contribute at large momenta.

Effective Field Theory:
Tail of the Momentum Distribution

Now consider RHS of

$$\langle \psi^\dagger(\tfrac{1}{2}\mathbf{r})\psi(-\tfrac{1}{2}\mathbf{r})\rangle = \sum_n c_n(\mathbf{r})\langle \mathcal{O}_n(0)\rangle$$

Calculate expectation values of relevant local operators in 2-body scattering states:

$$\langle \tfrac{1}{4}g_2^2\psi^\dagger\psi^\dagger\psi\psi(0)\rangle = m^2\mathcal{A}^2(p^2/m)$$

This matches the p-dependence of the r^1 term from the LHS. So, the expansion coefficient of $\langle \tfrac{1}{4}g_2^2\psi^\dagger\psi^\dagger\psi\psi(0)\rangle$ is

$$-\frac{r}{8\pi} \quad \Longrightarrow$$

This term can contribute at large momenta!

Effective Field Theory:
Tail of the Momentum Distribution

Putting these together:

$$\psi^\dagger(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\psi(\mathbf{R}-\tfrac{1}{2}\mathbf{r}) = \psi^\dagger\psi(\mathbf{R}) + \tfrac{1}{2}\mathbf{r}\cdot[\psi^\dagger\nabla\psi(\mathbf{R}) - \nabla\psi^\dagger\psi(\mathbf{R})] + \cdots$$
$$- \frac{r}{8\pi}\,\tfrac{1}{4}g_2^2\psi^\dagger\psi^\dagger\psi\psi(\mathbf{R}) + \cdots$$

Insert into

$$n(\mathbf{k}) = \int \mathrm{d}^3R \int \mathrm{d}^3r e^{-i\mathbf{k}\cdot\mathbf{r}} \langle \psi^\dagger(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\psi(\mathbf{R}-\tfrac{1}{2}\mathbf{r})\rangle$$

Only terms in the second line can contribute at large $\mathbf{k}$.

$$\int \mathrm{d}^3r \left(\frac{r}{8\pi}\right) e^{i\mathbf{k}\cdot\mathbf{r}} = -\frac{1}{k^4} \Longrightarrow \begin{aligned} n(\mathbf{k}) &\to \frac{1}{k^4}\int \mathrm{d}^3R\,\langle\tfrac{1}{4}g_2^2\,\psi^\dagger\psi^\dagger\psi\psi(\mathbf{R})\rangle + \cdots \\ &= \frac{1}{k^4}C_2 + \cdots \end{aligned}$$

Effective Field Theory:
Tail of the Momentum Distribution

Including the 3-atom interactions:

$$n(\mathbf{k}) \rightarrow \frac{1}{k^4} C_2 + \frac{A \sin[2 s_0 \ln(k/\kappa_*) + \phi]}{k^5} C_3 + \cdots$$

Just like 2-component fermions!

