Influence of trap anisotropy and dimensionality on perturbative effective 2- and 3-body interactions

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Universality in Few-Body Systems
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Collaborators

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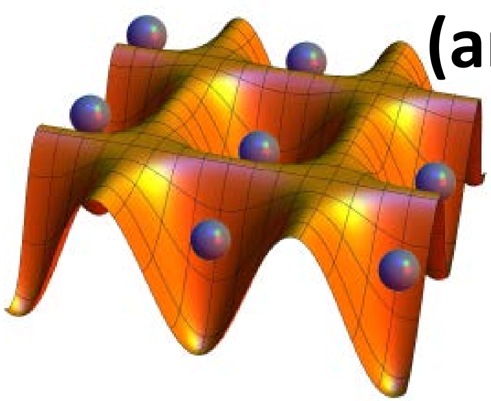
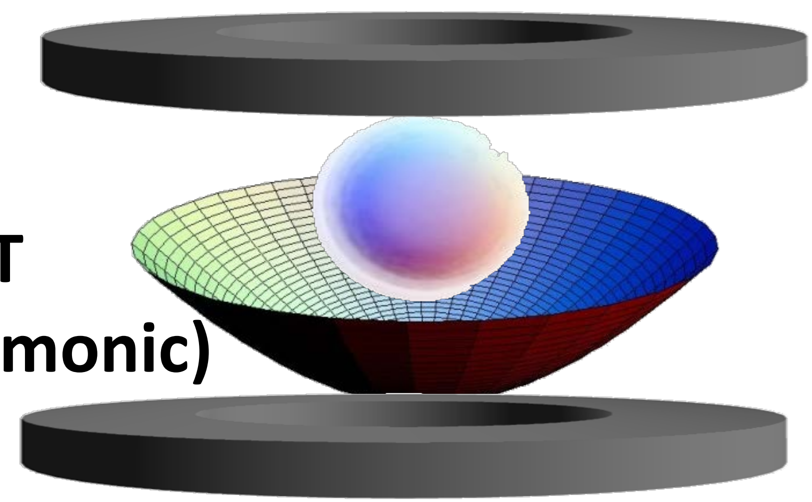
Outline

- **Many-body physics in optical lattices**
- Few-body physics in optical lattices
- Effective interactions in harmonic traps
- Effective interactions in anisotropic harmonic traps
- Effective interactions in 1D, 2D, (4D?)



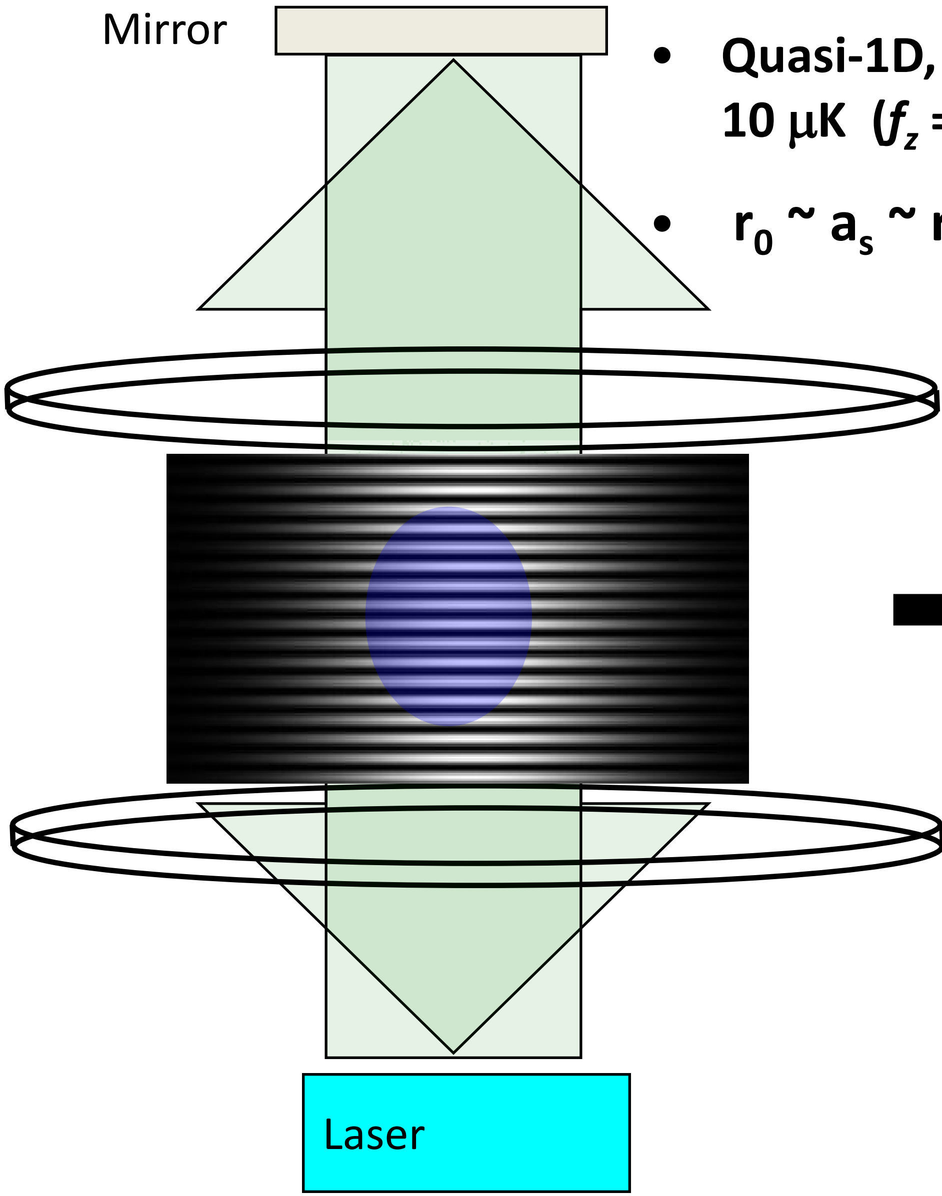
Optical lattice (anharmonic)

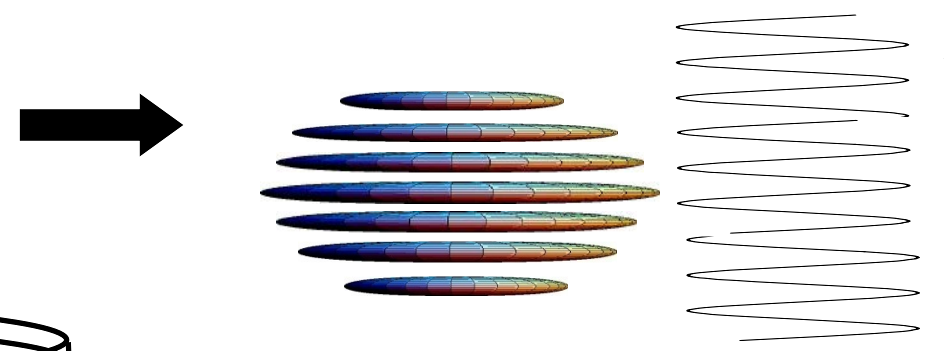
MOT (harmonic)



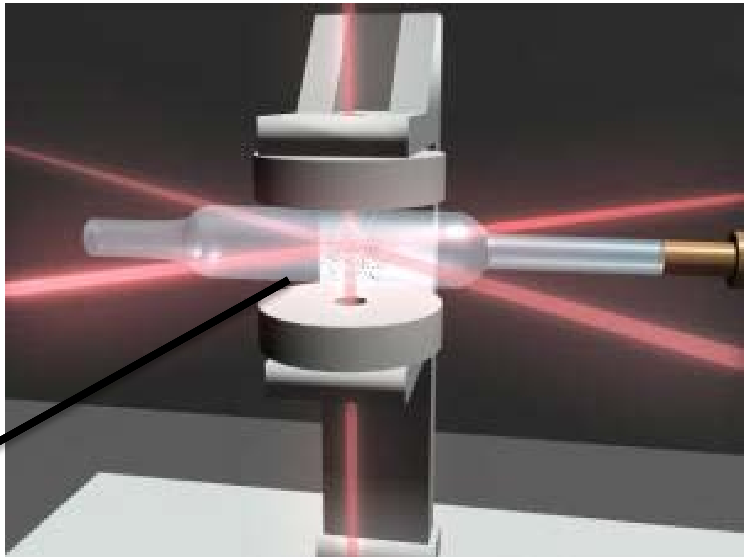
1D optical lattice

- **Quasi-1D, z motion frozen in ground state: $T < T_z \sim 10\ \mu K$ ($f_z = 10 - 100$ kHz)**
- **$r_0 \sim a_s \sim r_{eff} << l_z << l_x \sim l_y$**

