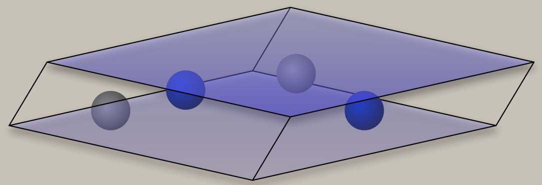
# UNIVERSAL BOUND STATES IN CONFINED GEOMETRIES

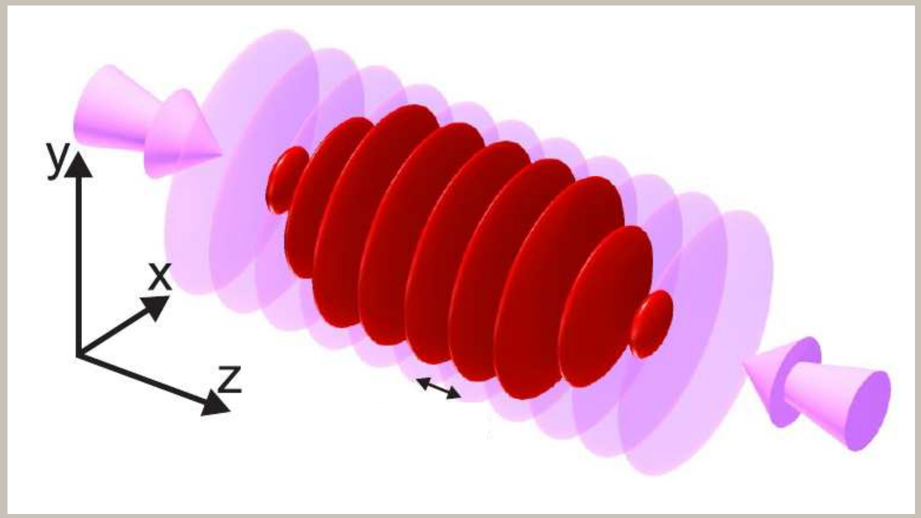
MEERA PARISH

*University of Washington, 11 April 2014*

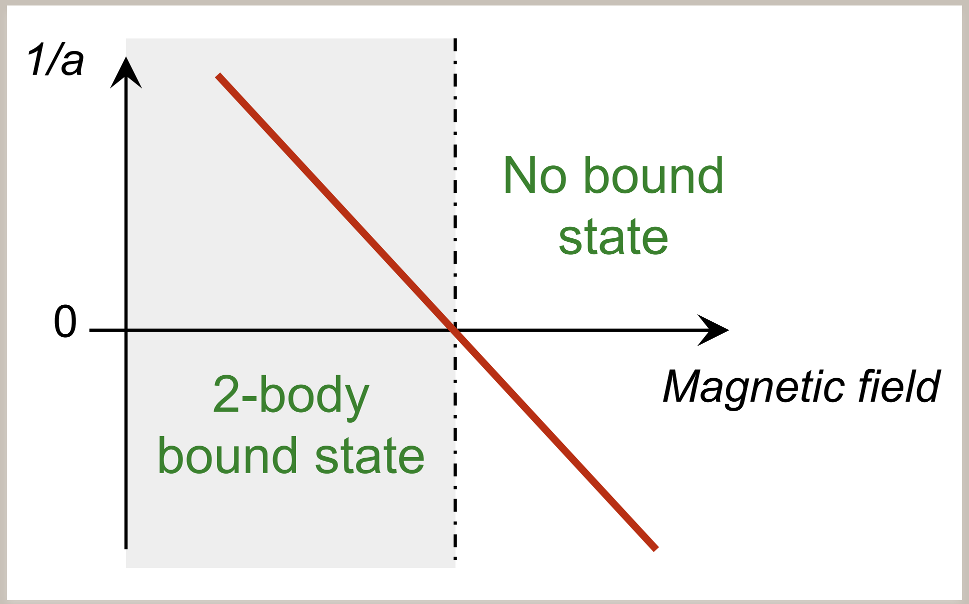
# Qu: How does confinement impact few-body bound states?



- Experimentally achievable in the cold-atom system
  - Tunable dimensionality & interactions:




*Quasi-2D geometry*

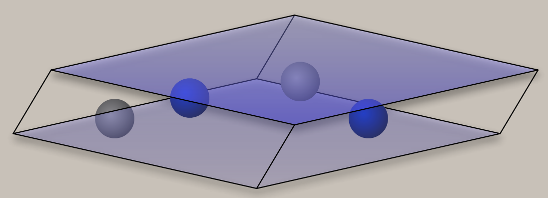



*Short-range s-wave interactions*

  - Different atomic species (fermions or bosons)

# OUTLINE

- Identical bosons in 3D – The Efimov effect
- Evolution towards 2D
  - Quasi-2D geometry

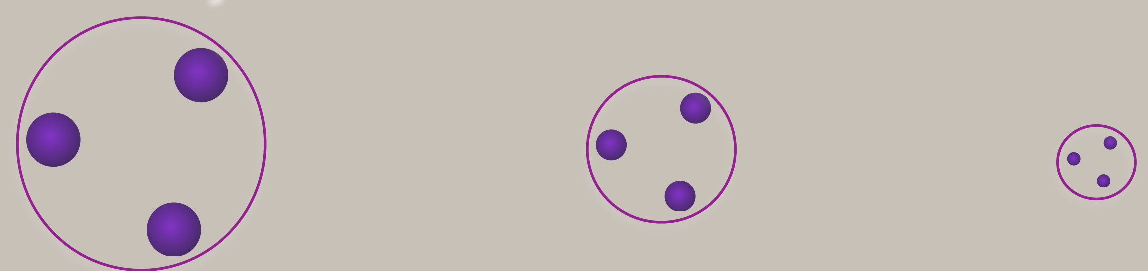


- Three-boson problem in quasi-2D
  - Trimer spectra & wave functions
  - Hyperspherical potentials
- Two-component quasi-2D Fermi system
- Conclusions

# Identical bosons in 3D

- V. Efimov (1970): Three identical bosons with resonant short-range interactions (1/*a*=0) support an infinite number of trimers
    - Experimental evidence: Gas of Cs atoms (Kraemer *et al*, Nature 2006)