Note: depends on κ_*

Terms which fall faster than $1/k^5$

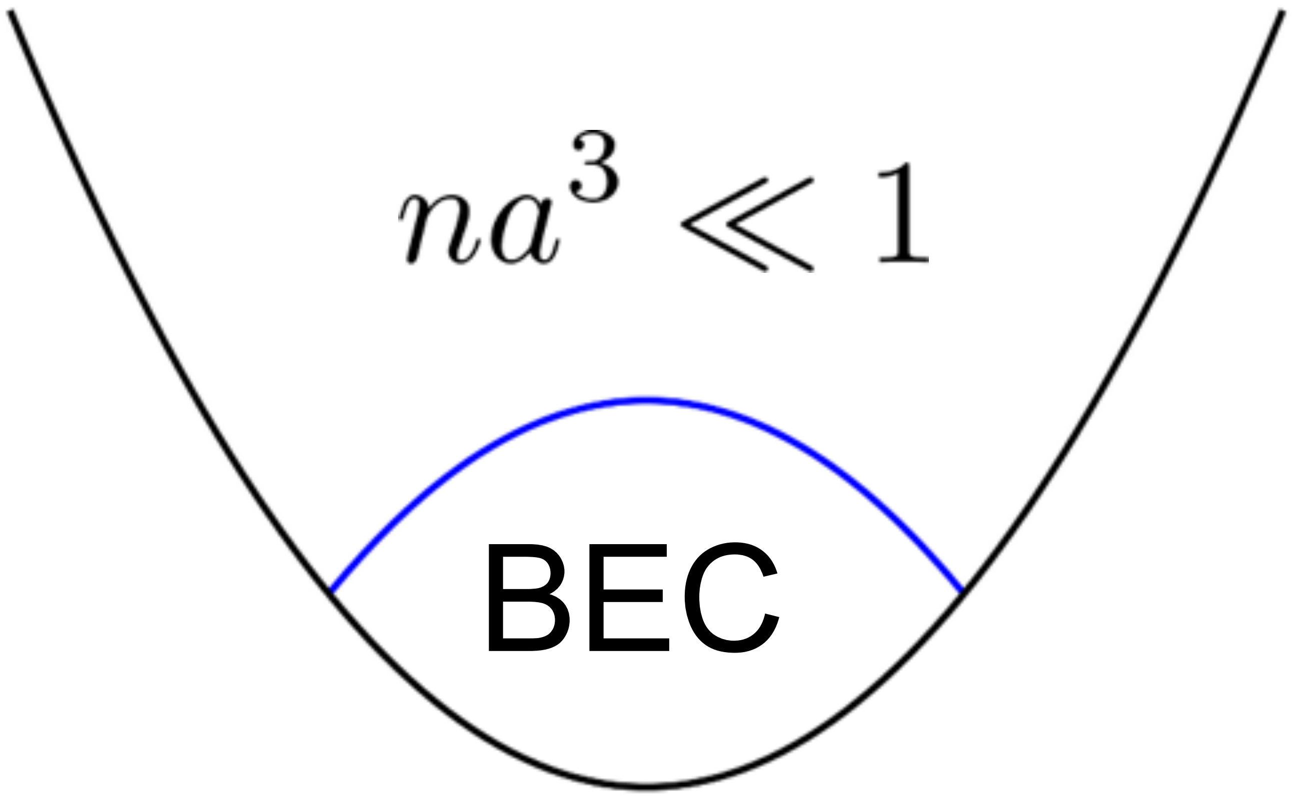
Comparing with (the) Experiment

- Description of experiment
- Loss rate measurement
- Measurement of momentum distributions
- Looking for universality
- Extracting the contacts

JILA experiment:
Description of experiment

JILA, Nature Physics 10, 116–119 (2014)

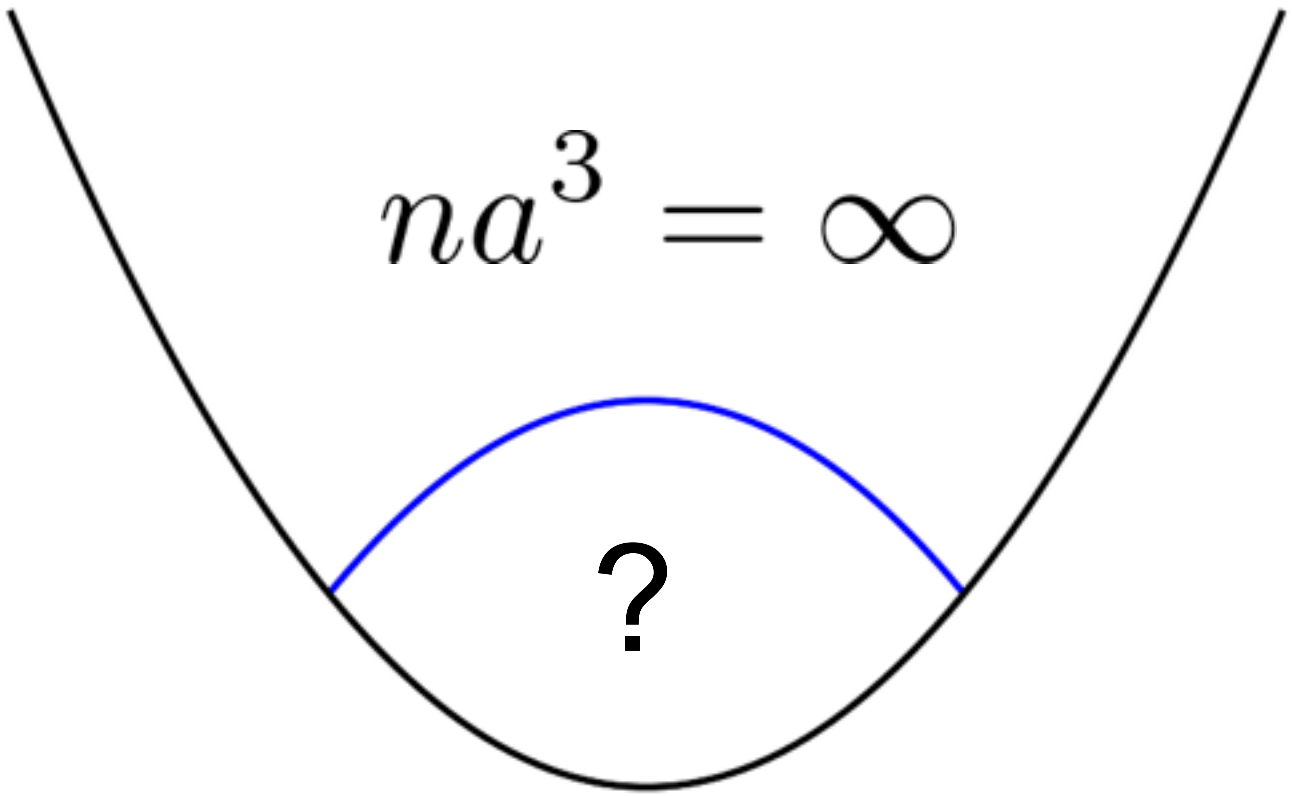
Step 1: Produce a dilute BEC of $^{85}\mathrm{Rb}$