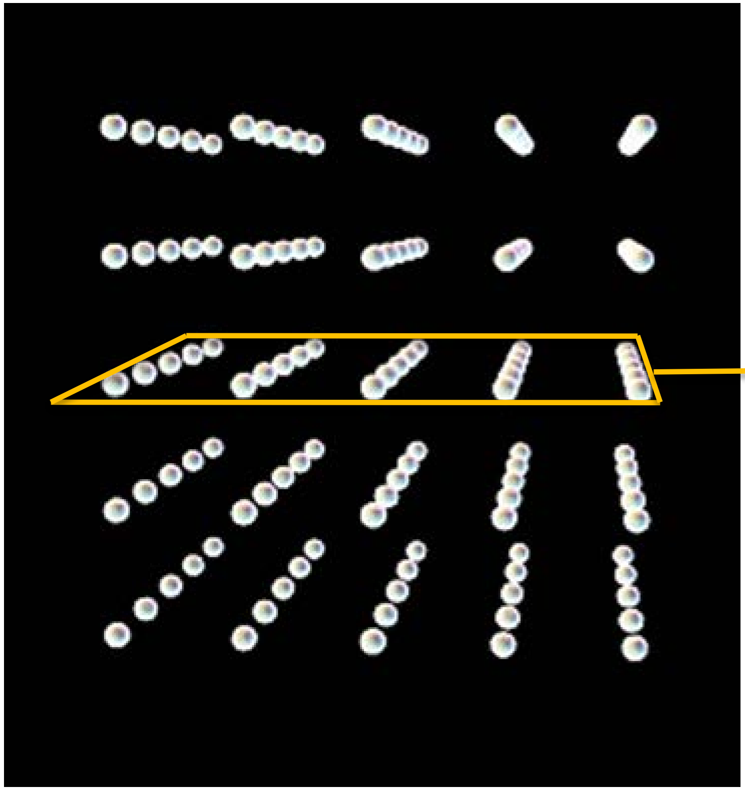


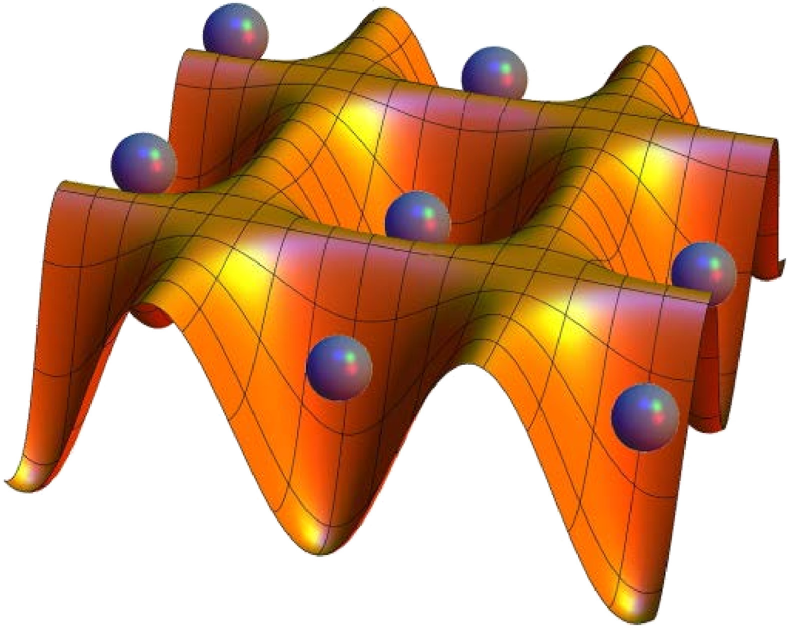


3D Optical Lattice



Typically assume atoms in lowest band, although higher-band physics can be very important.

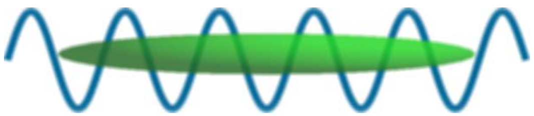




Equilibrium Lattice States

- Shallow lattice
- n/site unknown, phase defined

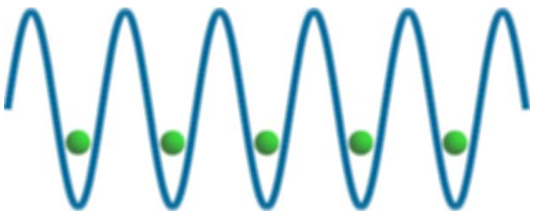
- Deep lattice
- e.g., 1 atom per lattice site, random phase



Superfluid

Each atom is in a superposition over all lattice sites.

$$s = U/J \quad \longleftrightarrow$$



Mott Insulator

Bose-Hubbard Hamiltonian: *single-mode per lattice site*

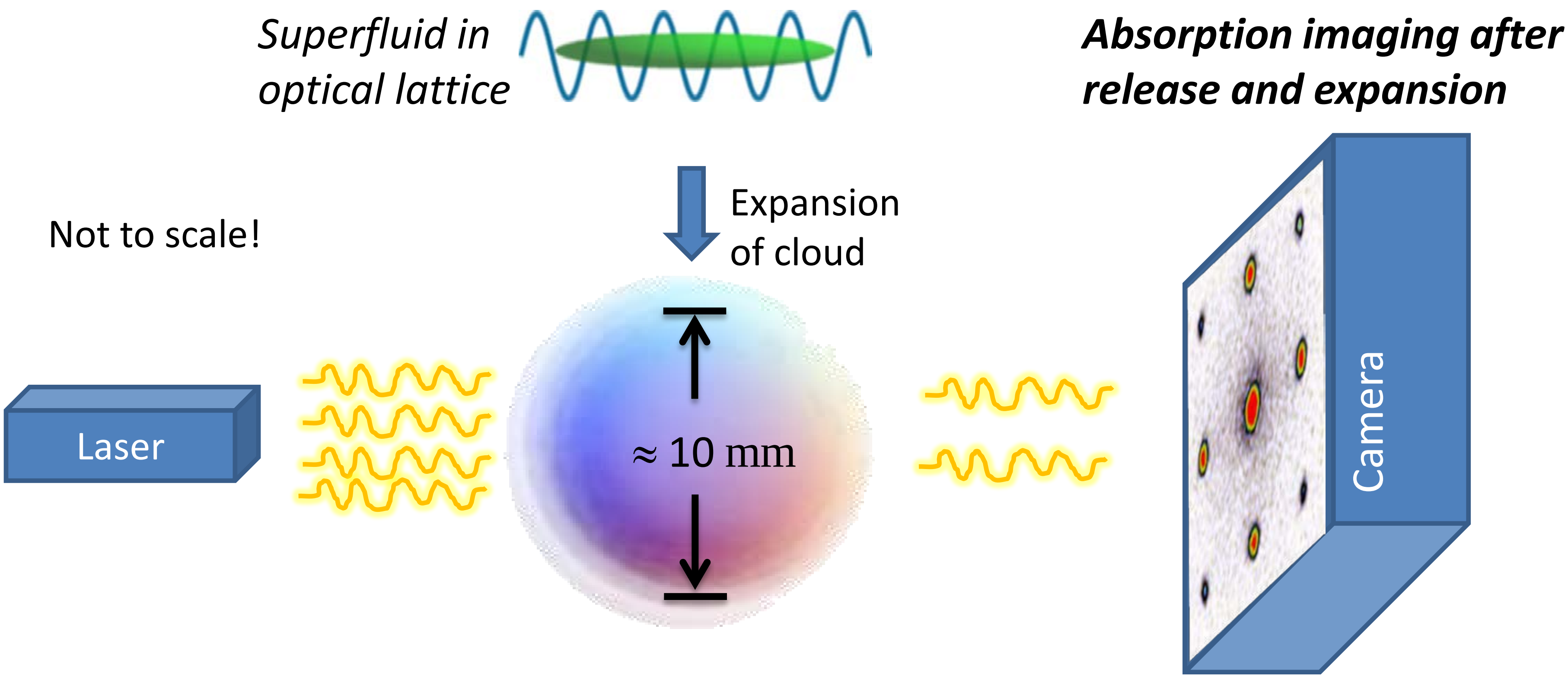
$$H = -J_{ij} \sum_{i,j} \hat{a}_i^{+} \hat{a}_j + \frac{1}{2} \sum_i U_i \hat{n}\left(\hat{n} - 1\right)$$

, *i* = lattice site index

Tunneling   Interactions

Quantum Phase Transition at $s = s_c$ (Zero Temperature Phase Transition)

<u>**Atom density measurement**</u> after release and expansion from lattice gives the ***momentum distribution*** at the instant of release.

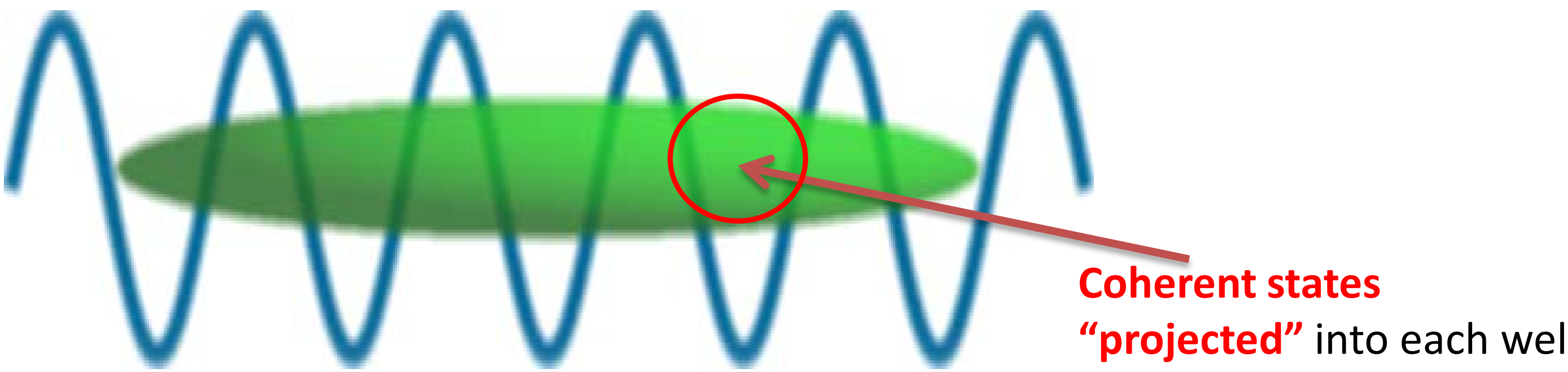


Distribution *when* atoms in $k = 0$ quasi-momentum at moment of release

Quenching from shallow to deep lattice

- Start with superfluid in shallow lattice

- Quickly increase lattice well depth (by increasing lattice laser intensities)



- Fast enough atoms don't have time to interact/tunnel (avoid Mott transition). Quantum field at each lattice site "frozen" in place.
- Atom number in each site remains unknown; each atom in superposition over all lattice sites.
- Slow enough that vibrational excitation from ground band to higher bands is minimal.

