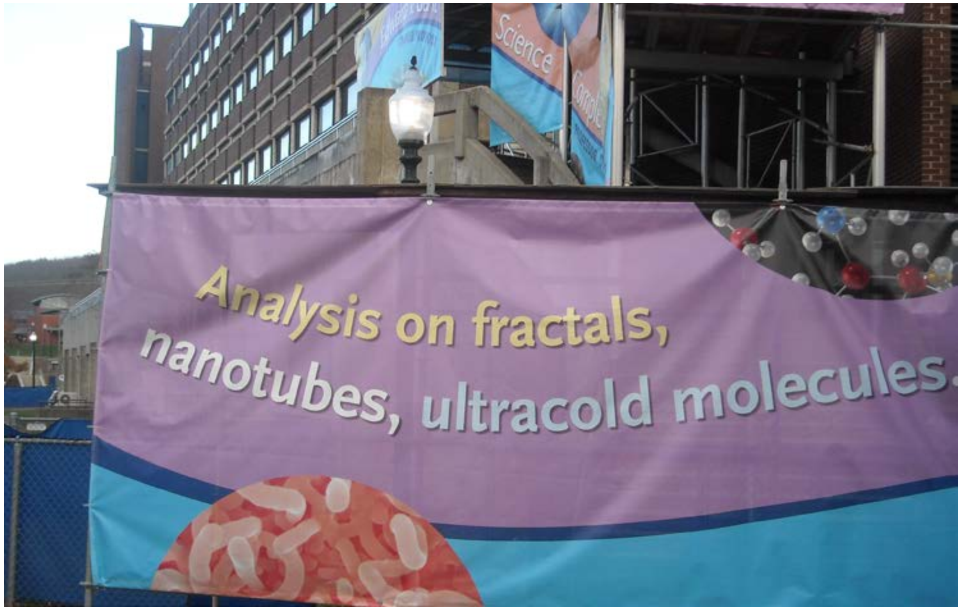
Efimov States as Fields on a Fractal

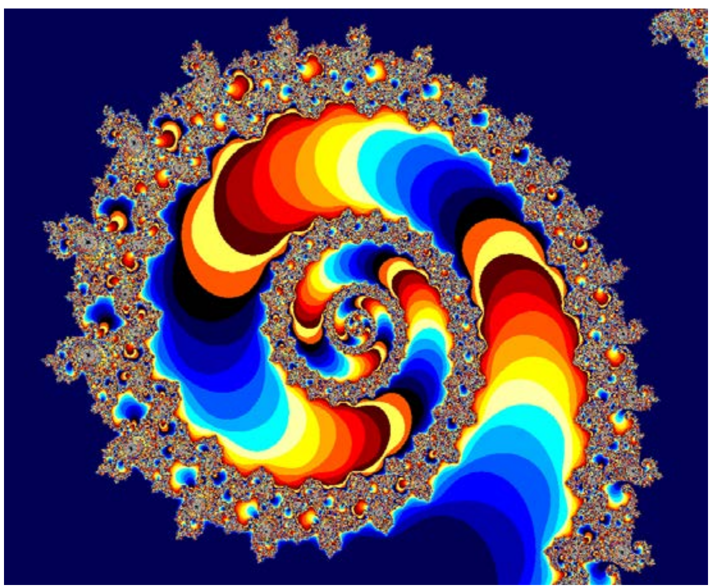


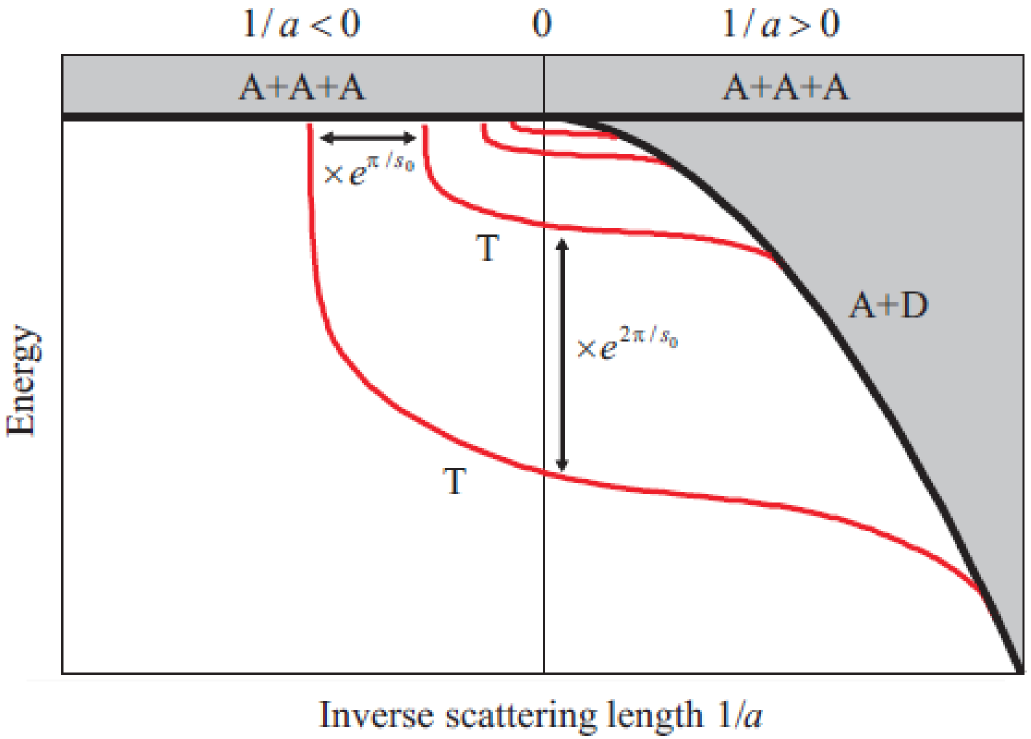
Ehoud Pazy, Physics Department, NRCN

INT Program , *Universality in Few-Body Systems*, 29.4.2014

Seemingly Un-Connected Topics

- Efimov states-Three body universal states recently experimentally observed
- Fields on fractals- essentially a theoretical object of research