$$\langle n \rangle = 5.5/\mu\mathrm{m}^3$$
$$(\langle n \rangle = 1.6/\mu\mathrm{m}^3)$$

JILA experiment:
Description of experiment

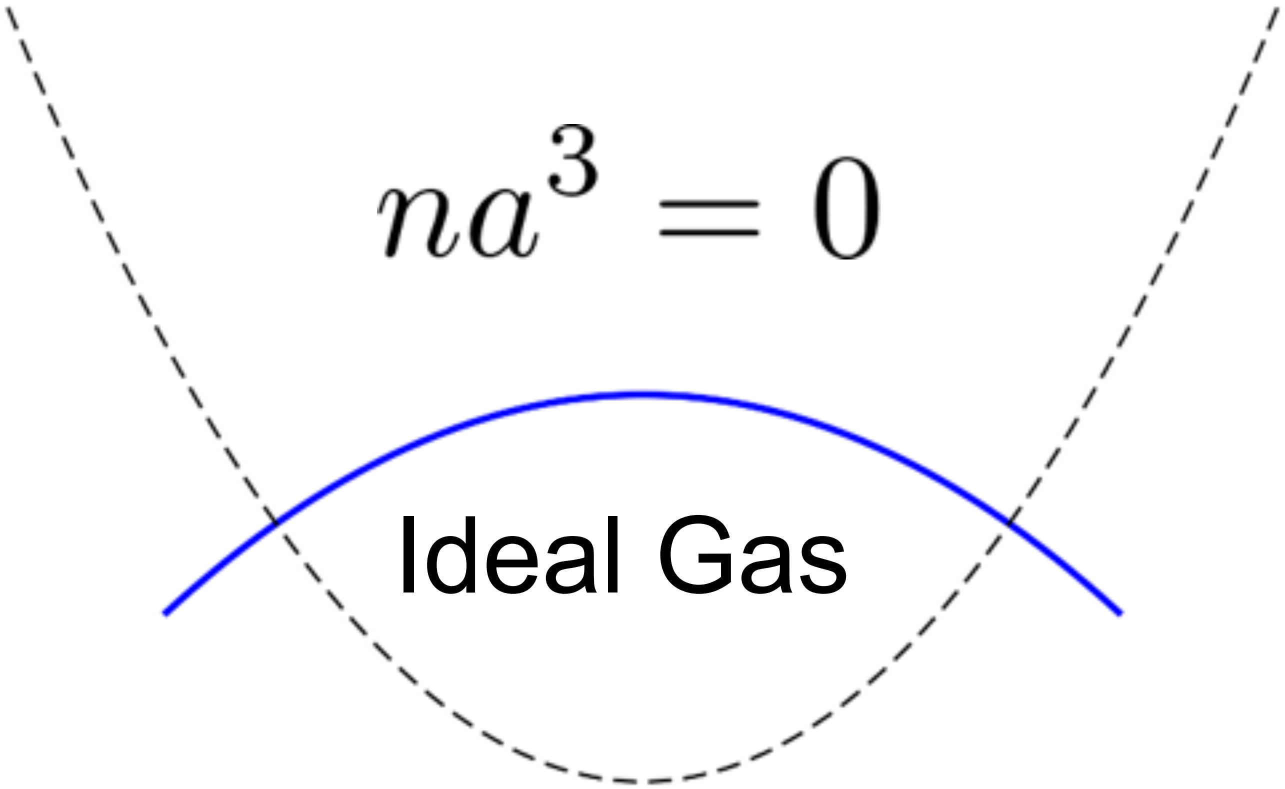
Step 2: ramp to unitarity,
then hold for a variable time



The JILA group observed that the spacial distribution didn't change dramatically after the ramp and hold.