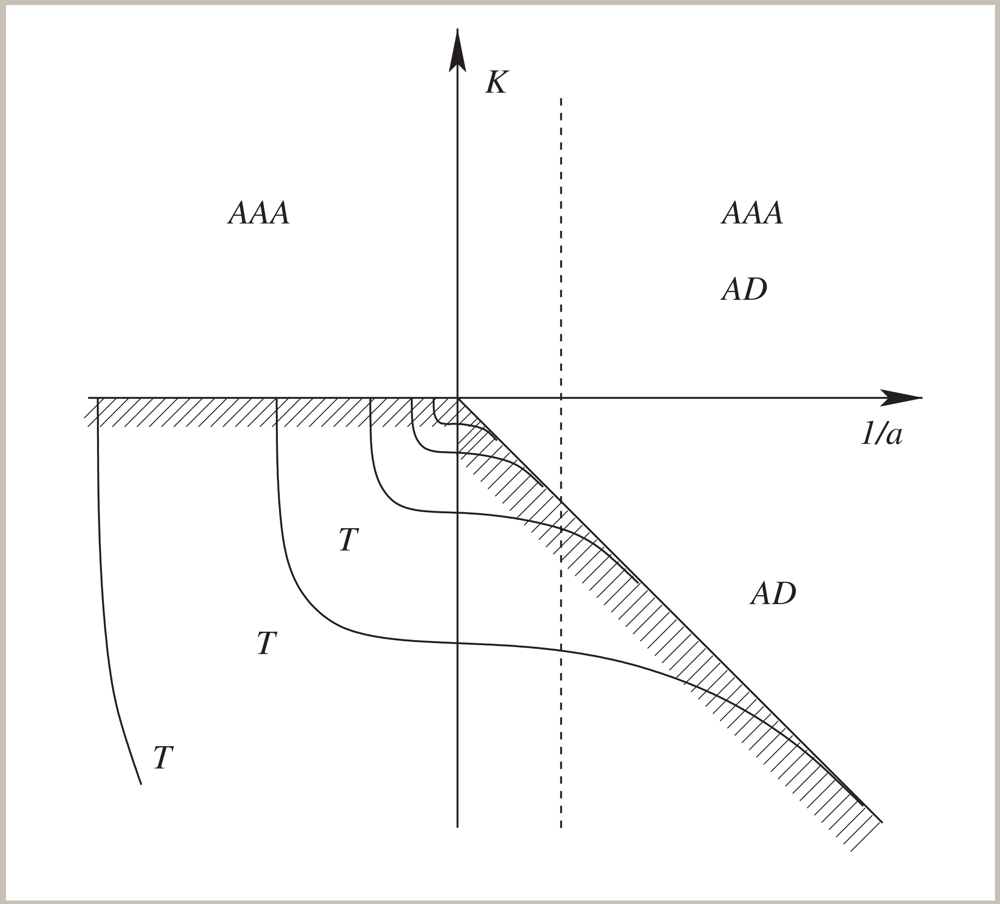
- Three-body problem has discrete scaling symmetry:



  - Trimers can be mapped onto another via transformation:

$$E \to \lambda_0^{-2n} E \qquad a \to \lambda_0^{n} a$$

- Trimer energies at resonance: $-\lambda_0^{-2n}\dfrac{\hbar^2\kappa_*^2}{m}$




*Braaten & Hammer, Phys. Rep. 2006*

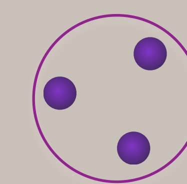
$\lambda_0 \simeq 22.7$

# Identical bosons in 2D

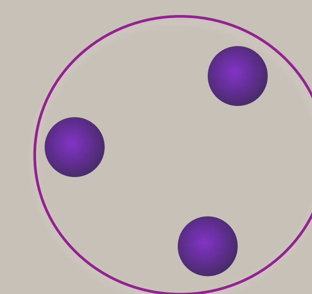
- Only *two* universal trimers:

  *Bruch & Tjon, PRA 1979*

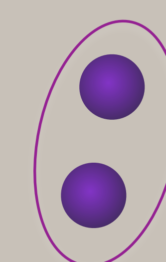
$$-16.5|E_b|$$



$$-1.27|E_b|$$



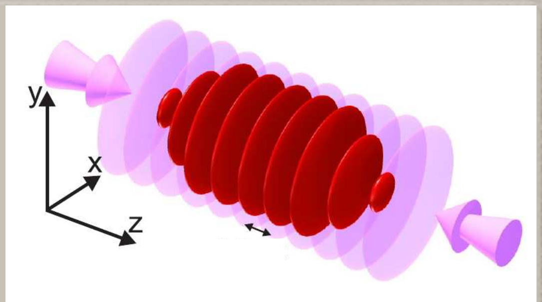
Two-body binding energy $E_b = -\dfrac{\hbar^2}{m a_{2D}^2}$



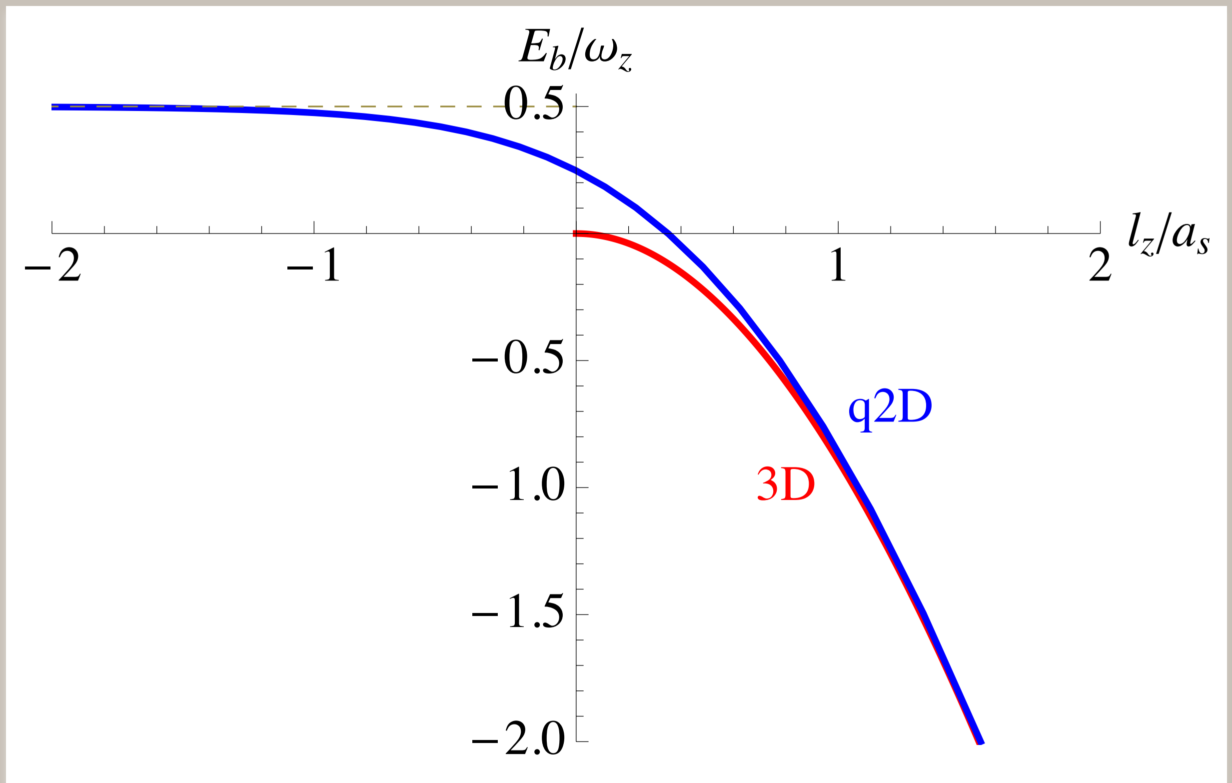
  - Dimers and trimers always exist for arbitrarily weak attractive interactions
  - Three-body problem exhibits a *continuous* scaling symmetry