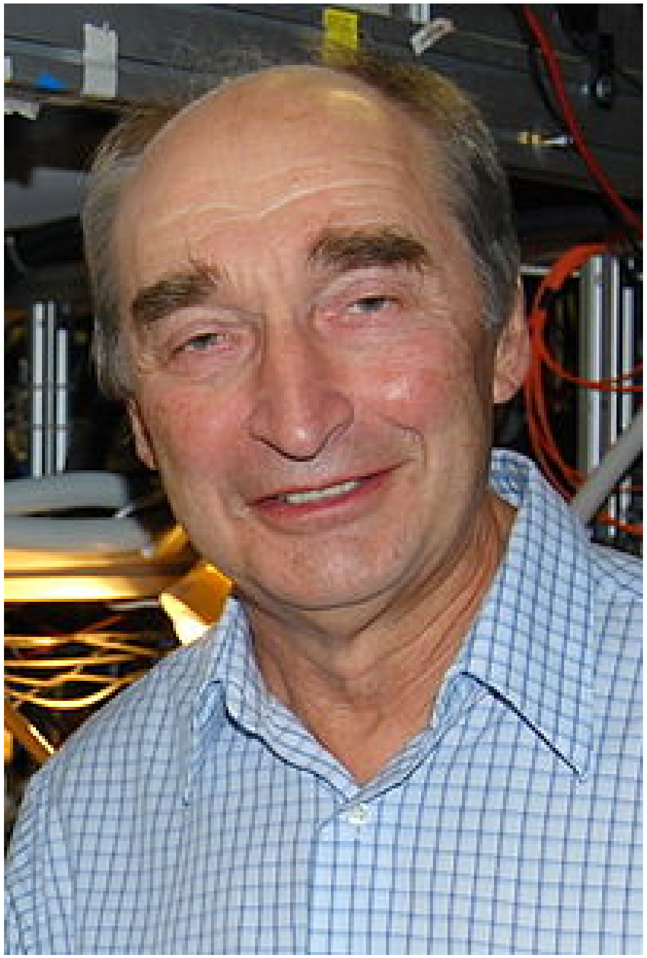




Outline

- Introduction
  - Some Efimov state properties
  - Discrete scaling functions
  - Hints on mathematical connections
- Efimov physics in terms of EFT
  - Neumann series solution of integral equation
  - Conditions for discrete scaling solution
- Scaling parameters and conditions on them
- Connecting to one dimensional physics
- Identifying the fractal structure
- Wild speculations
- Summary

Efimov States



In 1970 Vitaly Efimov discovered that in the limit where:

Two-body scattering length diverges $a \to \pm\infty$

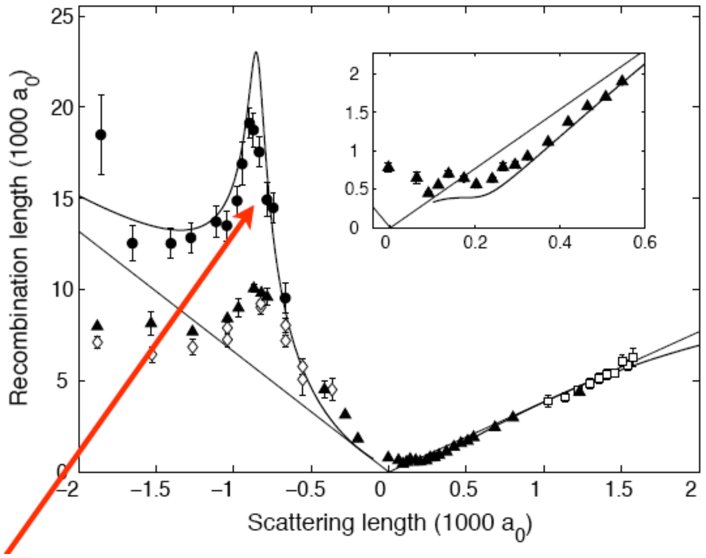
No two body bound state is found but there are an infinite number of three body (trimer) states

"From questionable to pathological to exotic to a hot topic"

Nature Physics **5**, 533 (2009)

The Experimental Search for Efimov States

- Experiments with ^{4}He atoms (T~0.3 mk)
- Nucleons: triton (pnn) and ^{3}He (ppn)
- Halo Nuclei
- Cold atom physics ^{133}Cs atoms 2005



Kraememr et al (Innsbruck)

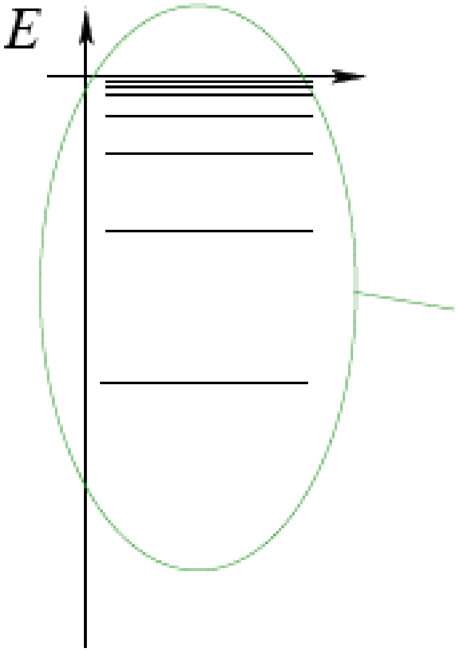
Properties of Efimov Trimers

Universal three body physics

The physics depends only
on the scattering length
and a further quantity called
the “Three body parameter”