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- **Few-body physics in optical lattices**
- Effective interactions in harmonic traps
- Effective interactions in anisotropic harmonic traps
- Effective interactions in 1D, 2D

Many-body → Few-Body Physics in Optical Lattices

$$\left|\Phi_{BEC}\right\rangle^{\otimes N} \xrightarrow[\sim 300\ \mu s]{\text{Fast Loading}} \left|\sum_{i=lattice}\psi_i\right\rangle^{\otimes N} \approx \prod_i \left|\alpha_i\right\rangle$$

All N atoms in single BEC state.

Superposition of ***each*** atom in ***every*** lattice site

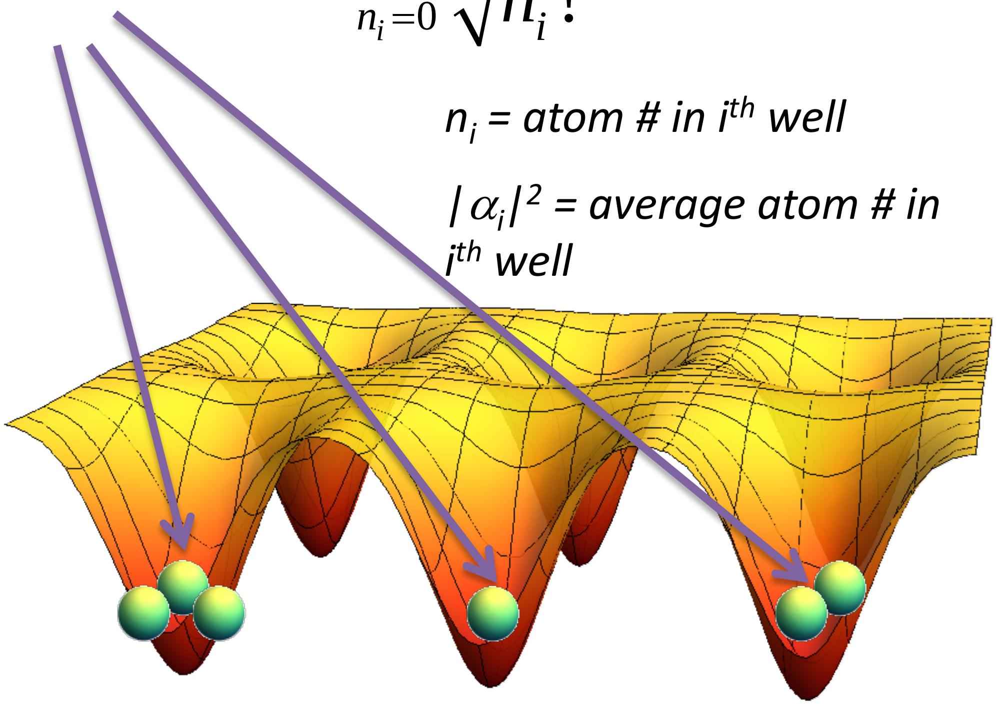
Poissonian number statistics (*coherent states*) in each unit cell.

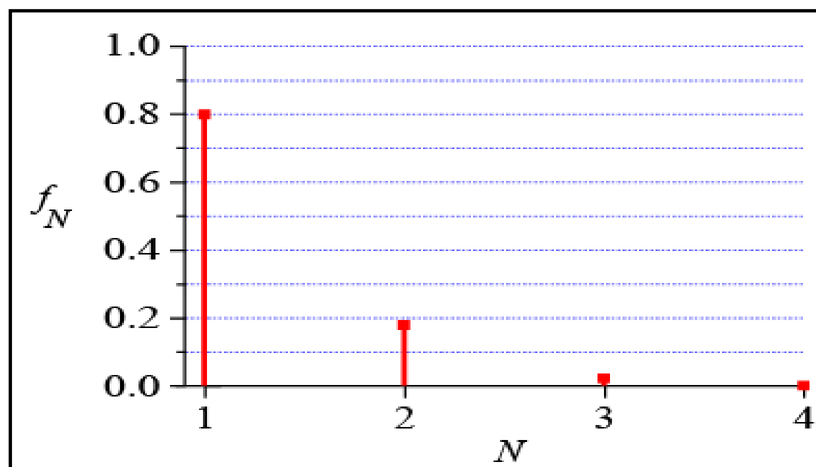
<u>Coherent states</u>

$$\left|\alpha_i\right\rangle = e^{-|\alpha_i|^2/2}\sum_{n_i=0}\frac{\alpha_i^{n_i}}{\sqrt{n_i!}}\left|n_i\right\rangle = e^{-\bar{n}/2}\left(0 + \alpha\ (\ ) + \frac{\alpha^2}{\sqrt{2}}\ (\ ) + \frac{\alpha^3}{\sqrt{3!}}\ (\ ) + \ldots\right)$$

n_i = atom # in i^{th} well

$|\alpha_i|^2$ = average atom # in i^{th} well



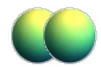


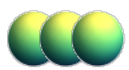
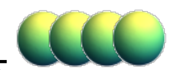
E.g. Data (NIST, Porto/Williams group)

Bose-Hubbard dynamics in deep lattice

$H = \frac{1}{2}U_2 a^\dagger a^\dagger a a + \text{~~tunneling~~}$ ⇒ $E_N = \frac{1}{2}U_2 N(N-1)$

Set 1-particle ground state energy to zero.

$|\psi(t)\rangle = e^{-\bar{n}/2}$ **(0 +** α  **+** $\frac{\alpha^2}{\sqrt{2}}$  $e^{-iU_2 t}$

+ $\frac{\alpha^3}{\sqrt{3!}}$  $e^{-i3U_2 t}$ **+** $\frac{\alpha^4}{\sqrt{4!}}$  $e^{-i6U_2 t}$ **+ …)**

3 pairs from 3 atoms

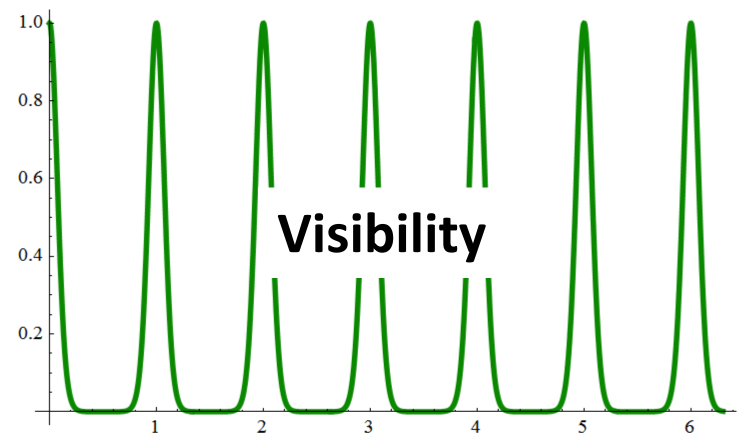
6 pairs from 4 atoms

Interference pattern visibility:

$$v(t_h) = |\langle\psi(t_h)|\hat{a}_0|\psi(t_h)\rangle|^2$$

$$= e^{-2\bar{n}[1-\cos(\tilde{U}_2 t_h/\hbar)]}$$

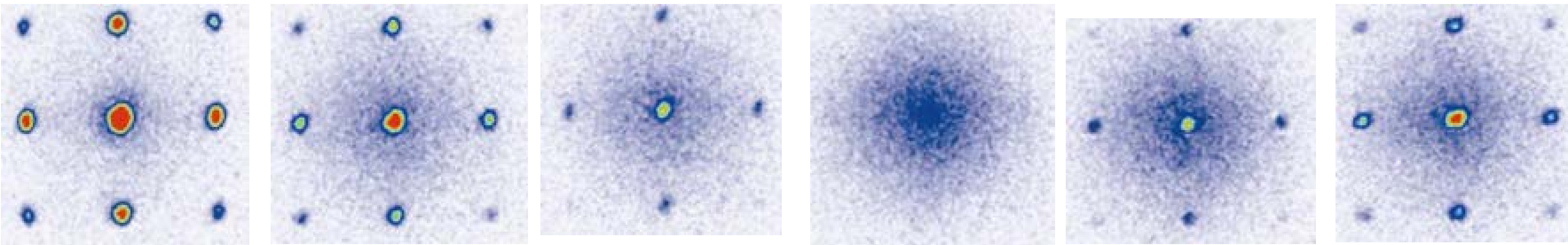
Predicts re-phasing every multiple of $t = h/U_2$

