Few-body universality: from Efimov effect to super Efimov effect

Yusuke Nishida (Tokyo Tech)

INT workshop on “few-body universality in atomic and nuclear physics”

May 12-16 (2014)

Plan of this talk

1. Universality of Efimov effect

   => Solid state physics

nature physics

ARTICLES

PUBLISHED ONLINE: 13 JANUARY 2013 | DOI: 10.1038/NPHYS2523

Efimov effect in quantum magnets

Yusuke Nishida*, Yasuyuki Kato and Cristian D. Batista



2. Novel few-body universality

   => Super Efimov effect

PRL **110,** 235301 (2013) PHYSICAL REVIEW LETTERS week ending 7 JUNE 2013

Super Efimov Effect of Resonantly Interacting Fermions in Two Dimensions

Yusuke Nishida,[1] Sergej Moroz,[2] and Dam Thanh Son[3]

[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[3]*Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637, USA*
(Received 18 January 2013; published 4 June 2013)



Few-body universality

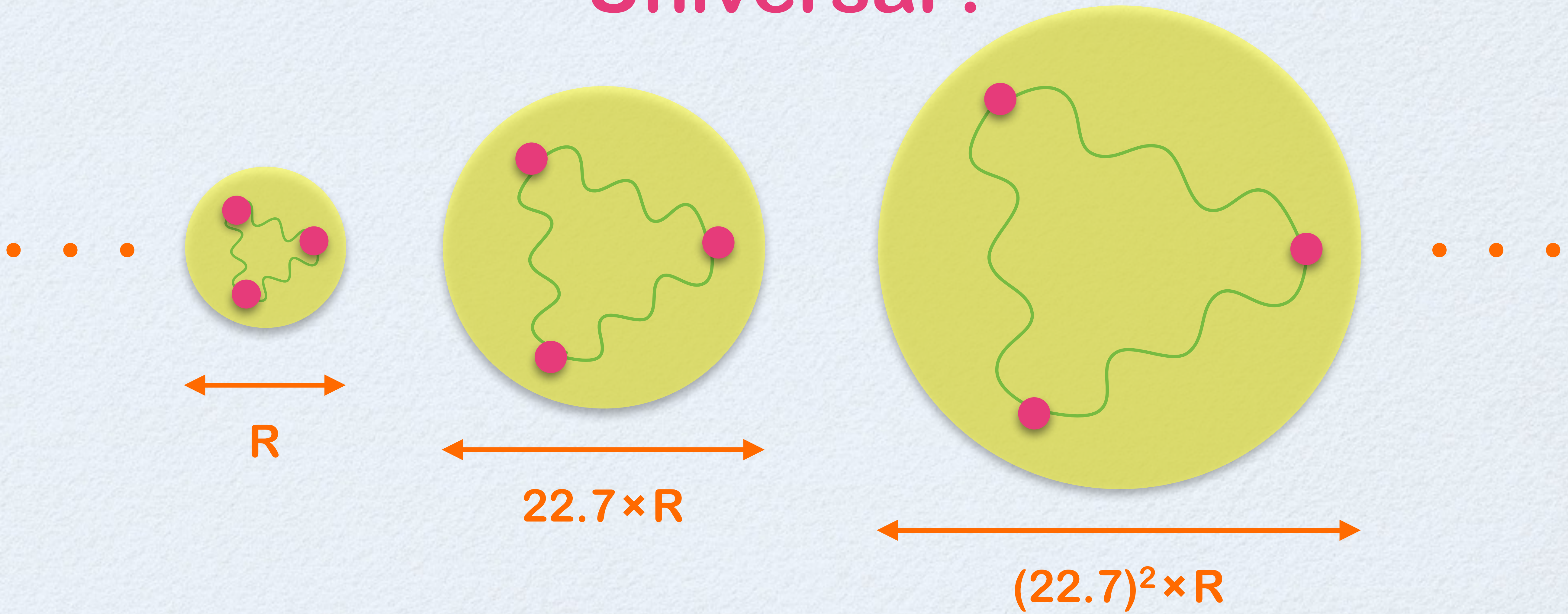
Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Universal !



Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Universal !

atomic physics

- helium atoms
- cold atoms
- …

nuclear physics

- nucleons
- halo nucleus
- …

Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Universal !

atomic physics

condensed matter

nuclear physics

Efimov effect in solid states ?

× electrons (fermions with long-range repulsion)

- bosonic collective excitations !?

Efimov effect in quantum magnets

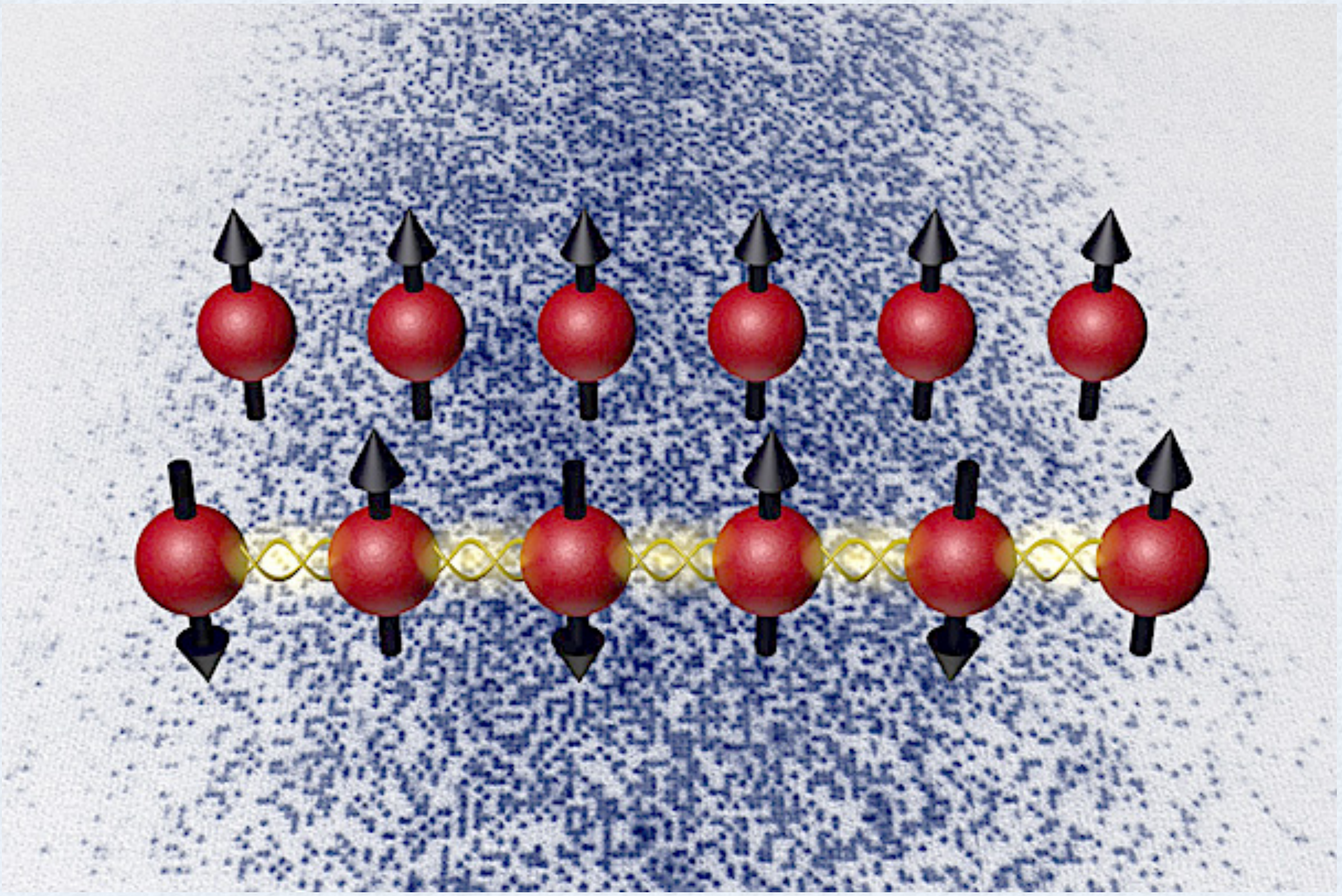
Quantum magnet

Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_{r}\Big[\sum_{\hat{e}}(J S_r^+ S_{r+\hat{e}}^- + J_z S_r^z S_{r+\hat{e}}^z) + D(S_r^z)^2 - B S_r^z\Big]$$

exchange anisotropy

single-ion anisotropy



Quantum magnet

Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_{r}\left[\sum_{\hat{e}}\left(J S_r^+ S_{r+\hat{e}}^- + J_z S_r^z S_{r+\hat{e}}^z\right) + D(S_r^z)^2 - B S_r^z\right]$$

exchange anisotropy — single-ion anisotropy

Spin-boson correspondence

fully polarized state (B→∞) — No boson = vacuum

N spin-flips — N bosons = magnons

Quantum magnet

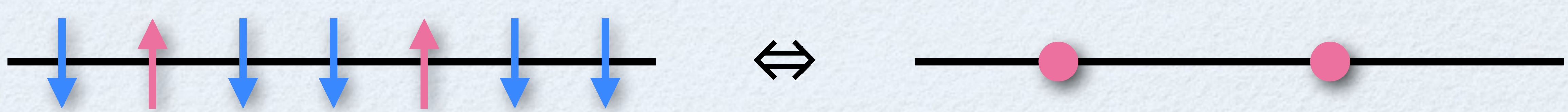
Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_{r}\Big[\sum_{\hat{e}}(J S_r^+ S_{r+\hat{e}}^- + J_z S_r^z S_{r+\hat{e}}^z) + D(S_r^z)^2 - B S_r^z\Big]$$

xy-exchange coupling
⇔ hopping

single-ion anisotropy
⇔ on-site attraction

z-exchange coupling
⇔ neighbor attraction



N spin-flips ⇔ N bosons = magnons

Quantum magnet

Anisotropic Heisenberg model on a 3D lattice

$$H = -\sum_{r}\left[\sum_{\hat{e}}(J S_r^+ S_{r+\hat{e}}^- + J_z S_r^z S_{r+\hat{e}}^z) + D(S_r^z)^2 - B S_r^z\right]$$

xy-exchange coupling
⇔ hopping

single-ion anisotropy
⇔ on-site attraction

z-exchange coupling
⇔ neighbor attraction

Tune these couplings to induce
scattering resonance between two magnons
=> Three magnons show the Efimov effect

