Universality from a lattice

Amy N. Nicholson
University of Maryland

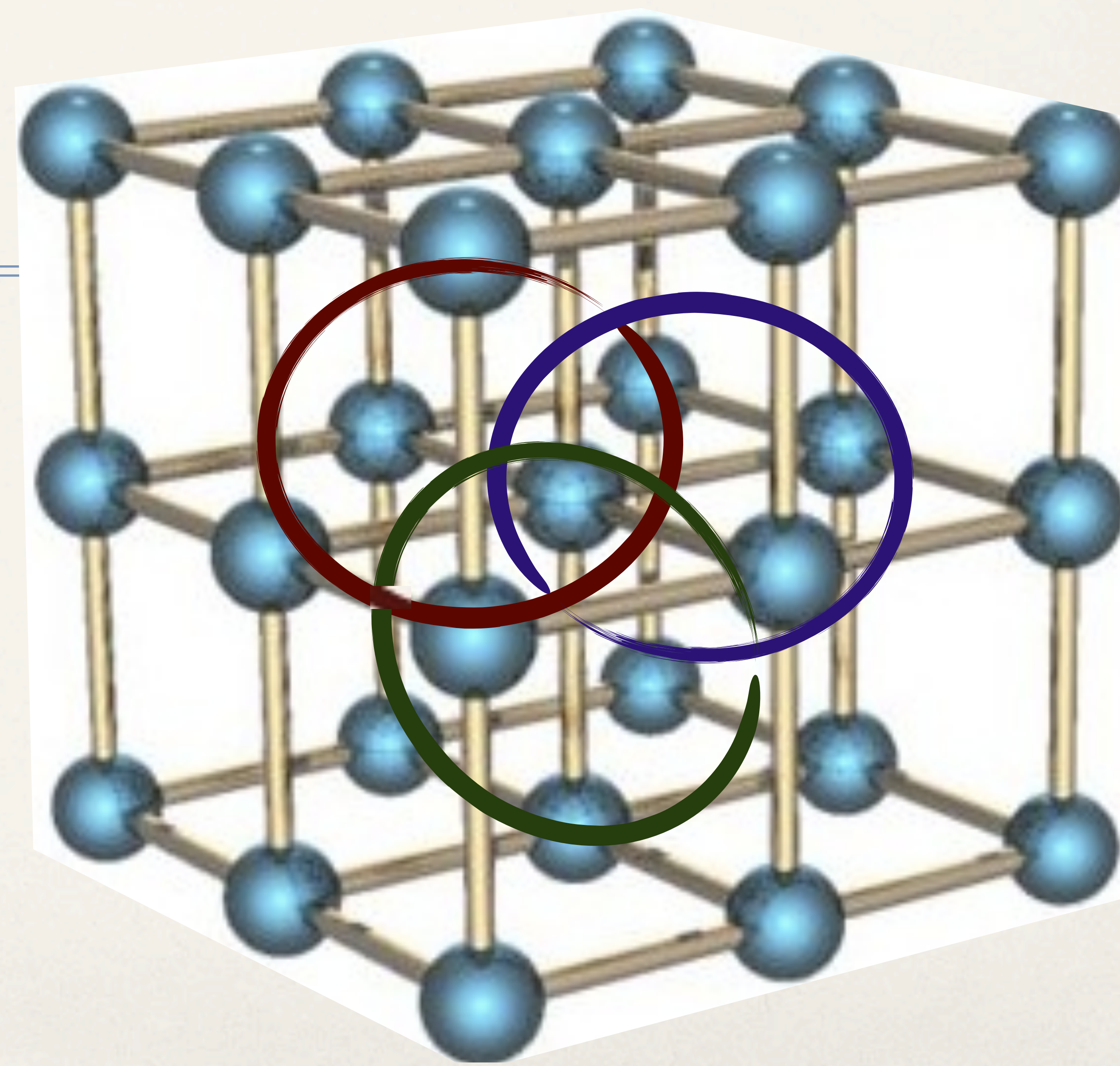
INT, May, 2014

Few-body Universality

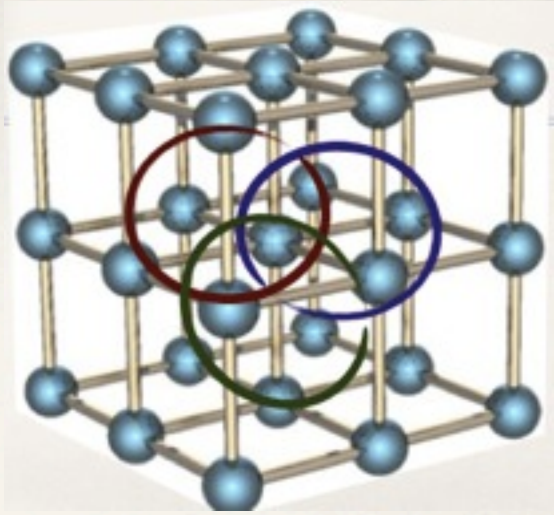
In Atomic And Nuclear Physics:

Recent Experimental And

Theoretical Advances

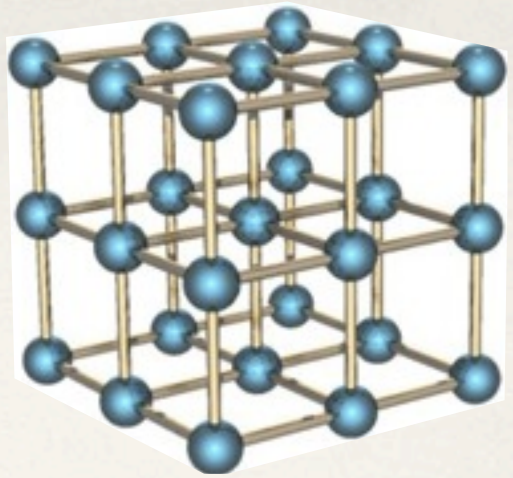


Universality from a lattice



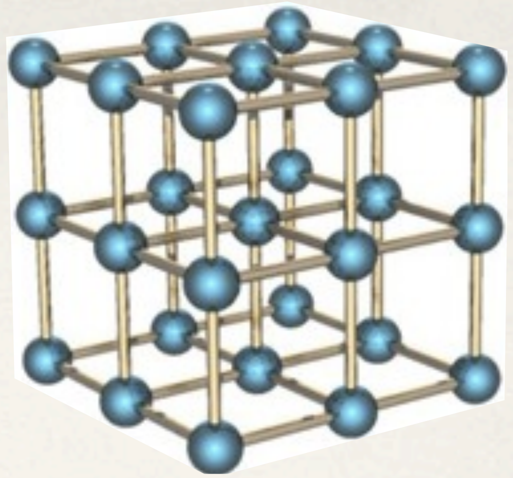
- Part I: Lattice theory for particles at unitarity
  - Results for few/many unitary fermions
- Part II: Statistical distributions
  - “Universal” distributions
- Part III: Predicting the spectrum of universal 2N-body clusters from 2-particle noise

Lattice Methods



- Start with path integral
- Integrate out fermions
- Generate configurations using Monte Carlo methods
- Calculate observables

Lattice Methods



- Start with path integral

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\int d\tau d^3x \left[\mathcal{L}(\phi) + \psi^\dagger K(\phi)\psi\right]}$$

- Discretize and integrate out fermions
- Generate configurations using Monte Carlo methods
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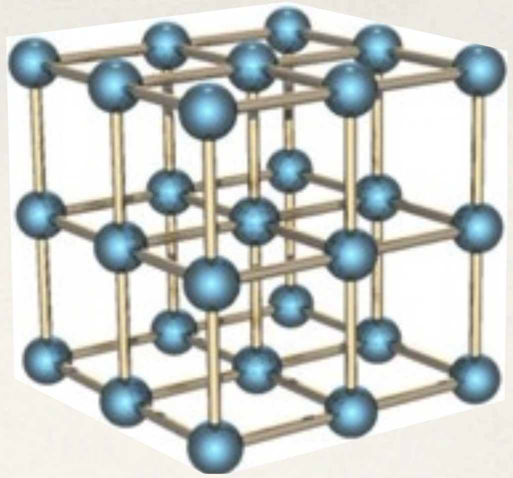
Lattice Methods



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$$Z = \int [d\phi] \det K(\phi) e^{-S_\phi}$$

- Generate configurations using Monte Carlo methods
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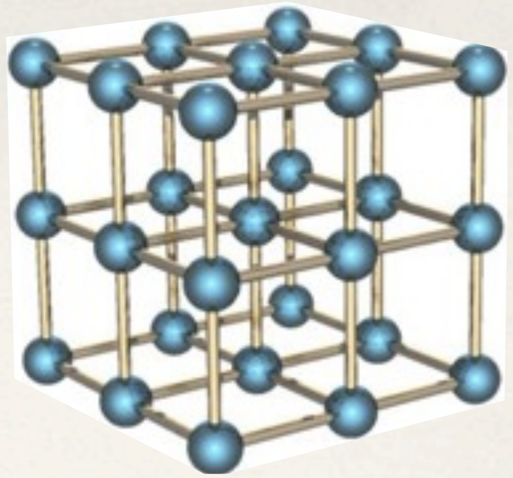
Lattice Methods



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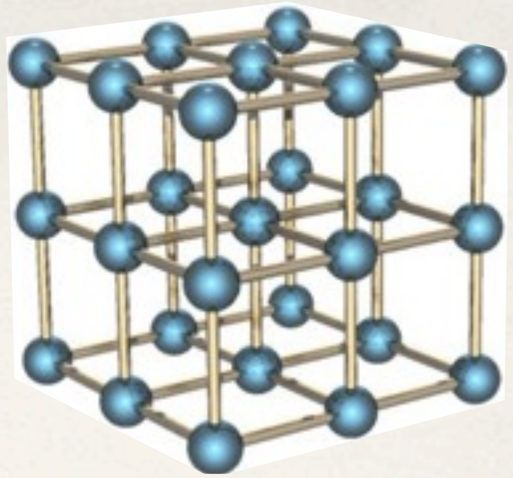
$\rho(\phi)$

- Calculate observables

Lattice Methods



- Start with path integral
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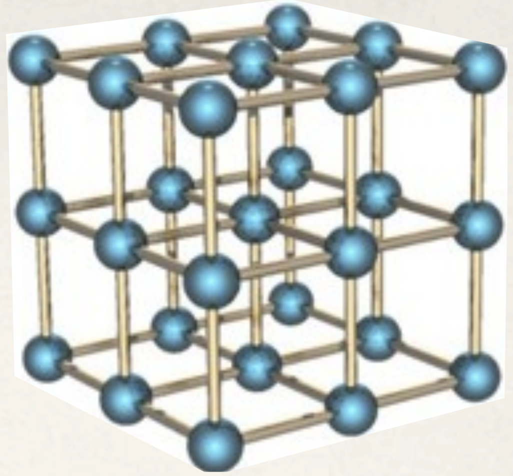
- Calculate observables

$$\frac{1}{Z}\int [d\phi]\, \rho(\phi)\mathcal{O}(\phi, x, \tau) \approx \frac{1}{N_{cfg}}\sum_{n=1}^{N_{cfg}} \mathcal{O}(\phi_n, x, \tau)$$

Lattice Methods



- Start with path integral
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- Calculate observables

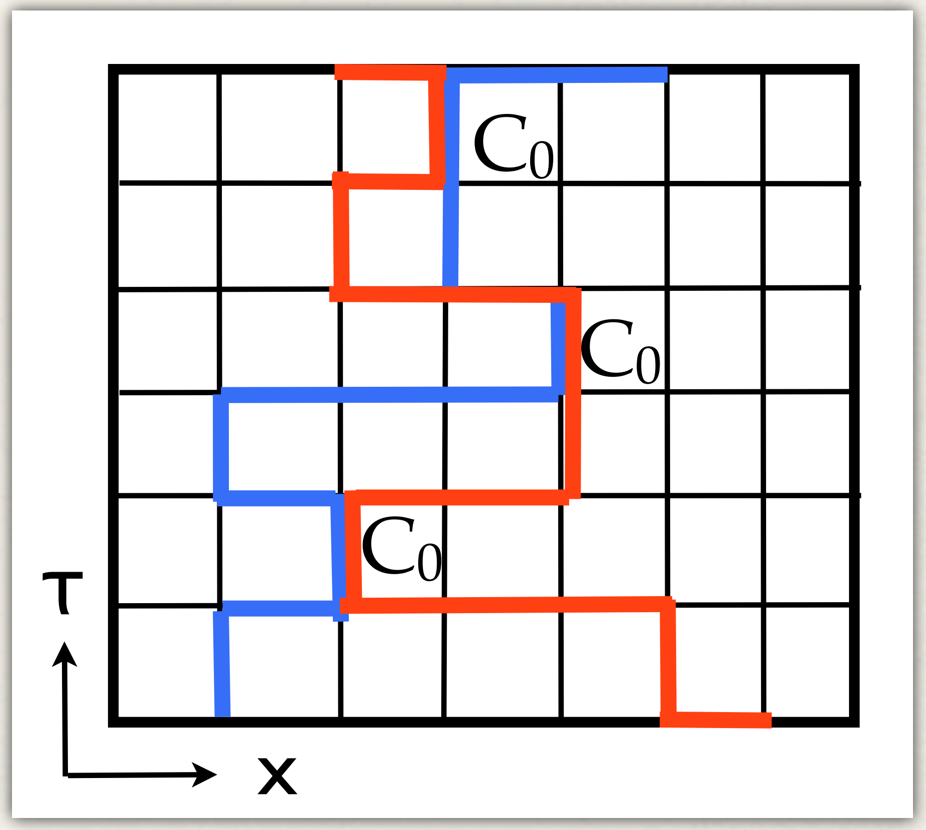
$$\langle 0|\psi(\tau)\psi^\dagger(0)|0\rangle = \frac{1}{Z}\int [d\phi]\,\rho(\phi)S(\phi,\tau) \underset{\tau\to\infty}{\longrightarrow} \mathcal{Z}e^{-E_0\tau}$$

Lattice Methods

Lattice version:
Chen & Kaplan, 2003

An example:

$$\mathcal{L} = \bar{\psi}\left(\partial_\tau - \frac{\nabla^2}{2M}\right)\psi + C_0(\bar{\psi}\psi)^2$$



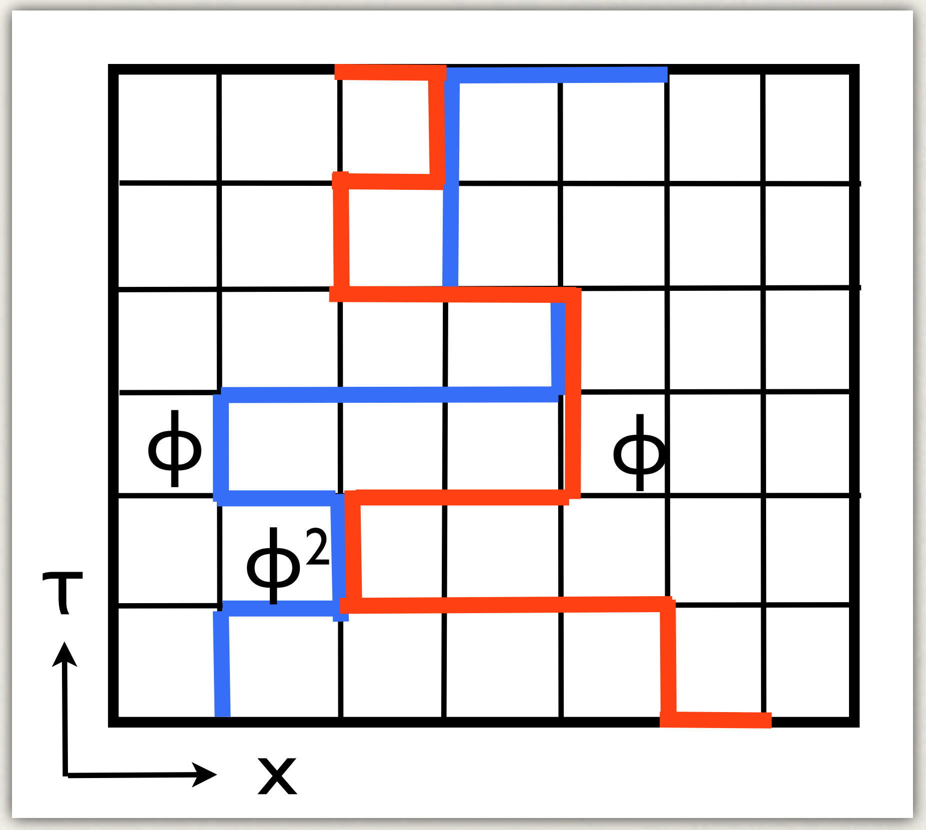
Lattice Methods

Lattice version:
Chen & Kaplan, 2003

$\phi \in Z_2$

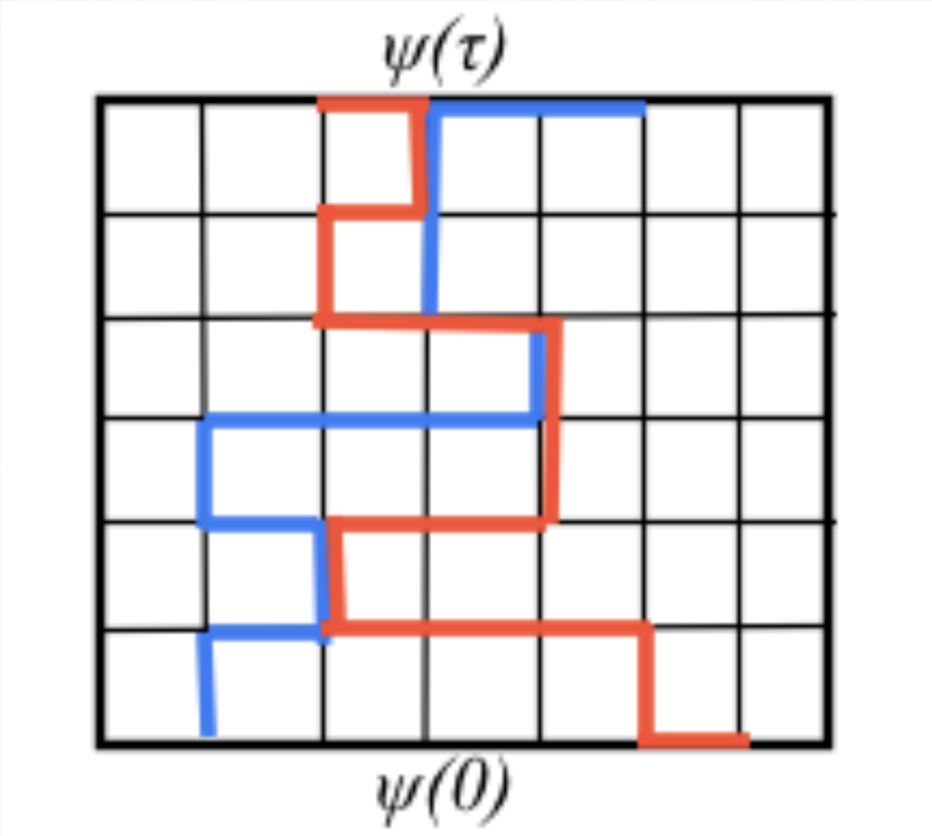
Must be bilinear in fermion fields

$$\mathcal{L} \to \bar{\psi} \underbrace{\left(\partial_\tau - \frac{\nabla^2}{2M} + \sqrt{C_0}\phi \right)}_{K(\phi)} \psi$$