Details of the short range
interaction become
irrelevant

Infinitely many bound states

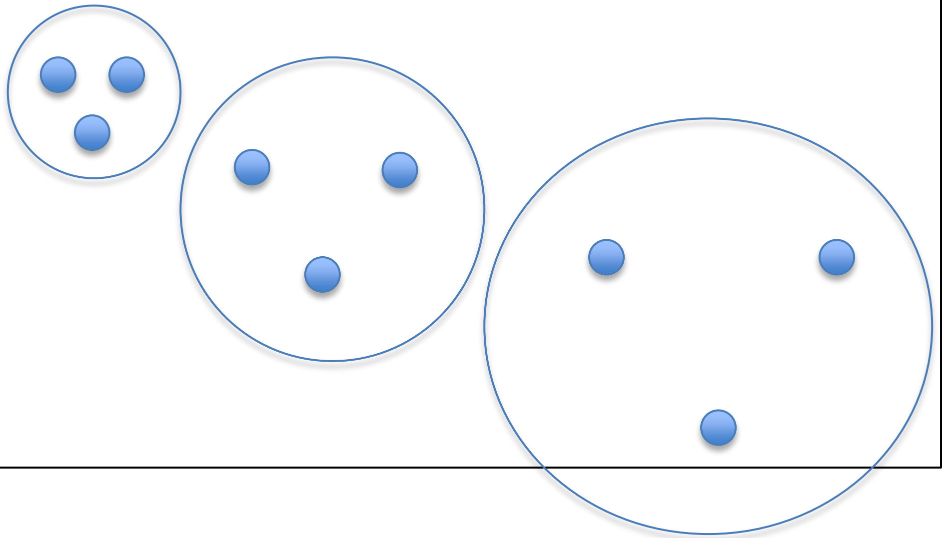


Properties of Efimov Trimers

Discrete scaling:

binding energies $\lambda^{-2} = e^{-2\pi/s_0}$

sizes differ $\lambda = e^{\pi/s_0}$



The scattering amplitude is a log periodic function

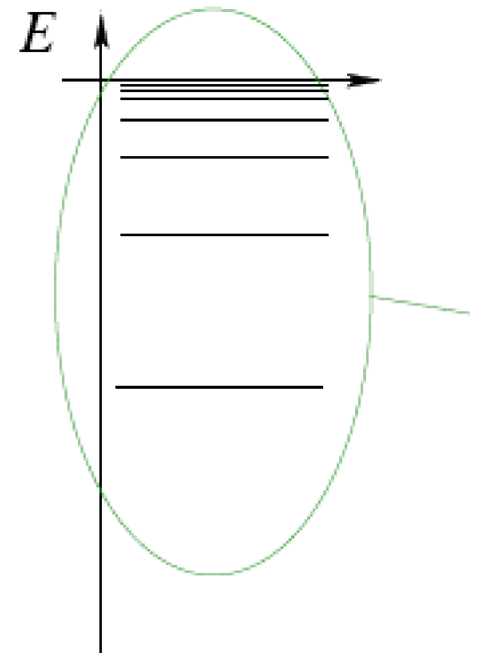
$$a(p) = A\cos\left(s_0 \ln\frac{p}{\Lambda} + \delta\right)$$

The Efimov Spectrum

$$E_T^{(n)} \to \left(e^{-2\pi/s_0}\right)^{n-n^*} \frac{\hbar^2 \kappa_*^2}{m}$$

$$n \to \infty \qquad a = \pm\infty$$

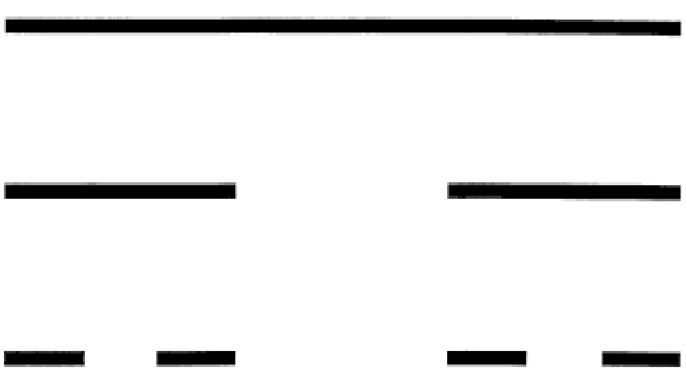
- Shows a discrete symmetry
- Is infinite
- Universal