Two-magnon resonance

Schrödinger equation for two magnons

$$E\Psi(r_1, r_2) = \Big[SJ \sum_{\hat{e}} (2 - \nabla_{1\hat{e}} - \nabla_{2\hat{e}})$$

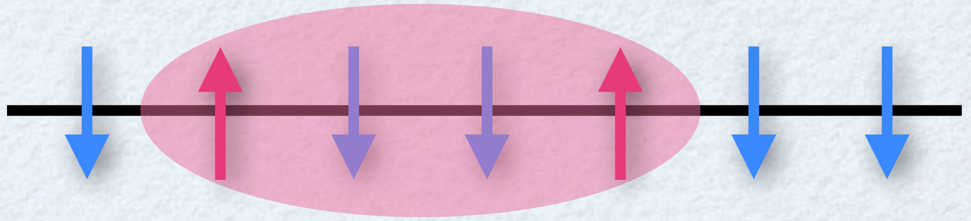
← hopping

$$+ J \sum_{\hat{e}} \delta_{r_1,r_2} \nabla_{2\hat{e}} - J_z \sum_{\hat{e}} \delta_{r_1,r_2+\hat{e}} - 2D\delta_{r_1,r_2} \Big] \Psi(r_1, r_2)$$

neighbor/on-site attraction

Scattering length between two magnons

$$\lim_{|r_1 - r_2| \to \infty} \Psi(r_1, r_2)\Big|_{E=0} \to \frac{1}{|r_1 - r_2|} - \frac{1}{a_s}$$



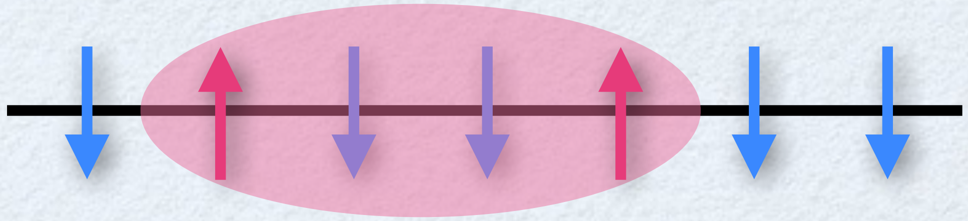
Two-magnon resonance

Scattering length between two magnons

$$\frac{a_s}{a} = \frac{\frac{3}{2\pi}\left[1 - \frac{D}{3J} - \frac{J_z}{J}\left(1 - \frac{D}{6SJ}\right)\right]}{2S - 1 + \frac{J_z}{J}\left(1 - \frac{D}{6SJ}\right) + 1.52\left[1 - \frac{D}{3J} - \frac{J_z}{J}\left(1 - \frac{D}{6SJ}\right)\right]}$$

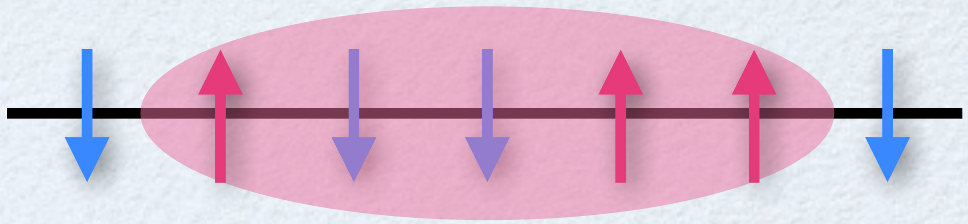
→ **Two-magnon resonance** ($a_s \to \infty$)

- $J_z/J = 2.94$ (spin-1/2)
- $J_z/J = 4.87$ (spin-1, D=0)
- $D/J = 4.77$ (spin-1, ferro $J_z=J>0$)
- $D/J = 5.13$ (spin-1, antiferro $J_z=J<0$)
- …



Three-magnon spectrum

At the resonance, three magnons form bound states with binding energies E_n



- Spin-1/2

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-2.09×10^{-1}	—
1	-4.15×10^{-4}	22.4
2	-8.08×10^{-7}	22.7

- Spin-1, D=0

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-5.16×10^{-1}	—
1	-1.02×10^{-3}	22.4
2	-2.00×10^{-6}	22.7

- Spin-1, $J_z=J>0$

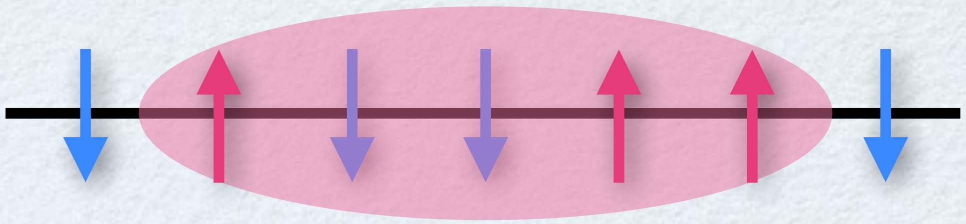
n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-5.50×10^{-2}	—
1	-1.16×10^{-4}	21.8

- Spin-1, $J_z=J<0$

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-4.36×10^{-3}	—
1	-8.88×10^{-6}	22.2

Three-magnon spectrum

At the resonance, three magnons form bound states with binding energies E_n



- Spin-1/2

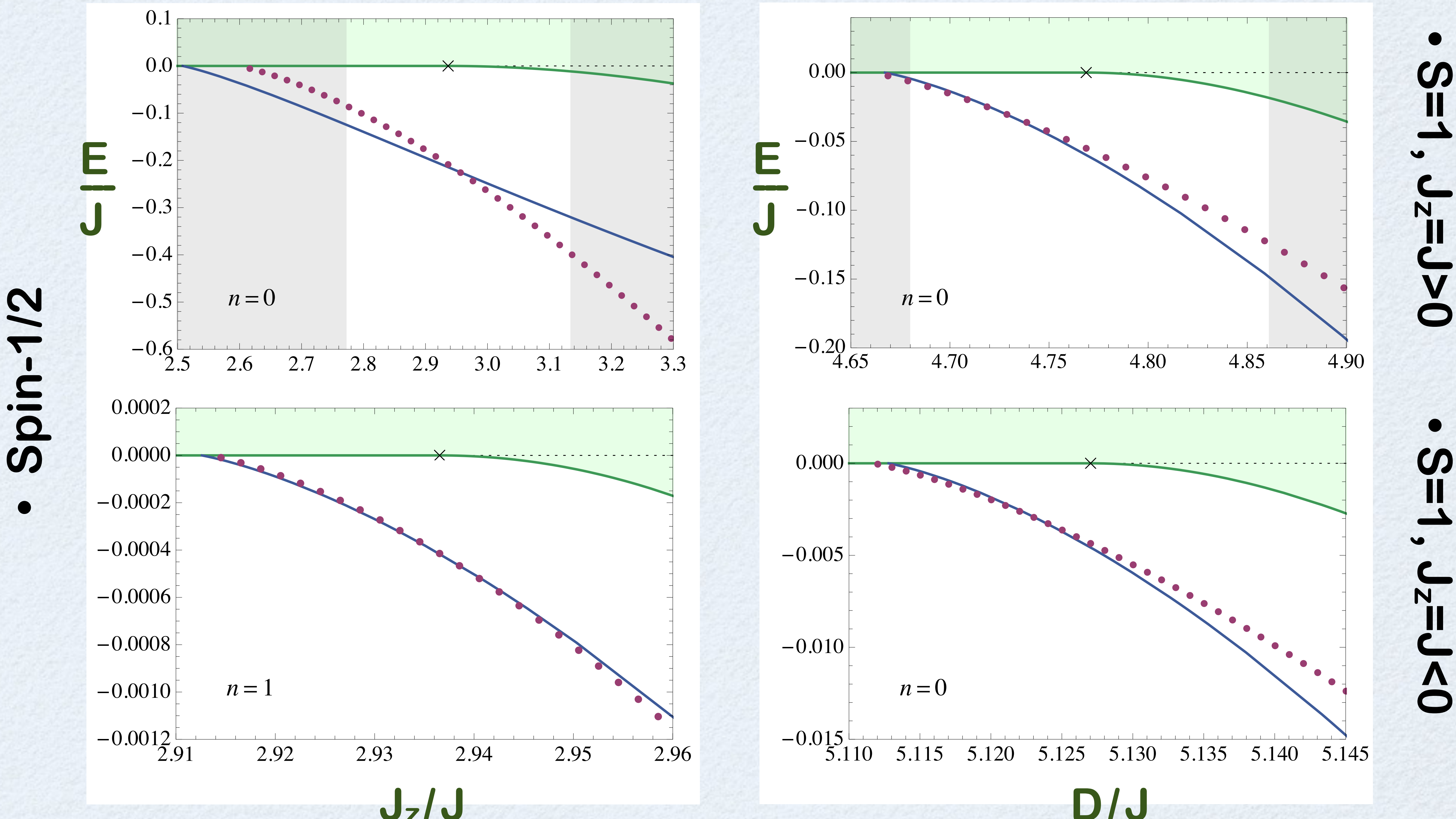
n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-2.09×10^{-1}	—
1	-4.15×10^{-4}	22.4
2	-8.08×10^{-7}	22.7