$$D = 1 - \frac{\nabla^2}{2M}, \qquad X(\tau) = 1 - \sqrt{C}\phi(\mathbf{x}, \tau)$$

$$K(\phi) = \begin{pmatrix} D & X(0) & 0 & 0 & \cdots & 0 \\ 0 & D & X(1) & 0 & \cdots & 0 \\ 0 & 0 & D & X(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & D & X(T-1) \\ 0 & 0 & 0 & 0 & \cdots & D \end{pmatrix}$$



Features:

- Iterative propagator

$$K^{-1}(\tau, 0) = D^{-1}X(\tau-1)D^{-1}\cdots D^{-1}X(0)D^{-1}$$

can use FFTs to give exact free propagator

- Det[K] = constant → never have to compute!

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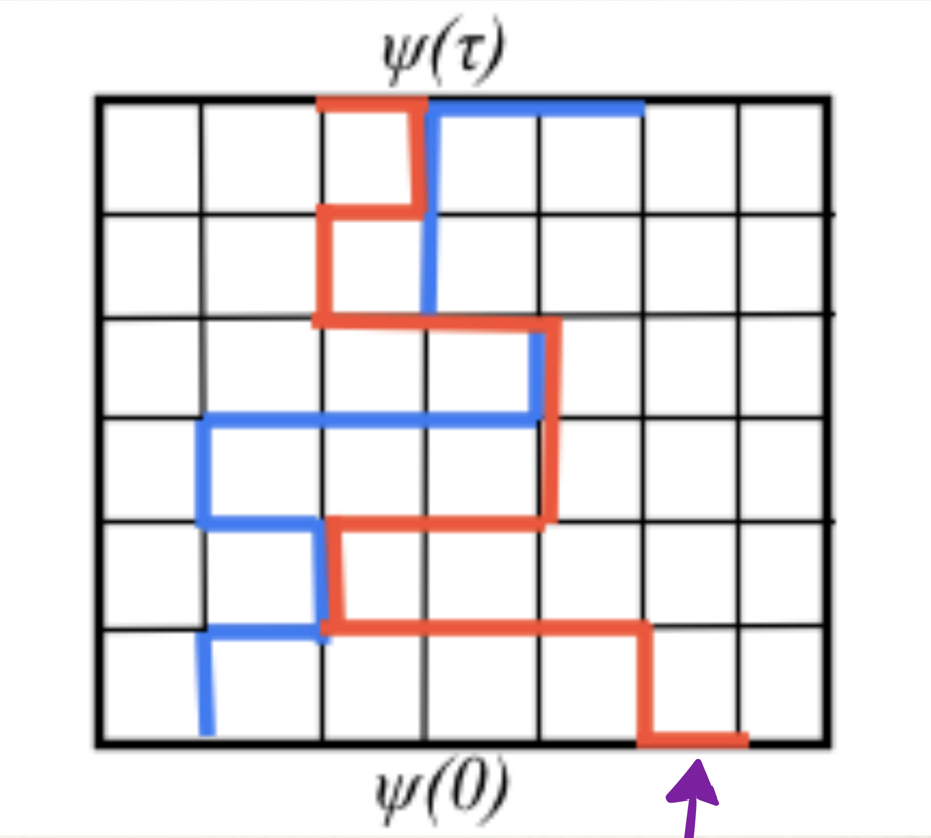

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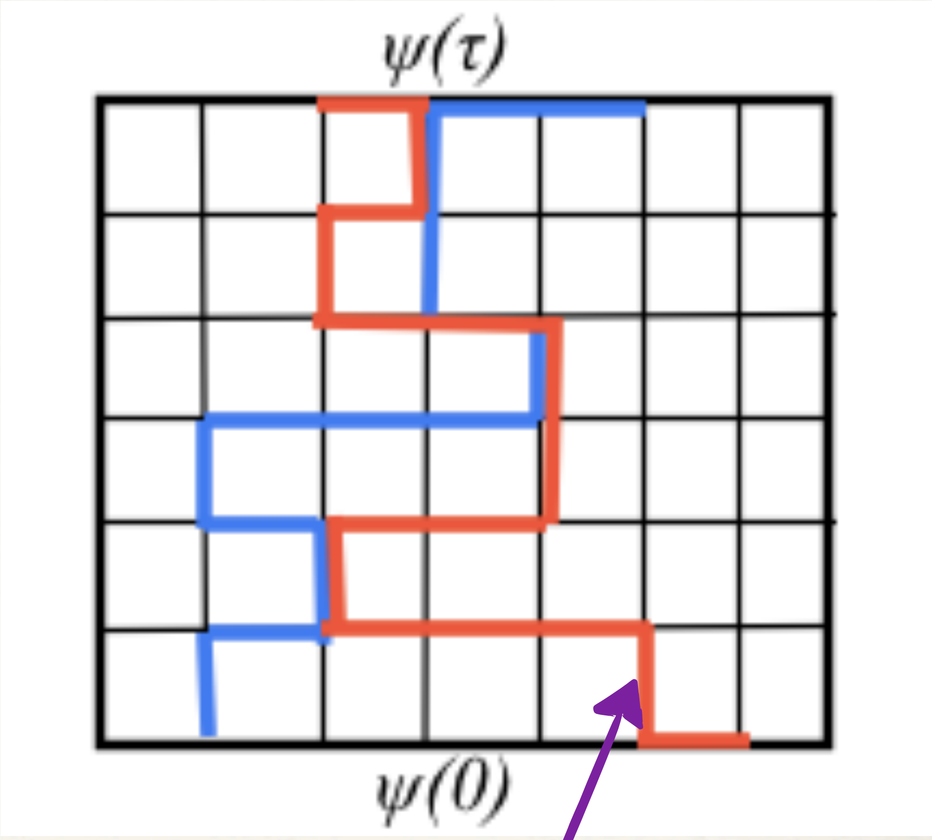

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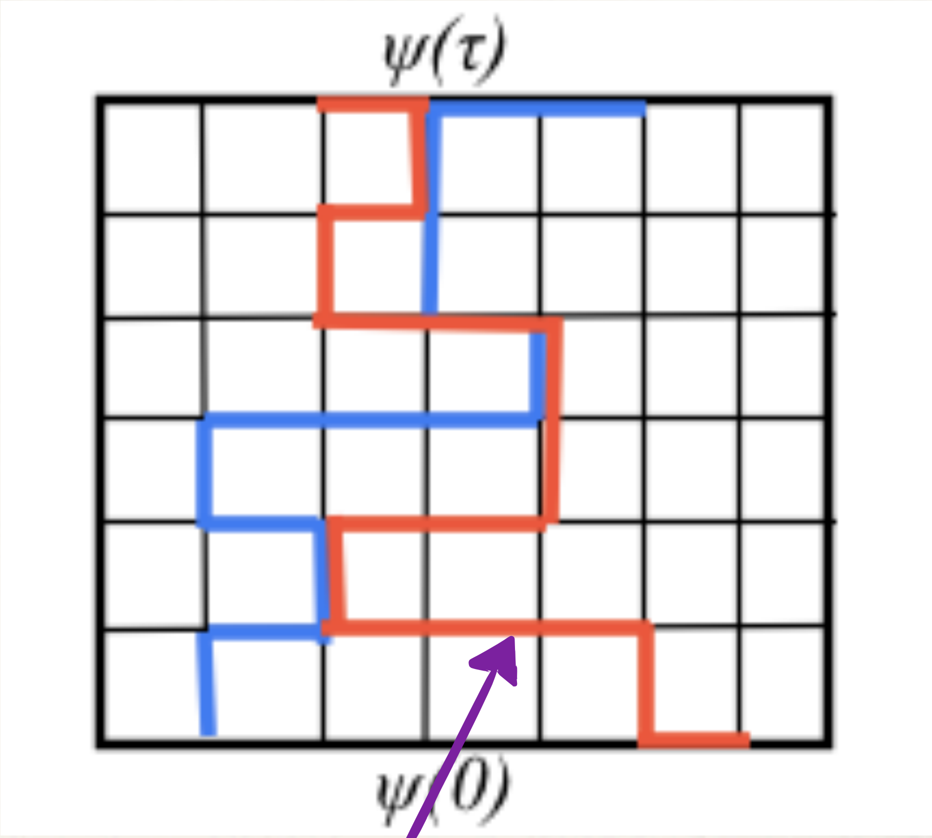

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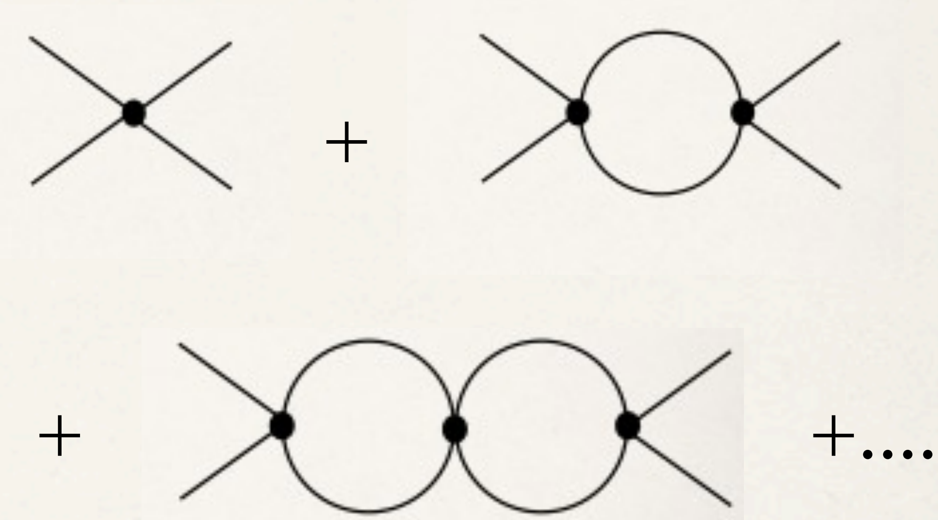
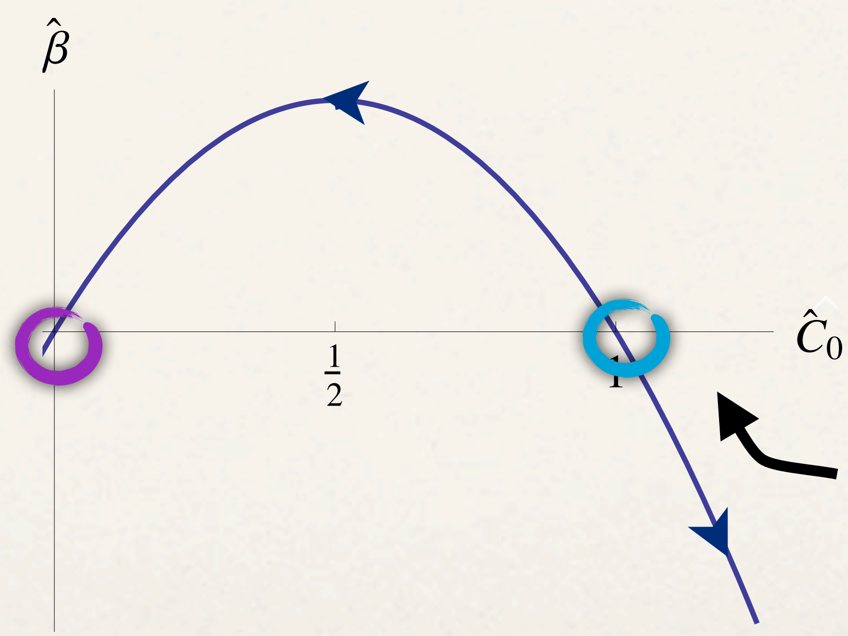
Unitary fermions

$$\mathcal{L}_{\rm int} = \frac{C_0}{4}\left(\psi^\dagger\psi\right)^2$$

$$\hat{C}_0 \equiv -\frac{M\mu}{4\pi}C_0 = \frac{\mu}{\mu + \frac{1}{a}}$$



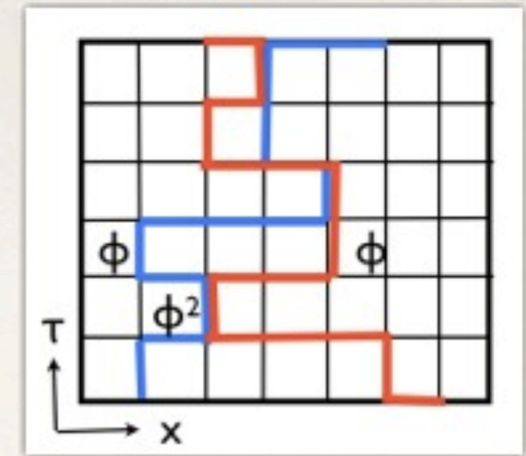
$$\hat{\beta} = \mu\frac{\partial \hat{C}_0}{\partial \mu}$$



Trivial IR fixed point: free particles

Nontrivial fixed point: unitarity

Removing lattice artifacts

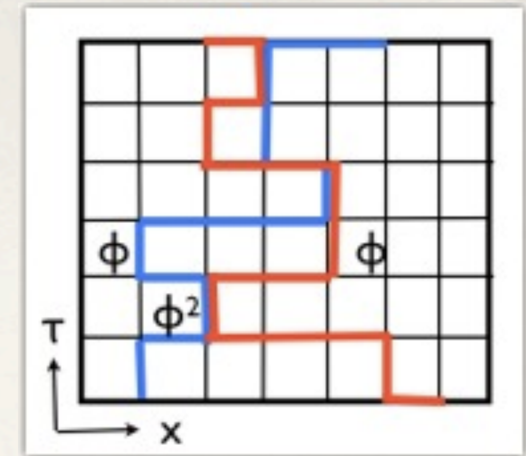


$$X(\tau) = 1 - \sqrt{C_0}\phi(\tau)$$

$$X(\tau, p, p') = 1 - \sqrt{\sum_n C_n \mathcal{O}_n\left[(p-p')^2\right]\phi(\tau)}$$