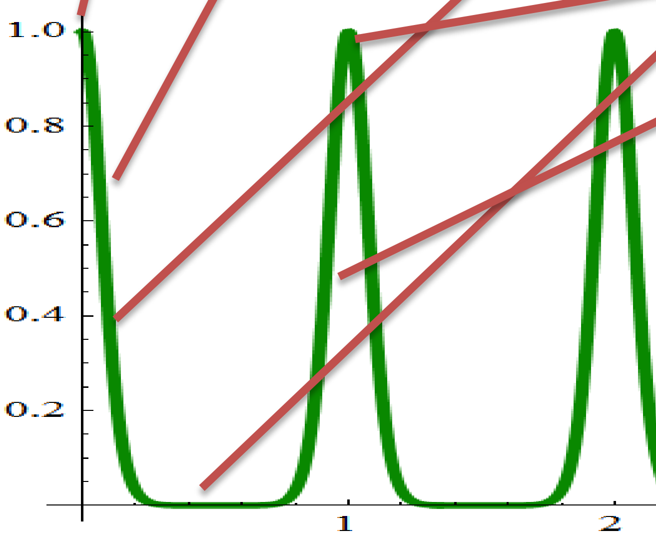


Lattice hold time

Collapse and Revival

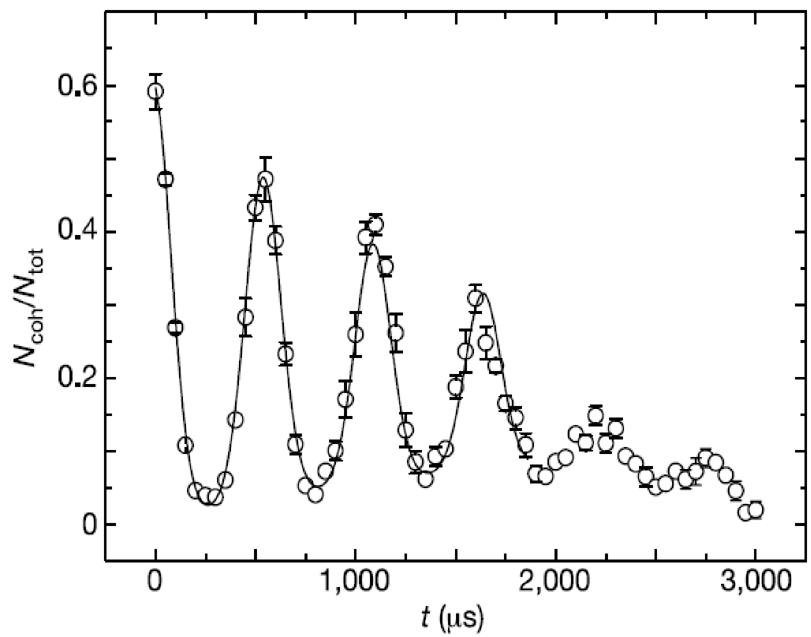
Lattice hold time





(Units of time = h/U_2)

e.g. Rb87 with
ω *~ 30 kHz,*
U_2 *~ 2kHz*

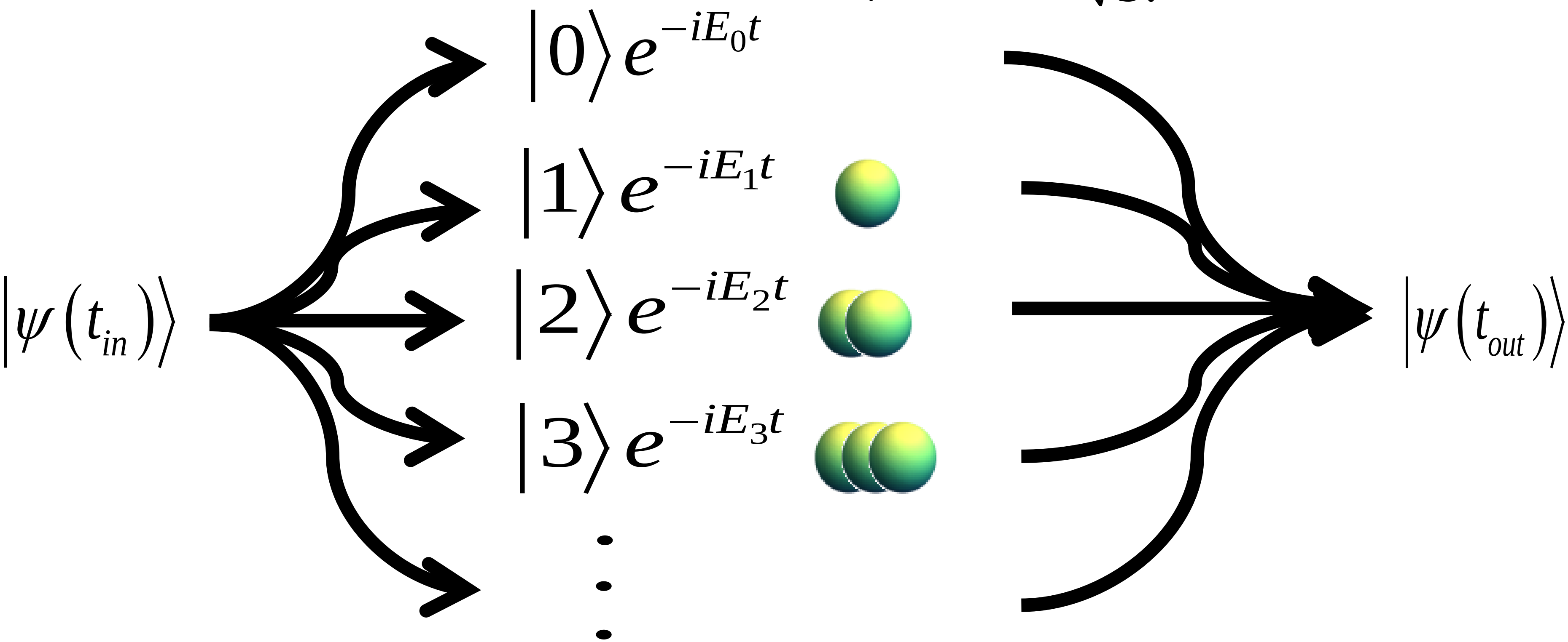


Greiner et al, Nature 419, 51 (2002); similar results at NIST Strabley et al, (2006)

"Multibody interaction interferometer"

Input state:

$$\left|\psi(0)\right\rangle = \left|\alpha\right\rangle = \quad e^{-\bar{n}/2} \ (0 + \alpha \ + \frac{\alpha^2}{\sqrt{2}} \ + \frac{\alpha^3}{\sqrt{3!}} \ + \ldots)$$

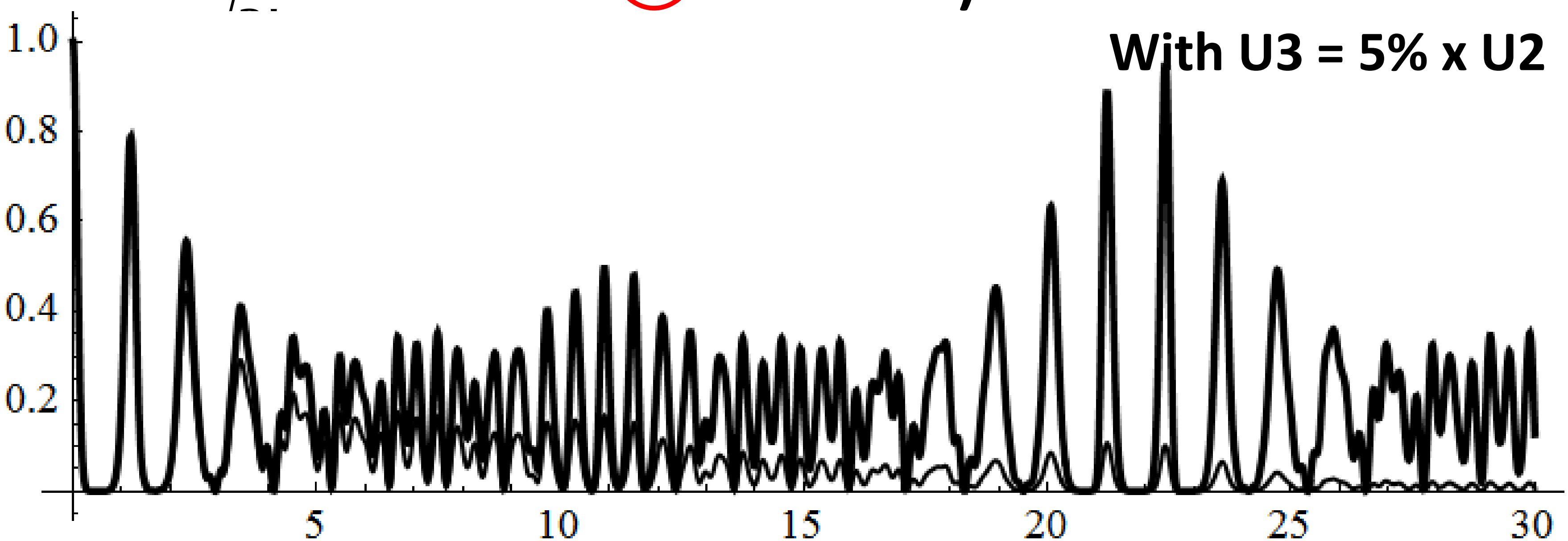


Each number state evolves "independently" while in lattice. They interfere after release and time-of-flight (TOF) expansion.