- Spin-1, D=0

n	E_n/J	$\sqrt{E_{n-1}/E_n}$
0	-5.16×10^{-1}	—
1	-1.02×10^{-3}	22.4
2	-2.00×10^{-6}	22.7

➡ Universal scaling law by ~ 22.7 confirms they are Efimov states !

Three-magnon spectrum



Agree with universal prediction : $E_n = -\lambda^{-2n}\frac{\kappa_*^2}{m}F\left(\frac{\lambda^n}{\kappa_* a_s}\right)$

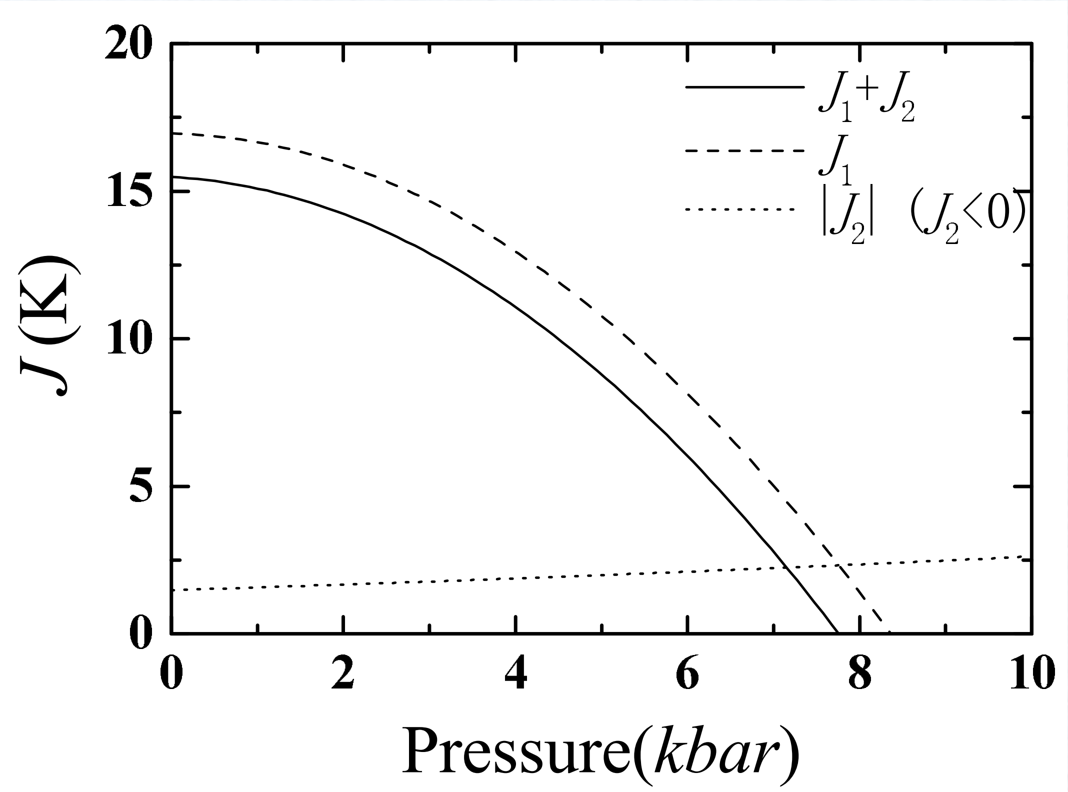
Toward experimental realization

1. Find a good compound
   whose anisotropy is close to the critical value

   E.g. Ni-based organic ferromagnet with D/J~3 (critical 4.8)

   R. Koch et al., Phys. Rev. B 67, 094407 (2003)

C.f. TDAE-C_{60}



T. Kawamoto et al, JPSJ (2001)

2. Tune the exchange coupling
   with pressure to induce
   the two-magnon resonance

3. Observe the Efimov states
   of three magnons with

- absorption spectroscopy
- inelastic neutron scattering
- electron spin resonance

[see Y.N., PRB88, 224402 (2013)]

Find interested experimentalists !

Atomic vs magnetic systems

Efimov effect: universality, discrete scale invariance, RG limit cycle

atomic physics

nuclear physics

condensed matter

Efimov effect in quantum magnets induced by

- exchange anisotropy
- single-ion anisotropy
- spatial anisotropy
- frustration


nature physics | ARTICLES

PUBLISHED ONLINE: 13 JANUARY 2013 | DOI: 10.1038/NPHYS2523

Efimov effect in quantum magnets

Yusuke Nishida*, Yasuyuki Kato and Cristian D. Batista



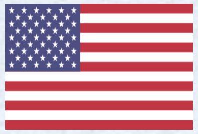
Atomic vs magnetic systems

Efimov effect: universality, discrete scale invariance, RG limit cycle

atomic physics

nuclear physics

condensed matter

Atomic BEC (1995 ) — Magnon BEC (1999 )

Efimov effect (2006 ) — Efimov effect (201? ?)

Atomic vs magnetic systems

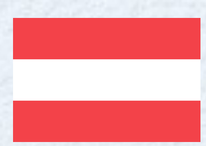
Efimov effect: universality, discrete scale invariance, RG limit cycle

atomic physics

condensed matter

Atomic BEC (1995 )

Magnon BEC (1999 )

Efimov effect (2006 )

Efimov effect (201? ?)

New link between atomic and magnetic systems

Unitary magnon gas ?!

Novel universality: Super Efimov effect

Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Efimov effect in other systems ?

No, only in 3D with s-wave resonance

	s-wave	p-wave	d-wave
3D	O	x	x
2D	x	x	x
1D	x	x	

Y.N. & S.Tan, Few-Body Syst

Y.N. & D.Lee Phys Rev A

Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Different universality in other systems ?

Yes, super Efimov effect in 2D with p-wave !

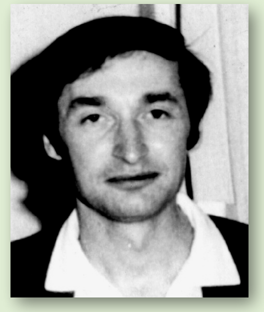
	s-wave	p-wave	d-wave
3D	O	x	x
2D	x	!x!	x
1D	x	x	

Y.N. & S.Tan, Few-Body Syst

Y.N. & D.Lee Phys Rev A

Few-body universality

Efimov effect



- 3 bosons
- 3 dimensions
- s-wave resonance

exponential scaling

$$E_n \sim e^{-2\pi n}$$

Super Efimov effect

- 3 fermions
- 2 dimensions
- p-wave resonance

"doubly" exponential

$$E_n \sim e^{-2e^{3\pi n/4}}$$

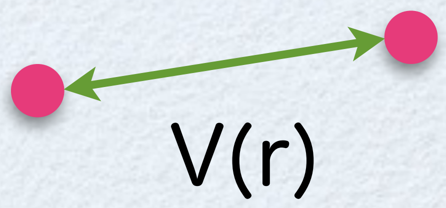
PRL 110, 235301 (2013) PHYSICAL REVIEW LETTERS week ending 7 JUNE 2013

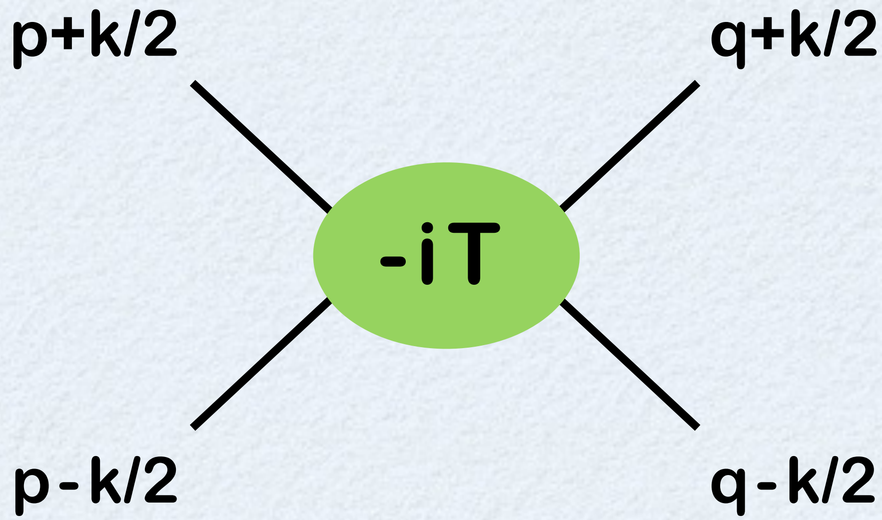
Super Efimov Effect of Resonantly Interacting Fermions in Two Dimensions