JILA experiment:
Description of experiment

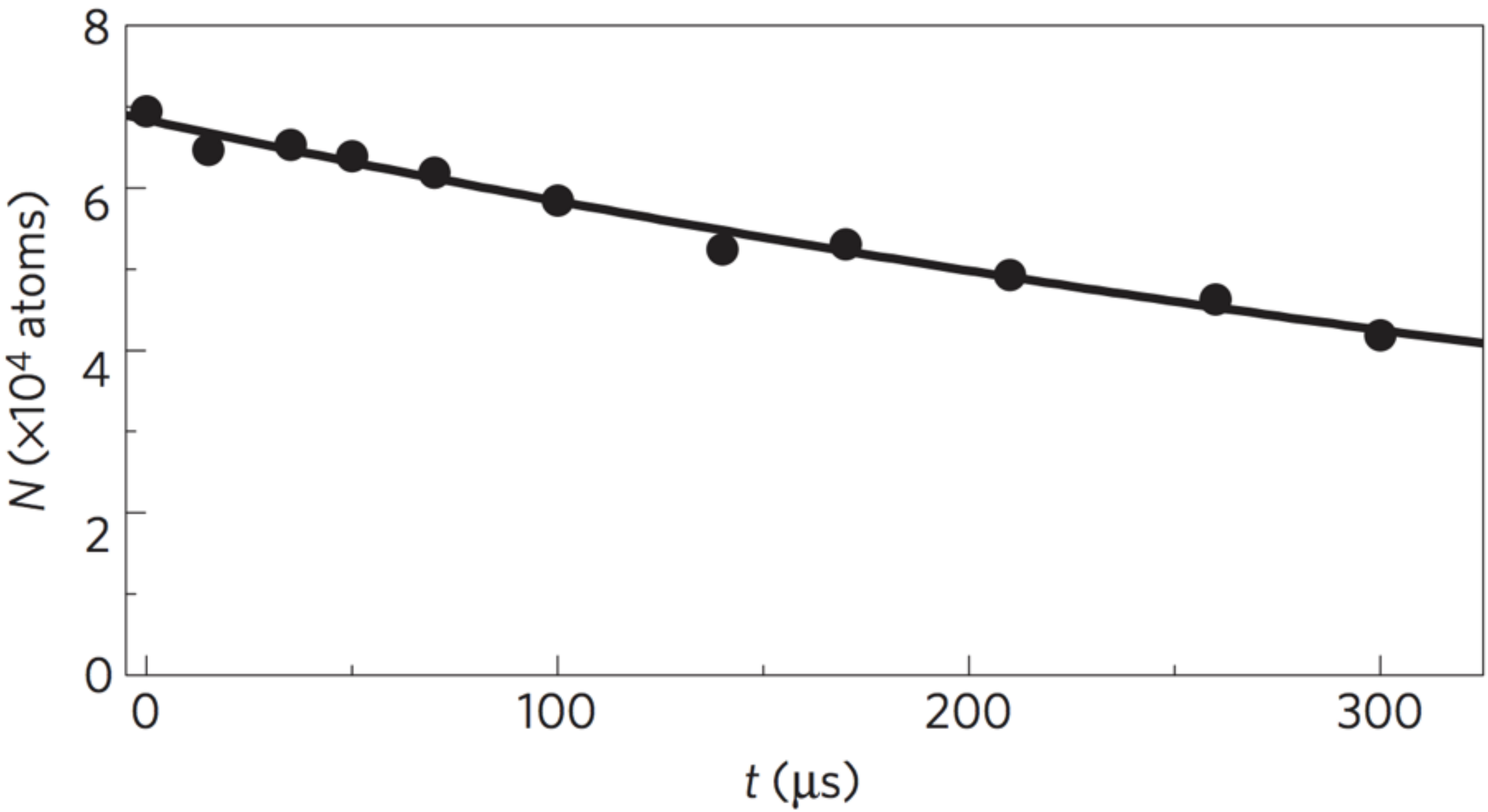
Step 3: turn off trapping potential and interactions



