Universality in FBS, INT
Seattle, April 25, 2014

(On) Complete universal description of the three-body spectrum of two-component fermions

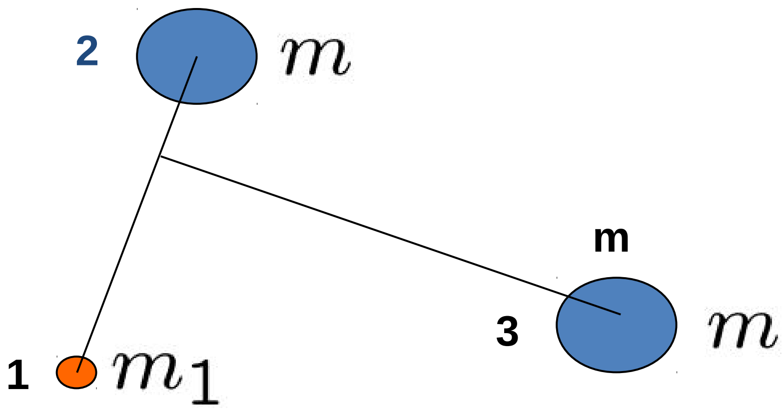
O. I. Kartavtsev (JINR)

In collaboration with A. V. Malykh

To begin with two points:

- Gratitude to organizers
- The work is neither completely finished nor even published

Two identical particles (fermions) and one distinct particle. Zero-range interactions between different particles.



One of the basic and simplest systems

Few-body dynamics in two-species mixtures of ultra-cold quantum gases

- Some history & Overview
- Technical details
- Three-body boundary condition
- Very simple model
- Numerical results
- Similar systems & more particles

Efimov effect takes place in $L^P = 1^-$ states for $m/m_1 > 13.6069657$.

Nuclear Physics **A210** (1973) 157 – 188

ENERGY LEVELS OF THREE RESONANTLY INTERACTING PARTICLES

V EFIMOV

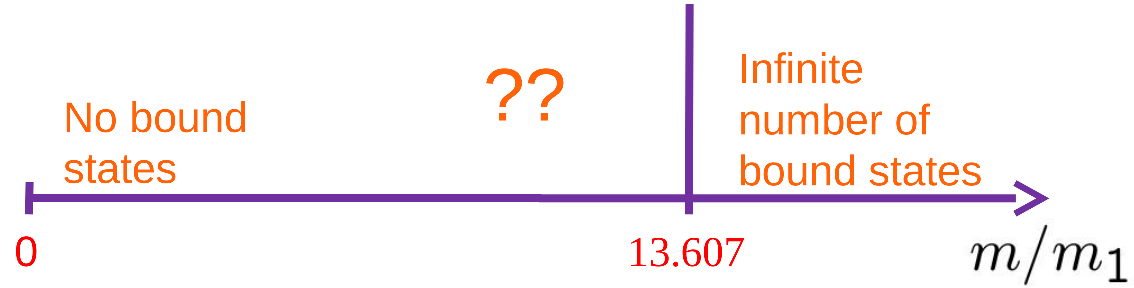
PHYSICAL REVIEW A **67**, 010703(R) (2003)

Three-body problem in Fermi gases with short-range interparticle interaction

D. S. Petrov

What's below critical mass ratio $m/m_1 = 13.6069657$?

For example



One or two bound states below critical mass ratio for positive two-body scattering length $a > 0$,
No bound states for $a < 0$.

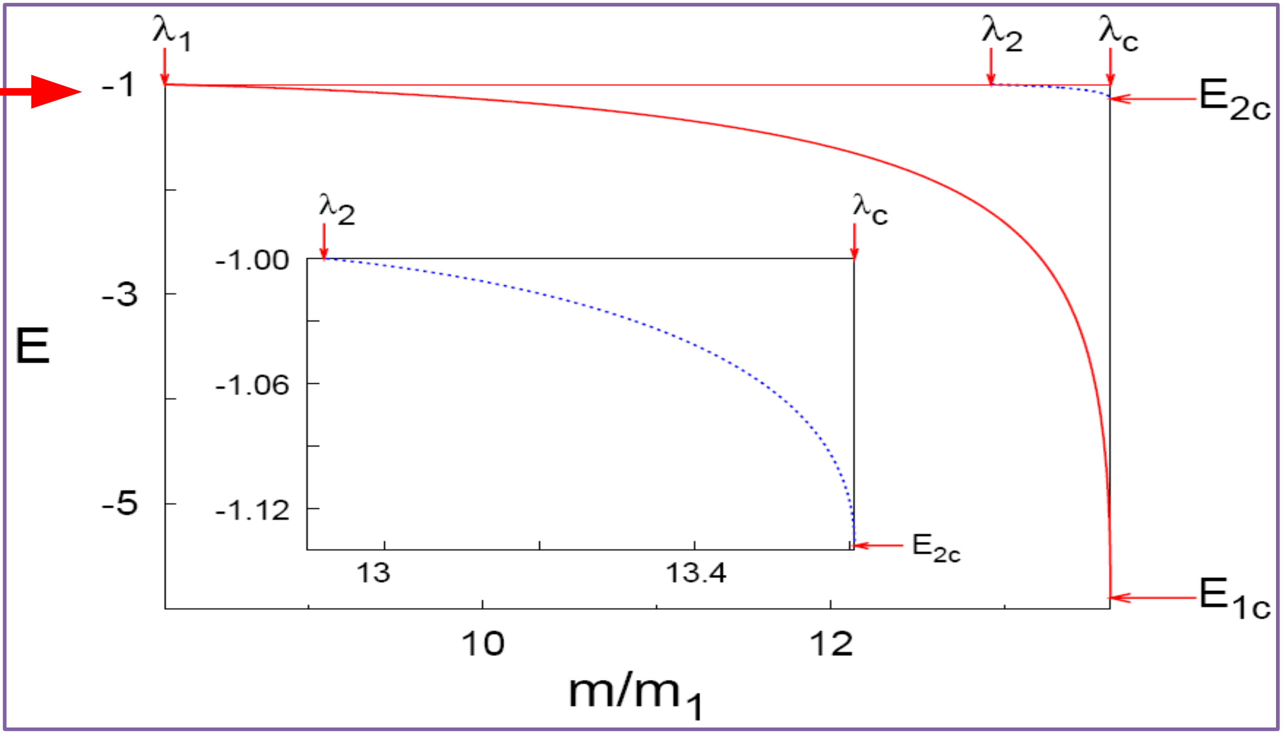
J. Phys. B: At. Mol. Opt. Phys. **40** (2007) 1429–1441

Low-energy three-body dynamics in binary quantum gases

O I Kartavtsev and A V Malykh

$$\lambda_1 = 8.17262, \lambda_2 = 12.9174$$

2-body threshold



-1.1375

-5.895

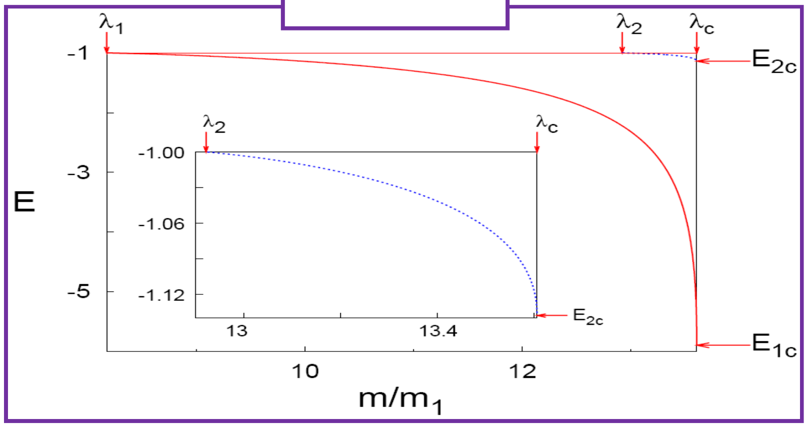
Similar picture for higher angular momenta and for two identical particles being bosons