Yusuke Nishida,[1] Sergej Moroz,[2] and Dam Thanh Son[3]
[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[3]*Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637, USA*
(Received 18 January 2013; published 4 June 2013)

P-wave scattering in 2D

Two fermions with short-range potential





⇒ Effective range expansion

Cf. H.-W. Hammer & D. Lee
Ann. Phys. 325, 2212 (2010)

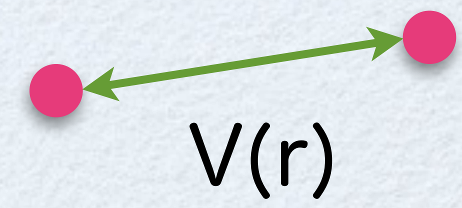
$$-iT = \frac{2i}{m}\,\frac{\vec{p}\cdot\vec{q}}{-\frac{1}{a} - \frac{m\varepsilon}{\pi}\ln\left(-\frac{\Lambda^2}{m\varepsilon}\right) + \sum_{n=2}^{\infty} c_n\,(m\varepsilon)^n}$$

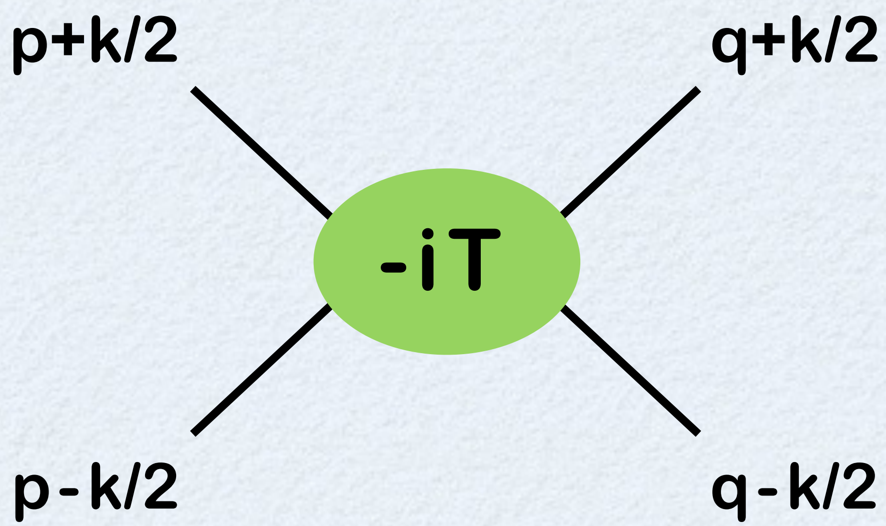
scattering “length” (a) — effective “range” (Λ^2)

collision energy $\varepsilon = E - \dfrac{k^2}{4m} + i0^+$

P-wave scattering in 2D

Two fermions with short-range potential





⇒ Effective range expansion

Cf. H.-W. Hammer & D. Lee
Ann. Phys. 325, 2212 (2010)

$$-iT = \frac{2i}{m} \frac{\vec{p} \cdot \vec{q}}{-\frac{1}{a} - \frac{m\varepsilon}{\pi} \ln\left(-\frac{\Lambda^2}{m\varepsilon}\right) + \sum_{n=2}^{\infty} c_n (m\varepsilon)^n}$$

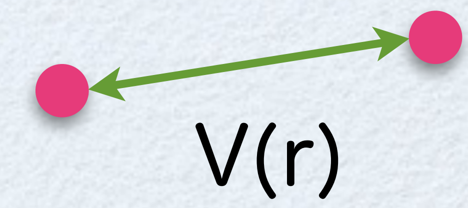
resonance
($a \to \infty$)

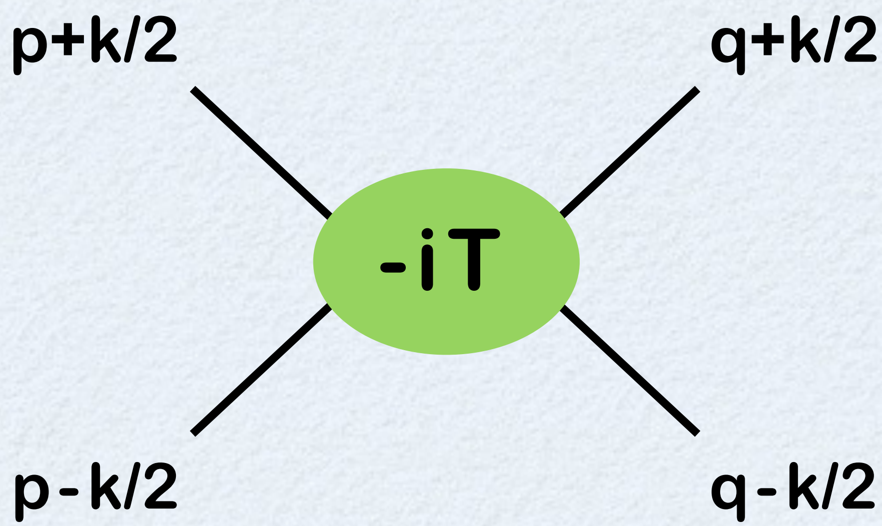
low-energy
($\varepsilon \to 0$)

collision energy $\varepsilon = E - \frac{k^2}{4m} + i0^+$

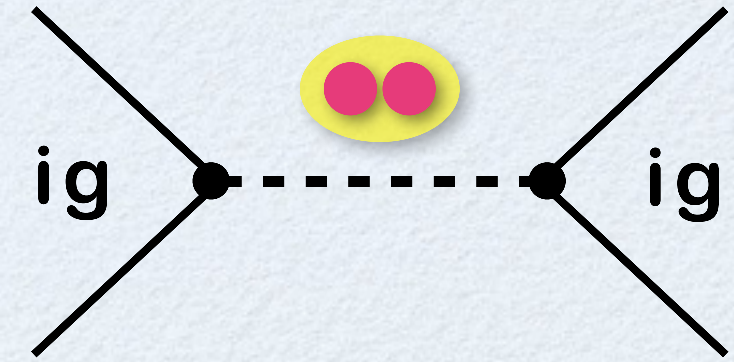
P-wave scattering in 2D

Two fermions with short-range potential



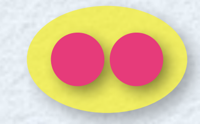


resonance → low-energy



⇒ Effective range expansion

$$-iT \to -\frac{2\pi\,\vec{p}\cdot\vec{q}}{m^2\ln\left(-\frac{\Lambda^2}{m\varepsilon}\right)} \times \frac{i}{E-\frac{k^2}{4m}+i0^+}$$

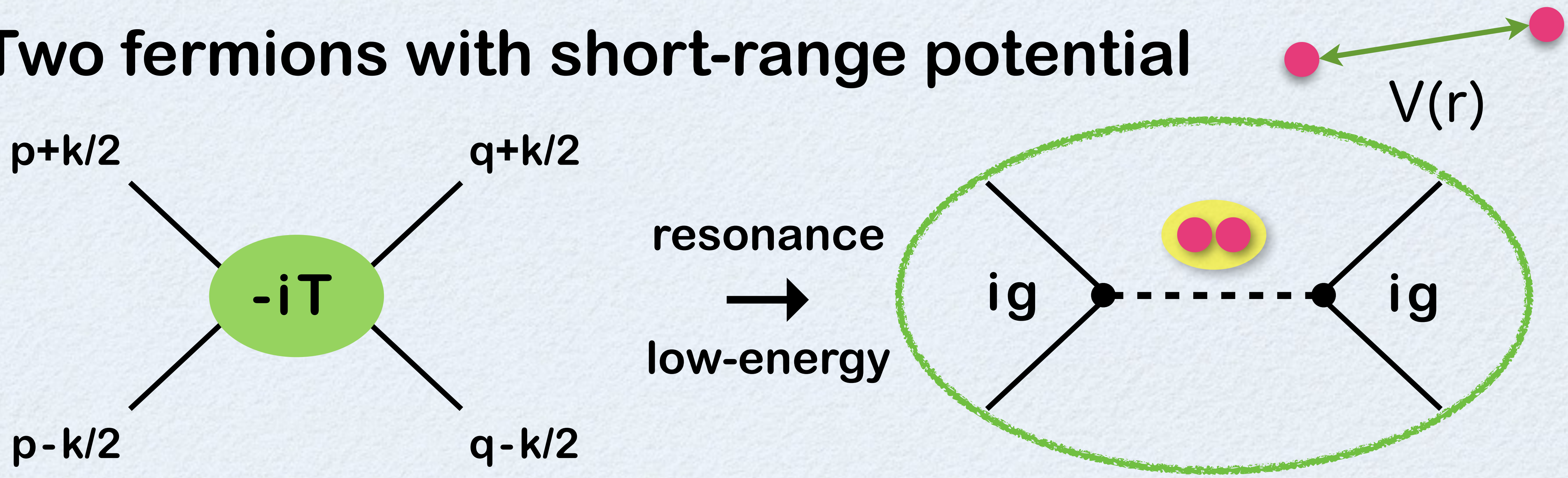
= $(ig)^2\, p\cdot q$ propagator of dimer 