Lüscher eigenvalues: Exact energies of 2 unitary fermions in a continuous box

Removing lattice artifacts



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Lüscher eigenvalues:
Exact energies of 2
unitary fermions in
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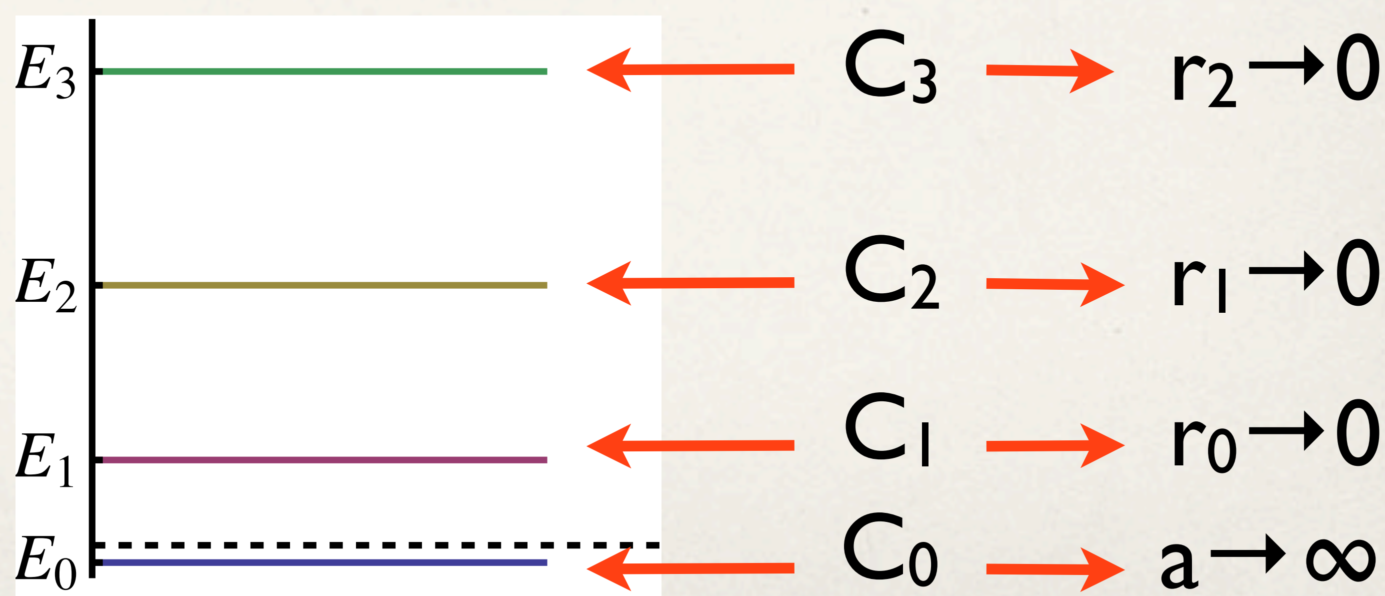
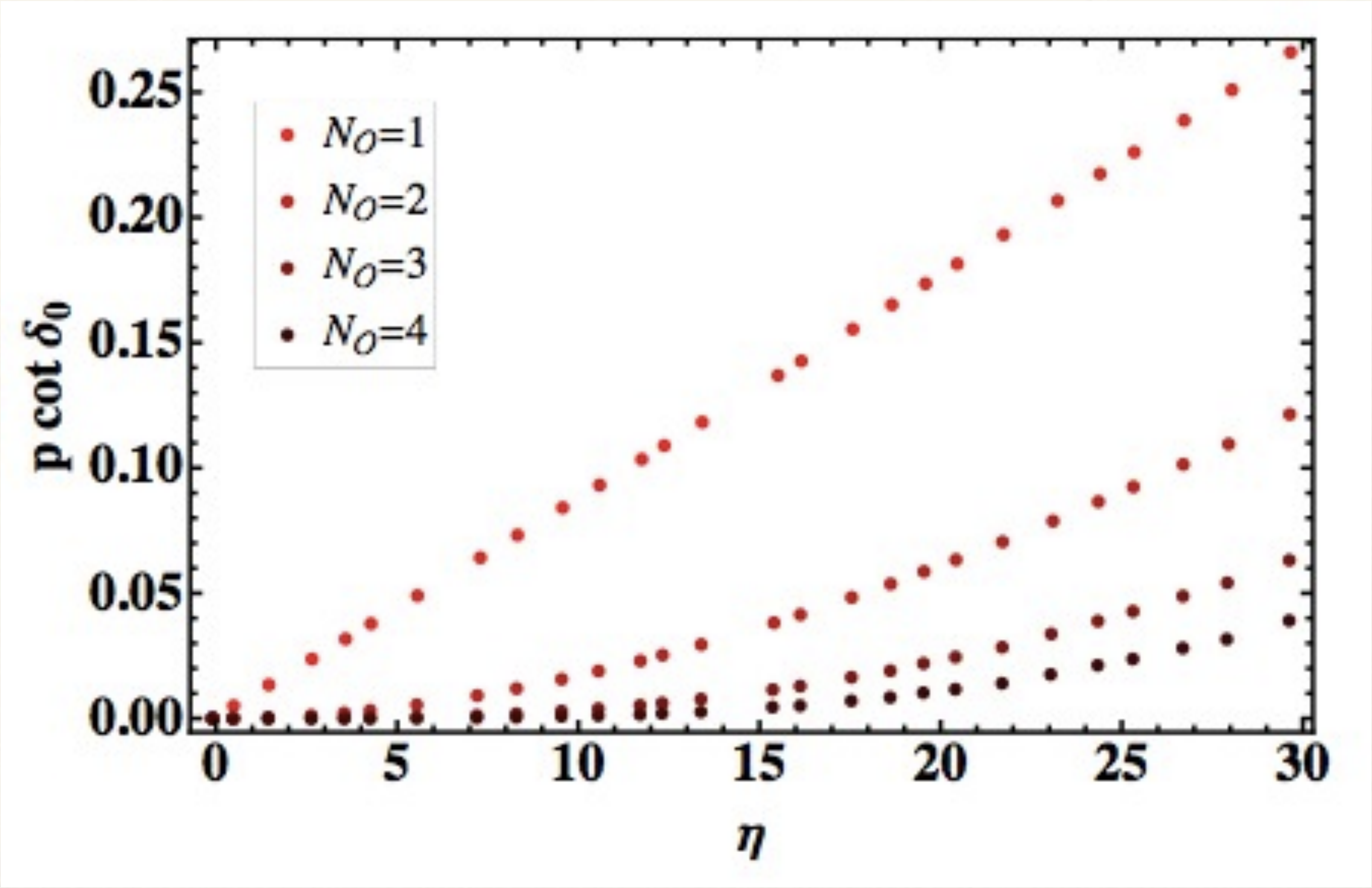
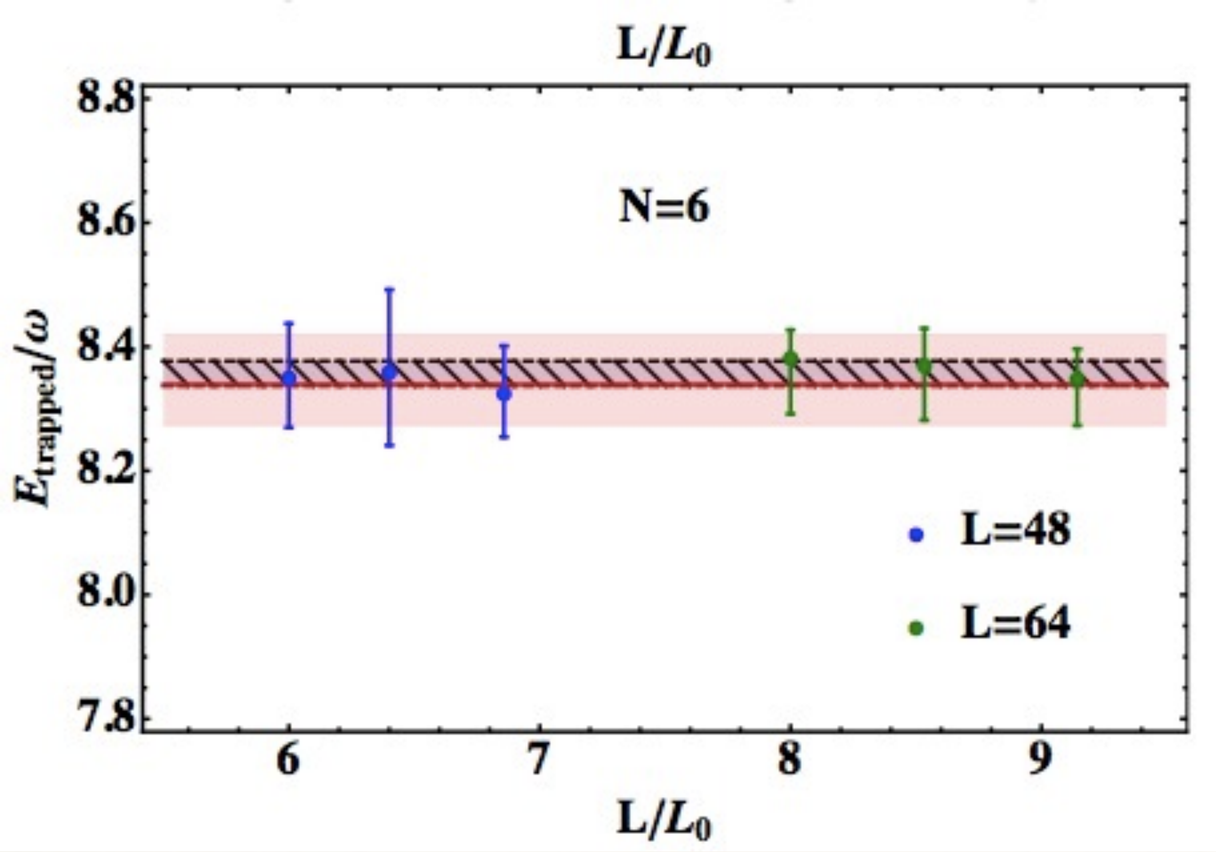


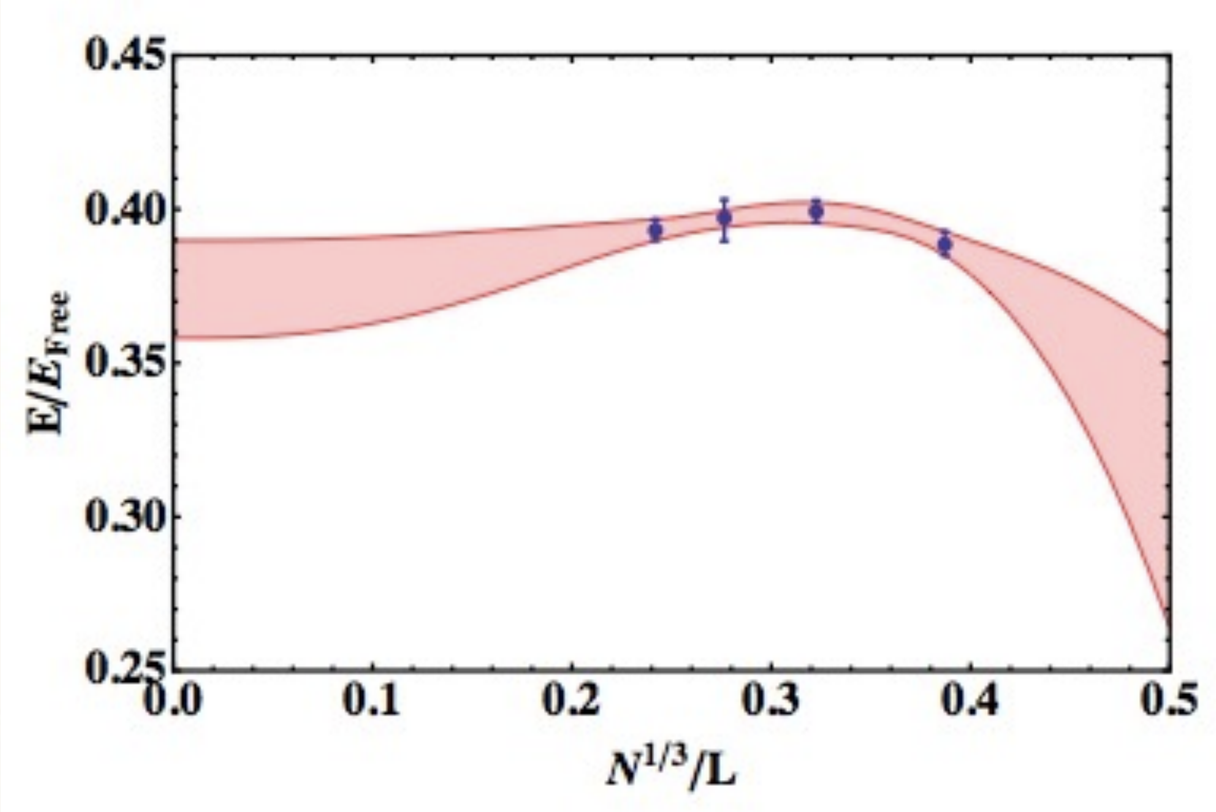
Removing lattice artifacts



Some results

M.G. Endres, D.B. Kaplan, J-W Lee, A.N.