Phase evolution with higher-body interactions

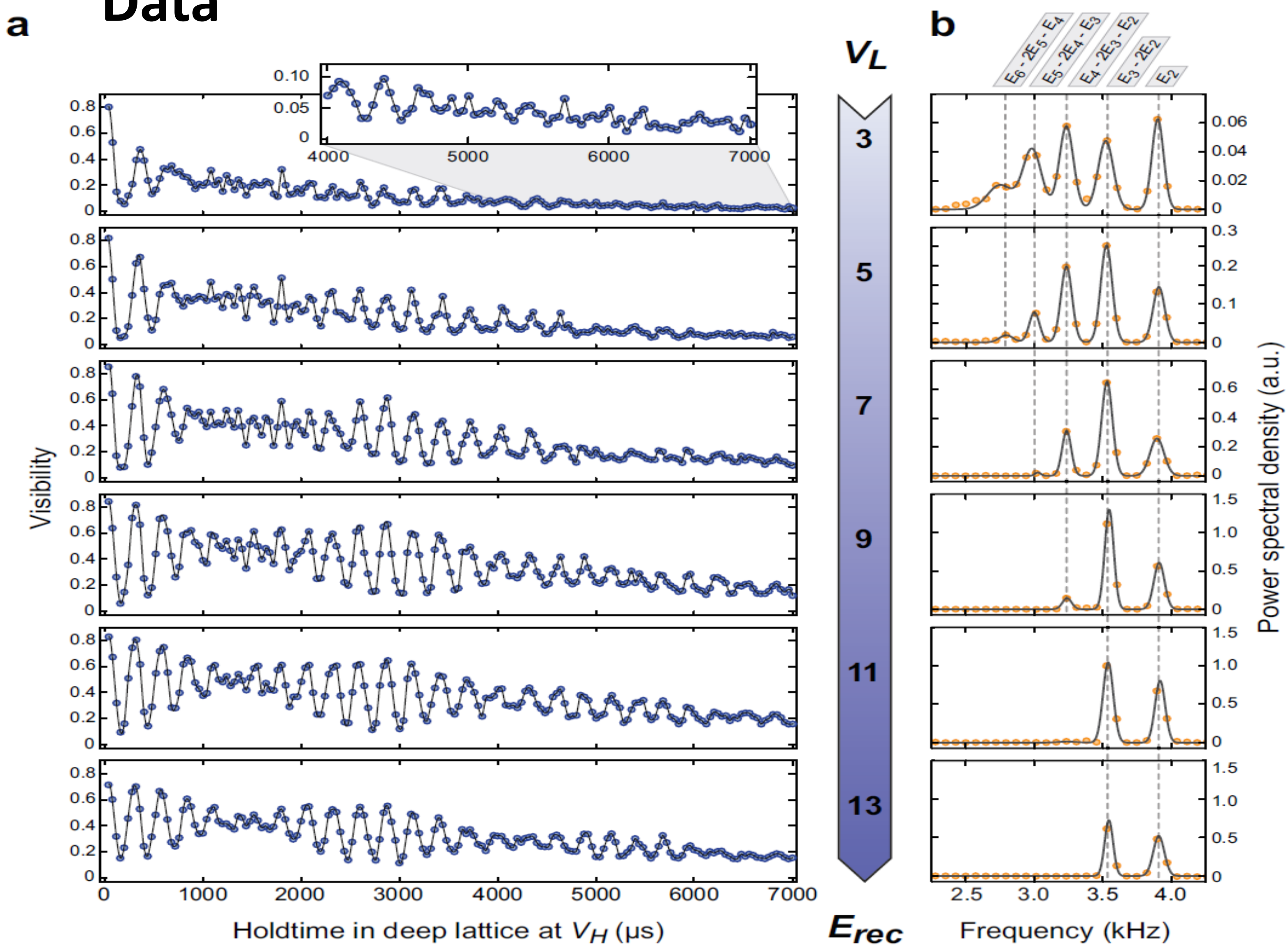
$$E_N = \frac{1}{2}U_2 N(N-1) + \frac{1}{6}U_3 N(N-1)(N-2) + \ldots$$

$$|\psi(t)\rangle = \sum_{N=0} b_N |N\rangle e^{-iE_N t} = e^{-\bar{n}/2}\left(0 + \alpha + \frac{\alpha^2}{\sqrt{2}} e^{-iU_2 t} + \frac{\alpha^3}{\sqrt{\ }} e^{-i(3U_2+U_3)t} + \ldots\right)$$



Johnson el al, NJP (2009); Tiesinga et al, PRA (2011).

Data



Will et al, Nature (2009)

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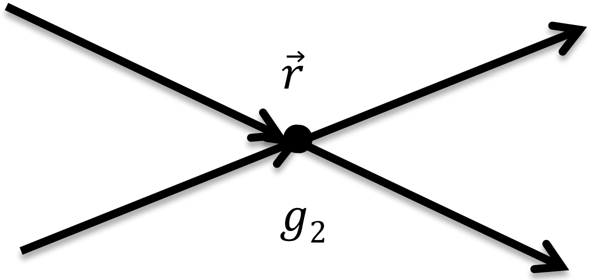
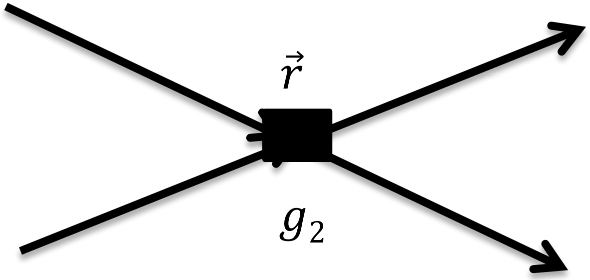
Low-energy field theory

- *N* particles, assume ***a/l << 1***.

$$H = \int \psi^\dagger H_0 \psi d\vec{r} + \frac{1}{2!} g_2 \int \psi_1^\dagger \psi_2^\dagger \; \delta(\vec{r}_1 - \vec{r}_2) \psi_1 \psi_2 d\vec{r}_1 d\vec{r}_2$$

$$+ \frac{1}{2!} g_{2,eff} \int \psi_1^\dagger \psi_2^\dagger \; \frac{1}{2} [\overleftarrow{\nabla}^2 \delta(\vec{r}_1 - \vec{r}_2) + \delta(\vec{r}_1 - \vec{r}_2) \vec{\nabla}^2] \psi_1 \psi_2 d\vec{r}_1 d\vec{r}_2$$





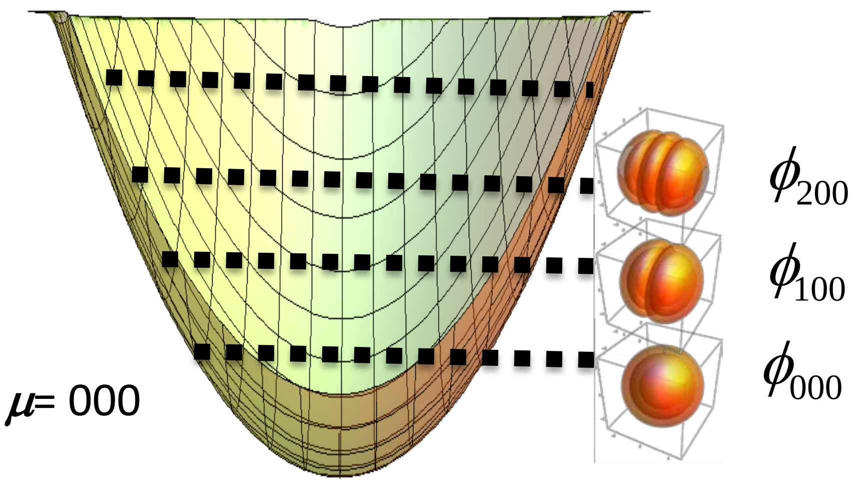
Johnson el al, NJP 14 053037 (2012).

Multimode Hamiltonian

Expand over single well harmonic oscillator wavefunctions

$$\psi(\vec{r}) = \sum_n a_n \phi_n(\vec{r})$$

a_μ **annihilates atom in mode** μ

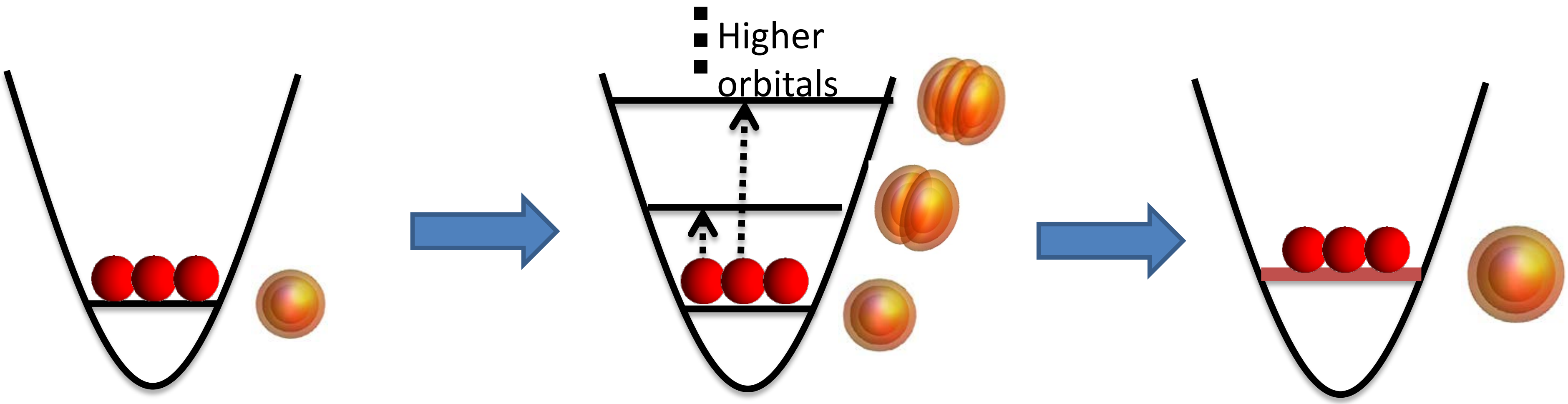


$$H = \sum_\mu E_\mu a_\mu{}^\dagger a_\mu + \frac{1}{2} g_2 \sum_{\mu\nu\lambda\rho} K_{\mu\nu\lambda\rho} a_\mu{}^\dagger a_\nu{}^\dagger a_\lambda a_\rho$$

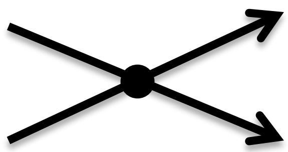
$$+ \frac{1}{2} g_{2,eff} \sum_{\mu\nu\lambda\rho} G_{\mu\nu\lambda\rho} a_\mu{}^\dagger a_\nu{}^\dagger a_\lambda a_\rho$$

$$K_{\mu\nu\lambda\rho} = \left(2\pi\sigma^2\right)^{3/2} \int \phi_\mu \phi_\nu \phi_\lambda \phi_\rho d\vec{r}, \quad K_{0000} = 1$$