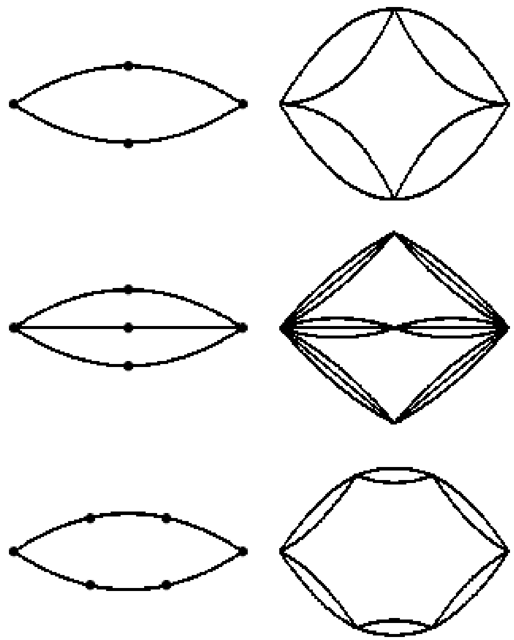
Defined by two parameters

The scaling parameter: $\boxed{\lambda_0 = e^{\pi/s_0}}$ The three body parameter: $\boxed{\kappa_*}$

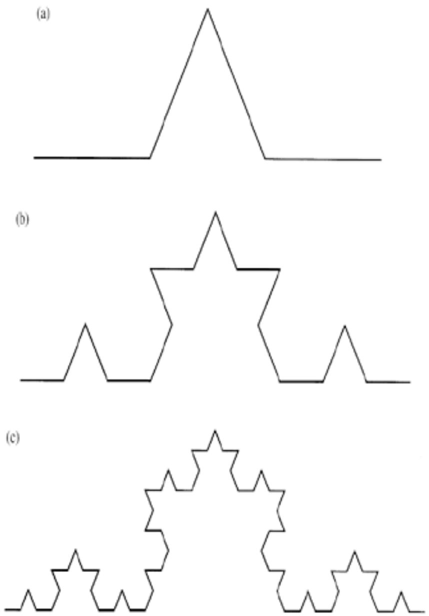
Quick Introduction to Discrete Scaling



Triadic Cantor Set



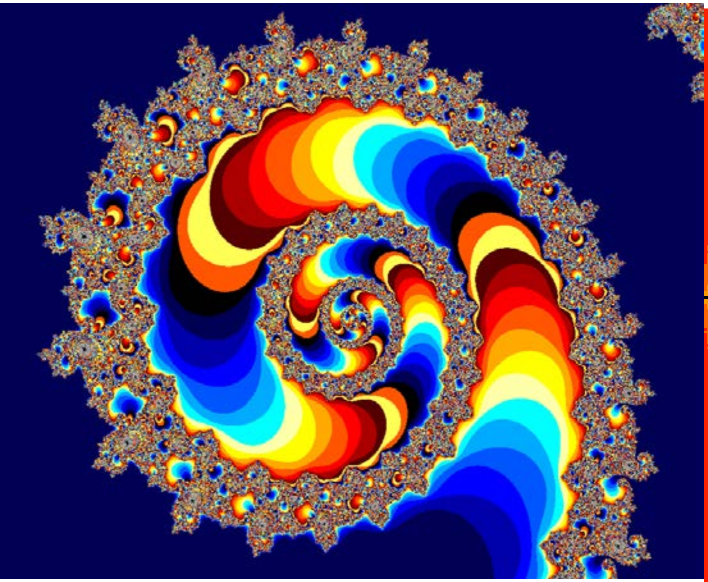
Diamond Fractals

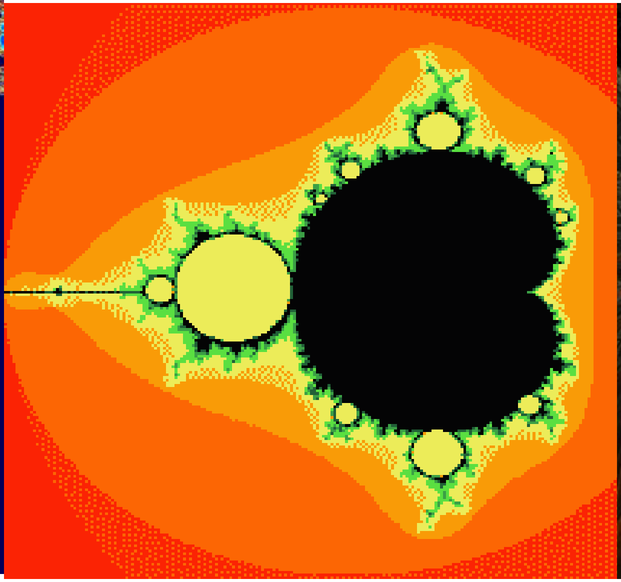


Koch Curve

A fractal is an iterative structure

Fractals Can be Very Fancy





Equation for Discrete Symmetry

- Equation for discrete symmetry

$$f(x) = g(x) + \frac{1}{b} f(ax)$$

a and b are scaling parameters

$g(x)$ some initial function

- Its solution is of the form

$$\boxed{f(x) = x^{\frac{\ln b}{\ln a}} G\left(\frac{\ln x}{\ln a}\right)}$$

where G is a periodic function

Equation for Discrete Symmetry Iterative Form

- Equation for discrete symmetry

$$f(x) = g(x) + \frac{1}{b} f(ax)$$

- The equation can be iterated to give

$$\boxed{f(x) = \sum_{n=0}^{\infty} b^{-n} g\left(a^n x\right)}$$

“Spectrum” and the Mellin Transform

Iterative solution:

$$f(x) = \sum_{n=0}^{\infty} b^{-n} g(a^n x)$$

Mellin transform:

$$M_f(s) \equiv \int_0^{\infty} dx \quad x^{s-1} f(x)$$

Mellin transform of
Iterative solution:

$$M_f(s) = M_g(s) \frac{ba^s}{ba^s - 1}$$

“Spectrum”:

$$\boxed{s_n = \frac{\ln(1/b)}{\ln a} + \frac{2i\pi n}{\ln a}}$$

Hints to the Connection to Quantum Fields on a Fractal

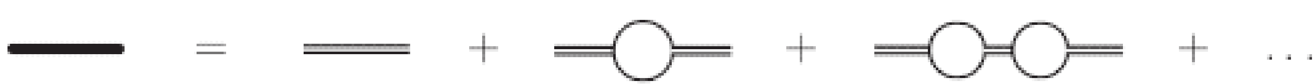
Efimov States	Fields on fractals
Discrete Symmetry Limit Cycle RG	Discrete Symmetry Limit Cycle RG
Log Periodic Functions	Log Periodic Functions
Infinite Spectrum	Infinite “Spectrum” (poles)