Pis'ma v ZhETF, vol. 86, iss. 10, pp. 713 – 717

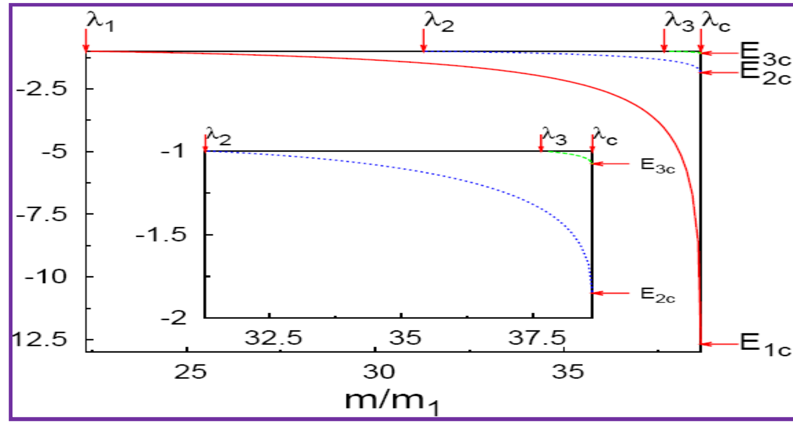
Universal description of the rotational-vibrational spectrum of three particles with zero-range interactions

O. I. Kartavtsev, A. V. Malykh

LP = 1-
Fermions

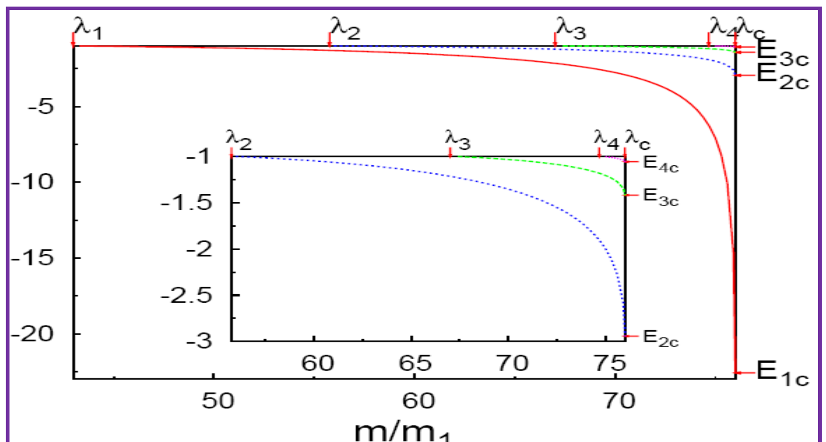


LP = 2+
bosons



a>0

Fermions: odd L, odd parity
Bosons: even L, even parity



LP = 3-
Fermions

Few more papers confirmed the above picture, e. g.

Few-Body Syst (2011) 51: 207–217
DOI 10.1007/s00601-011-0229-6

Shimpei Endo · Pascal Naidon · Masahito Ueda

Universal Physics of 2+1 Particles with Non-Zero Angular Momentum

J. Phys. B: At. Mol. Opt. Phys. **44** (2011) 215301 (8pp) doi:10.1088/0953-4075/44/21/215301

On the Efimov effect in higher partial waves

K Helfrich and H-W Hammer

All is truth, but is not all the truth

The problem to be handled is that one can construct a square-integrable solution for any energy in the mass-ratio interval

$$8.61857692\ldots < m/m_1 \leq 13.6069657\ldots$$

This means that the zero-range interactions model formally does not define a self-adjoint Hamiltonian.

To some extent, it is similar to Efimov situation, which takes place above critical mass ratio.

Origin is a singular behaviour near the triple-collision point.

Remark: certainly, there is no problem with one bound-state determined in a mass-ratio window

$$8.1726\ldots \leq m/m_1 \leq 8.6186\ldots$$

Problem of the solution's behaviour near the triple-collision point was discussed also by

PRL **100,** 090405 (2008) PHYSICAL REVIEW LETTERS week ending 7 MARCH 2008

Universal Fermi Gas with Two- and Three-Body Resonances

Yusuke Nishida, Dam Thanh Son, and Shina Tan

PHYSICAL REVIEW A **87**, 032713 (2013)

Nonuniversal bound states of two identical heavy fermions and one light particle

A. Safavi-Naini,[1,2,*] Seth T. Rittenhouse,[2,3] D. Blume,[2,4] and H. R. Sadeghpour[2]

Mathematical papers, e. g., by R. A. Minlos

MOSCOW MATHEMATICAL JOURNAL
Volume 11, Number 1, January–March 2011, Pages 113–127

ON POINT-LIKE INTERACTION BETWEEN n FERMIONS AND ANOTHER PARTICLE

R. MINLOS

International Scholarly Research Network
ISRN Mathematical Physics
Volume 2012, Article ID 230245, 18 pages
doi:10.5402/2012/230245

Research Article

On Pointlike Interaction between Three Particles: Two Fermions and Another Particle

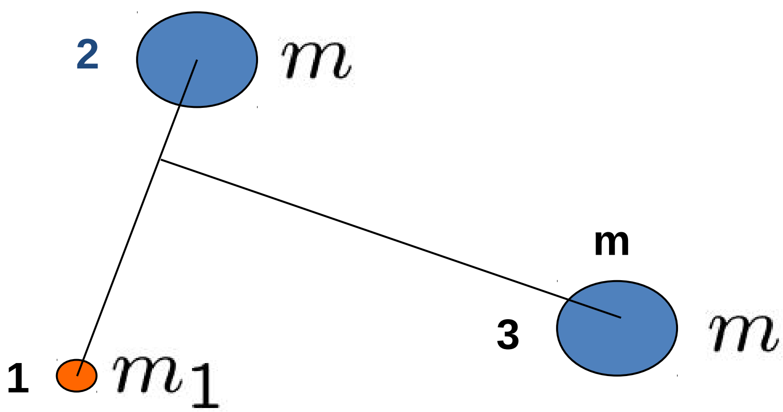
Robert Adol'fovich Minlos

Coordinates

$$\mathbf{x} = \sqrt{2\mu}\,(\mathbf{r}_2 - \mathbf{r}_1)$$
$$\mathbf{y} = \sqrt{2\tilde{\mu}}\left(\mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m\mathbf{r}_2}{m_1 + m}\right)$$

$$\mu = \frac{m m_1}{m + m_1}$$
$$\tilde{\mu} = \frac{m(m + m_1)}{m_1 + 2m}$$

$$x = \rho \sin\alpha,\; y = \rho\cos\alpha$$



Units

$$\hbar = 2\mu = |a| = 1$$

Unit two-body binding energy

Formal definition of a simple, pure mathematical, problem

$$(\Delta_{\mathbf{x}} + \Delta_{\mathbf{y}} + E)\,\Psi = 0$$

$$\lim_{r\to 0} \frac{\partial \ln(r\Psi)}{\partial r} = -\frac{1}{a}$$

- The wave function is antisymmetric under permutation of particles 2 and 3
- Two-body interactions defined as the boundary condition at zero inter-particle distance

Define solutions on the hypersphere

$$\Big[\frac{1}{\sin^2 2\alpha}\left(\sin^2 2\alpha\frac{\partial}{\partial\alpha}\right)+\frac{1}{\sin^2\alpha}\Delta_{\hat{\mathbf{x}}}+\frac{1}{\cos^2\alpha}\Delta_{\hat{\mathbf{y}}}$$

$$+\gamma_n^2(\rho)-4\Big]\Phi_n(\rho,\Omega)=0$$

$$\lim_{\alpha\to 0}\left[\frac{\partial\ln(\alpha\Phi_n)}{\partial\alpha}\pm\rho\right]=0$$

---

use in the expansion

$$\Psi=\rho^{-5/2}\sum_{n=1}^{\infty}\phi_n(\rho)\Phi_n(\rho,\Omega)$$

and results in hyper-radial equations

$$\left[\frac{d^2}{d\rho^2} - \frac{\gamma_n^2(\rho) - 1/4}{\rho^2} + E\right]\phi_n(\rho)$$

$$-\sum_{m=1}^{\infty}\left[P_{mn}(\rho) - Q_{mn}(\rho)\frac{d}{d\rho} - \frac{d}{d\rho}Q_{mn}(\rho)\right]\phi_m(\rho) = 0$$

where

$$Q_{nm}(\rho) = \left\langle \Phi_n \mid \frac{\partial \Phi_m}{\partial \rho} \right\rangle , \quad P_{nm}(\rho) = \left\langle \frac{\partial \Phi_n}{\partial \rho} \mid \frac{\partial \Phi_m}{\partial \rho} \right\rangle$$

Simple eigenvalue equation for $\gamma(\rho)$

$$\rho = \frac{1-\gamma^2}{\gamma}\tan\gamma\frac{\pi}{2} - \frac{2}{\sin 2\omega}\frac{\cos\gamma\omega}{\cos\gamma\frac{\pi}{2}} + \frac{\sin\gamma\omega}{\gamma\sin^2\omega\cos\gamma\frac{\pi}{2}}$$

$$\sin\omega = \frac{1}{1+m_1/m}$$

In particular, critical mass ratio is determined by

$$\frac{\pi}{2}\sin^2\omega_c - \tan\omega_c + \omega_c = 0$$

m/m_1 = 13.6069657...