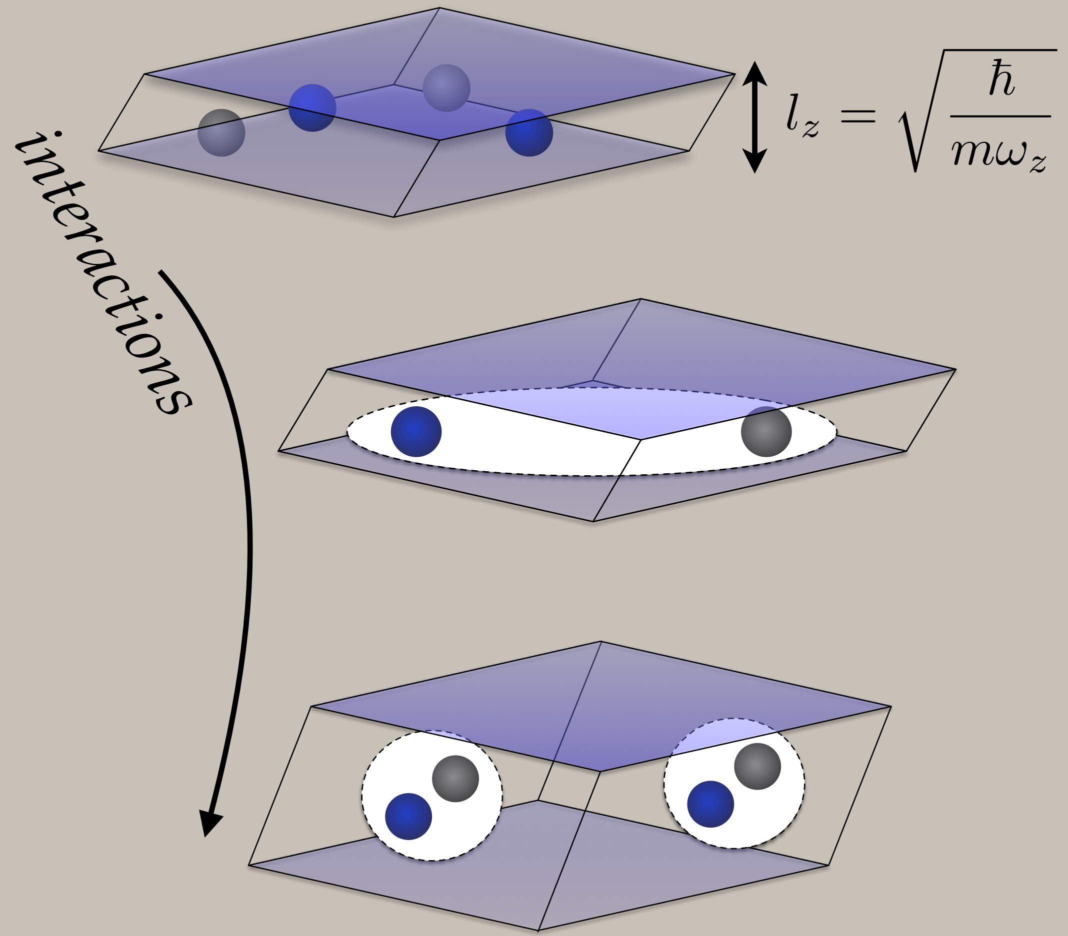
# Evolution from 3D to 2D?

# Quasi-2D system



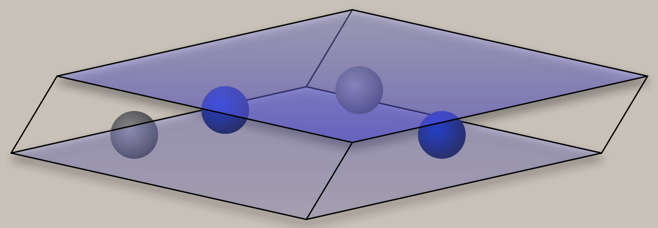

- Harmonic confinement along z: $V(z) = \frac{1}{2} m\omega_z^2 z^2$
  - Generated by optical lattice or trap in experiment
  - Bose gas is kinematically 2D when $k_B T \ll \hbar\omega_z$
- Two-body problem:



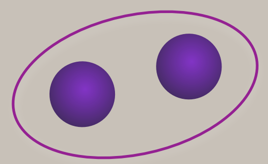




Petrov & Shlyapnikov, PRA 2001

# Quasi-2D system



- Confinement raises threshold of free atom continuum
- Always have a two-body bound state



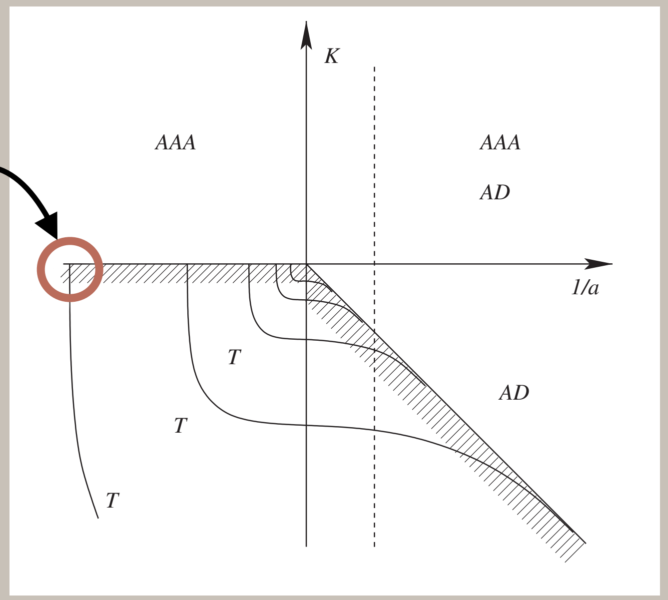
- Obtain 2D limit when interactions are weak $|a| \ll l_z$
  - Two-body binding energy $E_b = -\frac{B}{m\pi l_z^2} e^{\sqrt{2\pi} l_z / a}$

*Petrov & Shlyapnikov, PRA 2001*

Three-body problem characterised by 2 dimensionless parameters:

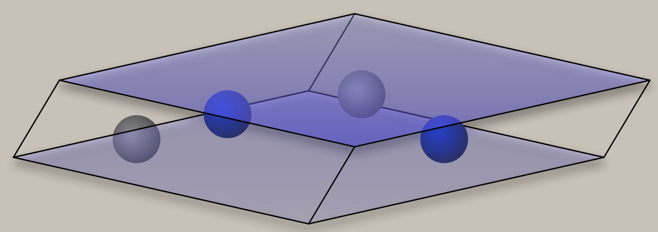
$|a_-|/a$ $\qquad \mathcal{C}_z \equiv |a_-|/l_z$

$1/a_-$



# Three bosons in quasi-2D

- Hamiltonian:



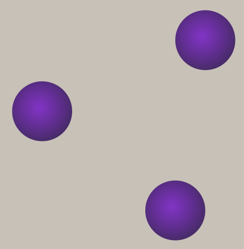
$$\hat{H} = \sum_{\mathbf{k},n} (\epsilon_{\mathbf{k}} + n\hbar\omega_z) a^\dagger_{\mathbf{k}n} a_{\mathbf{k}n}$$

$$+ \frac{1}{2} \sum_{\substack{\mathbf{k},n_1,n_2 \\ \mathbf{k}',n_3,n_4 \\ \mathbf{q}}} e^{-\mathbf{k}^2/\Lambda^2} e^{-\mathbf{k}'^2/\Lambda^2} \langle n_1 n_2 | \hat{g} | n_3 n_4 \rangle \; a^\dagger_{\mathbf{q}/2+\mathbf{k},n_1} a^\dagger_{\mathbf{q}/2-\mathbf{k},n_2} a_{\mathbf{q}/2-\mathbf{k}',n_3} a_{\mathbf{q}/2+\mathbf{k}',n_4}$$