“running” coupling

(logarithmic decrease toward low-energy $p/\Lambda \to 0$)

P-wave scattering in 2D

Two fermions with short-range potential



⇒ Low-energy effective field theory

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi + \sum_{\pm} \left[\phi_\pm^\dagger \left(i\partial_t + \frac{\nabla^2}{4m} \right) \phi_\pm \right.$$

$$\left. + g\, \phi_\pm^\dagger \psi \, (-i) \, (\nabla_x \pm i \nabla_y) \, \psi + \text{h.c.} \right]$$

dimer field $\Phi_\pm$ couples to two fermions ψ

with orbital angular momentum L=±1

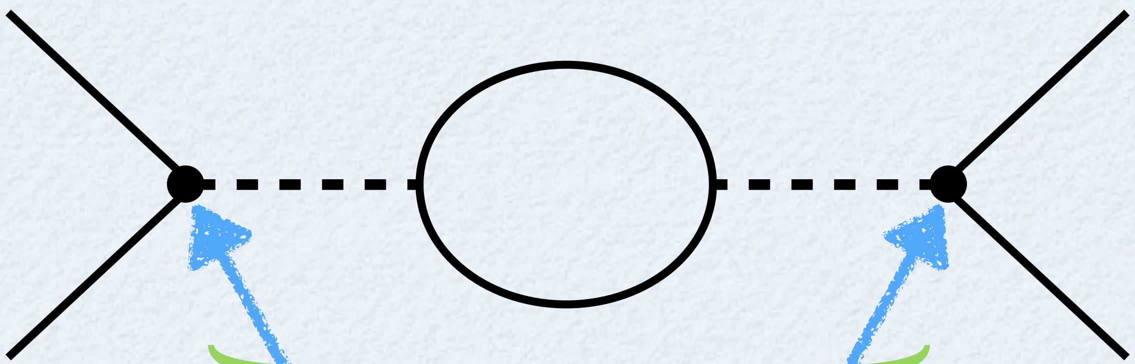
RG in 2-body sector

Low-energy effective field theory

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\boldsymbol{\nabla}^2}{2m} \right) \psi + \sum_{\pm} \Big[\phi_\pm^\dagger \left(i\partial_t + \frac{\boldsymbol{\nabla}^2}{4m} \right) \phi_\pm$$

$$+ g\, \phi_\pm^\dagger \psi\, (-i)\, (\nabla_x \pm i\nabla_y)\, \psi + \text{h. c.} \Big] + \cdots$$

marginal coupling (g) — irrelevant ($\cdots$)



$$\frac{1 - \frac{g^2}{\pi} \ln \frac{\Lambda}{e^{-s}\Lambda}}{E - \frac{k^2}{4m} + i0^+}$$

($e^{-s}\Lambda < p < \Lambda$ integrated out)

RG equation $\quad \dfrac{dg}{ds} = -\dfrac{g^3}{2\pi}$

$$\Rightarrow \quad g^2(s) = \frac{1}{\frac{1}{g^2(0)} + \frac{s}{\pi}} \to \frac{\pi}{s}$$

logarithmical decrease toward low-energy $s \to \infty$

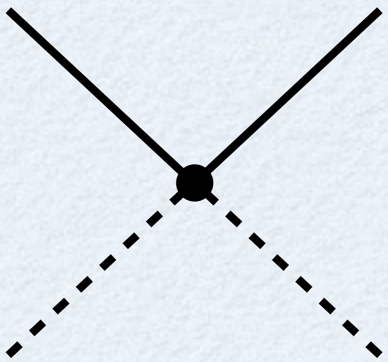
RG in 3-body sector

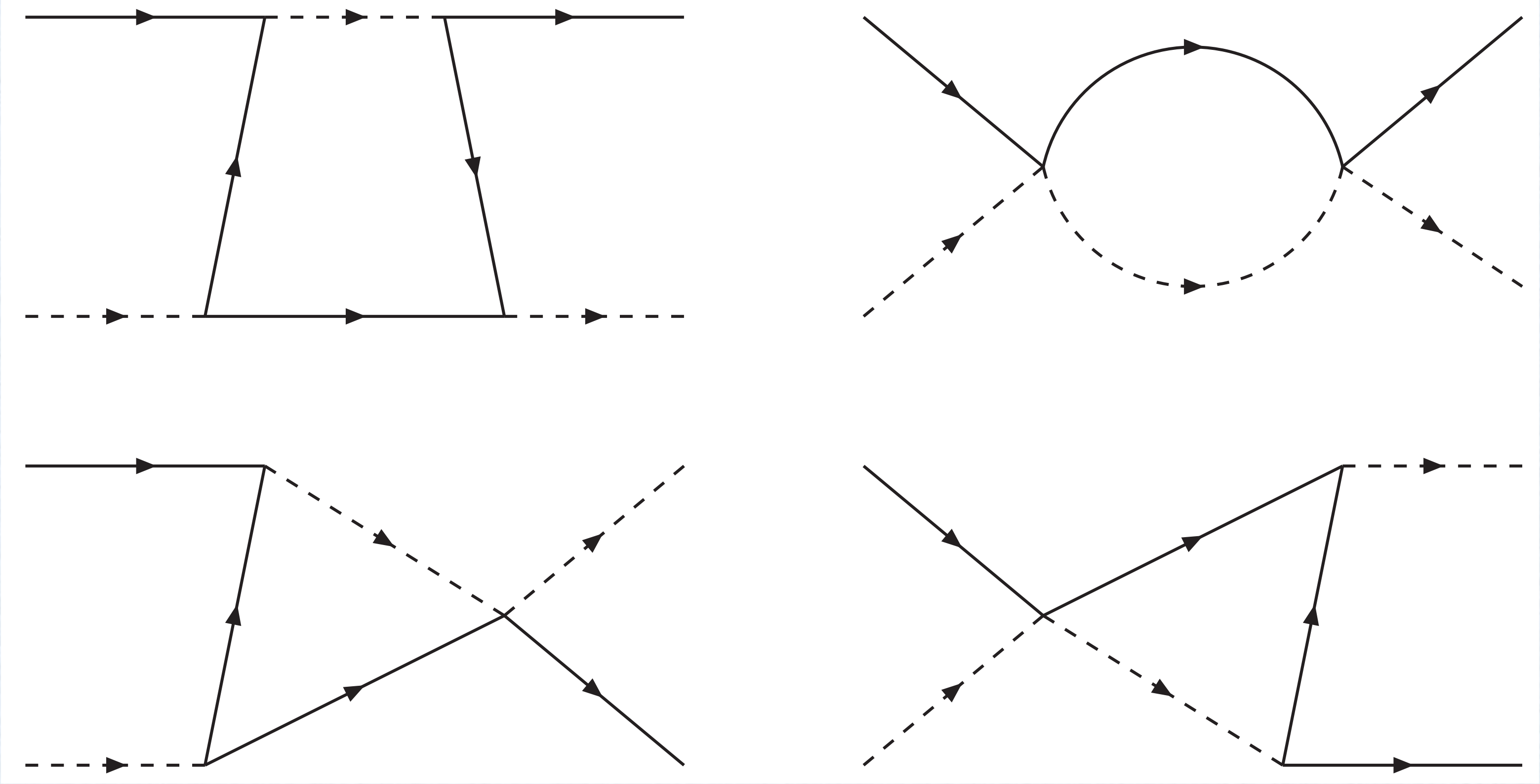
3-body problem ⇔ fermion + dimer scattering

$$\mathcal{L}_{3\text{-body}} = v_3 \sum_{a=\pm} \psi^\dagger \phi_a^\dagger \phi_a \psi + \cdots$$

irrelevant

marginal coupling renormalized by





⇒ RG equation $$\frac{dv_3}{ds} = \frac{16}{3\pi} g^4 - \frac{11}{3\pi} g^2 v_3 + \frac{2}{3\pi} v_3^2$$