6 fermions
in a harmonic trap

58 fermions
in a box

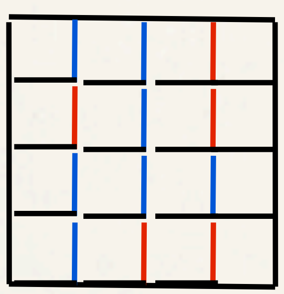
Blume & Daily, 2011

Statistical distributions

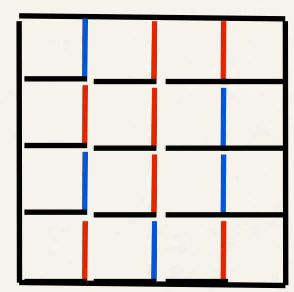
Statistical distributions

$$C(\tau) \approx \frac{1}{N_{cfg}} \sum_{i}^{N_{cfg}} C(\phi_i, \tau)$$

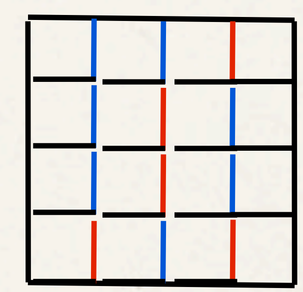
Φ_1



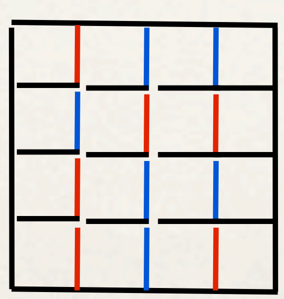
Φ_2



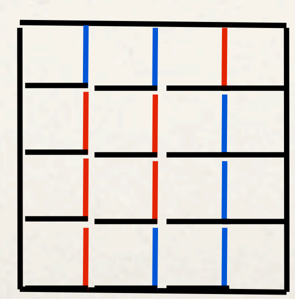
Φ_3



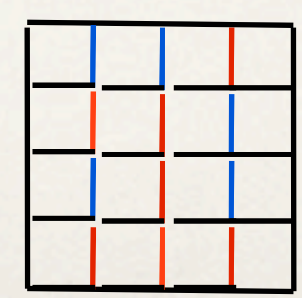
Φ_4



Φ_5

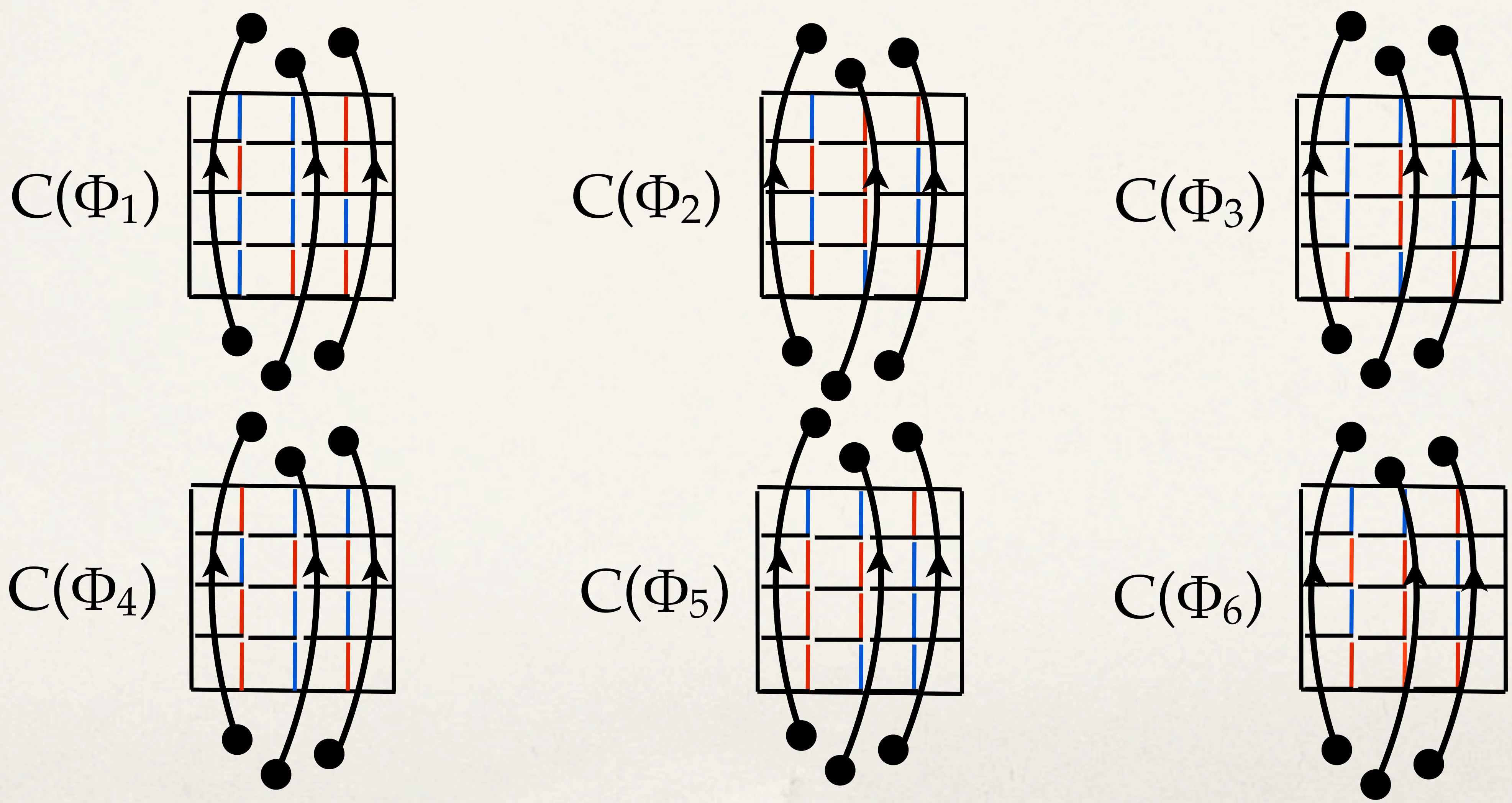


Φ_6

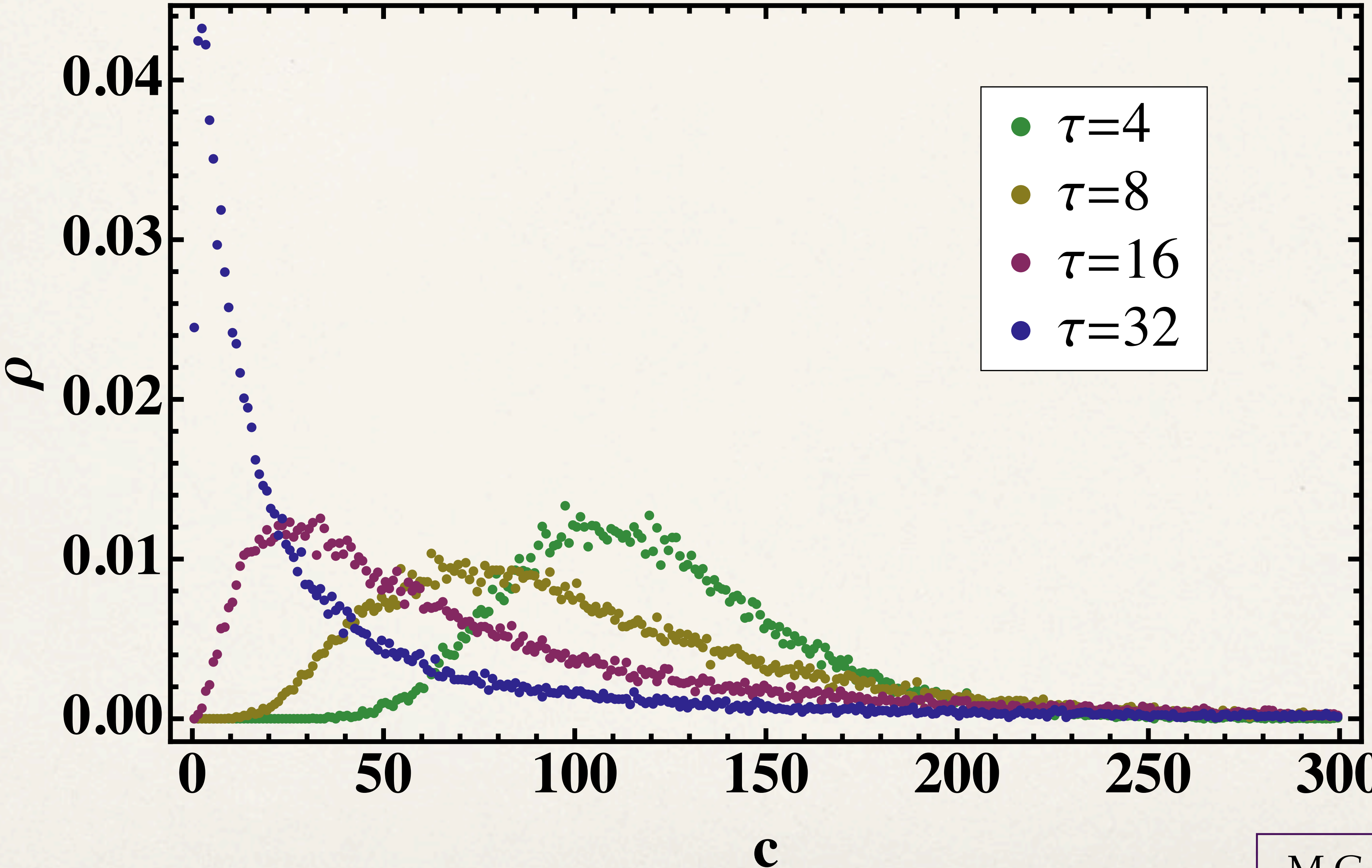


Statistical distributions

$$C(\tau) \approx \frac{1}{N_{cfg}} \sum_{i}^{N_{cfg}} C(\phi_i, \tau)$$



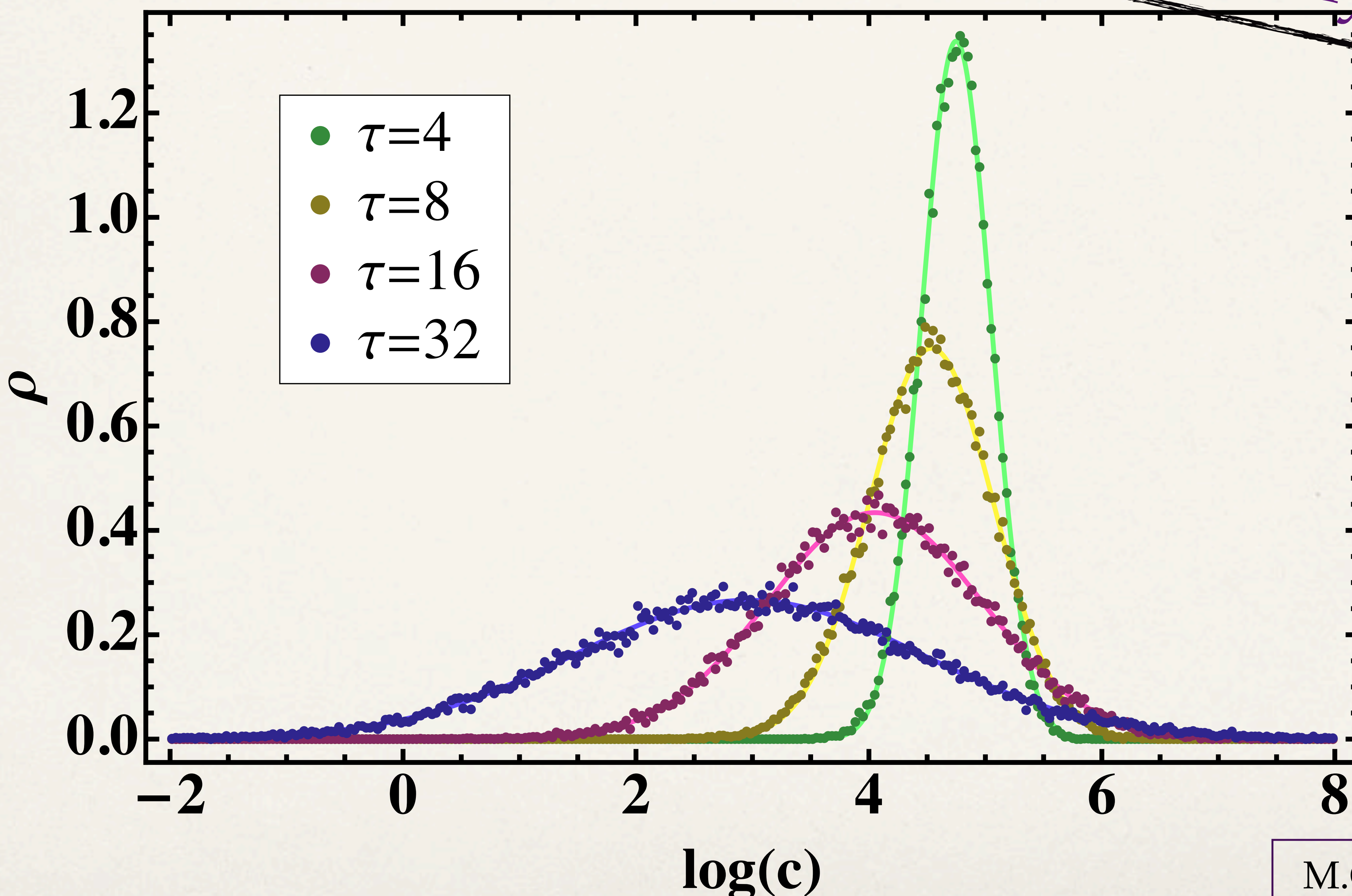
Unitary fermion correlators



M.G. Endres, D.B. Kaplan,
J-W Lee, A.N., 2011

Unitary fermion correlators

Log-normal: ln x is normally distributed



M.G. Endres, D.B. Kaplan, J-W Lee, A.N., 2011

Probability distributions

$$P(x) \propto \int [d\phi]\, e^{-S[\phi]} \delta(C[\phi] - x)$$

$$\Phi_C(s) \propto \int [d\phi]\, e^{-S[\phi] + isC[\phi]}$$

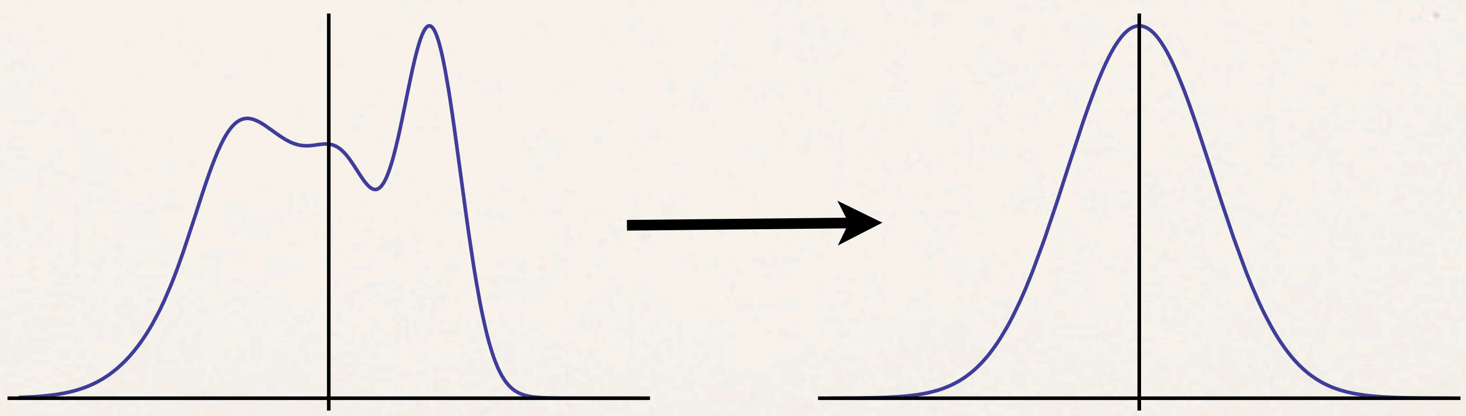
$$\ln \Phi_C(s) = \sum_{n=1}^{\infty} \frac{(is)^n}{n!} \kappa_n$$

nth cumulant is like
n-pt operator

"Universal" distributions

Central limit theorem: for random variables $x_1, x_2, \cdots, x_N$ drawn from distribution characterized by $\{\kappa_1(x), \kappa_2(x), \cdots\}$:

$$P\left(\frac{x_1 + x_2 + \cdots + x_N}{\sqrt{N}}\right) \xrightarrow[N \to \infty]{} \text{Gaussian}$$



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$$P\left(\frac{x_1 + x_2 + \cdots + x_N}{\sqrt{N}}\right) \xrightarrow[N \to \infty]{} \text{Gaussian}$$

Only first two cumulants (mean, variance) survive

$$\kappa_n\left(\frac{x_1 + x_2 + \cdots + x_N}{\sqrt{N}}\right) \sim N^{1-n/2}\kappa_n(x)$$

“Universal” distributions

Central limit theorem: for random variables $x_1, x_2, \cdots, x_N$ drawn from distribution characterized by $\{\kappa_1(x), \kappa_2(x), \cdots\}$:

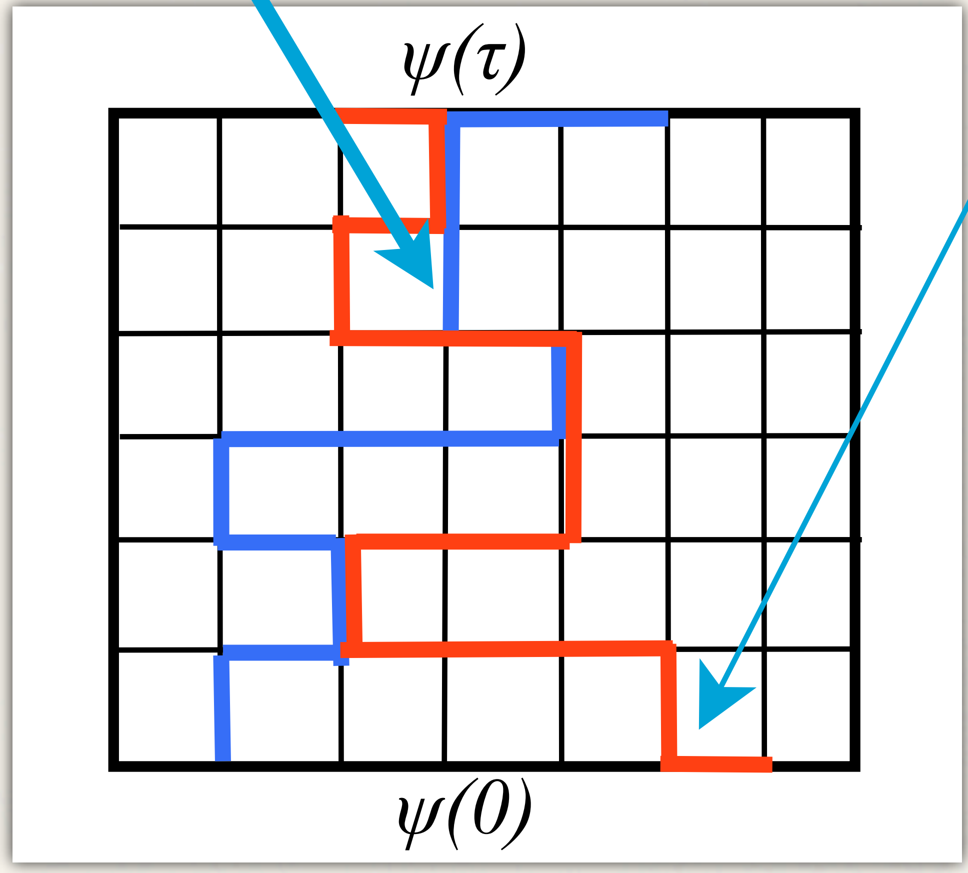
Similarly for log-normal:

$$\kappa_n\left(\ln(x_1 x_2 \cdots x_N)^{1/\sqrt{N}}\right) \sim N^{1-n/2}\kappa_n(\ln x)$$

$$P\left((x_1 x_2 \cdots x_N)^{1/\sqrt{N}}\right) \xrightarrow[N \to \infty]{} \text{Log-normal}$$

Propagator is a product of random variables!

$$K^{-1}(\tau,0) = D^{-1}X(\tau-1)D^{-1}\cdots D^{-1}X(0)D^{-1}$$



$$P\left((x_1 x_2 \cdots x_N)^{1/\sqrt{N}}\right) \xrightarrow[N\to\infty]{} \text{Log-normal}$$

Propagator is a product of random variables!