Efimov States through an Effective Field Theory

- Three body Bosonic Theory
- Effective Field Theory in term of a "di-baryon" two boson field "d"

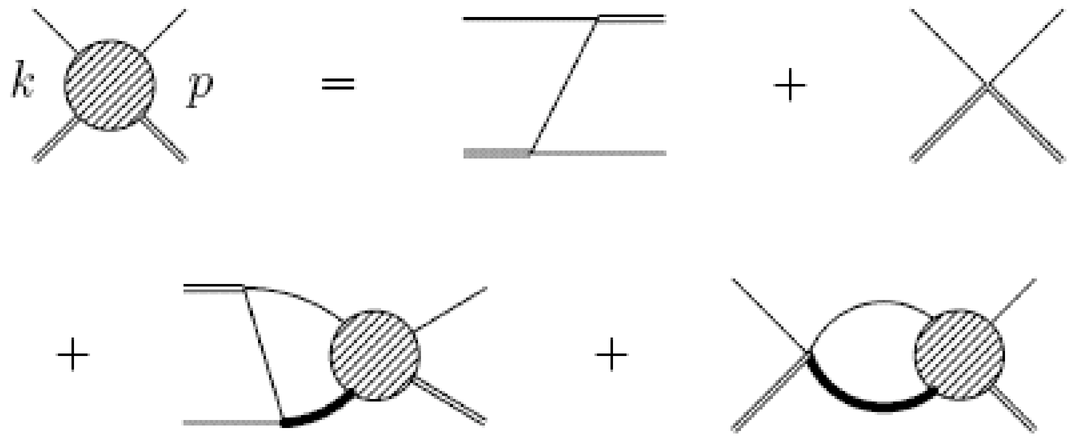
$$L = \psi^{+}\left(i\partial_{0} + \frac{\nabla^{2}}{2M} \right)\psi + \Delta d^{+} d - \frac{g}{\sqrt{2}}\left(d^{+}\psi\psi + H.c \right) + h d^{+} d \psi^{+} \psi$$

- Dressing of the di-atom propagator "d"



P. F Bedaque, H. –W. Hammer and U. Van Kolck PRL **80**, 463 (1999)

The Integral Equation



$$a_{ad}(p) = K(p,k) + \frac{2}{\pi}\int_0^{\Lambda} dq K(p,q) \frac{q^2}{q^2 - k^2 - i\varepsilon} a(q)$$

$$K(p,q) = \frac{2}{\sqrt{3}q} \ln\left(\frac{q^2 + pq + p^2}{q^2 - pq + p^2}\right)$$

P. F Bedaque, H. –W. Hammer and U. Van Kolck PRL **80**, 463 (1999)

Solution Asymptotic Homogeneous Case

The asymptotic homogeneous equation:

for $\frac{1}{a_2} << p << \Lambda$ but $k \sim \frac{1}{a_2}$

$$\boxed{a(p) = \frac{4}{\sqrt{3}\pi}\int_0^\infty \frac{dq}{q} a(q) \ln\left(\frac{q^2 + pq + p^2}{q^2 - pq + p^2}\right)}$$

Suggests an ansatz $a(p) \sim p^s$ this works if S satisfies

$$1 - \frac{8}{\sqrt{3}} \frac{\sin\left(\frac{\pi s}{6}\right)}{s\cos\left(\frac{\pi s}{2}\right)} = 0 \longrightarrow \boxed{S = \pm i S_o \quad S_0 = 1.0064}$$

P. F Bedaque, H. –W. Hammer and U. Van Kolck PRL **80**, 463 (1999)

Solution with Finite Cut-Off

- Returning to the equation with a finite cut-off ›
  The following solution is obtained

$$\boxed{a(p) = A\cos\left(s_0 \ln\frac{p}{\Lambda} + \delta\right)}$$

Making a Connection to Fractals Through the Integral Equation

$$a(p) = K(p,k) + \frac{2}{\pi}\int_0^{\Lambda} dq K(p,q) \frac{q^2}{q^2 - k^2 - i\varepsilon} a(q)$$

Writing out the formal solution in terms of a Neumann series

$$a(p) = K(p,k) + \sum_{n=1}^{\infty} \lambda^n \psi_n(p) \qquad \lambda = \frac{2}{\pi}$$

$$\psi_n(p) = \int_0^{\Lambda} \ldots \int_0^{\Lambda} K(p,q_1) K(q_1,q_2) \ldots K(q,k) dq_1 \ldots dq_n$$

Discrete Scale Self Similar Solution

- A self similar discrete scaling solution

$$a(p) = K(p,k) + \sum_{n=1}^{\infty} \lambda^n \psi_n(p) \qquad \lambda = \frac{2}{\pi}$$

$$\psi_n(p) = \int_0^{\Lambda} \ldots \int_0^{\Lambda} K(p,q_1) K(q_1,q_2) \ldots K(q,k)\, dq_1 \ldots dq_n$$

$$\boxed{a(p) = \sum_{n=1}^{\infty} \lambda^n K(\gamma^n p, k)}$$

- Can be found if $\boxed{\int_0^{\Lambda} dq K(p,q) K(q,k) = K(\gamma p, k)}$

The Connection

If $$\boxed{\int_0^{\Lambda} dq K(p,q) K(q,k) = K(\gamma p,k)}$$

Then

$$\boxed{a(p) = \sum_{n=1}^{\infty} \lambda^n K\left(\gamma^n p,k\right)}$$

$$\boxed{\begin{array}{l} \gamma \leftrightarrow a \\ \lambda^{-1} \leftrightarrow b \end{array}}$$

Questions

- What are the values of the scaling parameters?
- How to obtain the Efimov spectrum?
- What is the fractal like structure?