RG in 3-body sector

3-body problem ⇔ fermion+dimer scattering

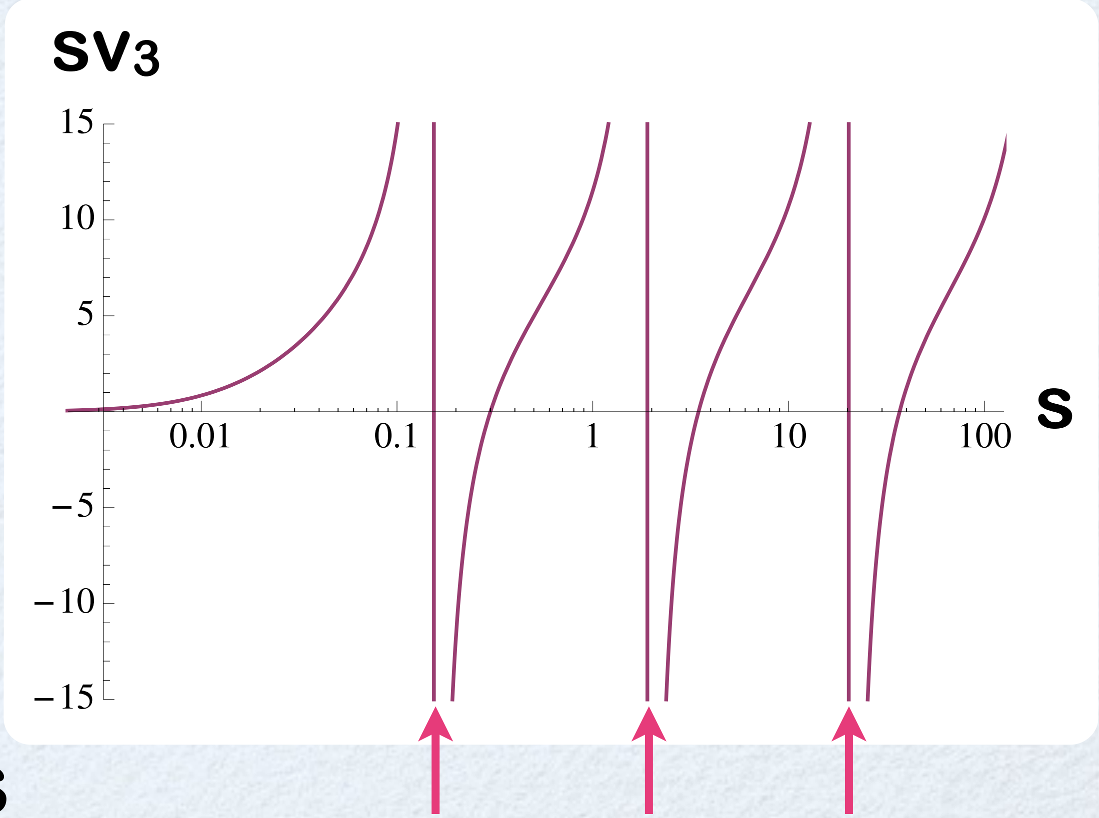
$$\mathcal{L}_{3\text{-body}} = v_3 \sum_{a=\pm} \psi^\dagger \phi_a^\dagger \phi_a \psi + \cdots$$

irrelevant

marginal coupling @ low-energy limit s→∞

$$v_3(s) \to \frac{2\pi}{s}\left\{1 - \cot\left[\frac{4}{3}(\ln s - \theta)\right]\right\}$$

non-universal



diverges at $\ln s = \frac{3\pi n}{4} + \theta$

=> characteristic energy scales

$$E_n \propto \frac{\Lambda^2}{m} e^{-2e^{3\pi n/4+\theta}}$$

Super Efimov effect !

Confirmation by model

$$H = \int \frac{dk}{(2\pi)^2} \frac{k^2}{2m} \psi_k^\dagger \psi_k$$

$$- v_0 \sum_{a=\pm} \int \frac{dk\,dp\,dq}{(2\pi)^6} \underbrace{\psi^\dagger_{\frac{k}{2}+p} \chi_a(p) \psi^\dagger_{\frac{k}{2}-p}}_{} \times \underbrace{\psi_{\frac{k}{2}-q} \chi_{-a}(q) \psi_{\frac{k}{2}+q}}_{}$$

resonance (a→∞)

$$\chi_\pm(p) = (p_x \pm i p_y)\, e^{-p^2/(2\Lambda^2)}$$

Spinless fermions with a separable potential

3-body binding energies $\lambda_n = \ln\ln(mE_n/\Lambda^2)^{-1/2}$

=> solve STM equation numerically

$$Z_a(p) = -\int \frac{dq}{2\pi} \frac{(p+2q)_{-a}\, e^{-(5p^2+5q^2+8p\cdot q)/(8\Lambda^2)}}{p^2+q^2+p\cdot q+\kappa^2} \times \frac{\sum_{b=\pm}(2p+q)_b\, Z_b(q)}{(\frac{3}{4}q^2+\kappa^2)\, e^{(\frac{3}{4}q^2+\kappa^2)/\Lambda^2}\, \mathrm{E}_1[(\frac{3}{4}q^2+\kappa^2)/\Lambda^2]}$$

Confirmation by model

Spinless fermions
with a separable potential

$$H = \int \frac{dk}{(2\pi)^2} \frac{k^2}{2m} \psi_k^\dagger \psi_k$$

$$- v_0 \sum_{a=\pm} \int \frac{dk\,dp\,dq}{(2\pi)^6} \psi^\dagger_{\frac{k}{2}+p} \chi_a(p) \psi^\dagger_{\frac{k}{2}-p} \times \psi_{\frac{k}{2}-q} \chi_{-a}(q) \psi_{\frac{k}{2}+q}$$

resonance (a→∞)

$$\chi_\pm(p) = (p_x \pm i p_y)\, e^{-p^2/(2\Lambda^2)}$$

3-body binding energies $\lambda_n = \ln\ln(mE_n/\Lambda^2)^{-1/2}$

n	λ_n	$\lambda_n - \lambda_{n-1}$
0	0.5632	—
1	2.770	2.207
2	5.078	2.308
3	7.430	2.352
4	9.785	2.355
5	12.141	2.356
∞	—	2.35619 ← $3\pi/4$

Super Efimov effect !

⇒ doubly exponential scaling $mE_n/\Lambda^2 \propto e^{-2e^{3\pi n/4+\theta}}$

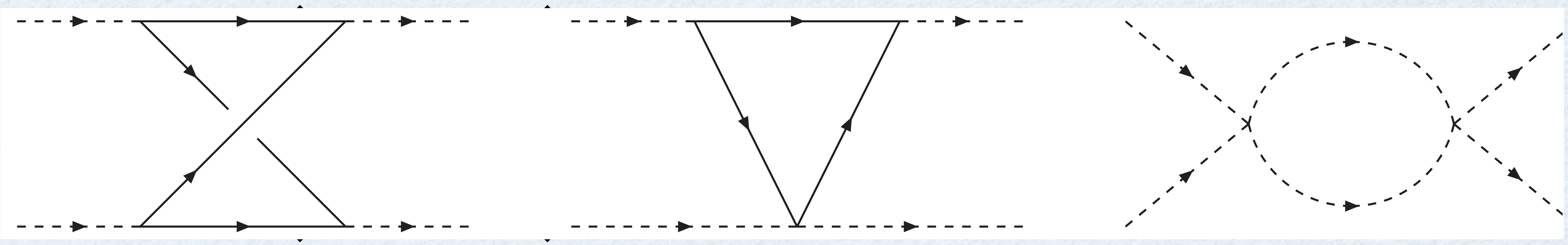
RG in 4-body sector

4-body problem ⇔ dimer+dimer scattering

$$\mathcal{L}_{4\text{-body}} = \sum_{a=\pm}\left[v_4 \phi_a^\dagger \phi_{-a}^\dagger \phi_{-a}\phi_a + v_4' \phi_a^\dagger \phi_a^\dagger \phi_a \phi_a\right] + \cdots$$

irrelevant

marginal couplings renormalized by



⇒ RG equations

$$\frac{dv_4}{ds} = -\frac{8}{\pi}g^4 + \frac{2}{\pi}g^2 v_3 - \frac{2}{\pi}g^2 v_4 + \frac{2}{\pi}v_4^2$$

$$\frac{dv_4'}{ds} = -\frac{4}{\pi}g^4 + \frac{2}{\pi}g^2 v_3 - \frac{2}{\pi}g^2 v_4' + \frac{2}{\pi}v_4'^2$$