Effective single-mode Hamiltonian



Single-orbital
Bose-Hubbard

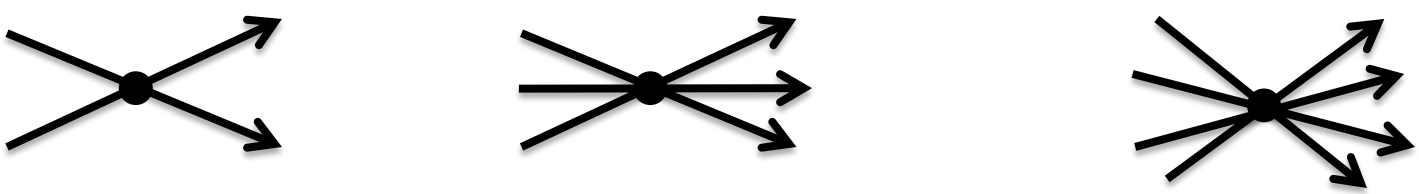


Multi-orbital
Bose-Hubbard

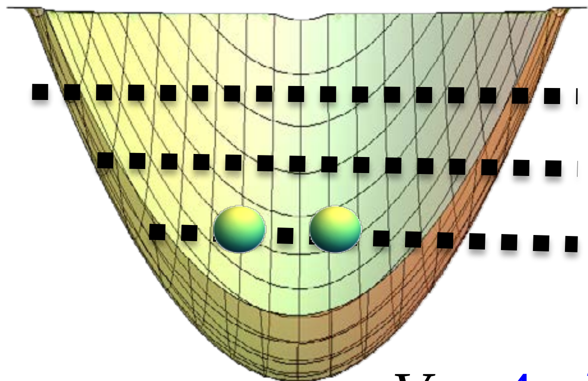
Effective single-orbital
Bose-Hubbard

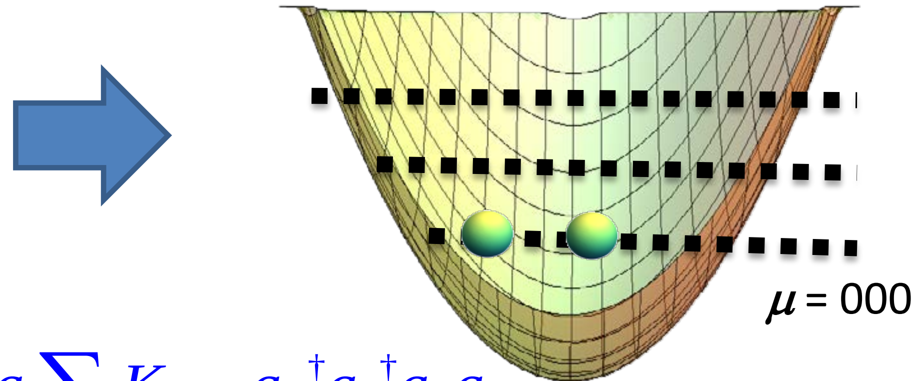
$$H_{s.m.} = \frac{1}{2} U_2 a_0^\dagger a_0^\dagger a_0 a_0 \qquad H_{m.m.} = \frac{1}{2} U_2 \sum_{nmlk} K_{nmlk} a_n^\dagger a_m^\dagger a_l a_k$$

$$\tilde{H}_{eff} = \frac{1}{2!}\tilde{U}_2 a_0^\dagger a_0^\dagger a_0 a_0 + \frac{1}{3!}\tilde{U}_3 a_0^\dagger a_0^\dagger a_0^\dagger a_0 a_0 a_0 + \frac{1}{4!}\tilde{U}_4 a_0^\dagger a_0^\dagger a_0^\dagger a_0^\dagger a_0 a_0 a_0 a_0 + \ldots$$



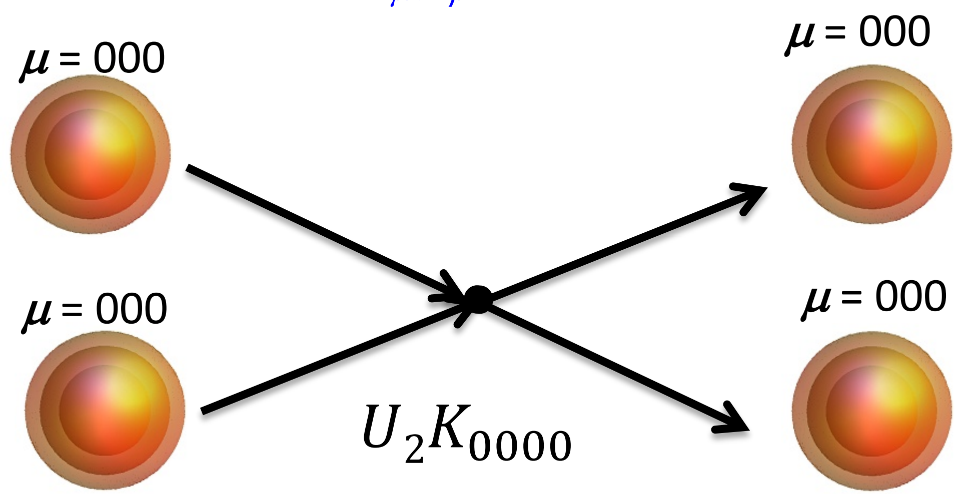
1^{st} order perturbation theory ("Mean field")





$$V = 4\pi \frac{\hbar^2}{m} a \sum_{\mu\nu\lambda\rho} K_{\mu\nu\lambda\rho} a_\mu^\dagger a_\nu^\dagger a_\lambda a_\rho$$

$\boldsymbol{a_0^\dagger a_0^\dagger a_0 a_0}$

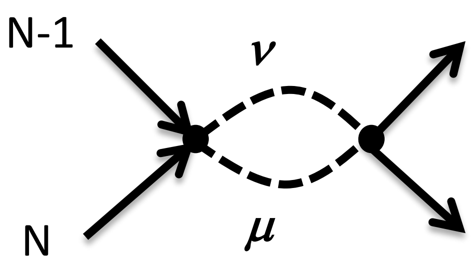


$$E^{(1)}(n) = \langle n|V|n\rangle = \frac{1}{2} g_2 K_{0000} \langle n|a_0^\dagger a_0^\dagger a_0 a_0|n\rangle = \frac{1}{2} U_2 n(n-1)$$

2^{nd} order perturbation theory

$$\delta E_N^{(2)} = -\sum_{\mu} \frac{\langle N|V|\mu\rangle\langle\mu|V|N\rangle}{\Delta E_{\mu}} \propto \sqrt{\Lambda}\, n(n-1)$$

Sum over intermediate excited (virtual) states



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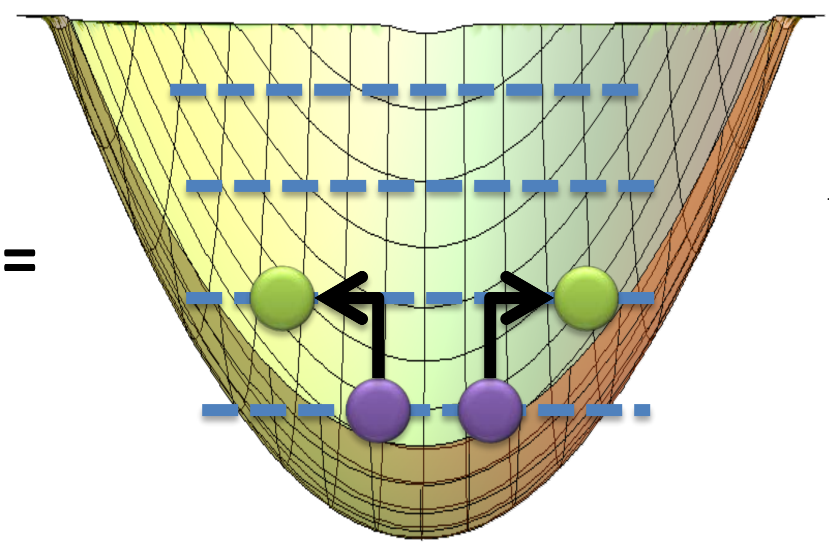
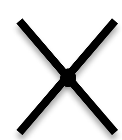