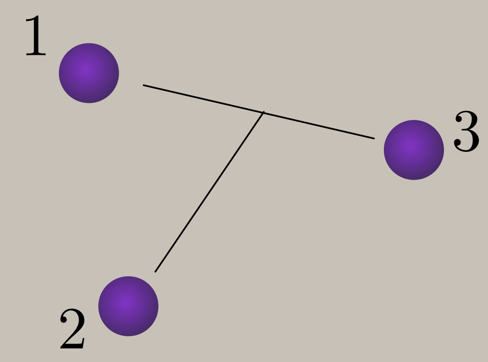
UV cut-off fixes $a_-$ in 3-body problem

- Trimer wave function:

$$\sum_{\substack{\mathbf{k}_1,\mathbf{k}_2 \\ n_1,n_2,n_3}} \psi^{n_1 n_2 n_3}_{\mathbf{k}_1 \mathbf{k}_2} a^\dagger_{\mathbf{k}_1,n_1} a^\dagger_{\mathbf{k}_2,n_2} a^\dagger_{-\mathbf{k}_1-\mathbf{k}_2,n_3} |0\rangle$$



# Three bosons in quasi-2D




✦ Simplifying the calculation:

$$\chi_{\mathbf{k}_2}^{NN_2} = g \sum_{\substack{\mathbf{k}_1, n_{13} \\ n_1 n_2 n_3}} e^{-k_1^2/\Lambda^2} f_{n_{13}} \langle N, N_2, n_{13} | n_1 n_2 n_3 \rangle \; \psi_{\mathbf{k}_1 \mathbf{k}_2}^{n_1 n_2 n_3}$$

$$\longrightarrow \sum_{k_z} e^{-k_z^2/\Lambda^2} \tilde{\phi}_{n_{13}}(k_z)$$

- Depends on only 2 parameters after dropping *N* (CoM)

- Wavefunction for atom-pair motion: $\psi(\boldsymbol{\rho}, Z) \equiv R^{3/2} \sum_{\mathbf{k}, N} e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \phi_N(Z) \chi_{\mathbf{k}}^N$

✦ Integral equation:

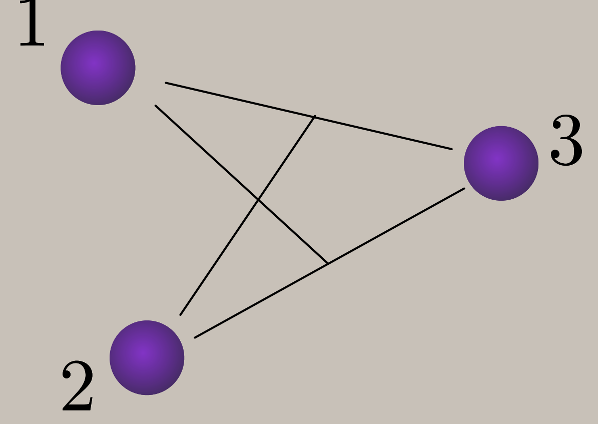
$$\mathcal{T}^{-1}(\mathbf{k}_1, E_3 - \epsilon_{\mathbf{k}_1} - N_1\omega_z)\, \chi_{\mathbf{k}_1}^{N_1} = 2 \sum_{\mathbf{k}_2, N_2 n_{23} n_{31}} \frac{f_{n_{23}} f_{n_{31}} \langle N_1 n_{23} | N_2 n_{31} \rangle e^{-(k_1^2 + k_2^2)/\Lambda^2} \chi_{\mathbf{k}_2}^{N_2}}{E_3 - \epsilon_{\mathbf{k}_1} - \epsilon_{\mathbf{k}_2} - \epsilon_{\mathbf{k}_1 + \mathbf{k}_2} - (N_1 + n_{23})\omega_z}$$

*Levinsen, Massignan & MMP, arXiv:1402.1859*

# Clebsch-Gordon coefficients

- The challenge: to describe the 3D regime with a discrete energy scaling of 515, we require > $515^3$ coefficients…

$$\langle N_1 n_{23} | N_2 n_{31} \rangle$$




$$e^{-i(2\pi/3)J_y}$$

- Use Schwinger's mapping of the 2D isotropic harmonic oscillator to the SU(2) representation of angular momentum algebra:

$$\mathbf{J} = \frac{1}{2} \begin{pmatrix} b_1^\dagger & b_2^\dagger \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

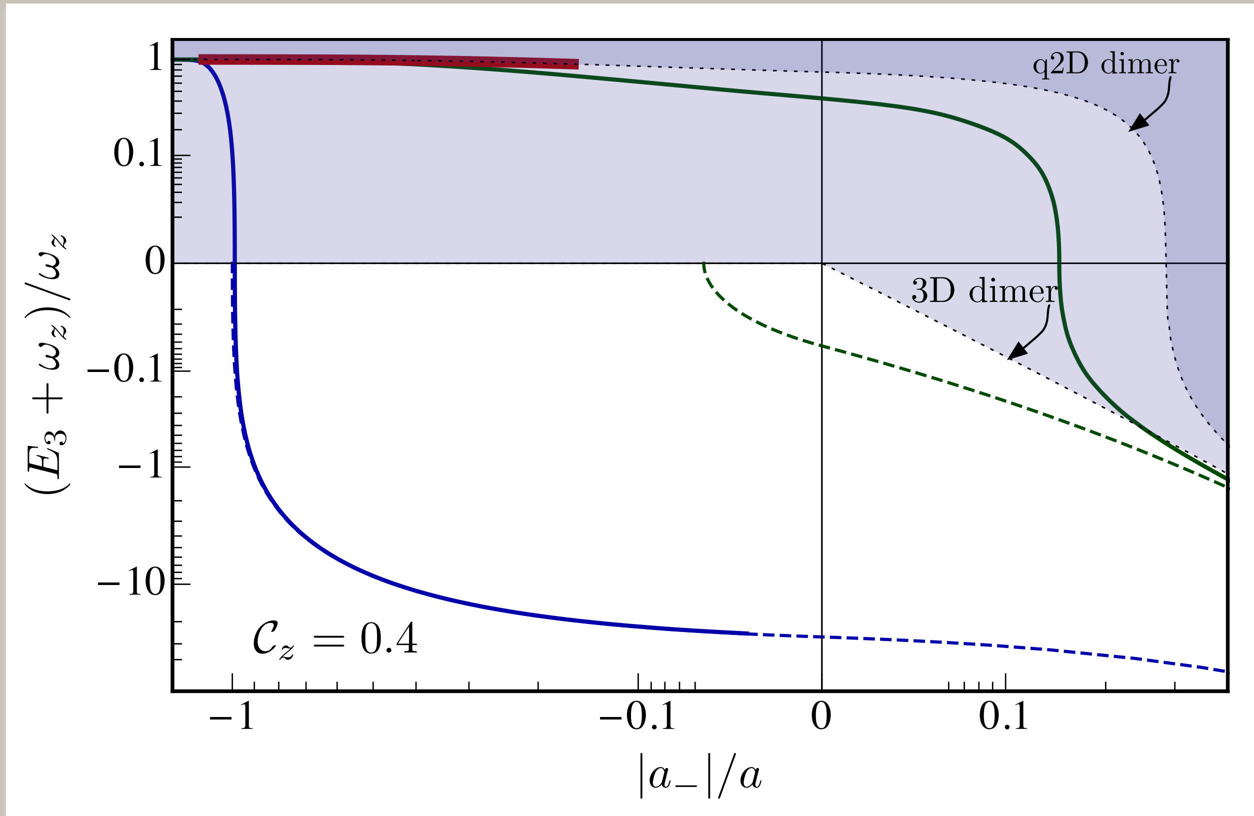
$$|\Psi(j,m)\rangle = |j+m, j-m\rangle$$