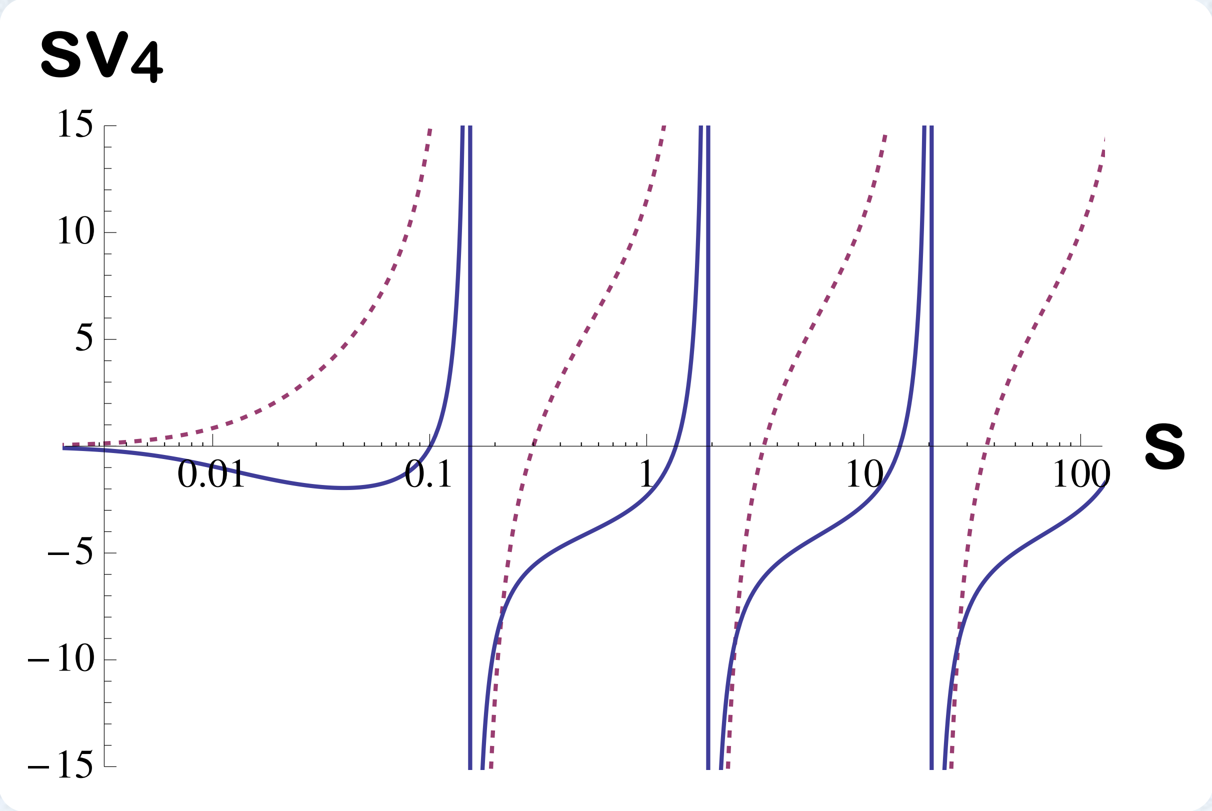
RG in 4-body sector

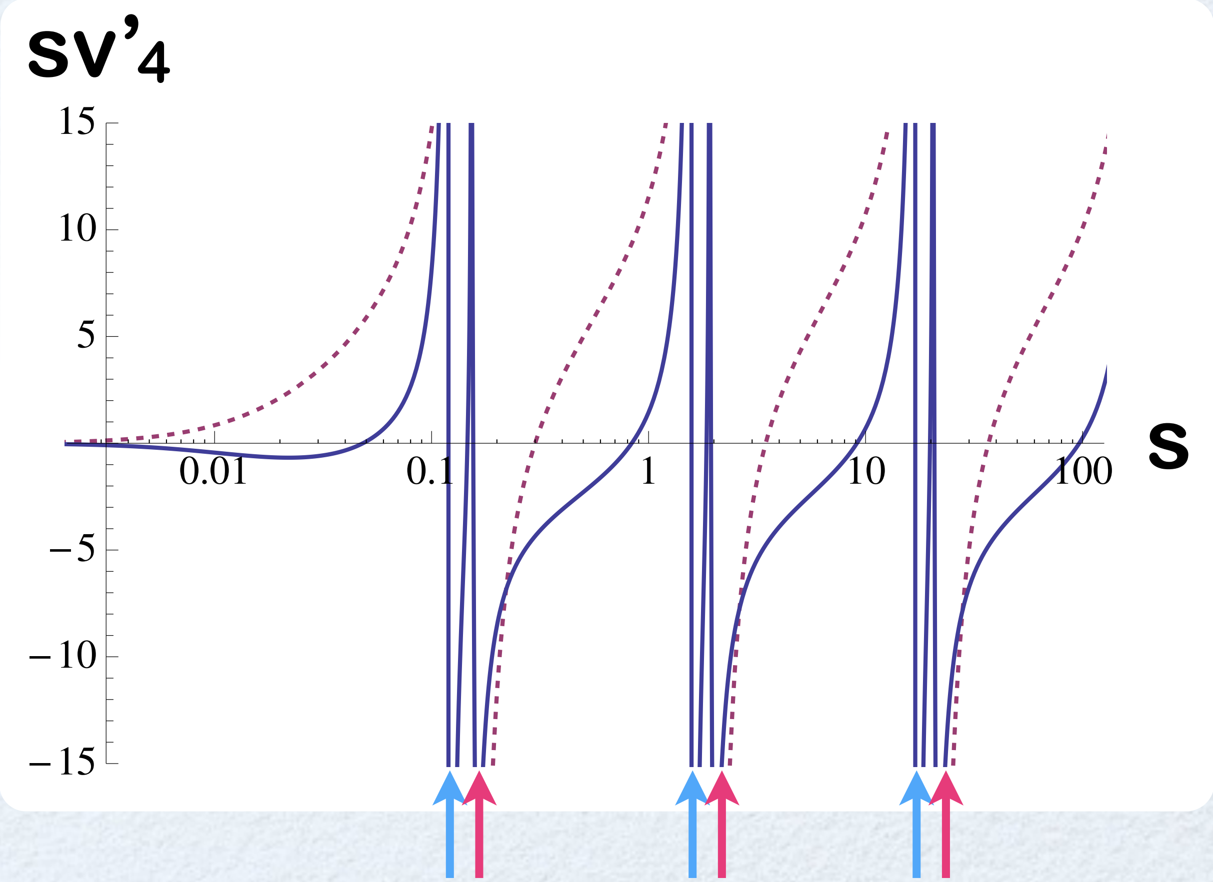
4-body problem ⇔ dimer+dimer scattering

$$\mathcal{L}_{4\text{-body}} = \sum_{a=\pm}\left[v_4 \phi_a^\dagger \phi_{-a}^\dagger \phi_{-a}\phi_a + v_4' \phi_a^\dagger \phi_a^\dagger \phi_a \phi_a\right] + \cdots$$

marginal couplings

irrelevant



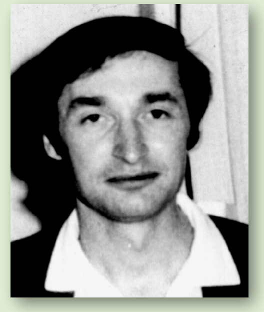


L=±2 tetramers attached to every trimer

with resonance energy $E_n \sim e^{-2e^{3\pi n/4+\theta-0.188}}$

Efimov vs super Efimov

Efimov effect



- 3 bosons
- 3 dimensions
- s-wave resonance

exponential scaling

$$E_n \sim e^{-2\pi n}$$

Super Efimov effect

- 3 fermions
- 2 dimensions
- p-wave resonance

"doubly" exponential

$$E_n \sim e^{-2e^{3\pi n/4}}$$

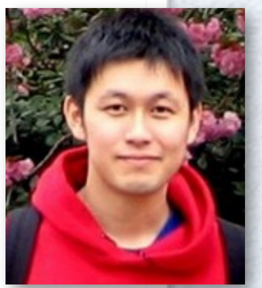
PRL **110**, 235301 (2013) PHYSICAL REVIEW LETTERS week ending 7 JUNE 2013

Super Efimov Effect of Resonantly Interacting Fermions in Two Dimensions

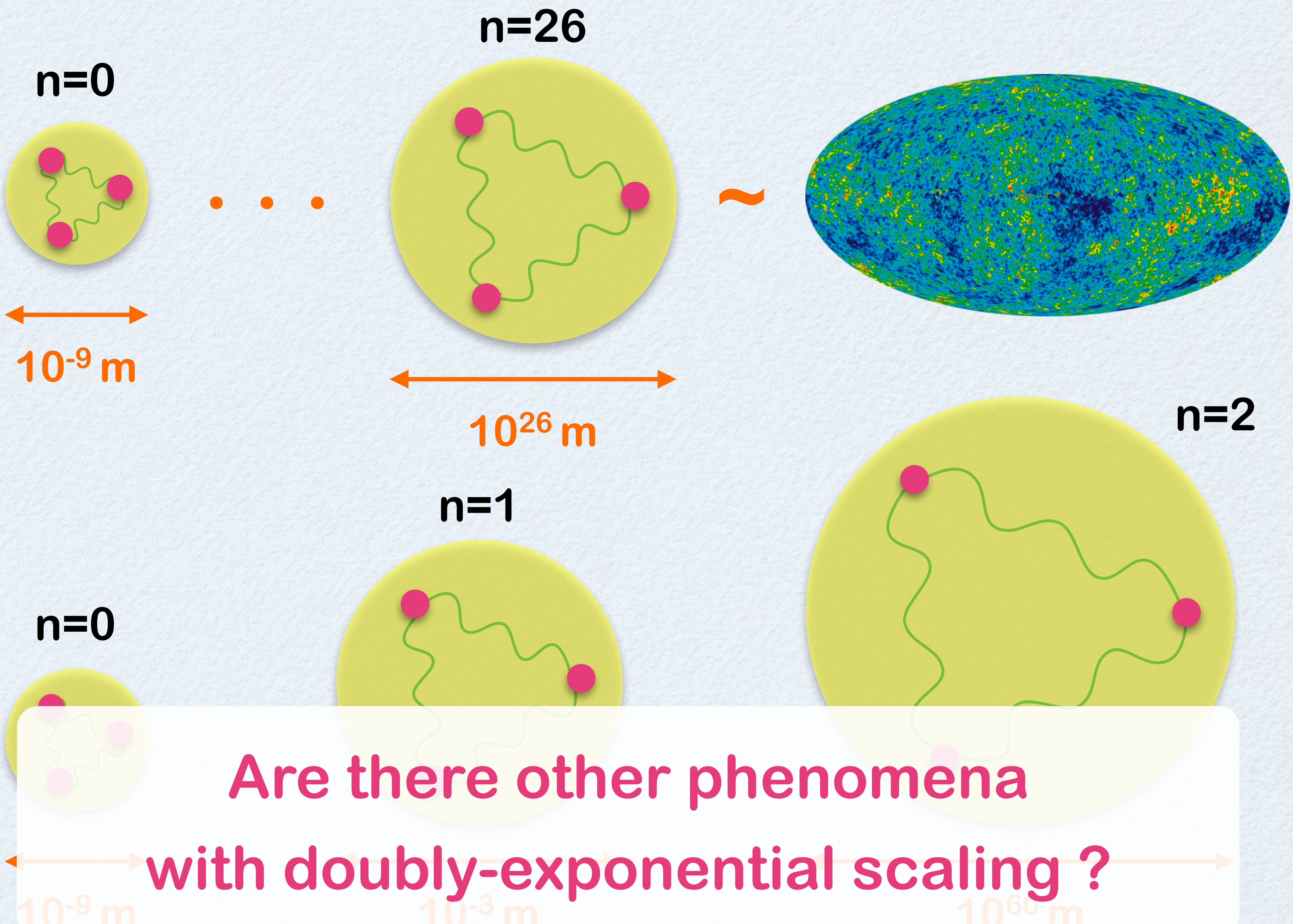
Yusuke Nishida,[1] Sergej Moroz,[2] and Dam Thanh Son[3]