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(sort of)

$$P\left((x_1 x_2 \cdots x_N)^{1/\sqrt{N}}\right) \xrightarrow[N \to \infty]{} \text{Log-normal}$$

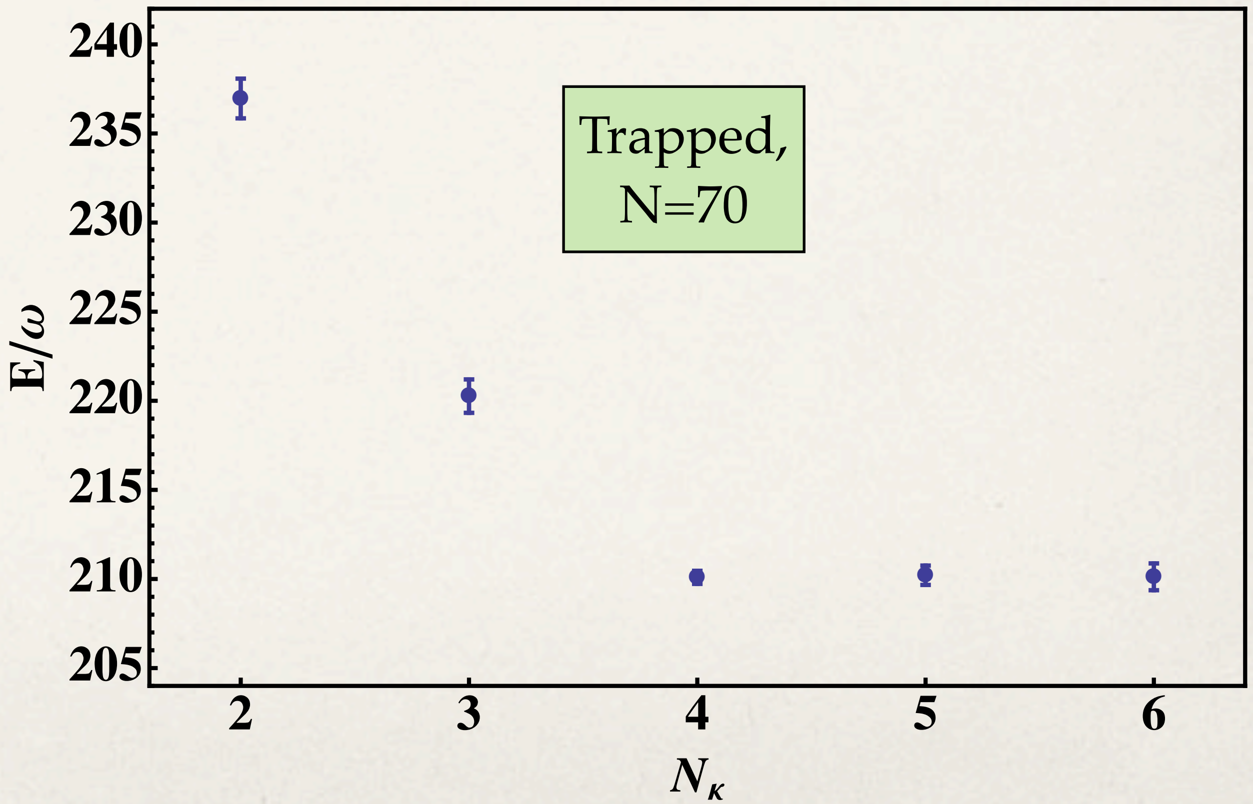
Cumulant expansion

- Expand ln ⟨C⟩ in terms of cumulants of ln C

$$\ln\langle C\rangle = \sum_{n=1}^{\infty} \frac{\kappa_n(\ln C)}{n!}$$

- Truncation at n=2 gives exact result if distribution is log-normal

- Contributions from higher cumulants reflect deviations from log-normal

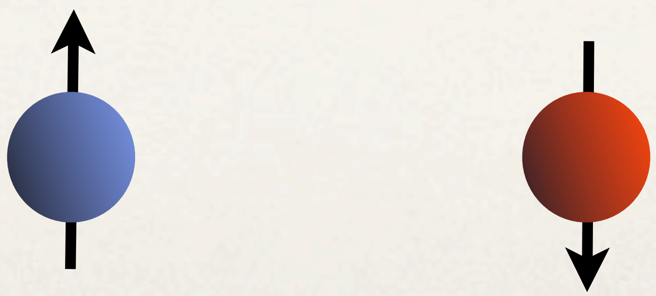


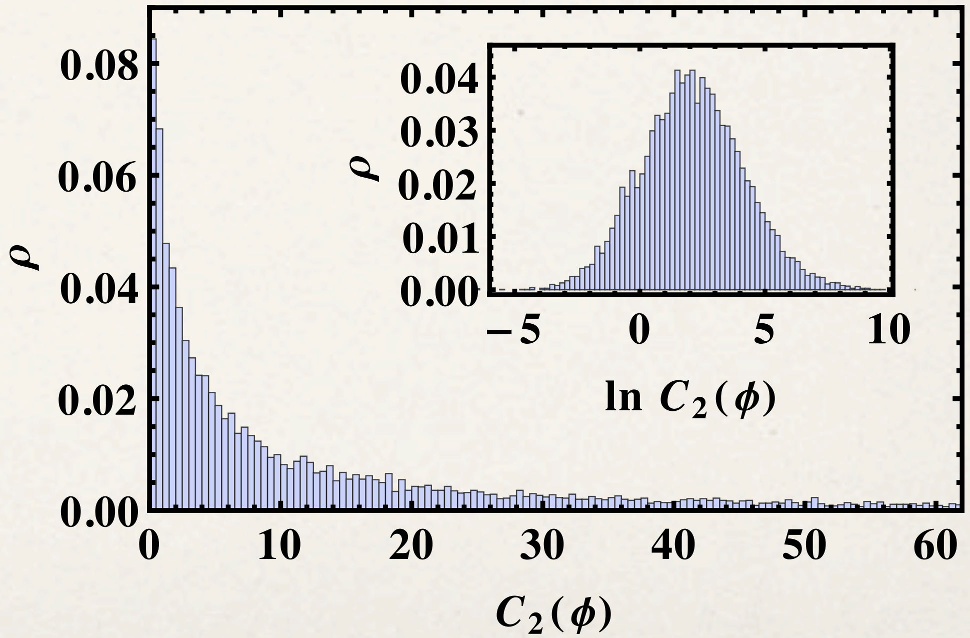
Log-normal distribution & unitarity

- How is the log-normal distribution related to unitarity?
- Can we learn anything new from this knowledge?

Look at simplest system:

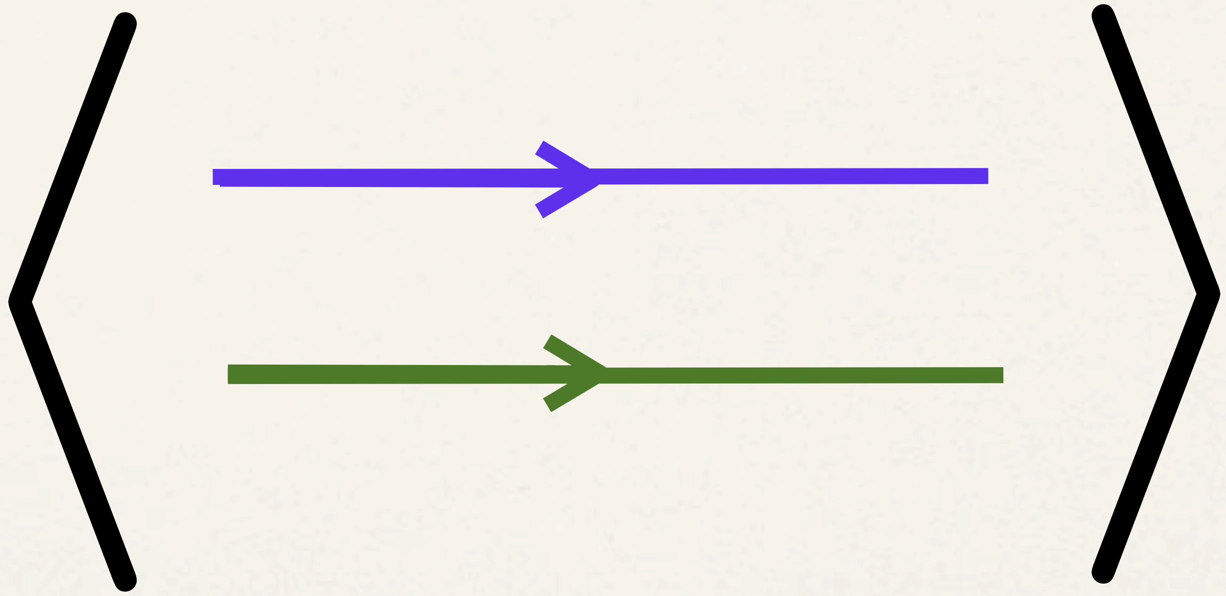
two particles at unitarity





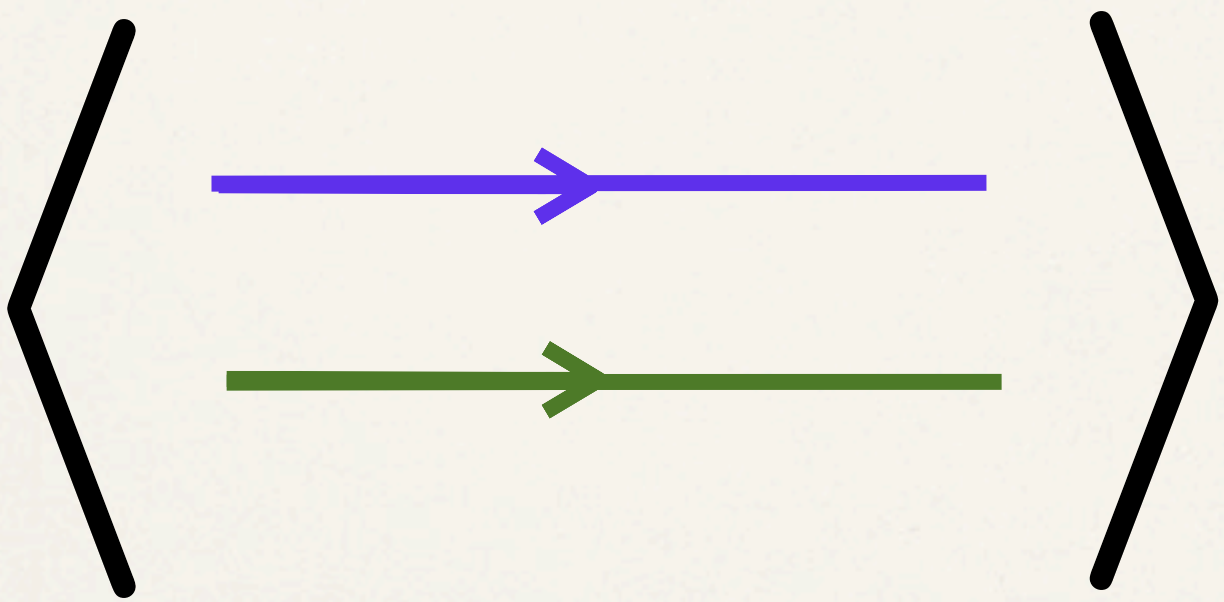
Moments of the 2-body correlator

$$C_2(\tau) = \langle [S(\phi,\tau)]^2 \rangle_\phi$$



Moments of the 2-body correlator

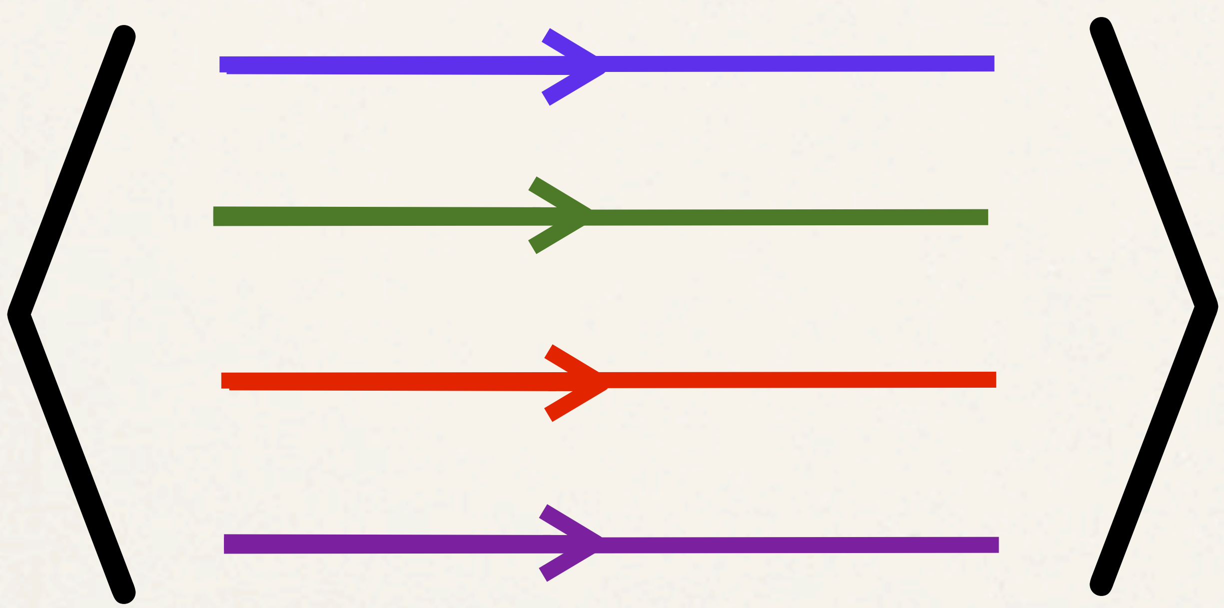
$$\mathcal{M}_1(\tau) = \langle [S(\phi,\tau)]^2 \rangle_\phi = C_2(\tau)$$



$$\xrightarrow[\tau\to\infty]{} \mathcal{Z}_2 e^{-E_0^{(2)}\tau}$$

Moments of the 2-body correlator

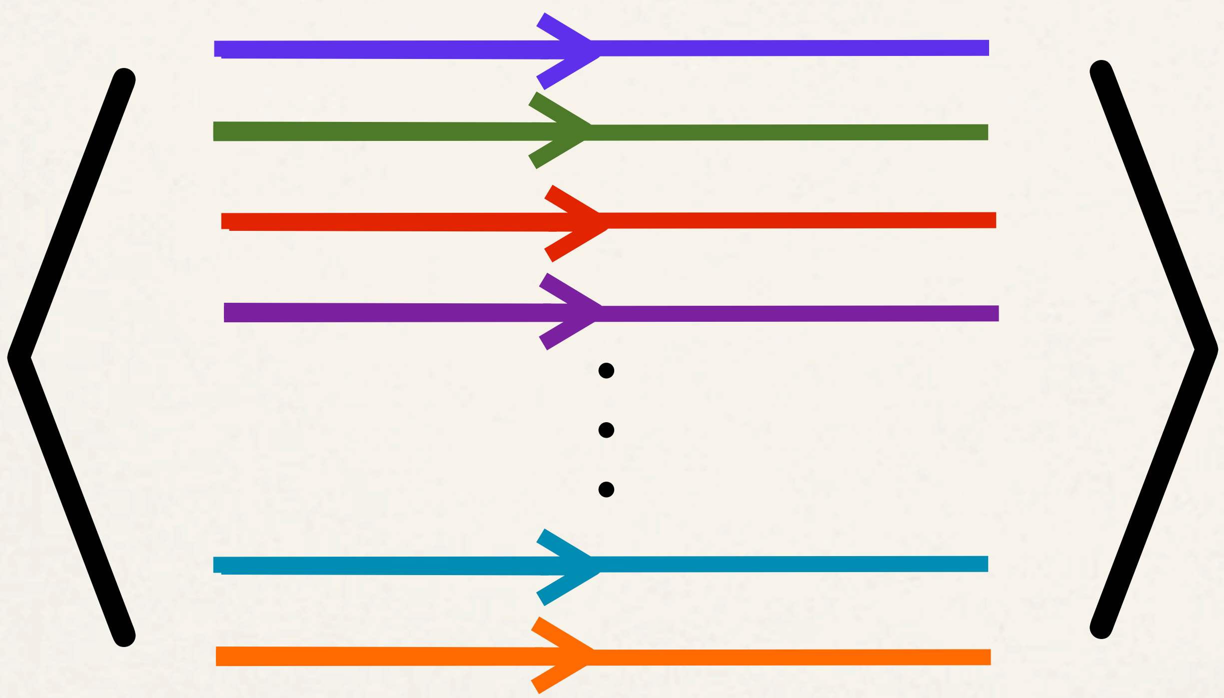
$$\mathcal{M}_2(\tau) = \langle [S(\phi,\tau)]^4 \rangle_\phi = C_4(\tau)$$



$$\underset{\tau\to\infty}{\longrightarrow} \mathcal{Z}_4 e^{-E_0^{(4)}\tau}$$

Moments of the 2-body correlator

$$\mathcal{M}_N(\tau) = \langle [S(\phi,\tau)]^{2N} \rangle_\phi = C_{2N}(\tau)$$



$$\xrightarrow[\tau\to\infty]{} \mathcal{Z}_{2N} e^{-E_0^{(2N)}\tau}$$

Moments of the 2-body correlator

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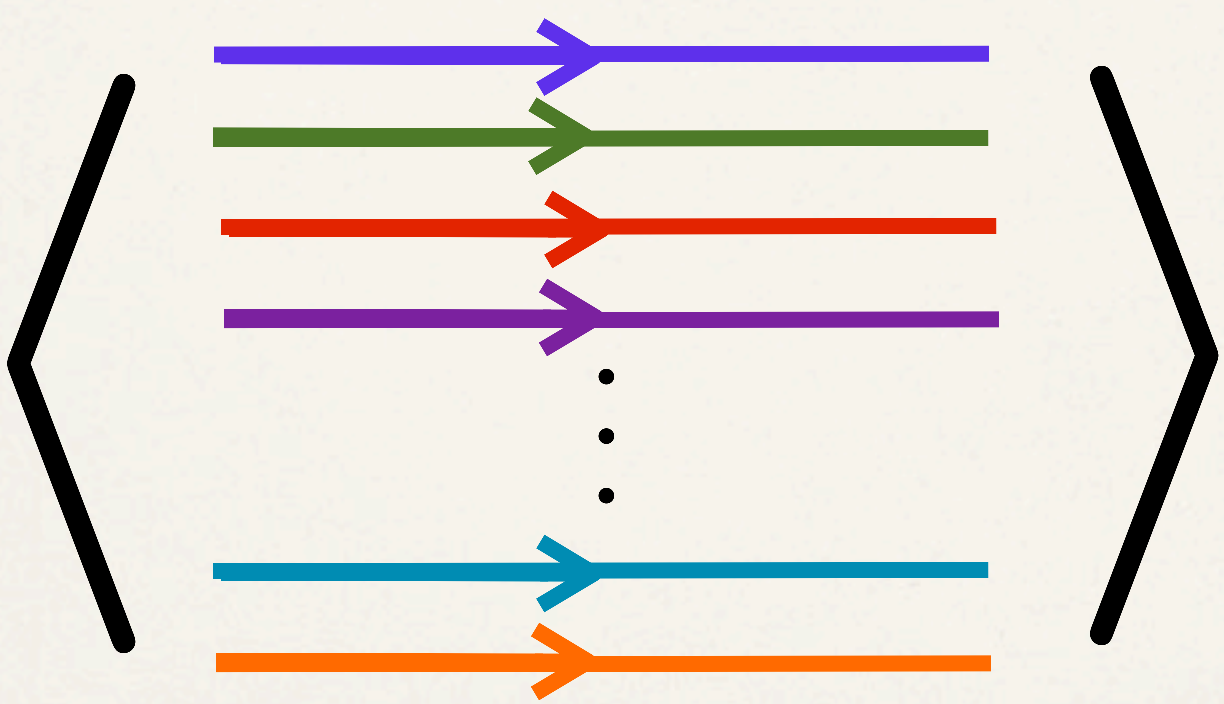

$$\xrightarrow[\tau\to\infty]{} \mathcal{Z}_{2N} e^{-E_0^{(2N)}\tau}$$

What about the partition function?