$\Lambda = \text{cutoff}$

Exclude $\mu = \nu = 0$

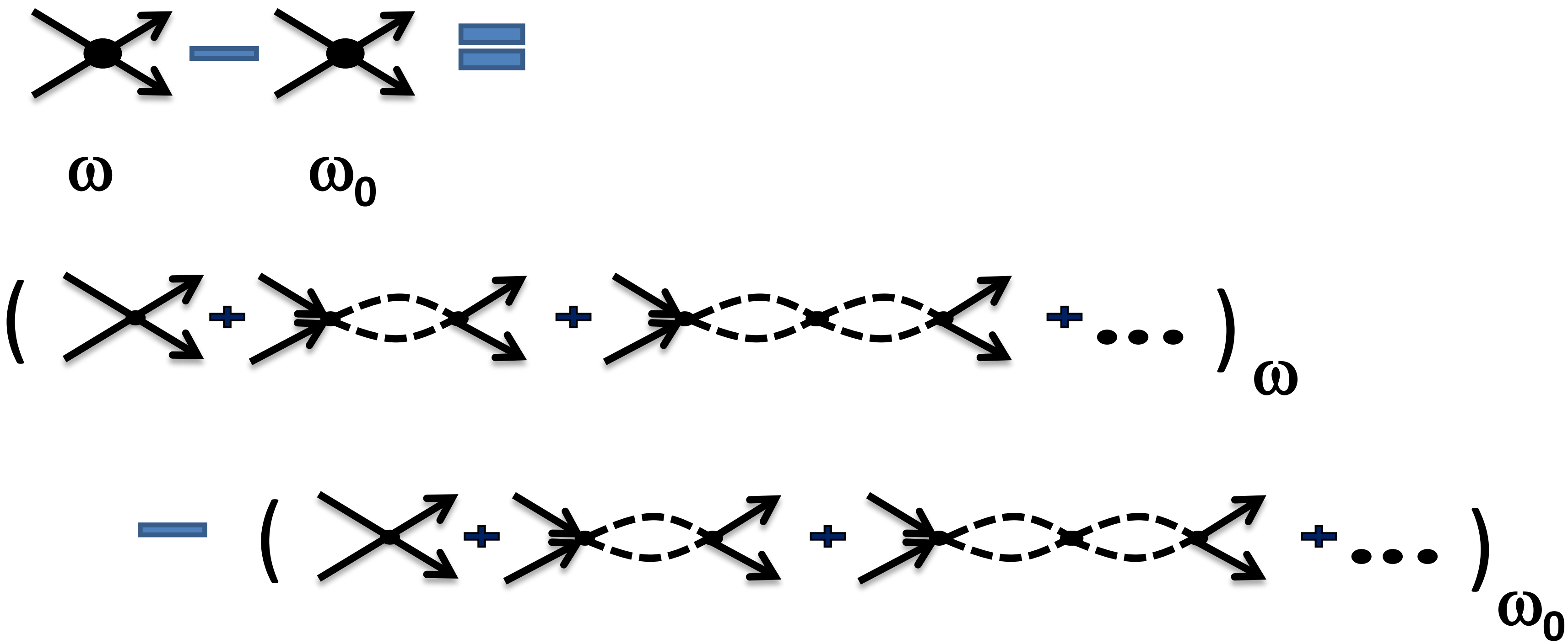
▪▪▪▪▪▪▪▪ = atom in vibrationally excited (virtual) intermediate state

▬▬▬ = atom in **μ = 0 state**



= U_2 interaction vertex

Renormalization



3rd order calculations: 2-body

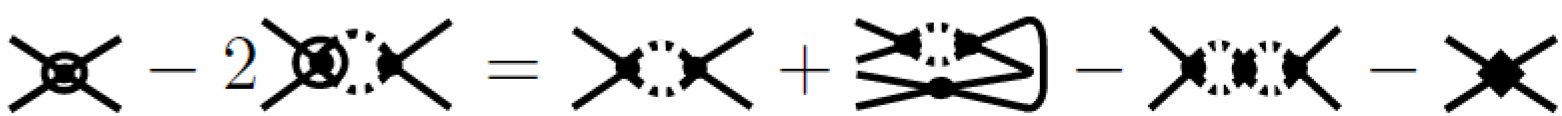
$U_2(\omega;\omega_0) =$  $+ \mathcal{O}(a_t^4)$

$$= \alpha_2^{(1)}\left(\frac{a_t(\omega_0)}{\sigma(\omega)}\right) - \beta_2^{(2)}(\omega)\left(\frac{a_t(\omega_0)}{\sigma(\omega)}\right)^2 + \alpha_2^{(1)}\left(\frac{a_{ct}(\omega_0)}{\sigma(\omega)}\right) - \alpha_{4,3}^{(3)}\left(\frac{a_t(\omega_0)}{\sigma(\omega)}\right)^3$$

$$+ \beta_2^{(3)}(\omega)\left(\frac{a_t(\omega_0)}{\sigma(\omega)}\right)^3 - 2\beta_2^{(2)}(\omega)\left(\frac{a_{ct}(\omega_0)}{\sigma(\omega)}\right)\left(\frac{a_t(\omega_0)}{\sigma(\omega)}\right) + \alpha_2^{(1,2)}\left(\frac{r_{eff}}{\sigma(\omega)}\right)\left(\frac{a_t(\omega_0)}{\sigma(\omega)}\right)^2$$

$$+ \mathcal{O}(a_t^4).$$

Counter-term renormalization condition



Renormalization

$$U_2(\omega) = c_2^{(1)}\left(\frac{a(0)}{l(\omega)}\right) + c_2^{(2)}\left(\frac{a(0)}{l(\omega)}\right)^2 + c_2^{(3)}\left(\frac{a(0)}{l(\omega)}\right)^3 + \ldots.$$

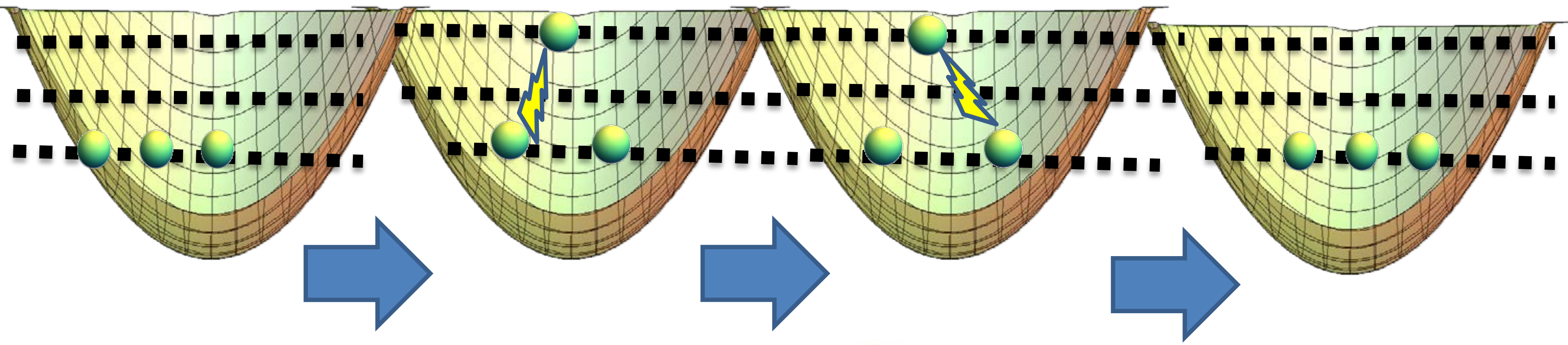
$$c_2^{(1)} = \left(\frac{2}{\pi}\right)^{1/2}$$

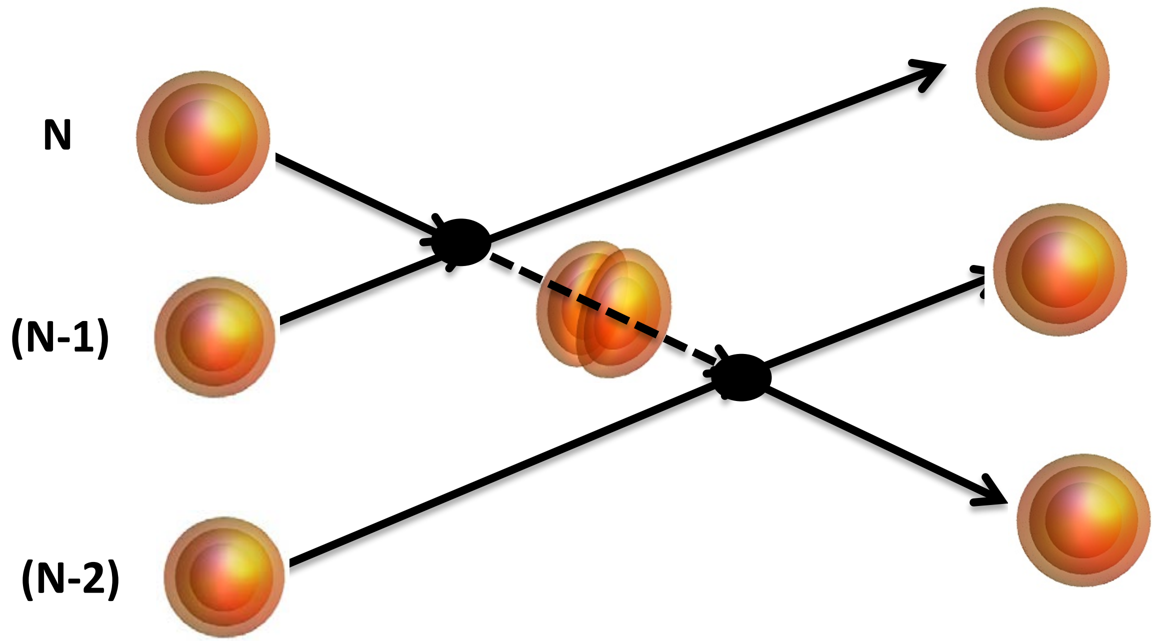
$$c_2^{(2)} = \left(\frac{2}{\pi}\right)\left(1-\log 2\right)(1-\sqrt{\omega_0/\omega})$$

$$c_2^{(3)} = \left(\frac{2}{\pi}\right)^{3/2}\left(1-\log 2\right)^2(1-\sqrt{\omega_0/\omega})^2 - \left(\frac{2}{\pi}\right)^{3/2}\left(\pi^2/24+\log 2-\frac{1}{2}\log^2 2\right)(1-\omega_0/\omega)$$

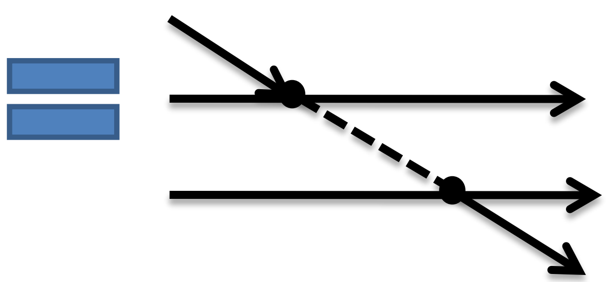
If w_0 = zero, (perturbatively) reproduces Busch et al.