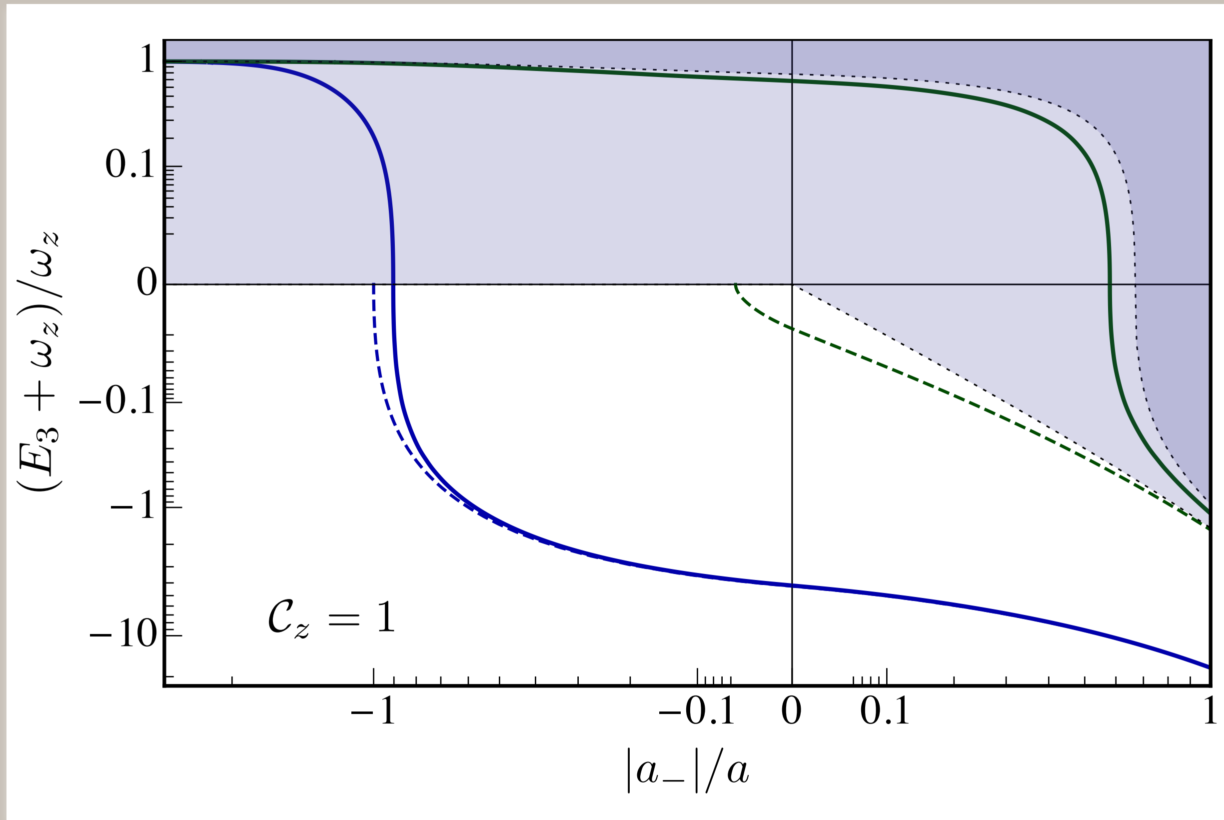
$$\langle N_1 n_{23} | N_2 n_{31} \rangle = d^{\left(\frac{N_1+n_{23}}{2}\right)}_{\frac{N_2-n_{31}}{2}, \frac{N_1-n_{23}}{2}}(2\pi/3)$$

*Wigner d-matrix:* $d^{(j)}_{m'm}(\beta) \equiv \langle \Psi(jm') | e^{-i\beta J_y} | \Psi(jm) \rangle$

# Trimer spectra

- Extra length scale $l_z$ removes weakest bound Efimov states
- Discrete scaling symmetry is only exists for $|a_-| \ll |a| \ll l_z$
- Moderate confinements $\mathcal{C}_z \sim 1$:



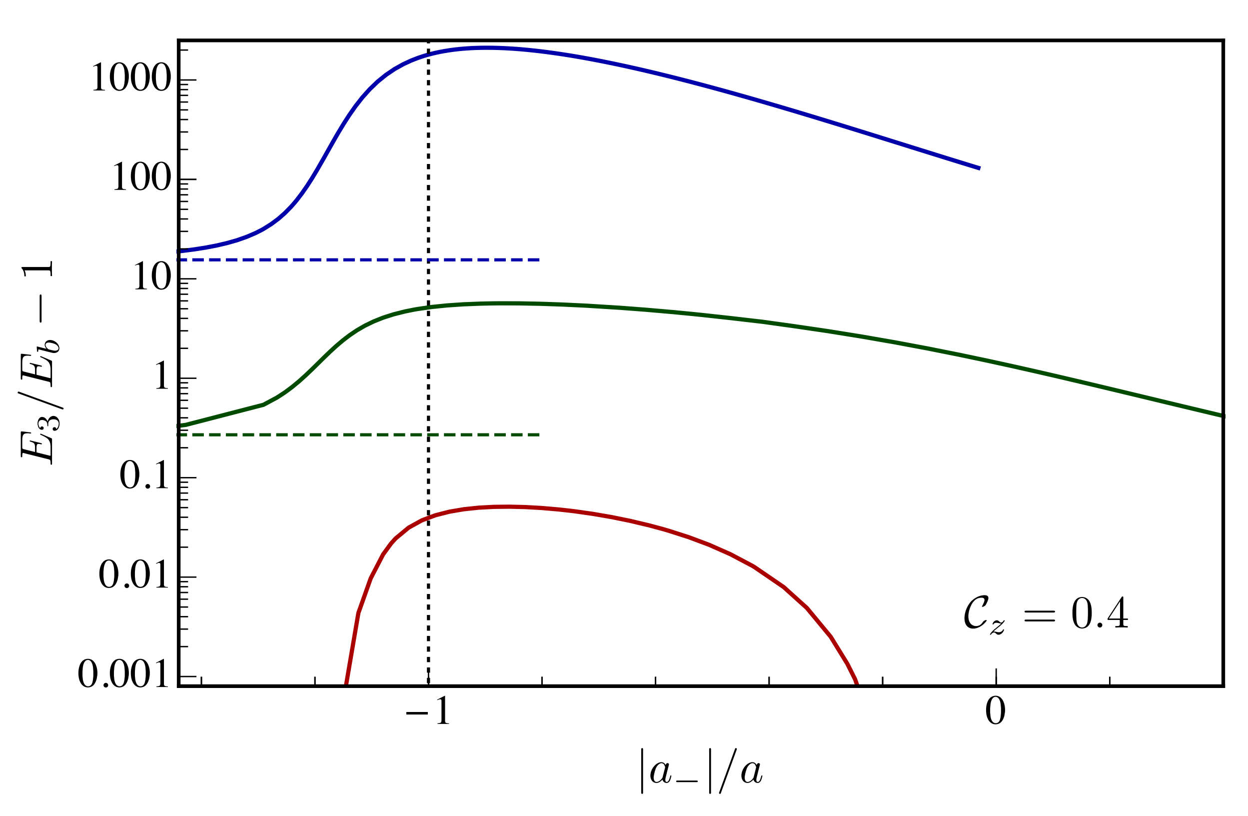



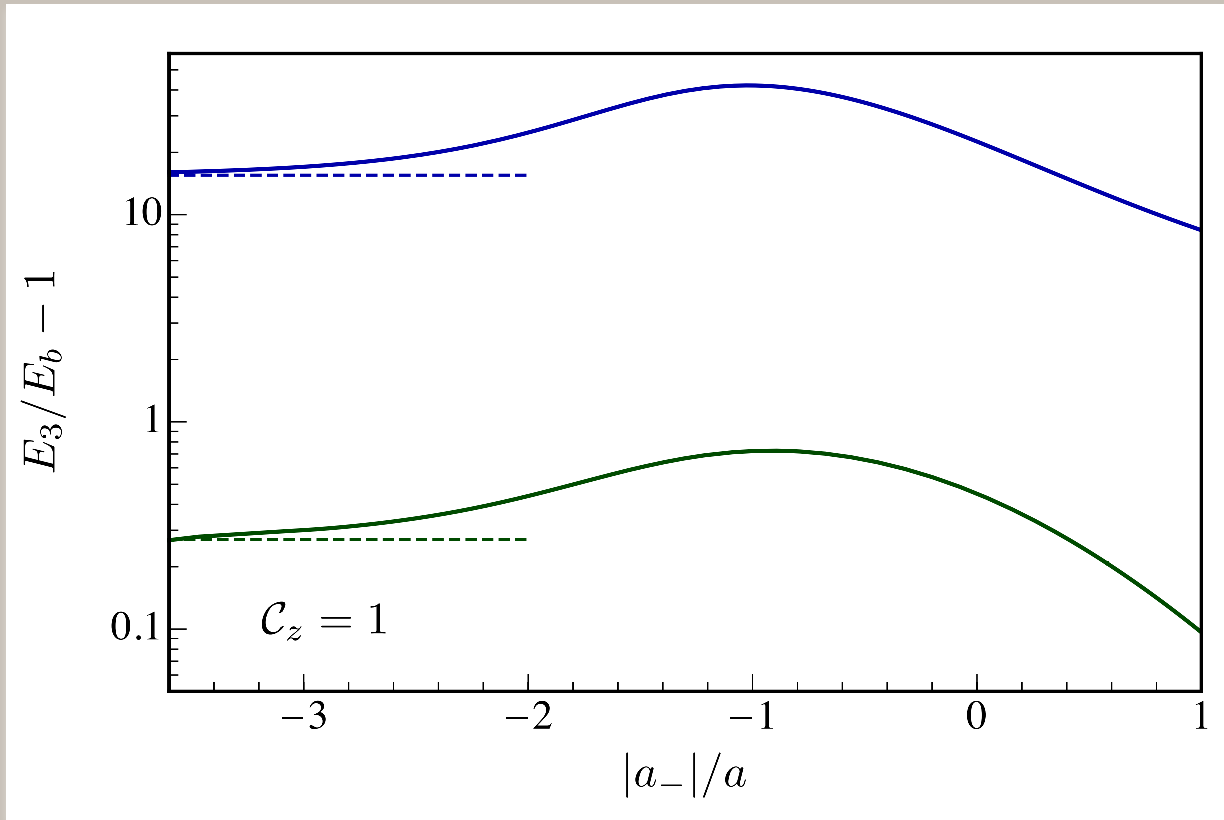

$^{133}$Cs: $\omega_z \approx 2\pi \times 5\text{kHz}$ $\omega_z \approx 2\pi \times 30\text{kHz}$

- Deepest trimer persists and exists above 3D continuum

$\mathcal{C}_z \equiv |a_-|/l_z$

# Trimer spectra



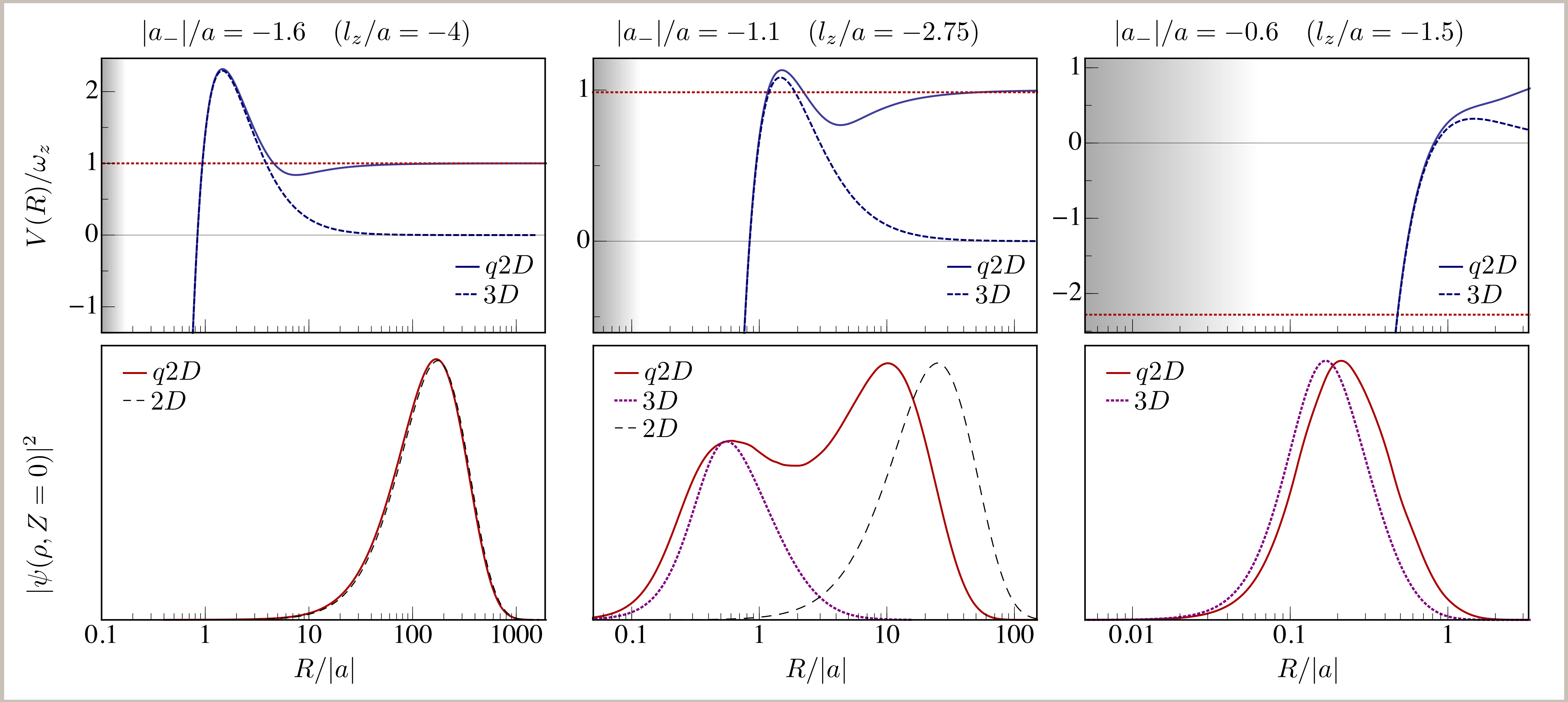

$^{133}$Cs: $\omega_z \approx 2\pi \times 5\text{kHz}$ $\omega_z \approx 2\pi \times 30\text{kHz}$