$$Z_\phi = \int d\phi \, (\det K)^2 \, e^{-\int d\tau d^3x \frac{1}{2}\phi^2}$$

Moments of the 2-body correlator

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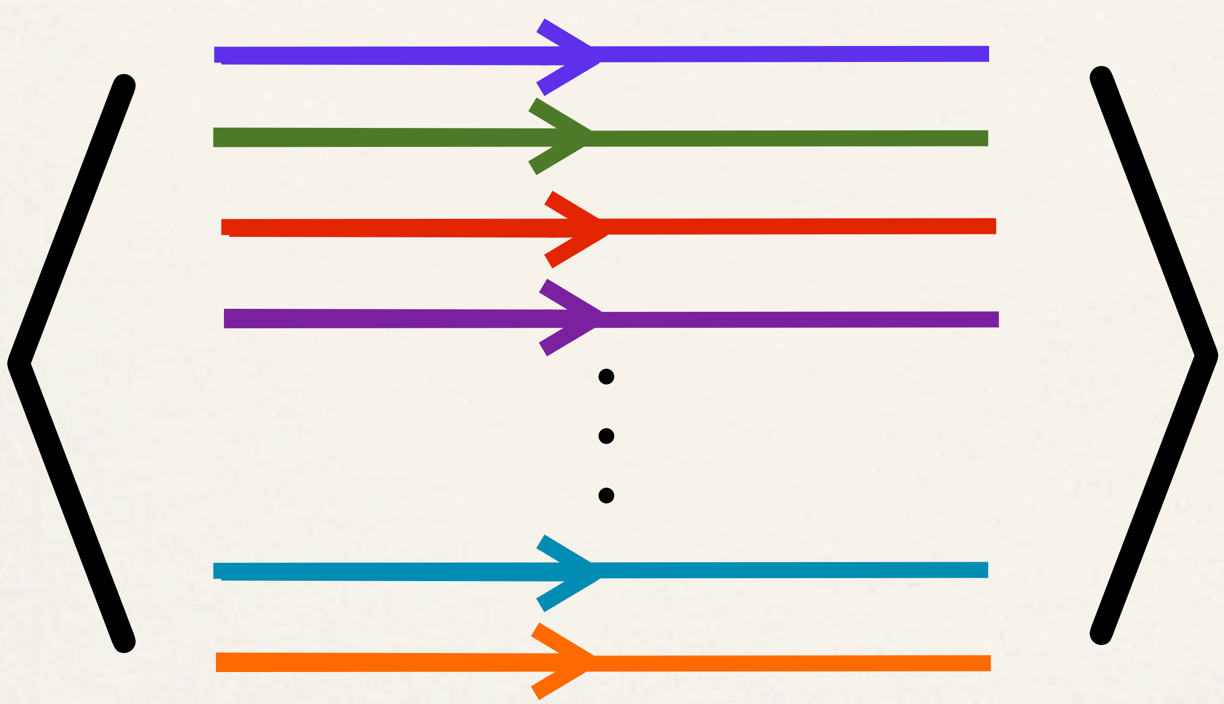

$$\underset{\tau\to\infty}{\longrightarrow} \mathcal{Z}_{2N} e^{-E_0^{(2N)}\tau}$$

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Moments of the 2-body correlator

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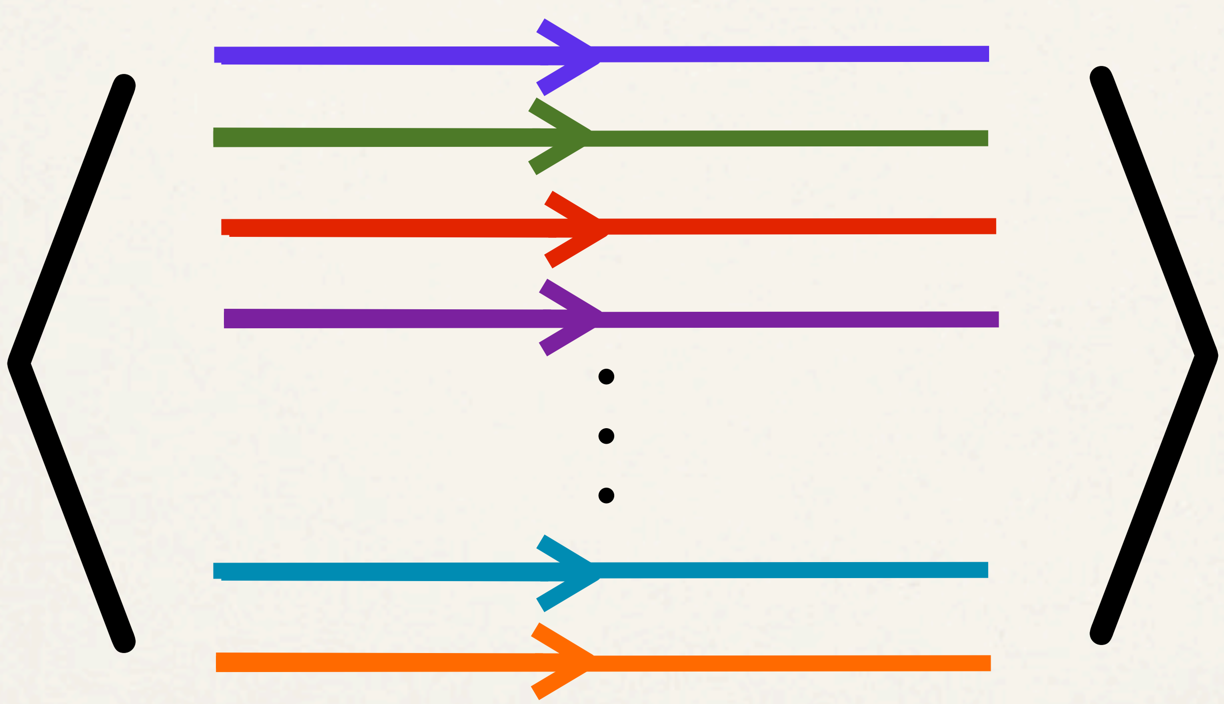


$$\xrightarrow[\tau\to\infty]{} \mathcal{Z}_{2N} e^{-E_0^{(2N)}\tau}$$

Energies correspond to physical states in 2N-flavor theory

Independent of number of fermion flavors

$$Z_\phi = \int d\phi e^{-\int d\tau d^3x \frac{1}{2}\phi^2}$$

Moments of the 2-body correlator

Moments depend on energies of 2N distinguishable particles at unitarity.

What do we know about these systems?

Efimov physics

Trimers

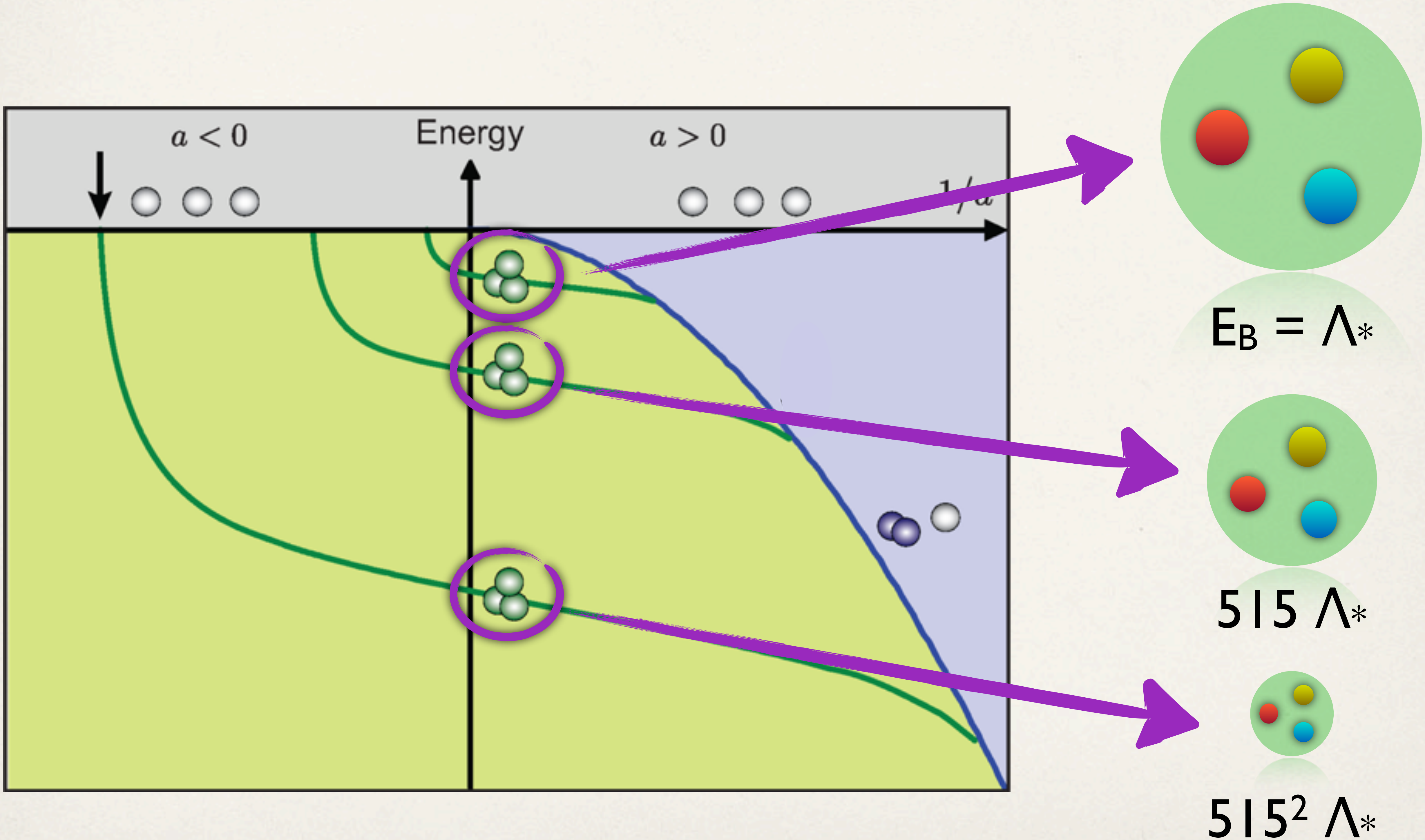
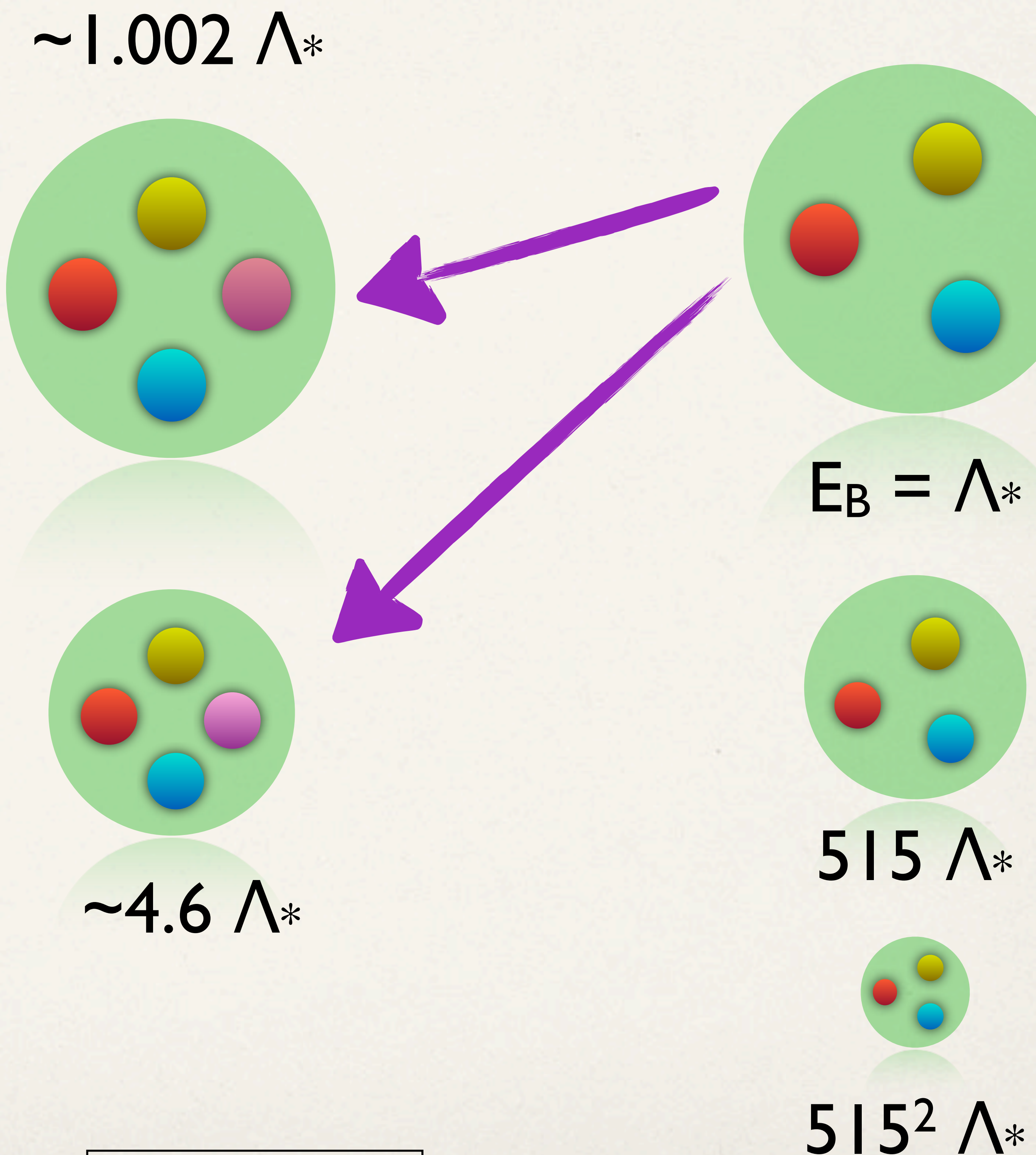


Figure: Ferlaino & Grimm, Physics 3,9 (2010)

Tetramers

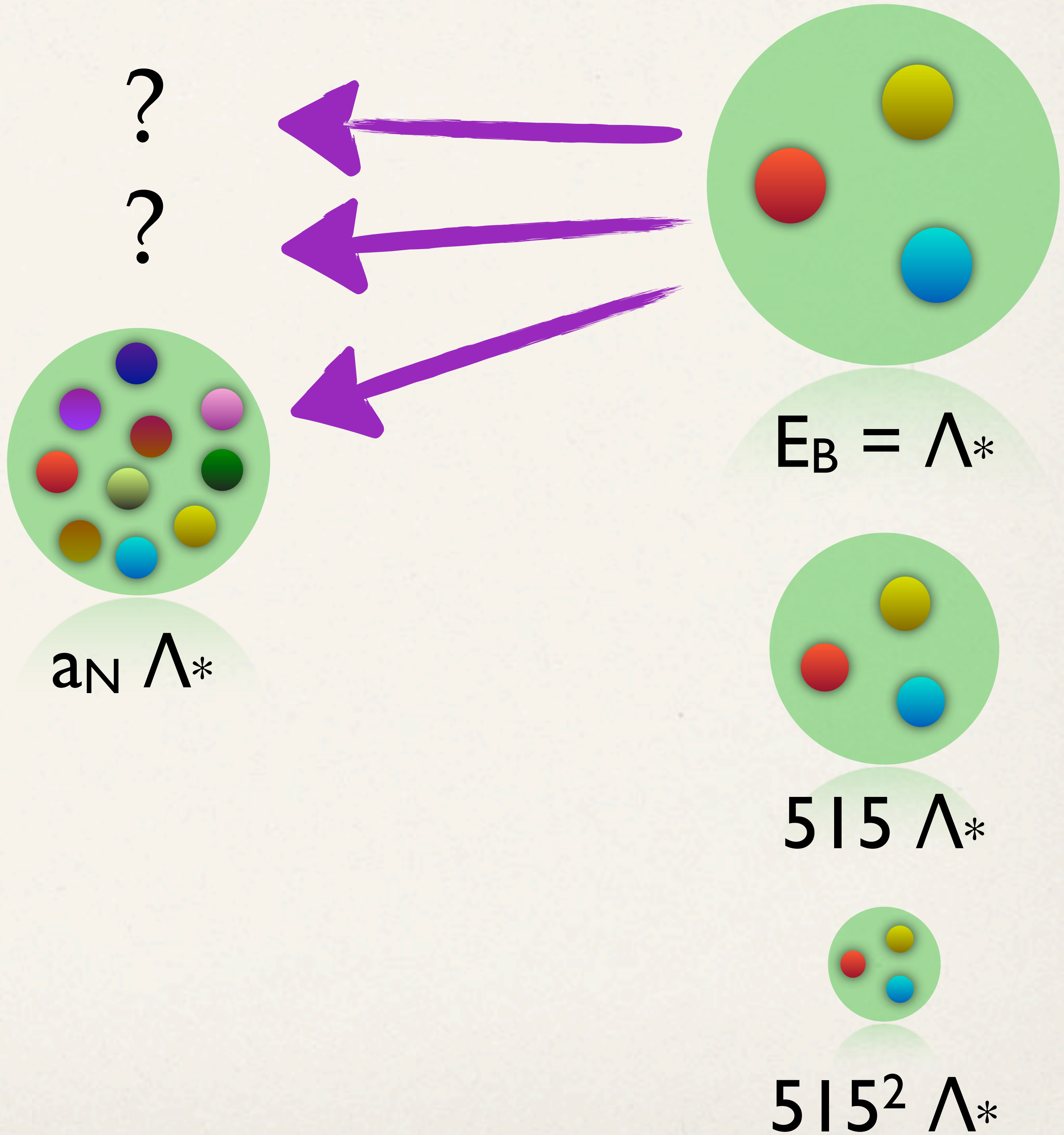
- 2 universal tetramers tied to each trimer
- All energies controlled by 3-body scale only
  - ➡ no new 4-body scale



Deltuva (2013)

N-body?