For brevity, the following notations will be used

$$\lim_{\rho\to 0} \gamma(\rho) \longrightarrow \gamma$$

$$q = \left[\frac{d\gamma^2(\rho)}{d\rho}\right]_{\rho\to 0}$$

Three-body boundary condition at $\rho \longrightarrow 0$

$$1/2 \;>\; \gamma \;>\; 0$$

$$\phi(\rho) \longrightarrow \rho^{1/2}\left(\rho^{\gamma} \mp b^{2\gamma}\rho^{-\gamma}\right)$$

b – generalized three-body scattering length (GTBSL)

Solution is uniquely defined by the GTBSL value

Sign convention: upper (-) for b > 0, lower (+) for b < 0

$1 > \gamma > 1/2$

$$\phi(\rho) \longrightarrow \rho^{1/2}\left[\rho^{\gamma} \mp b^{2\gamma}\rho^{-\gamma}\left(1 + \frac{q\rho}{1-2\gamma}\right)\right]$$

$\gamma = 1/2$

$$\phi(\rho) \longrightarrow \rho - b\,(1 + q\rho \log \rho)$$

$\gamma = 0$

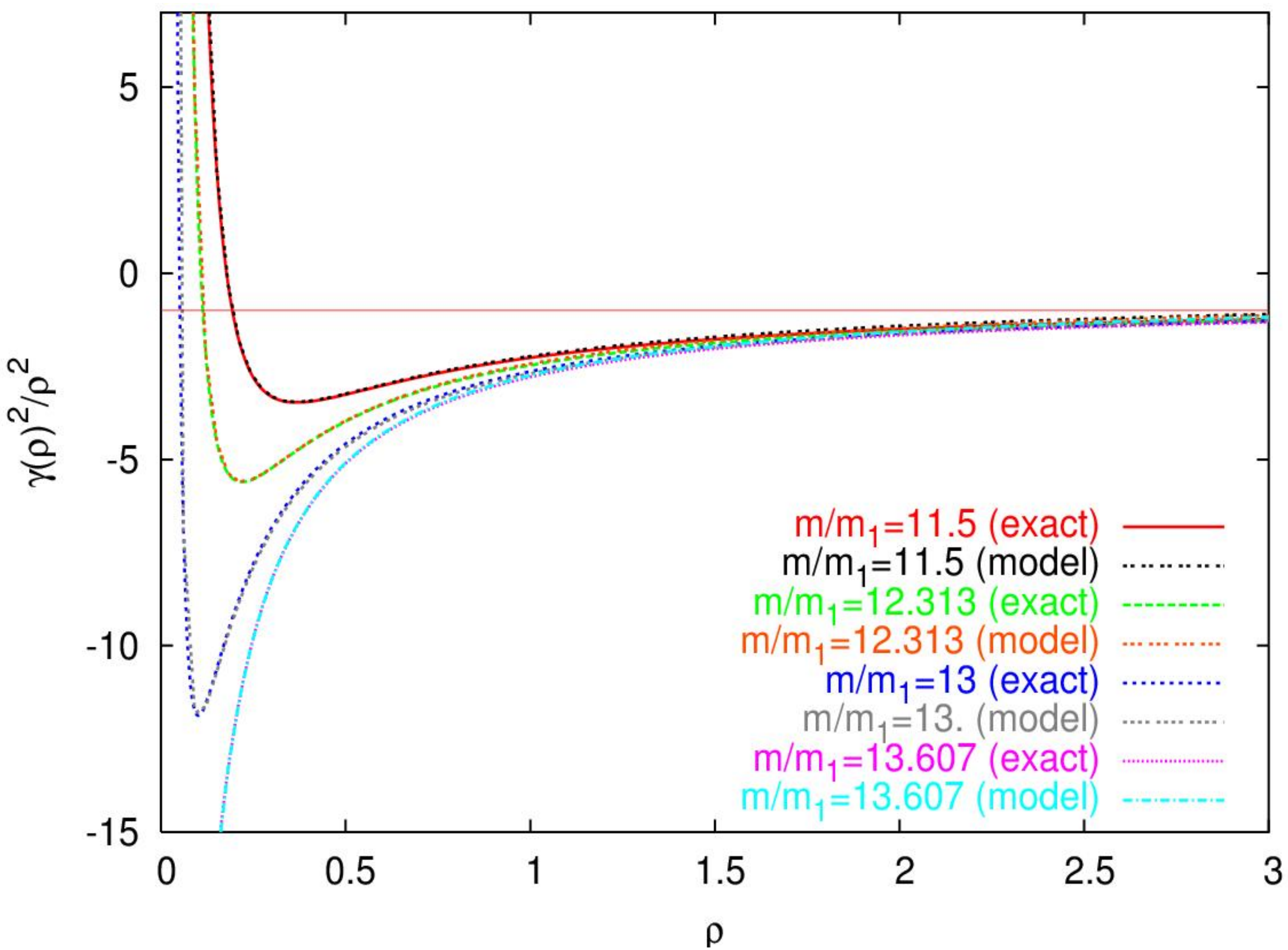
$$\phi(\rho) \longrightarrow \rho^{1/2} \log \frac{\rho}{b}$$