$$na^3 = 0$$

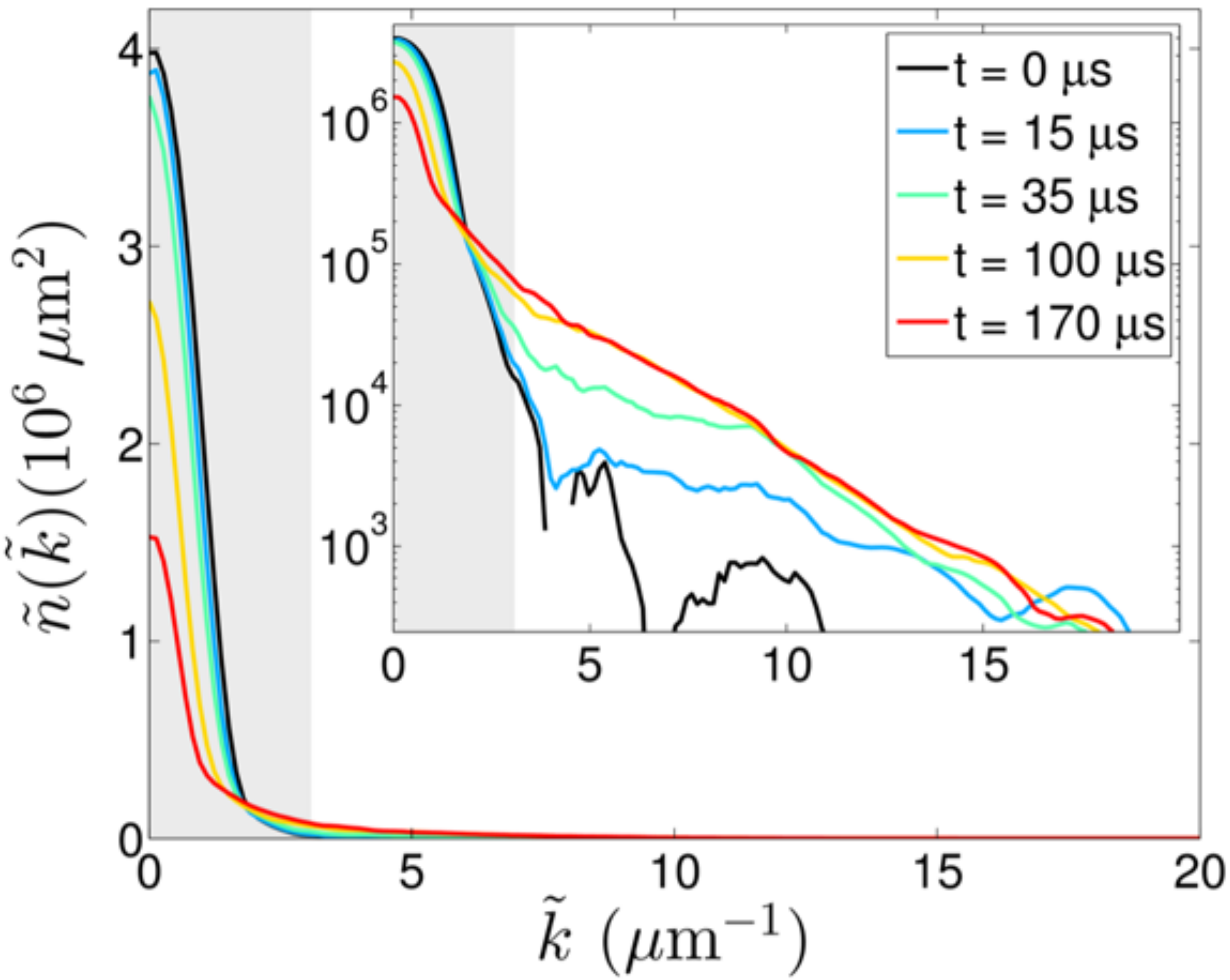
Step 4: allow the gas to expand then take a picture.