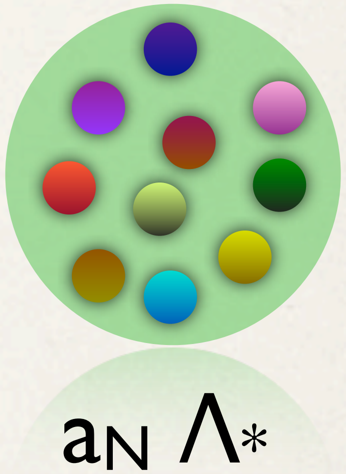
- Numerical calculations indicate persistence of scaling behavior
- Will be true as long as no new N-body operators become relevant



Moments of the 2-body correlator

$$E_0^{(2)} = 0$$

$$\mathcal{M}_N \xrightarrow[\tau\to\infty]{} \mathcal{Z}_N e^{-E_0^{(2N)}\tau}$$



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$$E_0^{(4)} = -a_4 \Lambda_*$$

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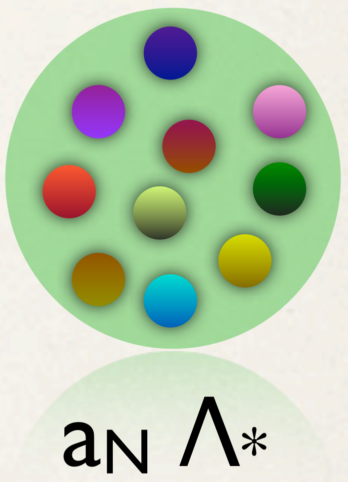

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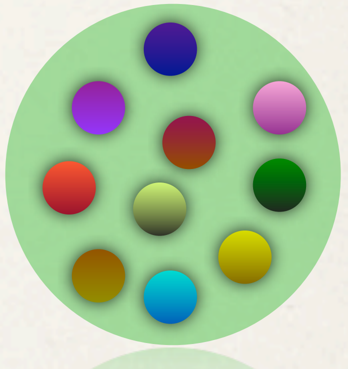

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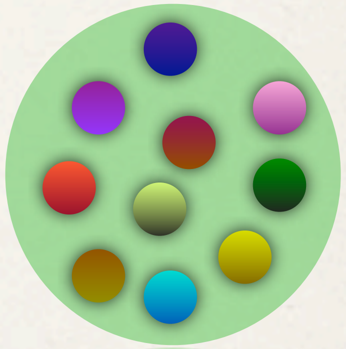
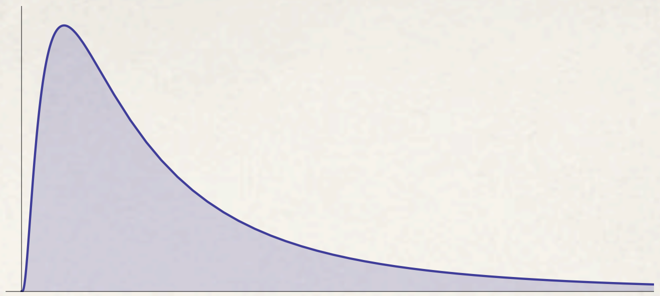


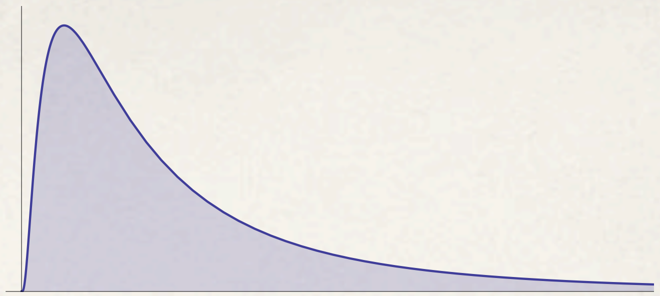
$$\mathcal{M}_N \to \mathcal{Z}_{2N} e^{a_{2N}\Lambda_* \tau}$$

Log-normal moments



$$\mathcal{M}_N = e^{N\mu + \frac{1}{2}N^2\sigma^2}$$

Log-normal moments



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$$\mathcal{M}_1 = 1 \quad \Rightarrow \quad \mu = -\frac{1}{2}\sigma^2$$

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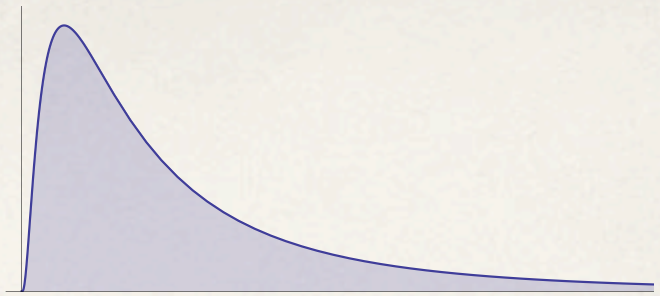
Log-normal moments



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Distribution controlled by a single scale, just like the Efimov spectrum!

Energy spectrum

If we know the distribution is log-normal, we can predict the universal constants!

$$E_0^{(2N)} = \frac{1}{2}N(N-1)E_0^{(4)}$$

Energy spectrum

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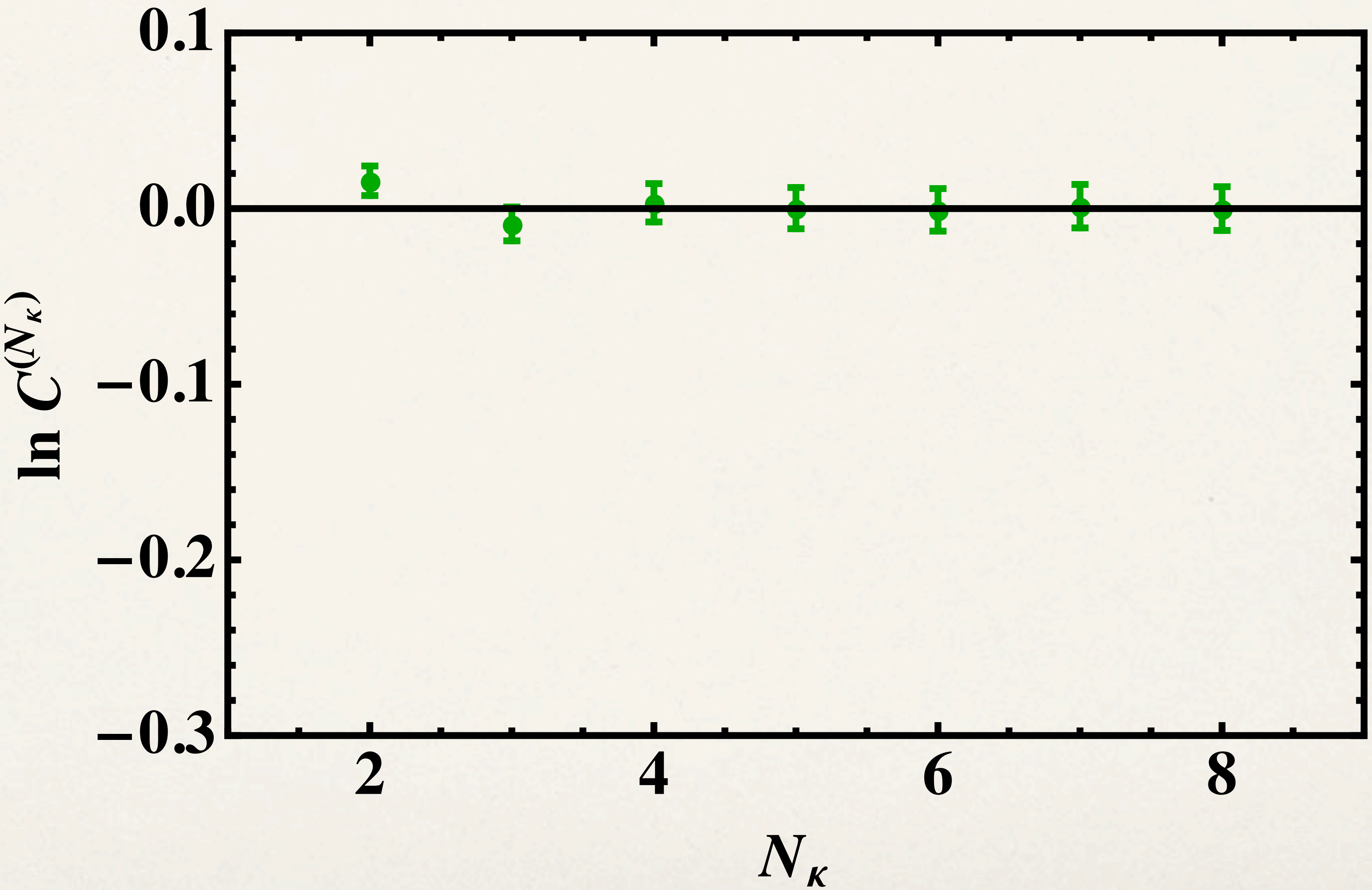
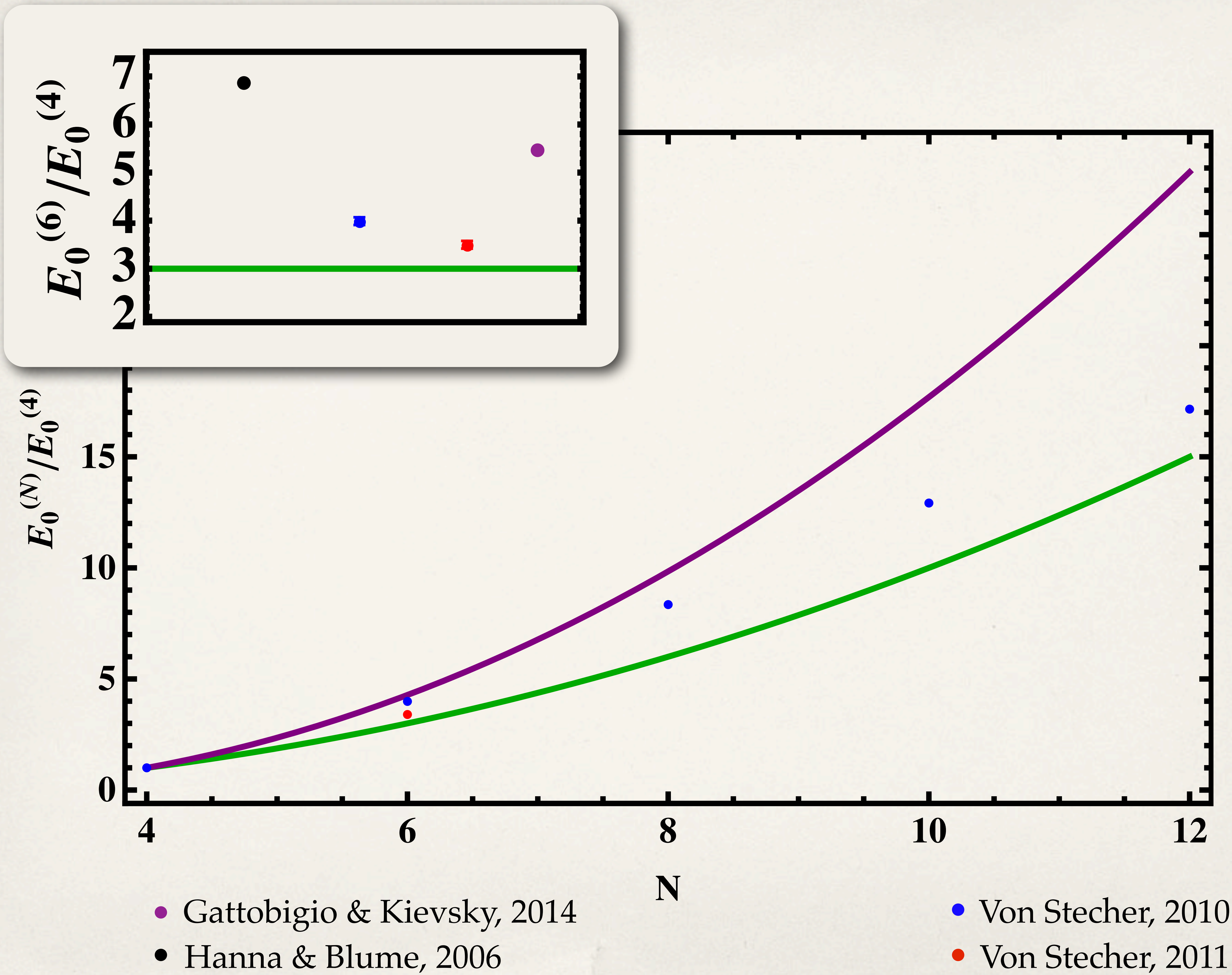
$$E_0^{(2N)} = \frac{1}{2}N(N-1)E_0^{(4)}$$

Caveat: for large N, states will be non-universal

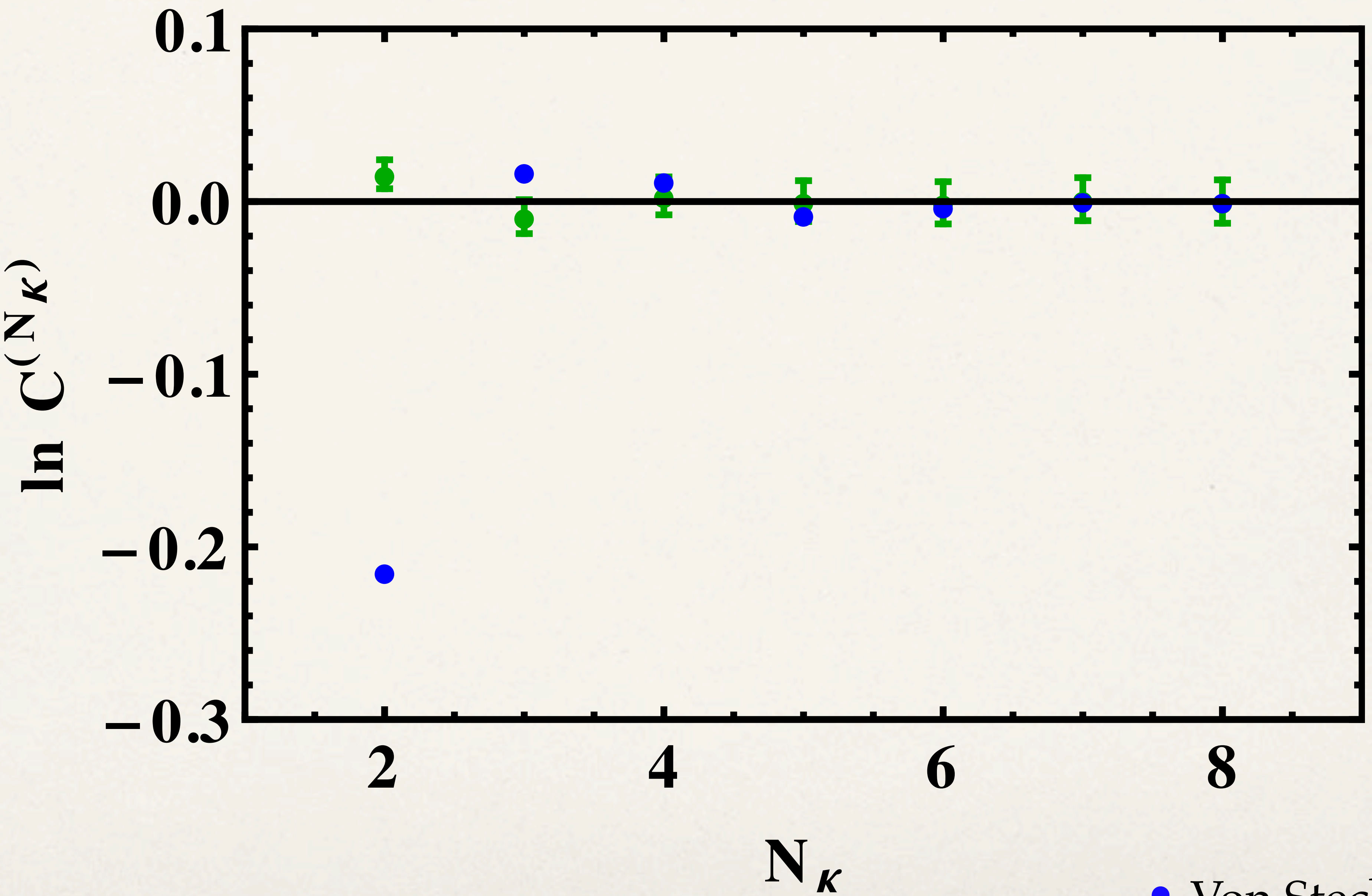
Cumulant expansion





$E_0^{(6)}/E_0^{(4)}$
7
6
5
4
3
2
$E_0^{(N)}/E_0^{(4)}$
15
10
5
0
4
6
8
10
12
N
Gattobigio & Kievsky, 2014
Hanna & Blume, 2006
Von Stecher, 2010
Von Stecher, 2011