$(b > 0)$

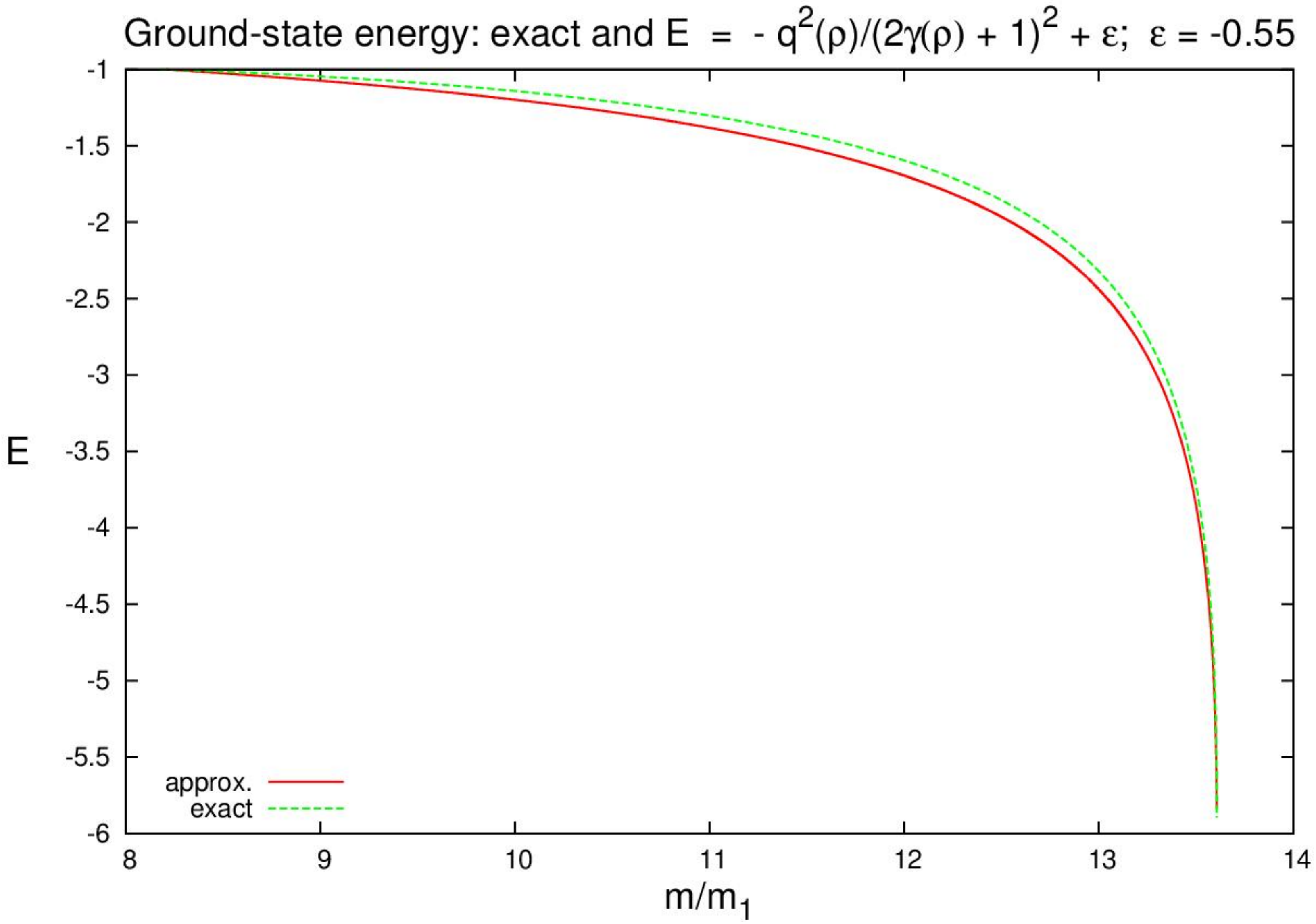
Simple model

- Take only one hyper-radial equation
- Retain only singular terms in the expansion of the effective potential in powers of hyper-radius
- Fit constant term to produce known (numerical) ground-state energy

$$V_{eff}(\rho) = \frac{\gamma^2 - 1/4}{\rho^2} + \frac{q}{\rho} + \varepsilon$$


$\gamma(\rho)^2/\rho^2$
ρ
5
0
-5
-10
-15
0
0.5
1
1.5
2
2.5
3
m/m_1=11.5 (exact)
m/m_1=11.5 (model)
m/m_1=12.313 (exact)
m/m_1=12.313 (model)
m/m_1=13 (exact)
m/m_1=13. (model)
m/m_1=13.607 (exact)
m/m_1=13.607 (model)

b = 0


Ground-state energy: exact and E = - q^2(ρ)/(2γ(ρ) + 1)^2 + ε; ε = -0.55
E
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5.5
-6
8
9
10
11
12
13
14
m/m1
approx.
exact

All essential dependencies on mass ratio and three-body boundary condition can be obtained from analysis of the simple equation:

$$\left(\frac{d^2}{dx^2} - \frac{\gamma^2 - 1/4}{x^2} - \frac{2\eta}{x} - 1\right)\phi = 0$$

$$\eta = \frac{q}{2\kappa}$$

$$E = -\kappa^2 + \varepsilon < -1$$

$$x = \kappa\rho$$

Infinite two-body scattering length $\rho \longrightarrow \infty$ (q --> 0)

$$\phi = x^{1/2} K_\gamma(x)$$

Taking x --> 0, for b > 0

$$\kappa = \frac{2}{b}\left[\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\right]^{\frac{1}{2\gamma}} \qquad E = -\frac{4}{b^2}\left[\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\right]^{\frac{1}{\gamma}}$$

It's similar to usual scattering length

Alternatively, one can define new three-body scattering length, e. g., by a condition $\kappa\tilde{b} = 1$

Positive two-body scattering length a > 0

$$\phi = x^{1/2+\gamma} e^{-x} \Psi\left(1/2+\gamma+\eta, 1+2\gamma; 2x\right)$$

Taking x --> 0

$$\left(\frac{q|b|}{\eta}\right)^{2\gamma} = \mp \frac{\Gamma(2\gamma)\Gamma\left(1/2-\gamma+\eta\right)}{\Gamma(-2\gamma)\Gamma\left(1/2+\gamma+\eta\right)}$$

Typical spectrum for a sum of long-range and short-range potentials

Positive two-body scattering length $a > 0$

$$\phi = x^{1/2+\gamma} e^{-x} \Psi\left(1/2 + \gamma + \eta, 1 + 2\gamma; 2x\right)$$

Taking x --> 0

$$\left(\frac{q|b|}{\eta}\right)^{2\gamma} = \mp \frac{\Gamma(2\gamma)\Gamma\left(1/2 - \gamma + \eta\right)}{\Gamma(-2\gamma)\Gamma\left(1/2 + \gamma + \eta\right)}$$

Typical spectrum for a sum of long-range and short-range potentials

GAME OVER

In two limits Coulomb spectrum appears

For b --> 0

$$\kappa_n = -\frac{q}{2n+2\gamma+1} \qquad E_n = -\frac{q^2}{(2n+2\gamma+1)^2} + \varepsilon$$

For infinite b

$$\kappa_n = -\frac{q}{2n-2\gamma+1} \qquad E_n = -\frac{q^2}{(2n-2\gamma+1)^2} + \varepsilon$$

Two states below threshold in both cases

$$\gamma = 0$$

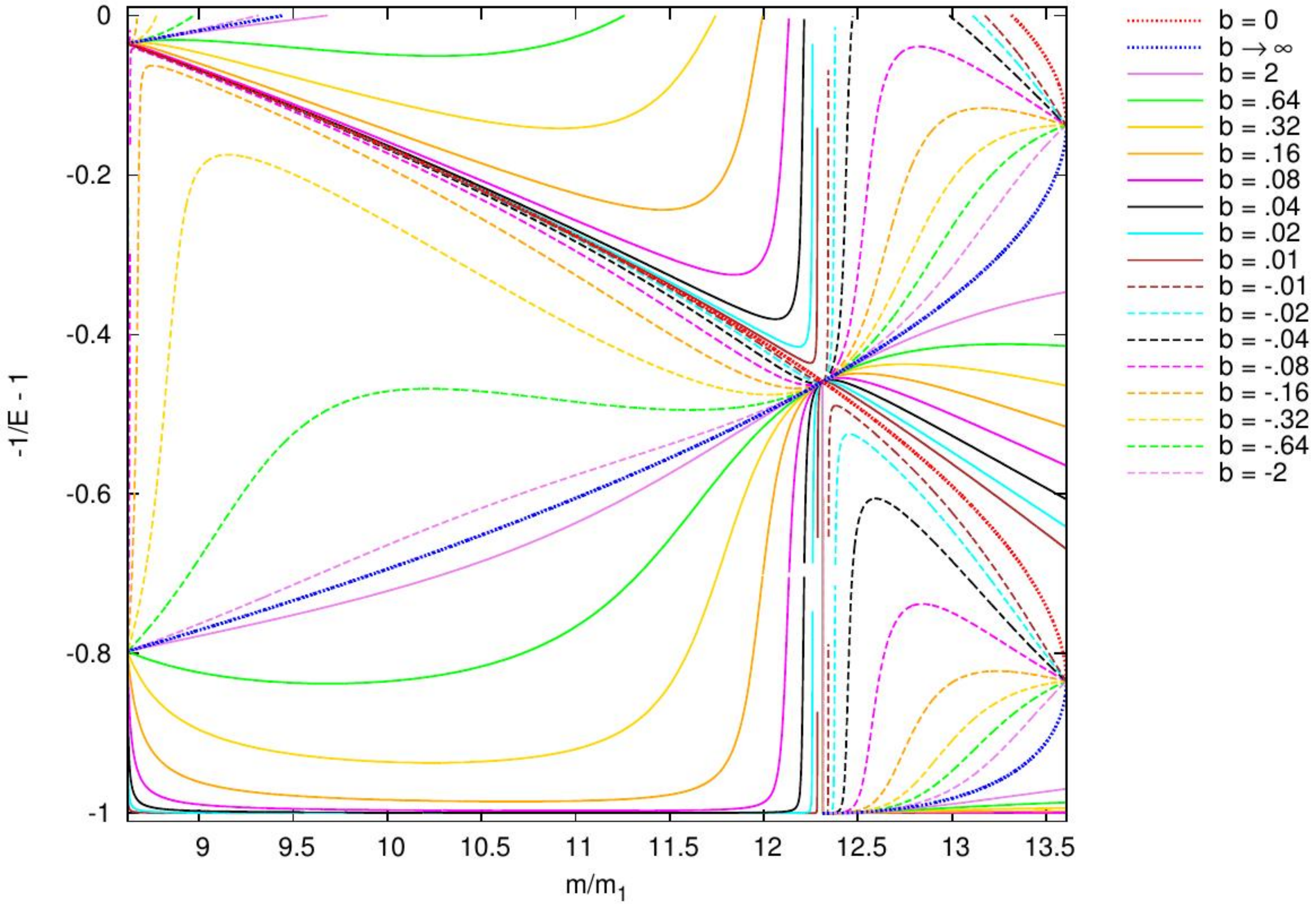
Solutions only for b > 0
Two or three bound states, which energies satisfy the equation:

$$\psi(\tfrac{1}{2} + \eta) + \log \frac{\eta}{qb} = 0$$

$$\gamma = 1/2$$

One or two bound states, which energies satisfy the equation:

$$\frac{\eta}{qb} + \psi(1 + \eta) + 2\gamma_C - 1 = 0$$


0
-0.2
-0.4
-0.6
-0.8
-1
-1/E - 1
9
9.5
10
10.5
11
11.5
12
12.5
13
13.5
m/m1
b = 0
b → ∞
b = 2
b = .64
b = .32
b = .16
b = .08
b = .04
b = .02
b = .01
b = -.01
b = -.02
b = -.04
b = -.08
b = -.16
b = -.32
b = -.64
b = -2

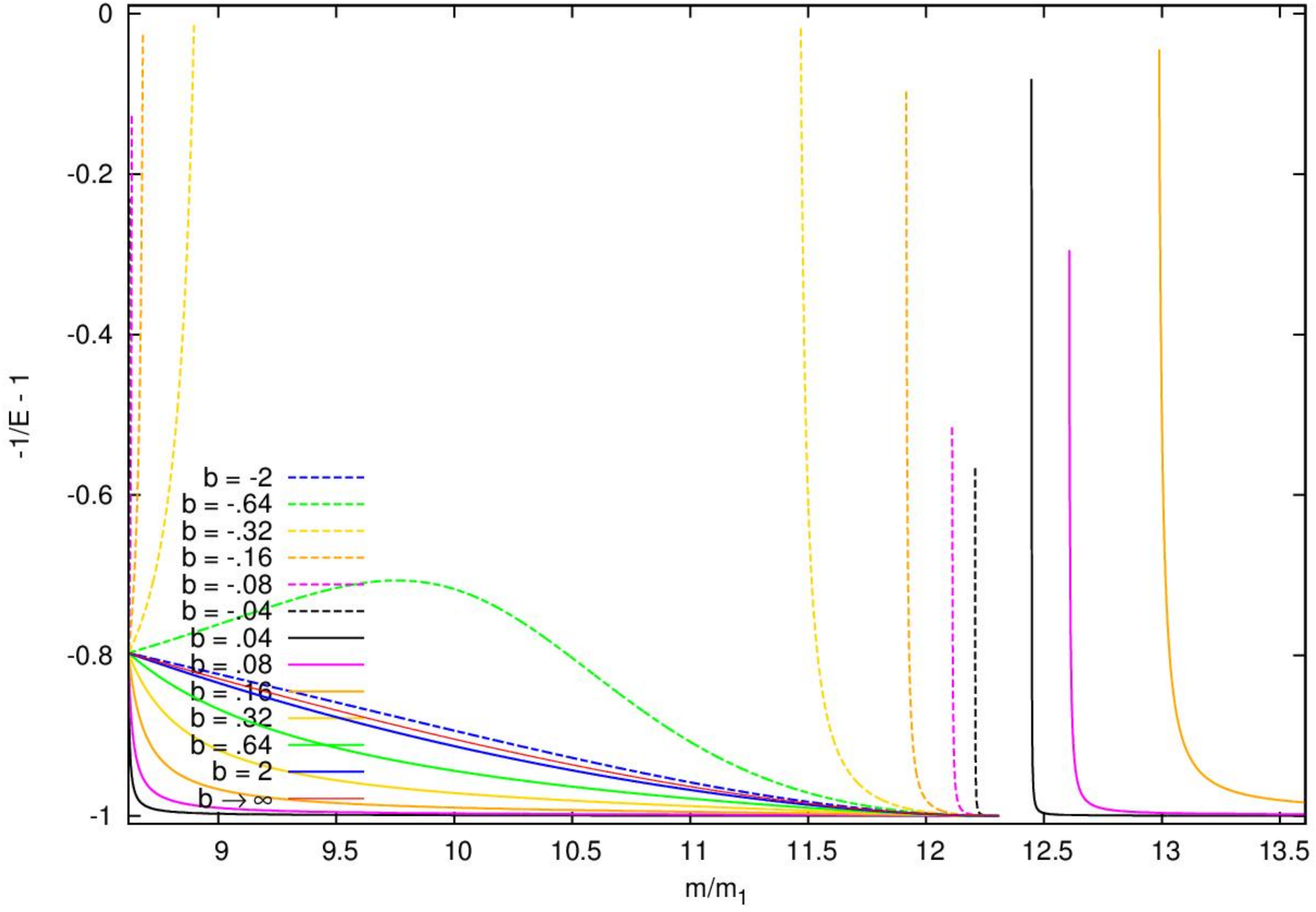
Negative two-body scattering length $a < 0$

$q > 0, \quad \eta > 0$

No bound states for $b = 0$,

One bound state for infinite b:

$$\kappa = \frac{q}{2\gamma - 1} \qquad E = -\frac{q^2}{(2\gamma-1)^2} + \varepsilon_1$$