- 2D limit recovered for small and negative scattering length
  - Two deepest trimers stabilised
- Appearance of 3rd trimer for weaker confinement
  - Spectrum exhibits avoided crossings

$\mathcal{C}_z \equiv |a_-|/l_z$

# Hyperspherical potentials

$$R^2 = r_1^2 + r_2^2 + r_3^2$$

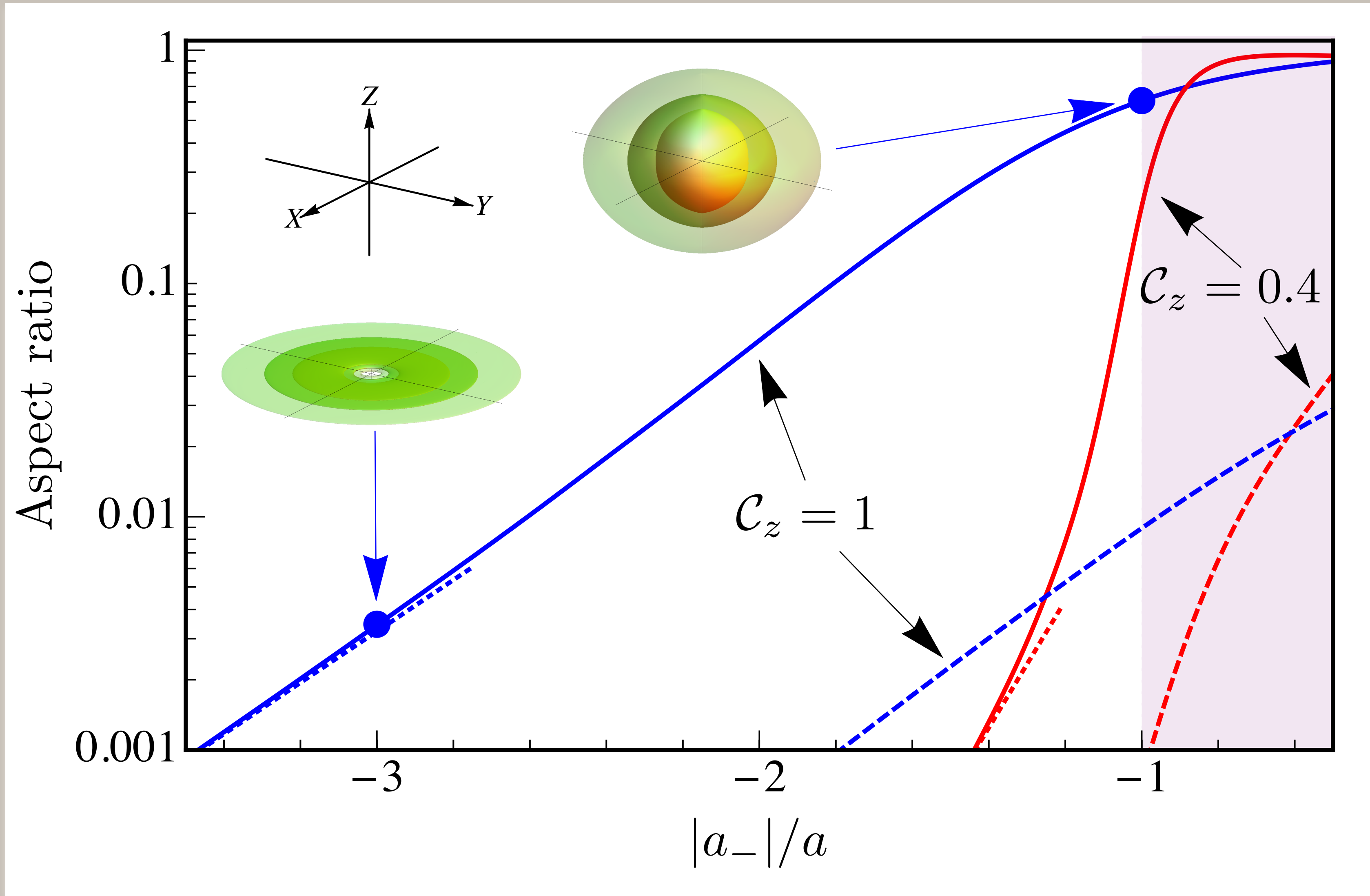



- When $l_z/a < -1$ potential has repulsive barrier $\sim 0.15/ma^2$
- For $R \gg l_z$, it resembles 2D hyperspherical potential
- Superposition of 2D and 3D trimers!

*2D theory: Nielsen, Fedorov & Jensen, Few-Body Systems 1999*

# Shape of the trimers

Efimov-like

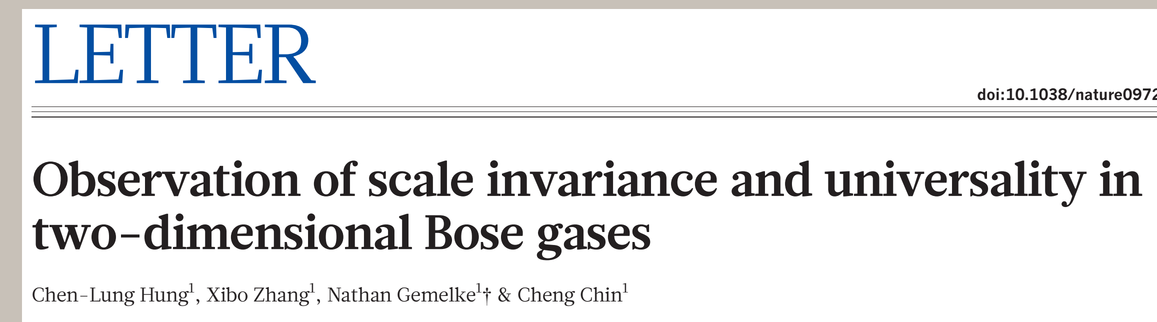



2D regime

*Levinsen, Massignan & MMP, arXiv:1402.1859*

# Experimental consequences

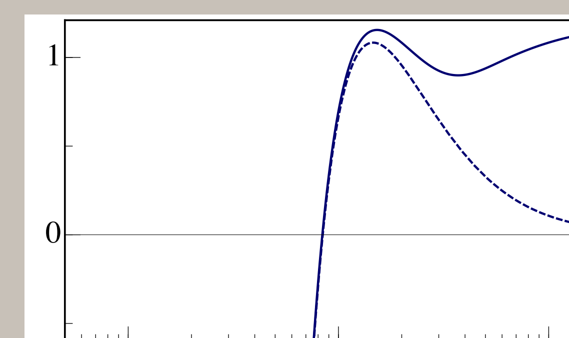
- Experiments are often performed at confinements even weaker than 5kHz
- 3D physics will thus impacts three-body correlations in realistic 2D gases when a < 0


LETTER

doi:10.1038/nature09722

**Observation of scale invariance and universality in two-dimensional Bose gases**

Chen-Lung Hung[1], Xibo Zhang[1], Nathan Gemelke[1]† & Cheng Chin[1]




- Confinement raises continuum by $\hbar\omega_z$
- Trimer resonance & loss peak disappear once $l_z/|a_-| \lesssim 2.5$