JILA experiment:
Measurement of atom loss rate



$$t_{\text{loss}} \approx 600\mu s$$

JILA experiment:
Measurement of momentum distributions

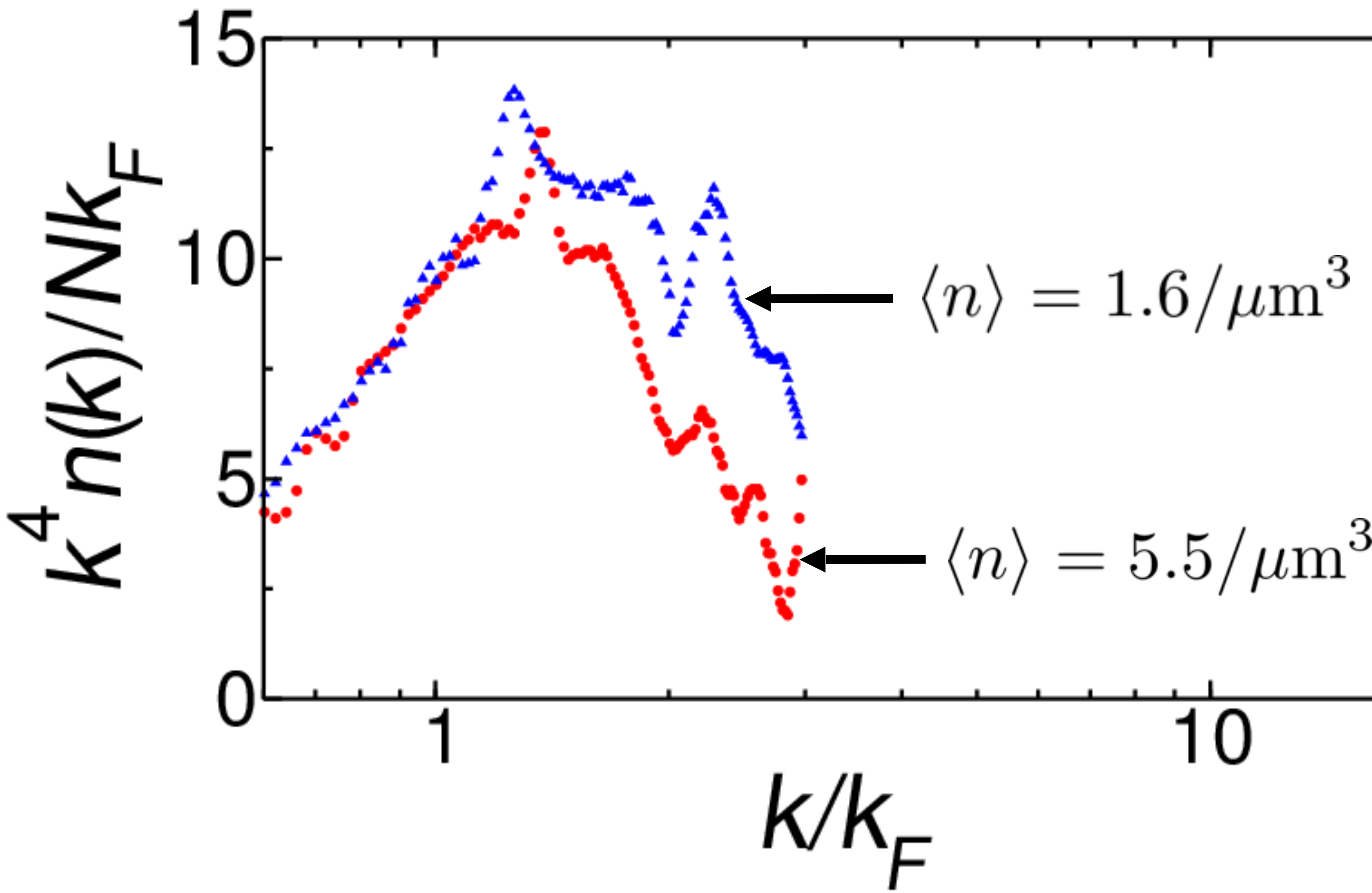


The momentum distributions saturate on a timescale

$$t_{\mathrm{sat}} \approx 100\mu s \ll t_{\mathrm{loss}}$$

JILA experiment:
Looking for universality

Scale $k^4 n(k)$ by appropriate powers of $k_F \equiv (6\pi^2 \langle n \rangle)^{1/3}$