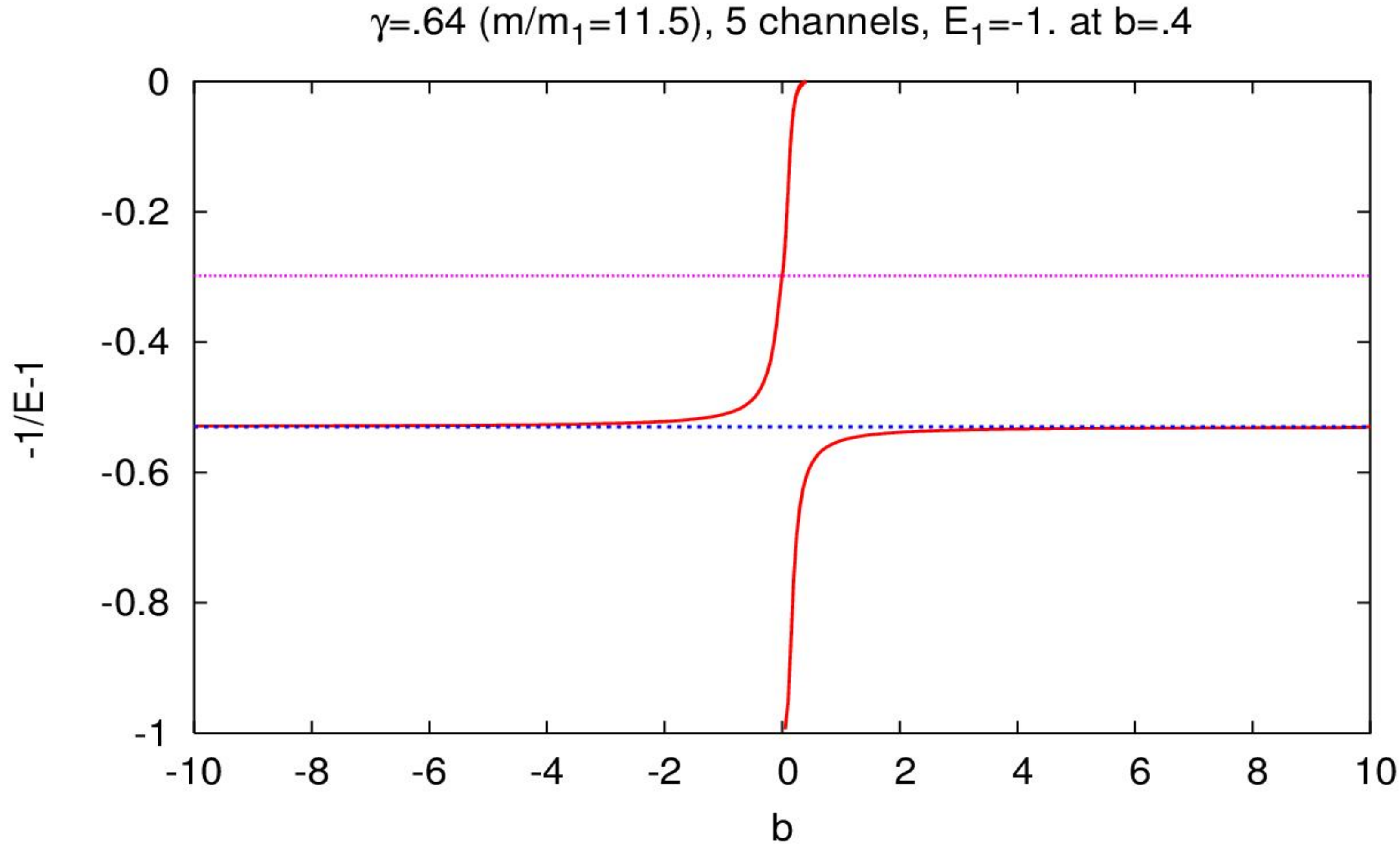

-1/E - 1
m/m_1
0
-0.2
-0.4
-0.6
-0.8
-1
9
9.5
10
10.5
11
11.5
12
12.5
13
13.5
b = -2
b = -.64
b = -.32
b = -.16
b = -.08
b = -.04
b = .04
b = .08
b = .16
b = .32
b = .64
b = 2
b → ∞

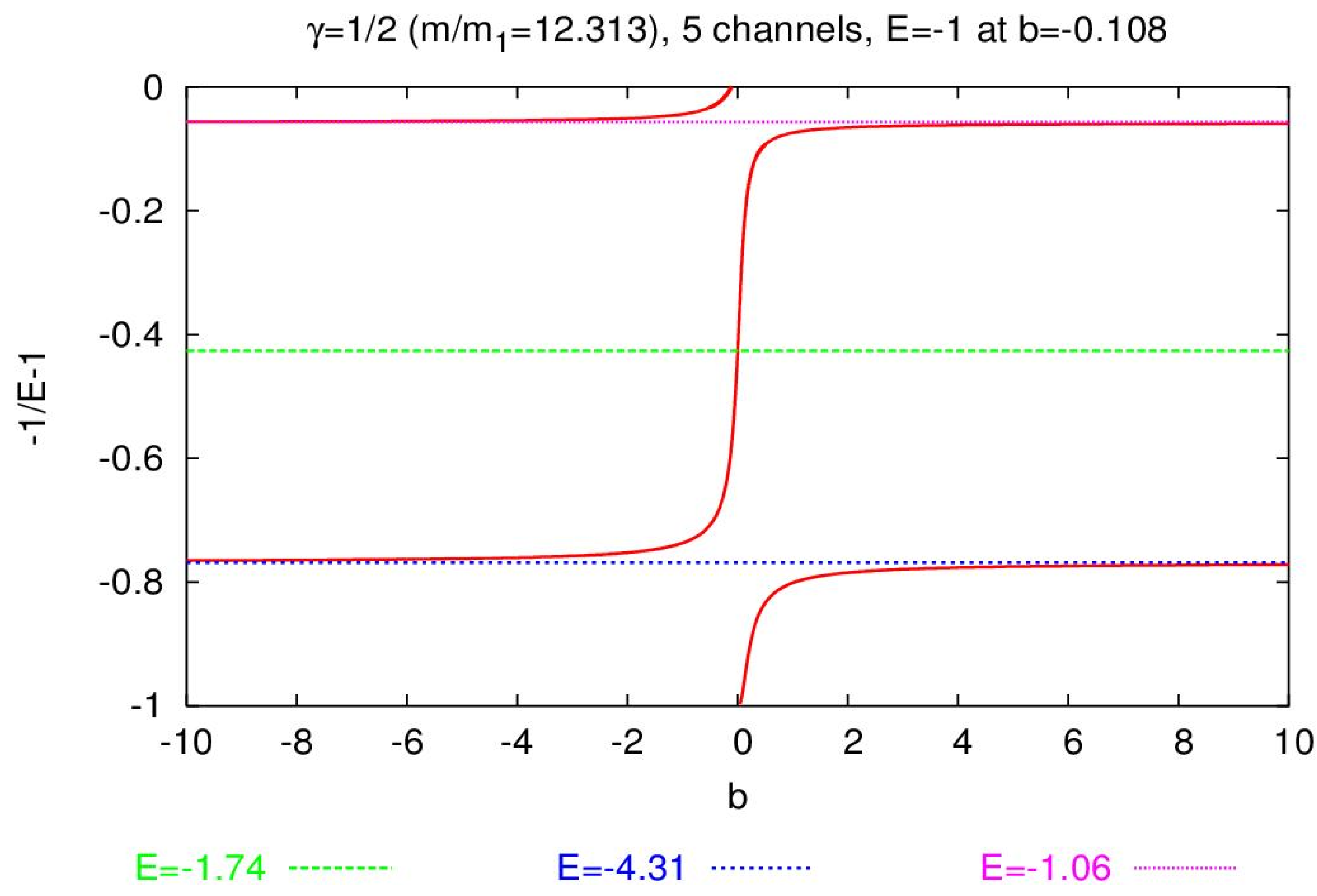
There are from one to three bound states for different values of mass-ratio and GTBSL b.
Not surprising: zero-range potential can change a number of state by one.