*Immediate consequence for the quest to observe discrete scaling symmetry:*

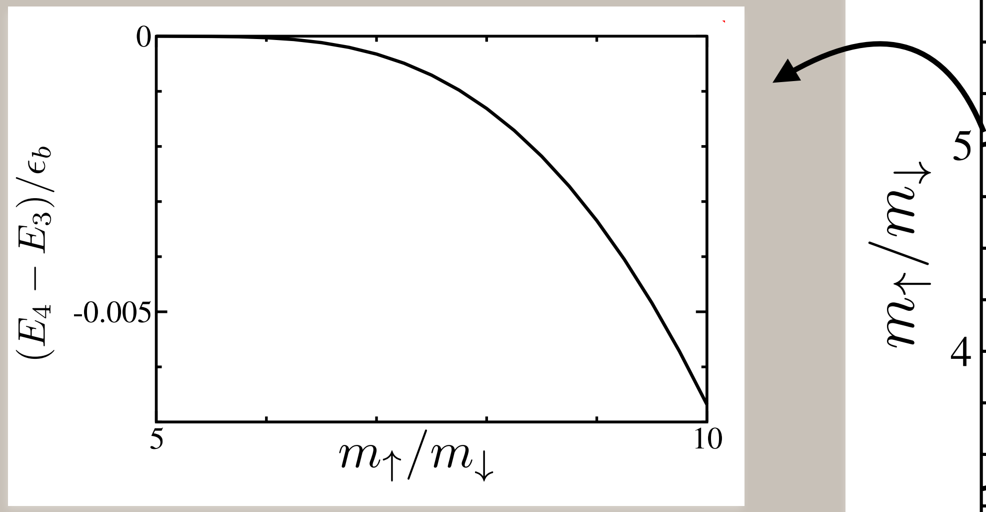
- *2nd trimer signature disappears once* $l_z/|a_-| \lesssim 22.7 * 2.5$*, corresponding to* $\omega_z \approx 2\pi \times 10\mathrm{Hz}$ *in the case of* $^{133}$Cs

*2D Bose gas experiments: Paris, Innsbruck, Chicago, Monash…*

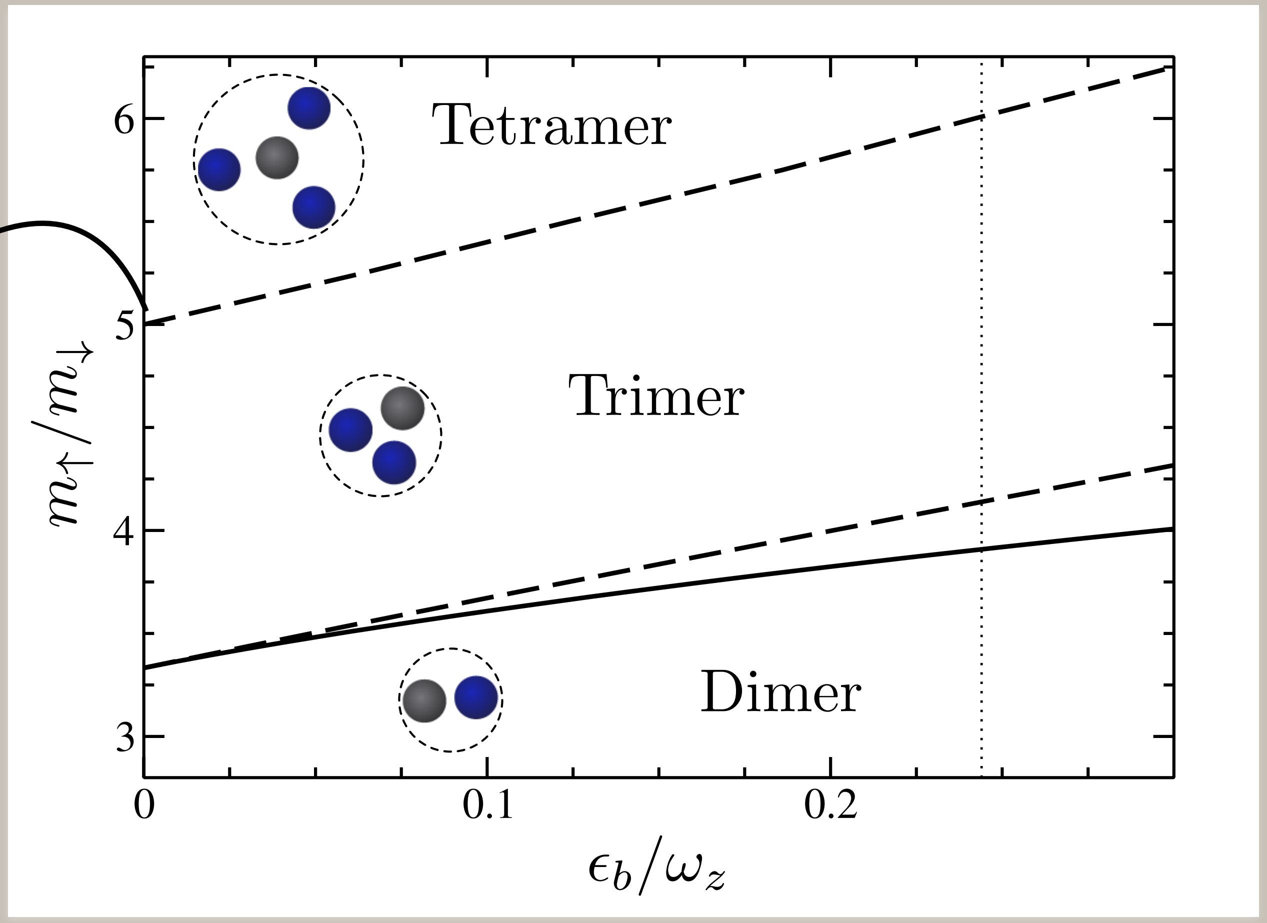
# Two-component Fermi systems

- Quasi-2D formalism can be generalised to more particles and mass-imbalanced fermions

- Existence of *universal* tetramer in 3+1 system
  – no dependence on UV cut-off




2D limit




Levinsen & MMP, PRL 110, 055304 (2013)

3D tetramer: Blume, PRL 2012

# Concluding remarks

- Quasi-2D confinement fundamentally impacts Efimov trimers
    - Deepest trimer remains 3D-like even under strong confinement
    - Hybridisation with 2D-like trimers stabilises the two deepest trimers for all negative scattering lengths
        - Use this to engineer more stable Efimov-like hybrid trimers?
    - Confinement is expected to have a similar effect on 4-body, 5-body etc. Efimov states
- Universal tetramers in the two-component Fermi system

# Acknowledgements

- Jesper Levinsen, Aarhus University
- Pietro Massignan, ICFO

*Levinsen, Massignan & MMP, arXiv:1402.1859*

*Levinsen & MMP, PRL 110, 055304 (2013)*

# Hyperspherical expansion

- Usual expression for the wave function:

$$\Psi(R,\Omega) = \frac{1}{R^{5/2}\sin(2\alpha_k)} \sum_{n=0}^{\infty} f_n(R)\Phi_n(R,\Omega)$$

- Further expand the angular part:

$$\Phi_n(R,\Omega) = \sum_{\mathbf{m}} \eta_{n\mathbf{m}}(R,\alpha_k)\; h_{\mathbf{m}}(R,\Omega)$$

$$h_{\mathbf{m}} = \tau_{m_1}(R\sin\alpha_k, \theta_{ij})\; \tau_{m_2}(R\cos\alpha_k, \theta_{k,ij})$$

where $\tau(X,\theta)$ obeys the equation:

$$\left(-\frac{l_z^2}{X^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{X^2}{l_z^2}\cos^2\theta\right)\tau = 2\mu\tau$$

- evolves into harmonic oscillator for $X \gg l_z$