Cumulant expansion



• Von Stecher, 2010

Other properties

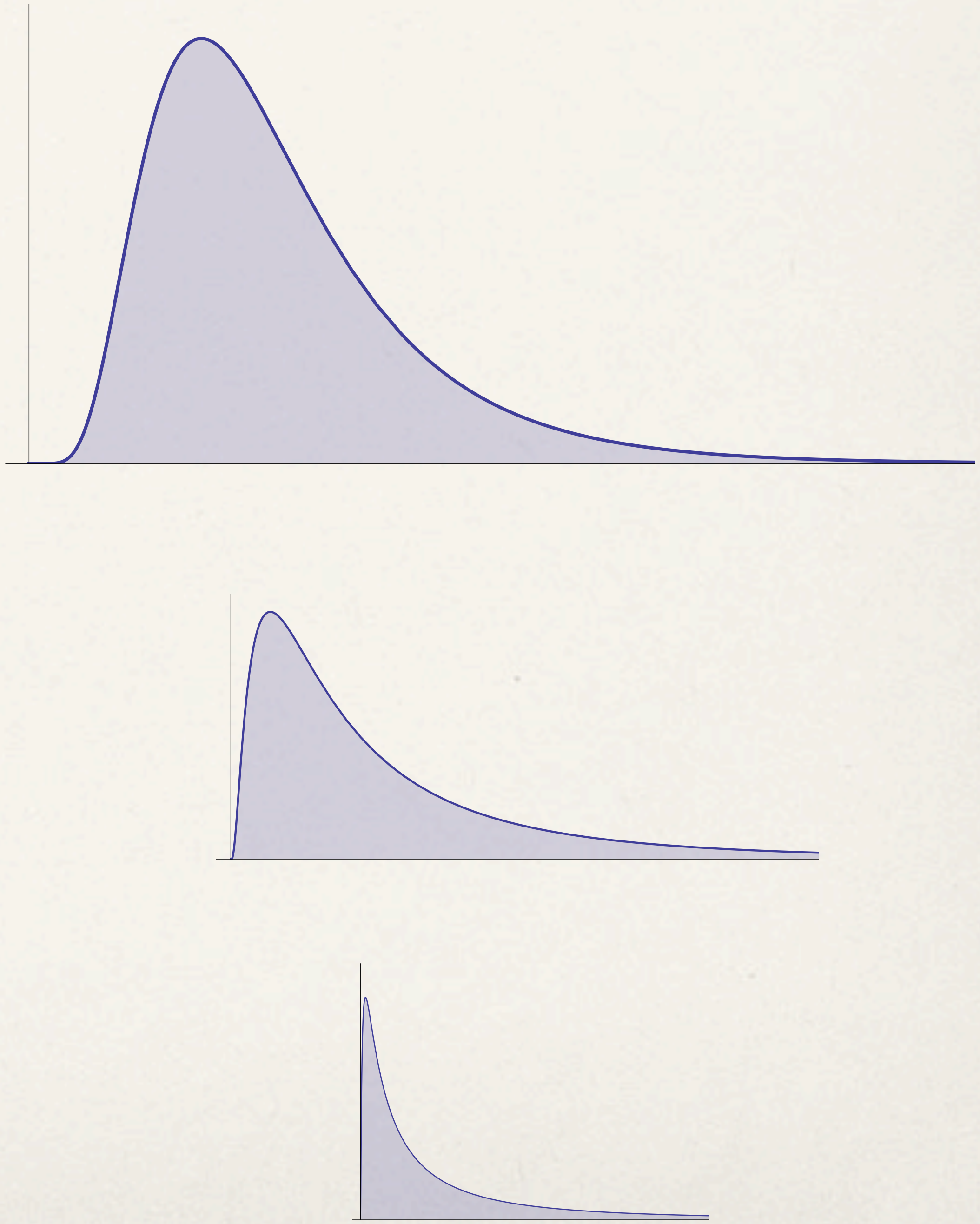
- Wavefunction overlap:

$$\mathcal{Z}_{2N} = \mathcal{Z}_4^{\frac{1}{2}N(N-1)}$$

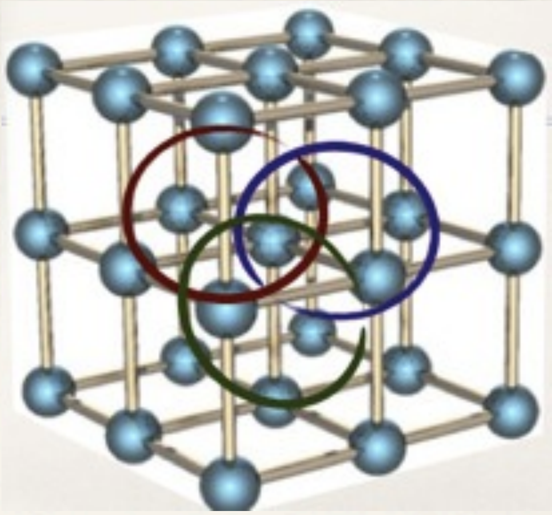
- Universal distribution for 2N particle correlators:

$$\mu = \frac{1}{2} N E_0^{(4)} \tau$$

$$\sigma^2 = -N^2 E_0^{(4)} \tau$$

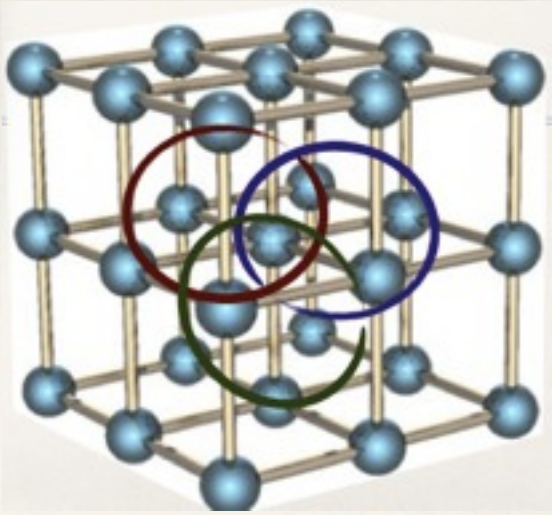


Conclusions



- Conformal systems can be reliably studied on a discrete lattice
    - Improvements may be made to systematically reduce the need for extrapolations to the continuum
- "Universal" distribution seems to have deep connection to Efimov physics for unitary fermion correlators
    - Can use probability distribution for 2-particle systems to predict 2N-body energies
    - N^2 dependence recently confirmed!

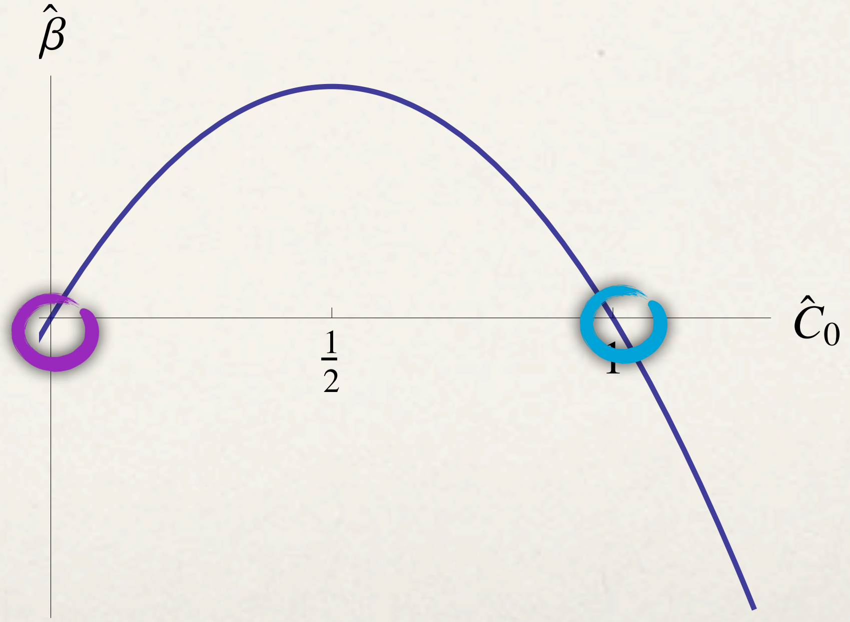
Conclusions



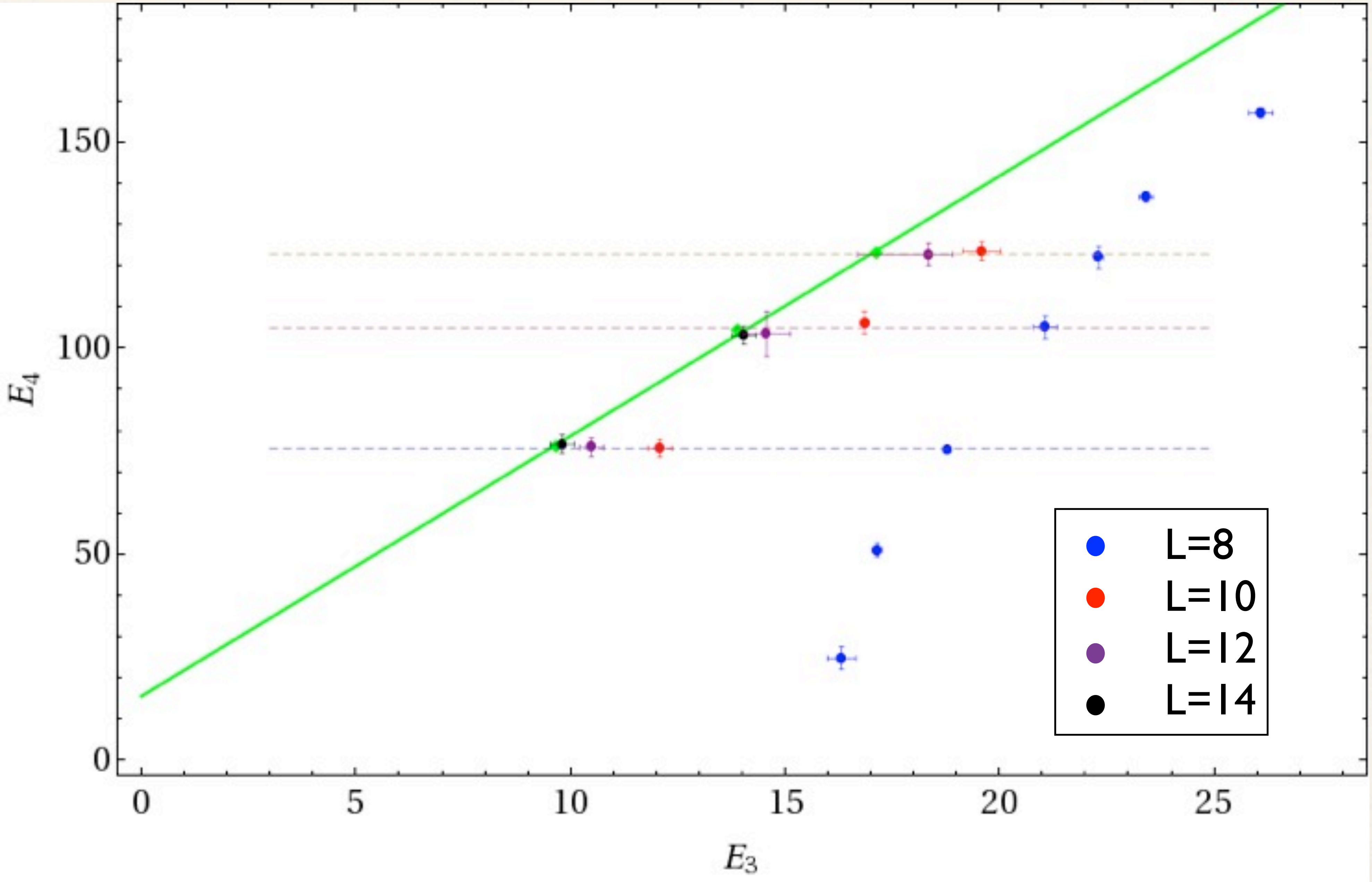
$$\Phi_C(s) \propto \int [d\phi]\, e^{-S[\phi]+isC[\phi]}$$

$$\ln \Phi_C(s) = \sum_{n=1}^{\infty} \frac{(is)^n}{n!} \kappa_n$$

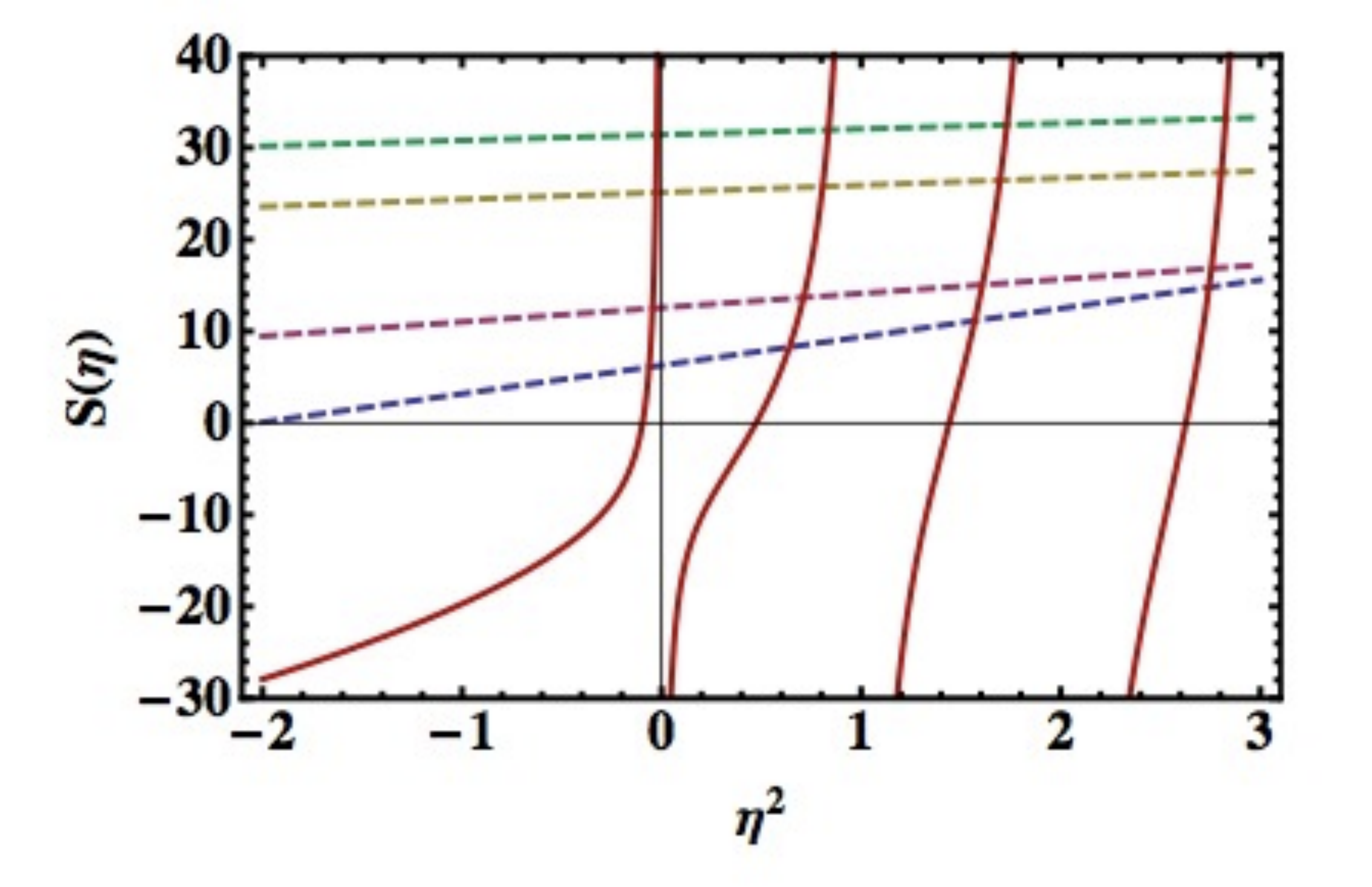
Q: Can an expansion around log-normal give us an analytic tool?



Finite volume



Lüscher



Cutoff dependence