Determining the Discrete Scaling Factor

- Performing a Mellin Transformation

$$M_f(s) \equiv \int_0^\infty dp\, p^{s-1} f(p)$$

on both sides of

$$\int_0^\Lambda dk K(q,k) K(k,p) = K(\gamma q, p)$$

The scaling factor γ is determined through

$$\boxed{\gamma^{-s} = M_{K\times p}(s)} \qquad M_{\mathrm{K}\times p}(s) = \int_0^\infty dp\, p^s K(1,p)$$

The Connection Between the Discrete Scaling Parameters

Mellin transform of the condition

$$\int_0^\Lambda dp K(q,p) K(p,k) = K(\gamma q, k)$$

$$\boxed{a^{-s} = M_{K\times p}(s)}$$

power solution for the integral eq.

$$1 - \frac{8}{\sqrt{3}} \frac{\sin\left(\frac{\pi s}{6}\right)}{s\cos\left(\frac{\pi s}{2}\right)} = 0$$

$$\boxed{b = M_{K\times p}(s)}$$

The two expressions are equivalent when

$$\boxed{a^s b = 1}$$

Instead of $s_n = \frac{\ln(1/b)}{\ln a} + \frac{2i\pi n}{\ln a}$

$$\boxed{S = \pm i S_0 \qquad S_0 = 1.0064}$$

An Alternative Method to Obtain the Scaling Factor

Discrete scale invariance

$$\boxed{a_{sc}(ap) = b a_{sc}(p)}$$

Power like solution to homogeneous integral eq.

$$\boxed{a_{sc}(p) \sim p^{s}}$$

$$\boxed{a^{-s} b = 1}$$

Instead of $s_n = \frac{\ln(1/b)}{\ln a} + \frac{2i\pi n}{\ln a}$

$$\boxed{S = \pm i S_0 \qquad S_0 = 1.0064}$$

Viewing Things in a Different Way

- Connecting to Bloch states
- Obtaining Efimov spectrum from Bohr-Sommerfeld quantization.

A Connection to Bloch States

$$f(p) = p^{-m} G\left(\frac{\ln(p)}{\ln(a)}\right) \qquad m = \frac{\ln(b)}{\ln(a)}$$

change coordinates to: $\boxed{\tilde{x} = \ln(p)}$

denoting : $k = im$

obtain a Bloch function $\boxed{f(\tilde{x}) = e^{ik\tilde{x}} G\left(\frac{\tilde{x}}{l}\right)}$

lattice constant $l = \ln(a)$

wave number $k = i\frac{\ln(b)}{\ln(a)}$

Connecting to Bloch states

Instead of $s_n = \frac{\ln(1/b)}{\ln a} + \frac{2i\pi n}{\ln a}$ only two values $s_n = \pm i s_0$

then we have $\boxed{i s_0 = -\frac{\ln(b)}{\ln(a)} + \frac{2\pi i n}{\ln(a)}}$

associating the wave number to $k = i\frac{\ln(b)}{\ln(a)}$

we get $s_0 = k + \frac{2\pi n}{\ln(a)}$

associating the lattice constant as $l = \ln(a)$

so s_0 can be considered as an effective crystal momentum

$$\boxed{s_0 = k + \frac{2\pi n}{l}}$$

Importing Results from Functions with Scaling Symmetry

Since the integral equation is now a scaling symmetry equation it's solution is given by

$$\boxed{f(x) = x^{\frac{\ln b}{\ln a}} G\left(\frac{\ln x}{\ln a}\right)} \quad a_{sc}(p) = p^m G\left(\frac{\ln(p)}{\ln(a)}\right) \quad m = -\frac{\ln(\lambda)}{\ln(a)}$$

since

$$G\left[\frac{\ln p}{\ln a}\right] = \sum_{n=-\infty}^{\infty} C_n \exp\left(2\pi i n \frac{\ln p}{\ln a}\right) \quad \ln a = \frac{2\pi n}{s_0}$$

$$\boxed{a_{sc}(p) \approx A \cos\left(s_0 \ln \frac{p}{\Lambda} + \delta\right)}$$

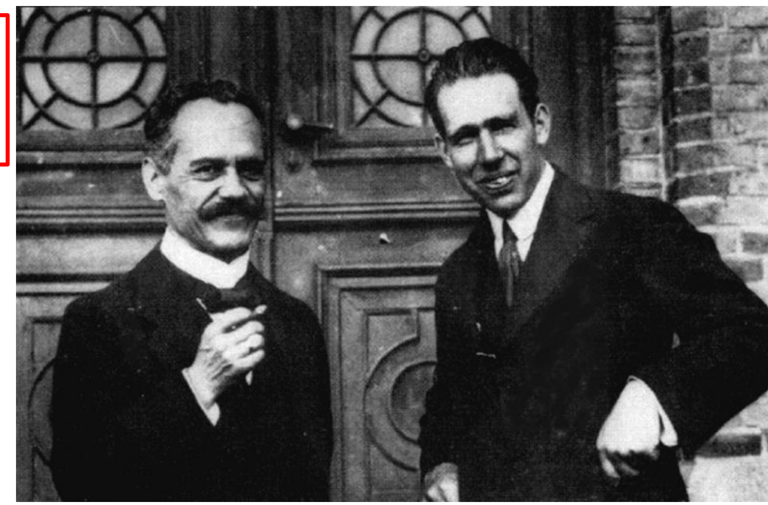
Spectrum from the Bhor-Sommerfeld Quantization