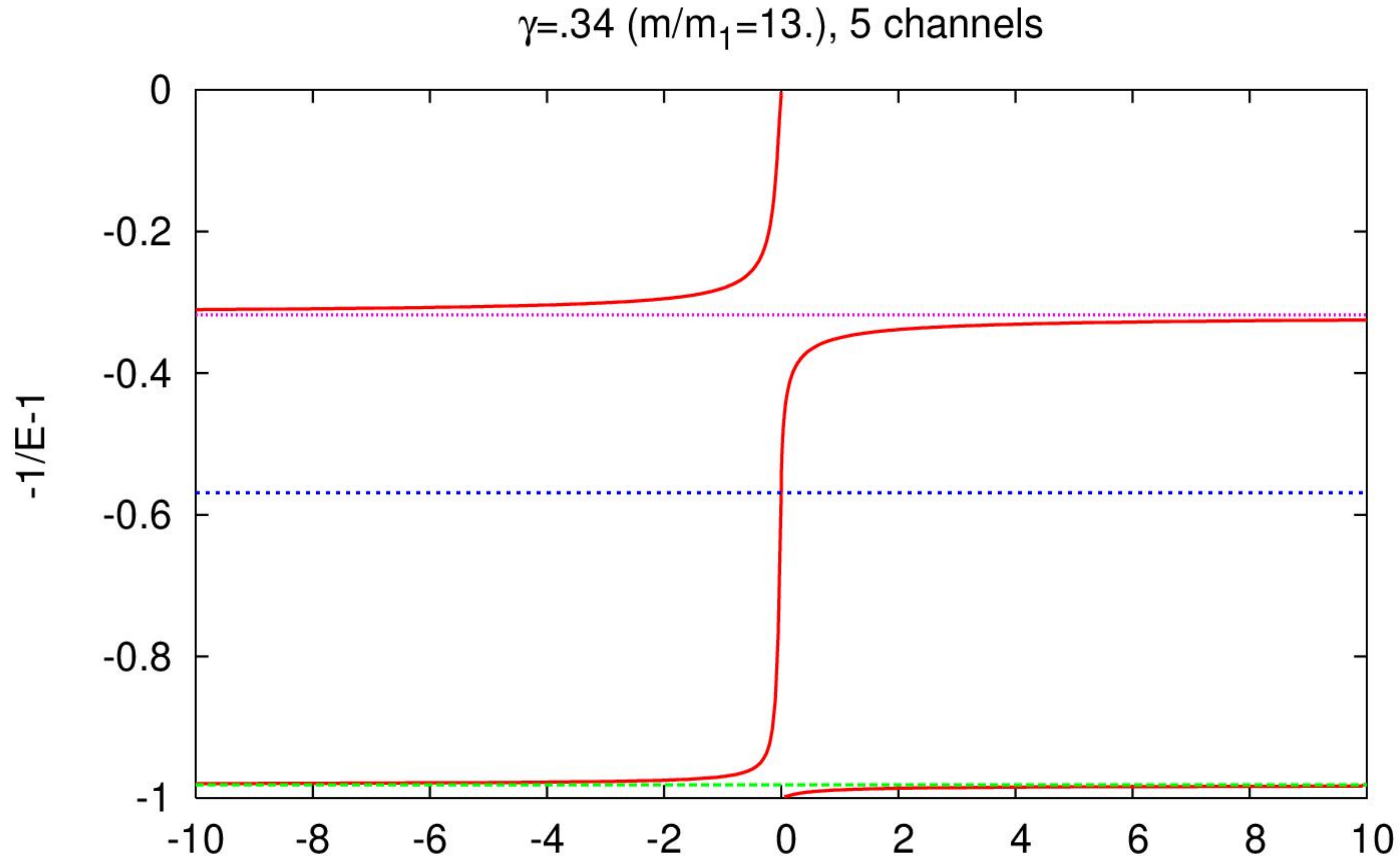
The solutions behave differently above, below, and exactly at mass ratio 12.3130993.
Not surprising: the most singular term of the effective potential change sign here.

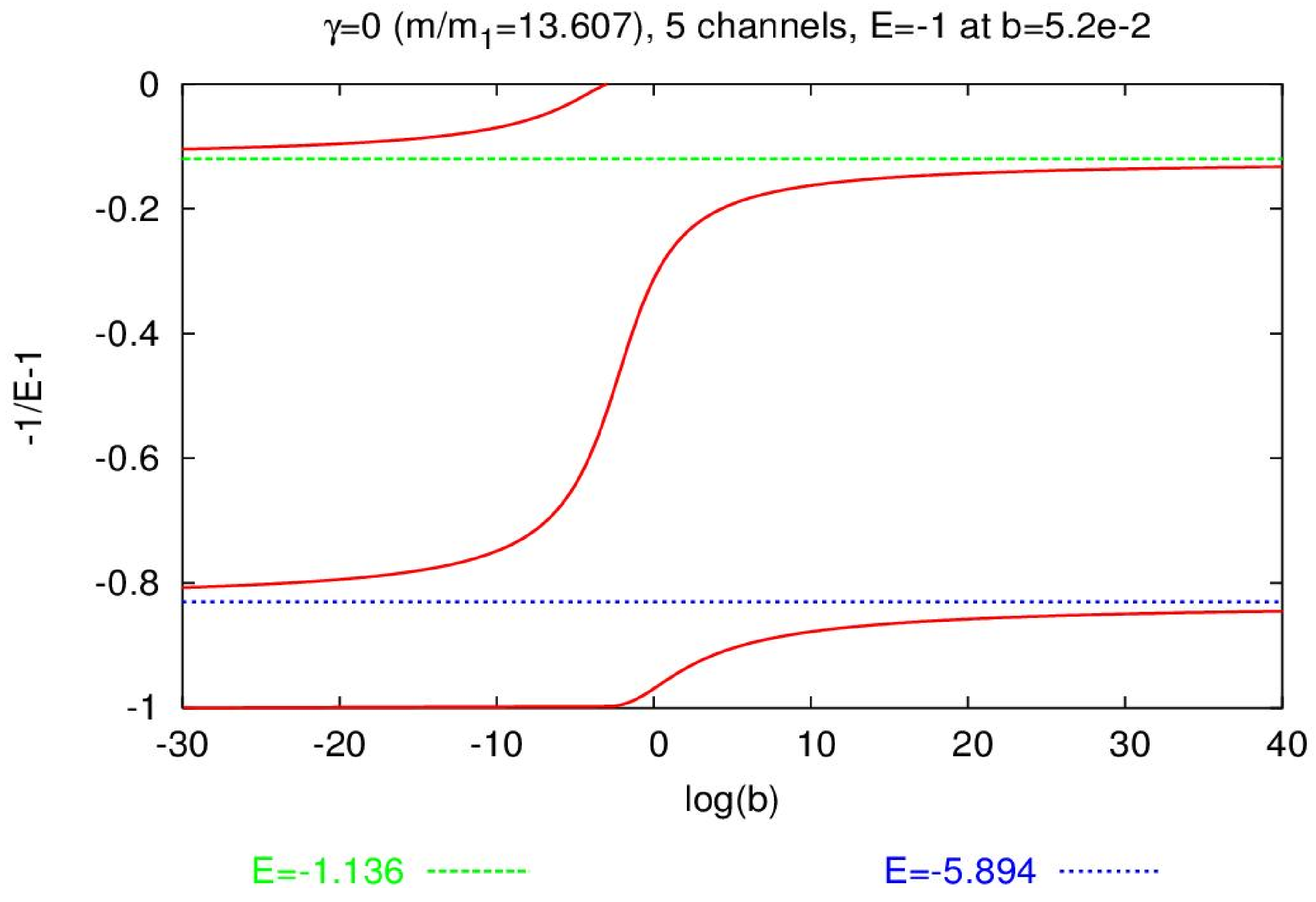
The solutions for zero and infinite GTBSL b becomes degenerate at this mass ratio.
This should persist in the exact calculations.

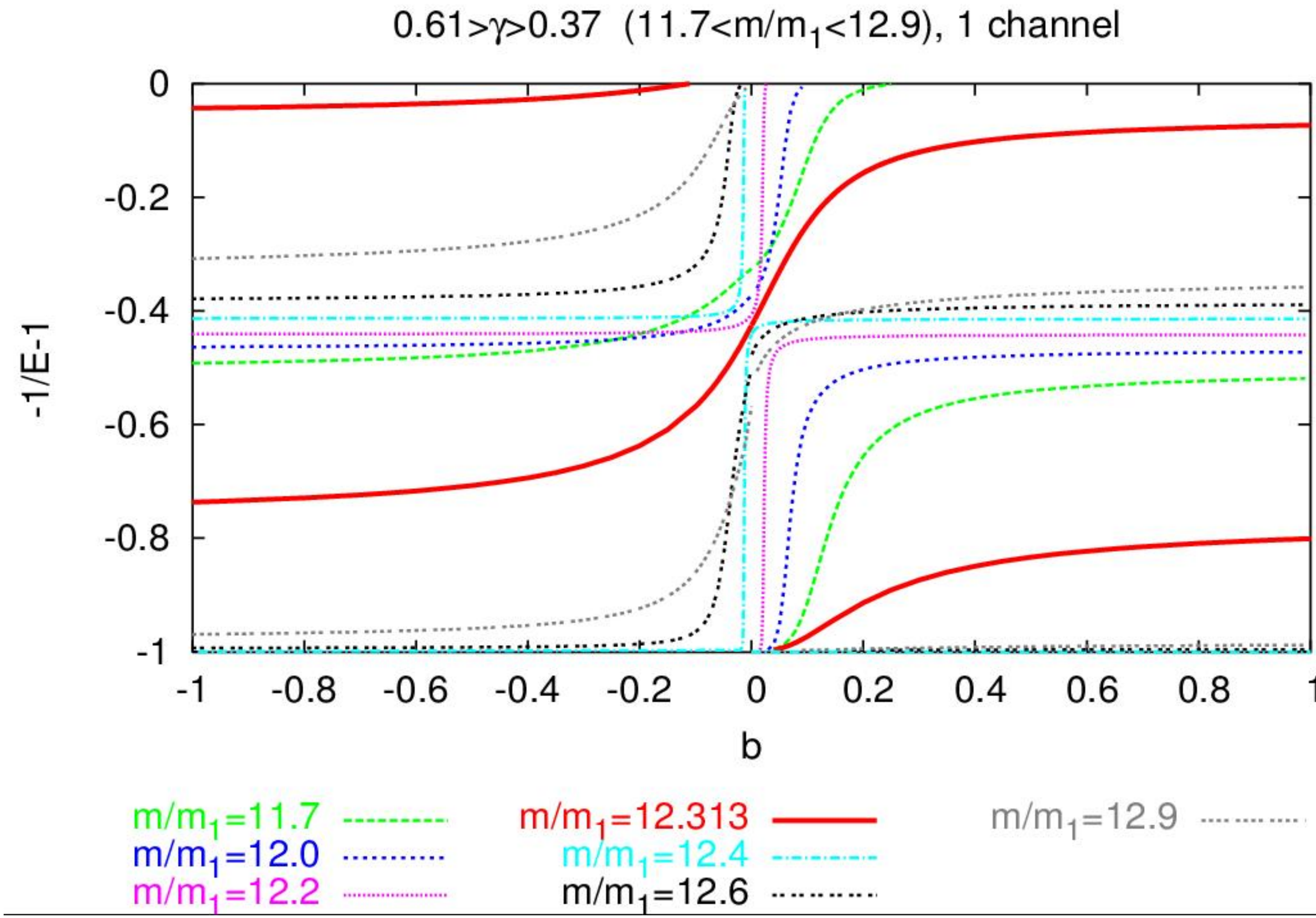
Numerical solution of hyper-radial equations