[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[3]*Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637, USA*

(Received 18 January 2013; published 4 June 2013)

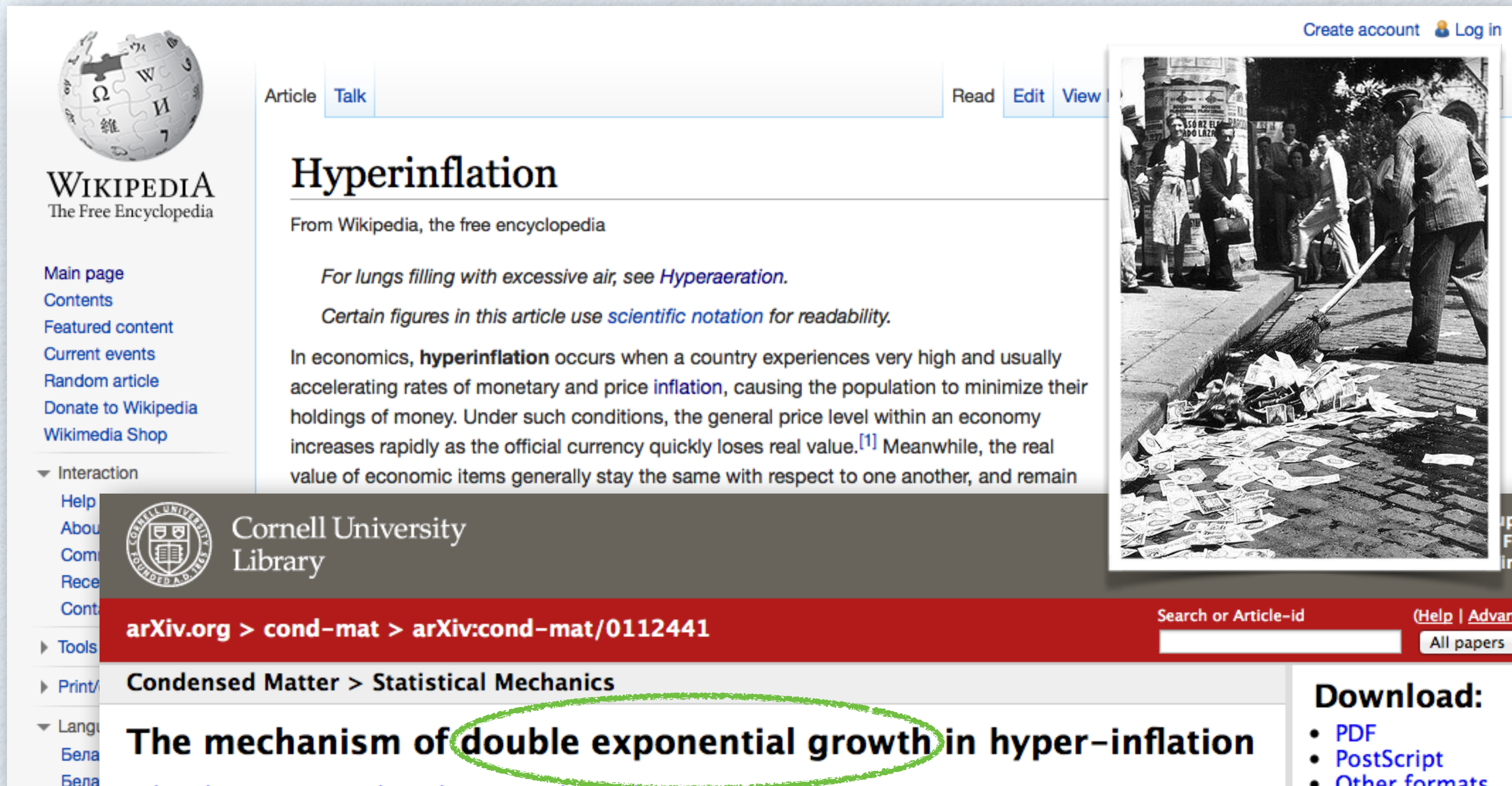


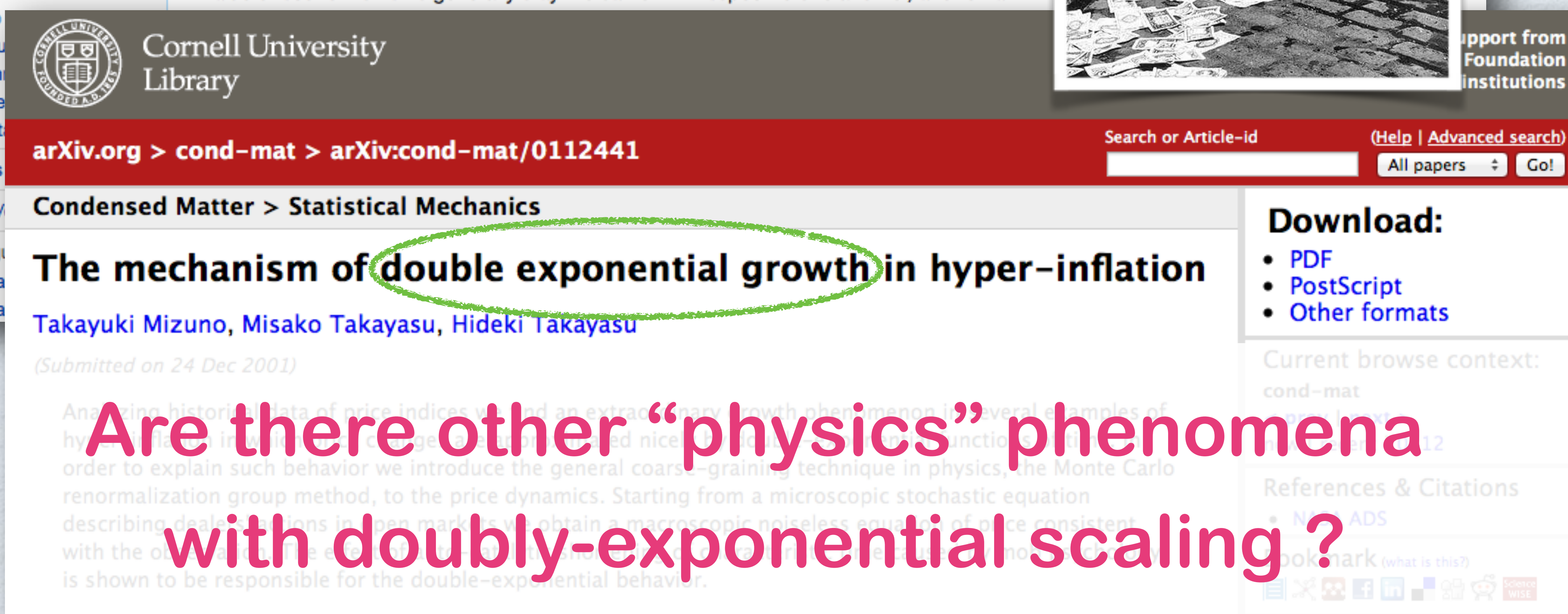
Efimov vs super Efimov



Are there other phenomena
with doubly-exponential scaling ?

Efimov vs super Efimov





Are there other “physics” phenomena with doubly-exponential scaling ?

Summary

Efimov effect: universality, discrete scale invariance, RG limit cycle

atomic physics

nuclear physics

condensed matter

✓ **Efimov effect in quantum magnets**

Y.N, Y.K, C.D.B, Nature Physics 9, 93-97 (2013)

✓ **Novel universality: Super Efimov effect**

Y.N, S.M, D.T.S, Phys Rev Lett 110, 235301 (2013)