S_3 plays the roll of effective momentum to obtain the spectrum we can use the Bohr-Sommerfeld quantization

$$\boxed{\oint p_i dq_i = 2\pi n_i}$$

$$s_0 \int_{\tilde{x}_1}^{\tilde{x}} d\tilde{x} = n\pi \longrightarrow \ln(p) = \frac{n\pi}{s_0}$$

$$\boxed{\tilde{x} = \ln(p)}$$



$$\boxed{E_T^{(n)} = -\left(e^{-2\pi/s_0}\right)^{n-n^*} \frac{\hbar^2 \kappa_*^2}{m}}$$

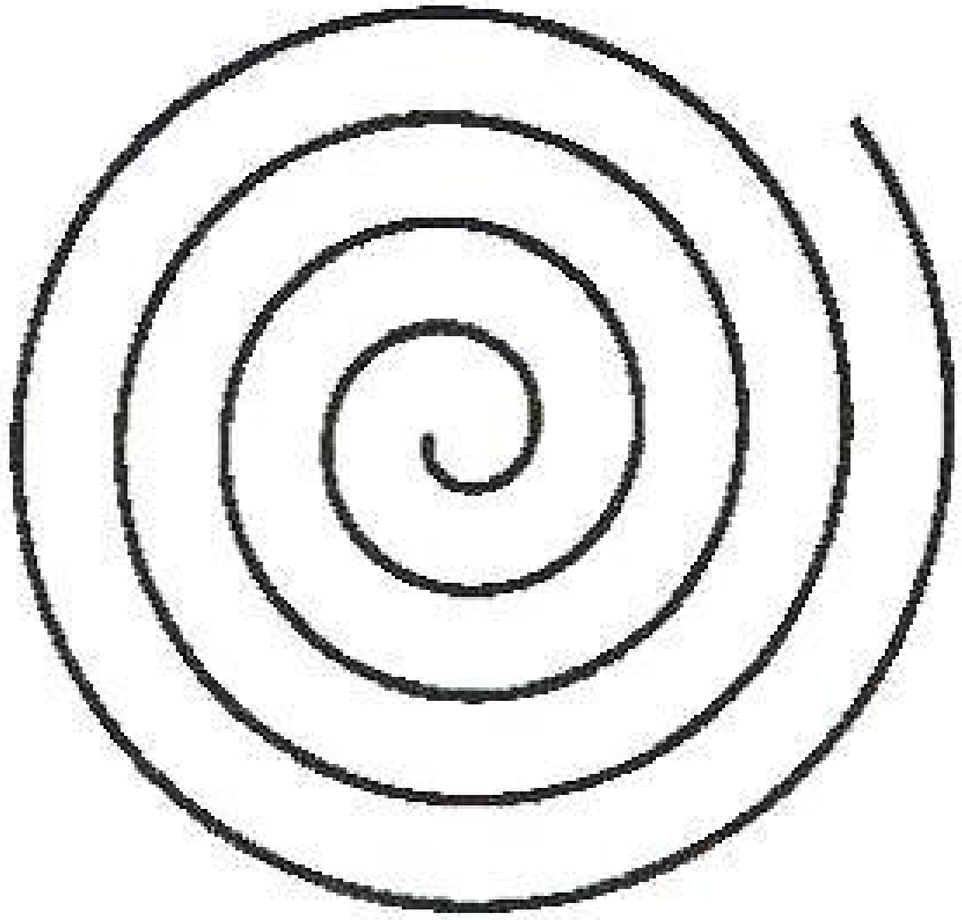
What is the Fractal?

It is (onal

Luke Rogers, Prof. of Mathematics
University of Connecticut

A Spiral

$a^t \qquad t \in R$

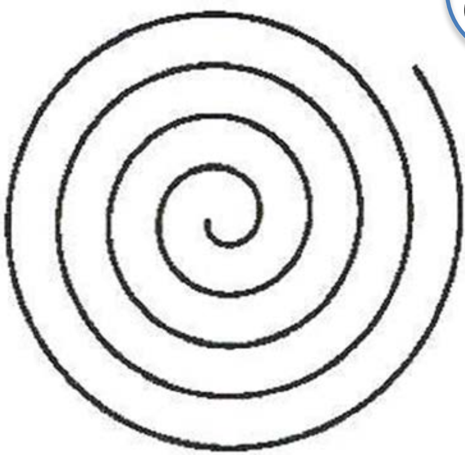


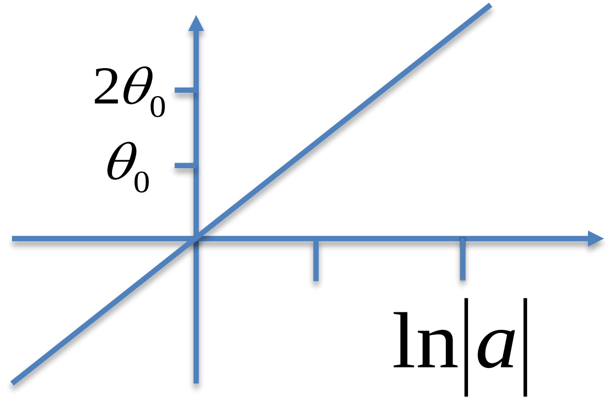
Parameterization

θ-distance along the curve

$$\exp[is_0 \ln(|a| e^{\theta/\theta_0} e^{i\theta})] = \exp[is_0 (\theta/\theta_0) \ln|a| - \theta s_0]$$