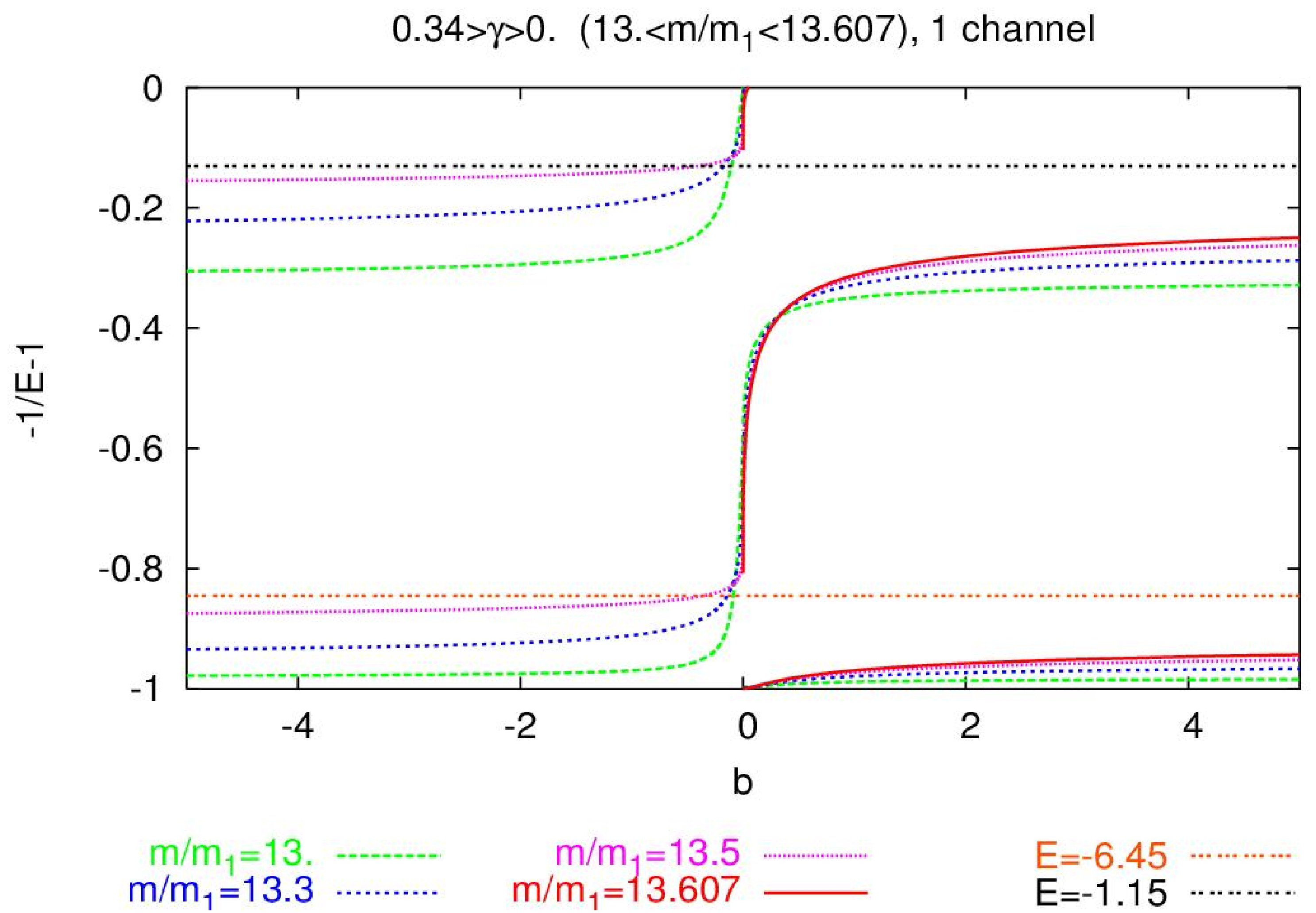



γ=1/2 (m/m1=12.313), 5 channels, E=-1 at b=-0.108
0
-0.2
-0.4
-0.6
-0.8
-1
-1/E-1
-10
-8
-6
-4
-2
0
2
4
6
8
10
b
E=-1.74
E=-4.31
E=-1.06


γ=.34 (m/m1=13.), 5 channels
-1/E-1
0
-0.2
-0.4
-0.6
-0.8
-1
-10
-8
-6
-4
-2
0
2
4
6
8
10


γ=0 (m/m1=13.607), 5 channels, E=-1 at b=5.2e-2
0
-0.2
-0.4
-0.6
-0.8
-1
-1/E-1
-30
-20
-10
0
10
20
30
40
log(b)
E=-1.136
E=-5.894


0.61>γ>0.37 (11.7<m/m1<12.9), 1 channel
0
-0.2
-0.4
-0.6
-0.8
-1
-1/E-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
b
m/m1=11.7
m/m1=12.0
m/m1=12.2
m/m1=12.313
m/m1=12.4
m/m1=12.6
m/m1=12.9


0.34>γ>0. (13.<m/m1<13.607), 1 channel
0
-0.2
-0.4
-0.6
-0.8
-1
-1/E-1
-4
-2
0
2
4
b
m/m1=13.
m/m1=13.3
m/m1=13.5
m/m1=13.607
E=-6.45
E=-1.15

- Two-parameter universal description

- Three-body zero-range potential can be (should be?) consistently introduced for mass-ratio interval [8.61857692, 13.6069657].

- These very considerations should be applied to the same system for higher angular momenta.

- Similar situation might arise in mixed dimensions

- Evident implications for the four-body (and many-body) problem.

Four-body (3 + 1)-system's spectrum is not bounded from below for mass ratio above 13.384

PRL **105,** 223201 (2010) PHYSICAL REVIEW LETTERS week ending 26 NOVEMBER 2010

Four-Body Efimov Effect for Three Fermions and a Lighter Particle

Yvan Castin,[1] Christophe Mora,[2] and Ludovic Pricoupenko[3]

This treatment allows us to shed light on the crossover from few bound states to infinite number of states near critical mass ratio

m/m_1 = 13.6069657.

Universality in FBS, INT
Seattle, April 25, 2014

PHYSICAL REVIEW A **86**, 062703 (2012)

Crossover trimers connecting continuous and discrete scaling regimes

Shimpei Endo,[1,*] Pascal Naidon,[2,3] and Masahito Ueda[1,3]