JILA experiment:
Looking for universality

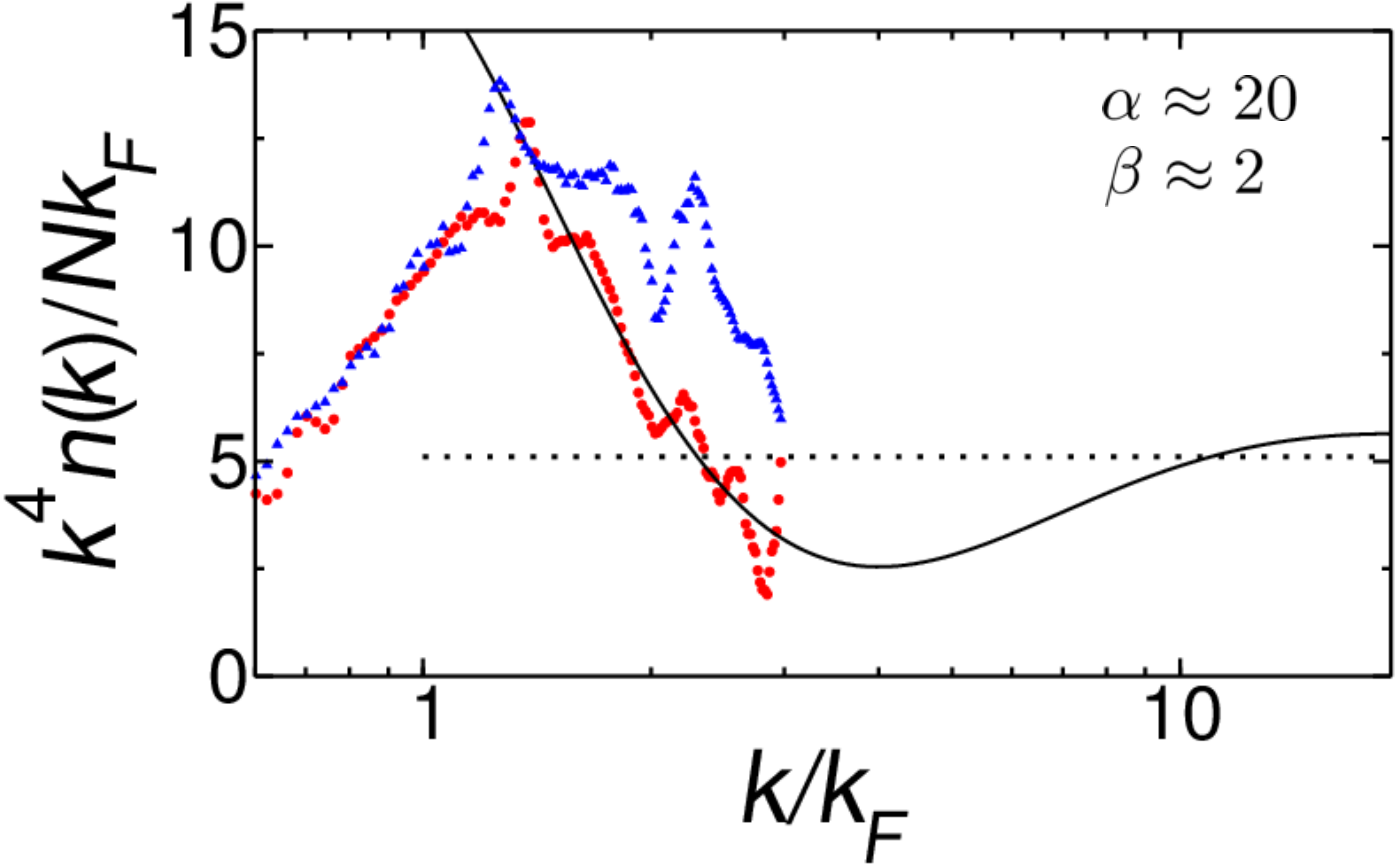
- The high-momentum tail of the scaled momentum distribution is

$$\frac{k^4 n(k)}{N k_F} \to \frac{\alpha \langle n^{1/3} \rangle}{k_F} + \frac{A \sin(2 s_0 \ln(k/\kappa_*) + \phi)}{k/k_F} \frac{\beta \langle n^{2/3} \rangle}{k_F^2}$$

- $\kappa_* = 1/(503\ a_0)$ determined from Efimov resonance
- Fit this for α and β to the tail of the measured momentum distribution with $\langle n \rangle = 5.5/\mu\mathrm{m}^3$

JILA experiment:
Extracting the contacts

Fit α and β to tail of the $\langle n \rangle = 5.5/\mu\mathrm{m}^3$ distribution:

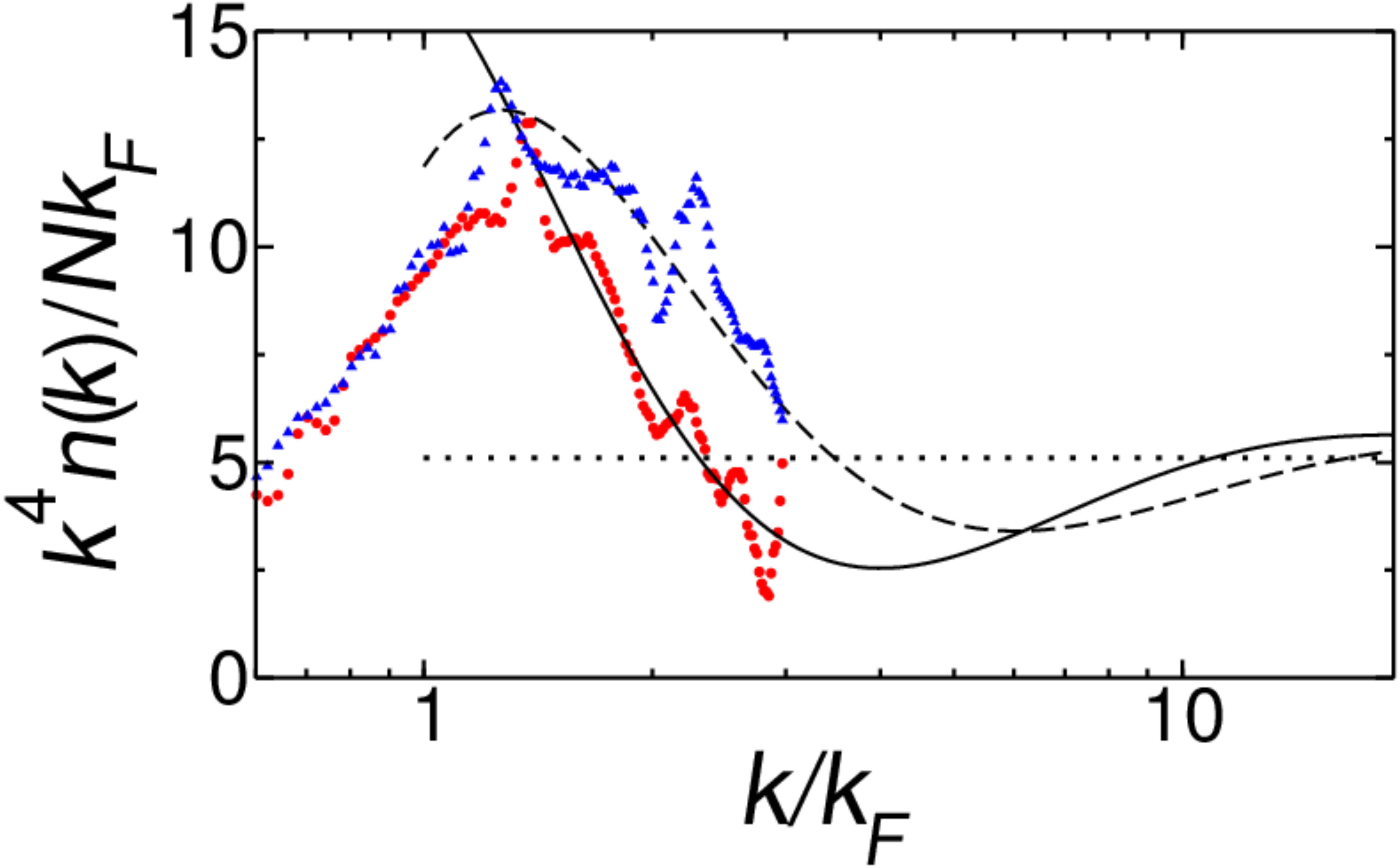


JILA experiment:
Extracting the contacts

- Noting that $\ln(k/\kappa_*) = \ln(k/k_F) + \ln(k_F/\kappa_*)$ we see that κ_* introduces a relative phase between distributions with different k_F
- Since α and β are universal numbers they should apply for the distribution with $\langle n \rangle = 1.6/\mu\mathrm{m}^3$

JILA experiment:
Extracting the contacts

Use fitted values of α and β from $\langle n \rangle = 5.5/\mu\mathrm{m}^3$ to predict momentum distribution with $\langle n \rangle = 1.6/\mu\mathrm{m}^3$:



JILA experiment:
Extracting the contacts

We find:

$$\mathcal{C}_2 = 20n^{4/3}$$

$$\mathcal{C}_3 = 2n^{5/3}$$

Calculations of $\mathcal{C}_2$:

$\mathcal{C}_2 = 10.3n^{4/3}$ Diederix, Van Heigst, and Stoof PRA 84, 033618 (2011)

$\mathcal{C}_2 = 32n^{4/3}$ Van Heugten and Stoof arXiv:1302.1792 (2013)

$\mathcal{C}_2 = 12n^{4/3}$ Skyes, Corson, D'Incao, Koller, Greene, Rey, Hazzard, and Bohn arXiv:1309.0828 (2013)

$\mathcal{C}_2 = 9.02n^{4/3}$ Rossi, Salasnich, Ancilotto, Toigo arXiv:1403.5145 (2014)

Calculations of $\mathcal{C}_3$: none!

Other Probes of the Contact

- Virial theorem
- Radio frequency spectroscopy

Other Probes of the Contact:
Virial Theorem

$$(T+U)-V=-\frac{1}{16\pi ma}C_2-\frac{1}{m}C_3$$

Determine C_2 from the slope near unitarity.

Determine C_3 from energy difference at unitarity.

Other Probes of the Contact:

Radio Frequency Spectroscopy

$$\Gamma(\omega) \longrightarrow \Omega^2 \left[\frac{1}{4\pi m^{1/3} \omega^{3/2}} C_2 + \frac{G_{\mathrm{rf}}(\omega)}{2m\omega^2} C_3 \right]$$

Rabi frequency

Log-periodic function of $\omega/m\kappa_*^2$

Conclusion

- EFT provides a convenient, powerful formalism for deriving universal relations for atomic systems.
- Measured tails of momentum distributions in the JILA experiment are consistent with logarithmic scaling violations predicted by universal relations.
- This agreement with the universal relations provides quantitative support for the claim by the JILA group that the state of matter they observed was a locally equilibrated unitary Bose gas.
- The virial theorem and rf transition rate provide other experimental probes of the contact.