$$a^{-s} b = 1$$





$$\theta_0 = \mathrm{Im}\{\ln a\}$$

Transform the plain wave eigenfunctions of the Laplacian

$$\exp[ikx] \Rightarrow \exp[ikz]$$

The phase of a

The scaling parameters: b is real but it seems that a has a phase

Express $a = e^{i\theta_0 + 2\pi n/s_0 + l}$ from $a^{\mp i s_0} b = 1$ $\quad e^{i l s_0} = 1$

$$a^{\theta_0 s_0} b = 1 \qquad\qquad a^{-\theta_0 s_0} b = 1$$

$$\cosh[s_0 \theta_0] = b$$

The phase of a

The scaling parameters : b is real but it seems a has a phase

$$a^{-s} b = 1 \longrightarrow e^{-is_0 \ln(a)} b = 1$$

$$e^{s_0 \operatorname{Im}\{\ln(a)\}} = b \qquad\qquad e^{is_0 \operatorname{Re}\{\ln(a)\}} = 1$$

$$S = \pm i S_0$$

$$\cosh\left[s_0 \operatorname{Im}\{\ln(a)\}\right] = b$$

$$\operatorname{Re}\{\ln(a)\} = \frac{2\pi n}{s_0}$$

s_3 plays the roll of effective momentum

Wild Speculations

Actually the quantization is $s_0 \int_{\tilde{x}_1}^{\tilde{x}} d\tilde{x} = (n+\delta)\pi$

Where δ results from the boundary conditions and determines the three body parameter $\boxed{\kappa_*}$

$$s_0 \int_{\tilde{x}_1}^{\tilde{x}} d\tilde{x} = (n+\delta)\pi \longrightarrow \ln(p) = \frac{n\pi}{s_0}$$

$$\downarrow$$

$$\boxed{\kappa_* = e^{-\pi\delta/s_0} \ln(\tilde{x}_1)} \longleftarrow \boxed{E_T^{(n)} = -\left(e^{-2\pi/s_0}\right)^{n-n^*} \frac{\hbar^2 \kappa_*^2}{m}}$$

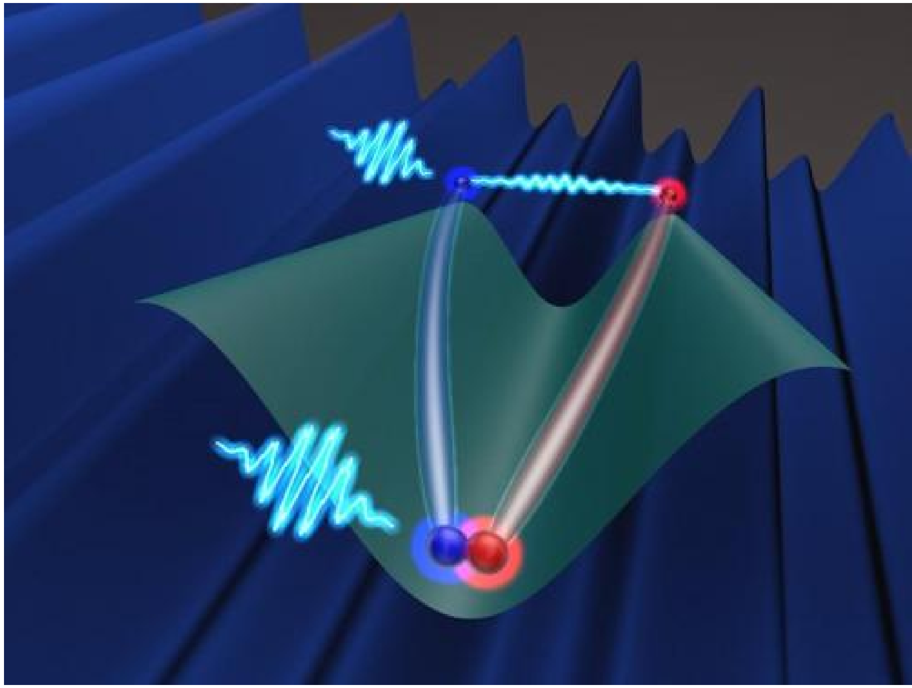
Wild Speculations

Can the non-trivial phase δ - $s_0 \int_{\tilde{x}_1}^{\tilde{x}} d\tilde{x} = (n+\delta)\pi$

Be connected top the Zak phase a non-trivial geometric phase obtained in solid.
The phase is called after it's discoverer Prof. Joshua Zak from the Technion.

Zak Phase

In 1989, Professor Zak identified the geometrical phases in the band theory of one-dimensional solids. When a particle travels "slowly" along the energy band and completes a closed loop it acquires a geometrical phase that has significant physical consequences for the properties of materials, which can be determined by the "quantum geometry" of the crystal.



Summary

- Obtained a known result through a different formalism.
- Sheds light on the connection between fields on fractals and the Skorniakov-Ter-Martirosian (SKM) equation.
- Obtained a physical realization of a “quantum field” on a “fractal”.