Effective 3-body Interaction





3 atoms in; 3 out



$$U_3^{(2)} = -\sum_{\vec{n}=n_x n_y n_z} \frac{\langle N | H_2 | \vec{n} \rangle \langle \vec{n} | H_2 | N \rangle}{\Delta E_{\vec{n}0}}$$

Effective 3-body Interaction Energy

$$\delta E^{(2)}{}_N = -\left(U_2\right)^2 \sum_{\mu \neq 0} \frac{K^2_{\mu 000} \left\langle N \middle| \hat{a}_0{}^{\dagger} \hat{a}_0{}^{\dagger} \hat{a}_0 \hat{a}_\mu \middle| \mu \right\rangle \left\langle \mu \middle| \hat{a}_\mu{}^{\dagger} \hat{a}_0{}^{\dagger} \hat{a}_0 \hat{a}_0 \middle| N \right\rangle}{4 \Delta E_\mu}$$

$$= -\beta \left(U_2\right)^2 n\left(n-1\right)\left(n-2\right)/6$$

Exact leading order, isotropic harmonic trap

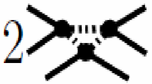
$$\beta = 4\sqrt{3} - 6 + 6\log\left(\frac{4}{2+\sqrt{3}}\right) \simeq 1.34\ldots.$$

* The factor n(n-1)(n-2)/6 counts the number of distinct triples.

The shift in energy is a 3-body effect.

Isotropic case

3rd order, 3-body

$U_3(\omega;\omega_0) = -6$  $+\,12$  $+\,12$  $-\,12$  $-\,6$  $-\,18$  $+\,\mathcal{O}(a_{\mathrm{t}}^4)$

$$= -6\alpha_3^{(2)}\left(\frac{a_{\mathrm{t}}(\omega_0)}{\sigma(\omega)}\right)^2 + 12\alpha_3^{(3)}\left(\frac{a_{\mathrm{t}}(\omega_0)}{\sigma(\omega)}\right)^3 + 12\beta_3^{(3)}(\omega)\left(\frac{a_{\mathrm{t}}(\omega_0)}{\sigma(\omega)}\right)^3$$

$$- 12\alpha_3^{(2)}\left(\frac{a_{\mathrm{ct}}(\omega_0)}{\sigma(\omega)}\right)\left(\frac{a_{\mathrm{t}}(\omega_0)}{\sigma(\omega)}\right) - 6\alpha_{4,3}^{(3)}\left(\frac{a_{\mathrm{t}}(\omega_0)}{\sigma(\omega)}\right)^3 - 18\alpha_5^{(3)}\left(\frac{a_{\mathrm{t}}(\omega_0)}{\sigma(\omega)}\right)^3 + \mathcal{O}(a_{\mathrm{t}}^4).$$

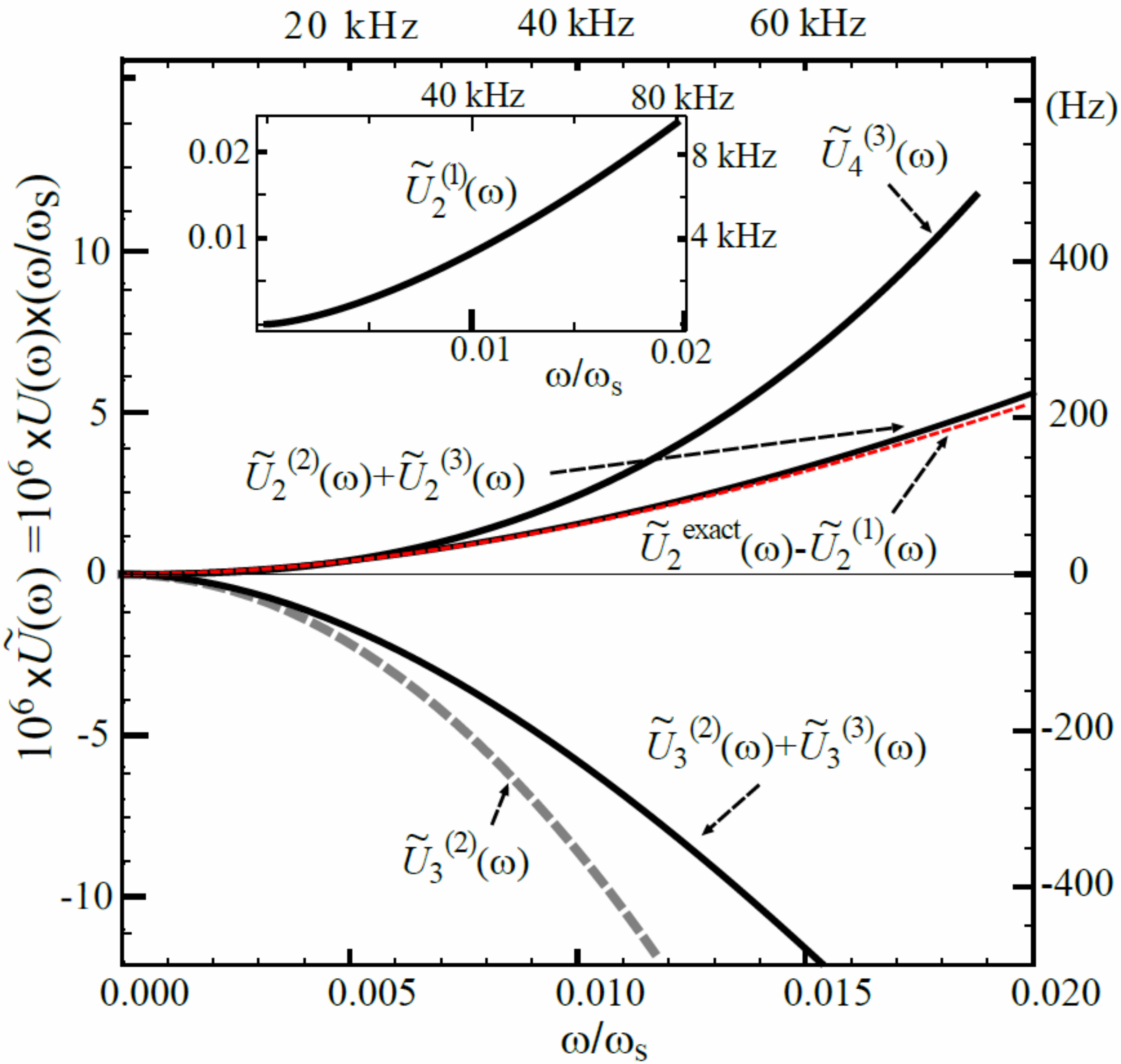
3rd order, 4-body

$U_4(\omega;\omega_0) = 48$  $+\,48$  $-\,72$  $+\,\mathcal{O}(a_{\mathrm{t}}^4)$

$$= c_4^{(3)}\left(\frac{a_{\mathrm{t}}(\omega_0)}{\sigma(\omega)}\right)^3 + \mathcal{O}(a_{\mathrm{t}}^4),$$

Johnson el al, NJP 14 053037 (2012).

“Running” interaction energies

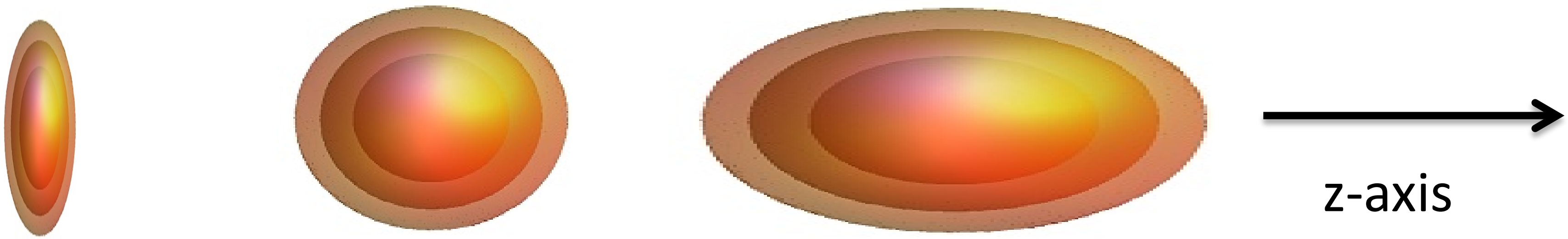


Outline

- Many-body physics in optical lattices
- Few-body physics in optical lattices
- Effective interactions in isotropic harmonic traps
- **Effective interactions in anisotropic harmonic traps**
- Effective interactions in 1D, 2D, (4D?)

Anisotropic harmonic potential

Fix $\omega_\perp$, vary ω_z. All energies in units of fixed $\hbar\omega_\perp$.



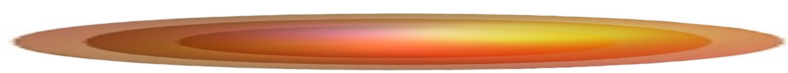
$\eta < 1$ $\quad$ $\eta = 1$ $\quad$ $\eta > 1$

z-axis

Anisotropy

$$\eta = \frac{\omega_\perp}{\omega_z} = \left(\frac{\sigma_z}{\sigma_\perp}\right)^2$$

1D limit (cigar)

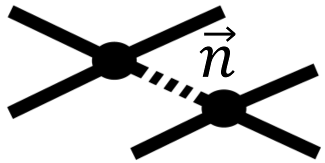


2D limit (pancake)

The perturbation theory breaks down before reaching 1D, 2D limits; work for the future…

Effective 3-body anistropic H.O.

$$U_3^{(2)} = -6$$



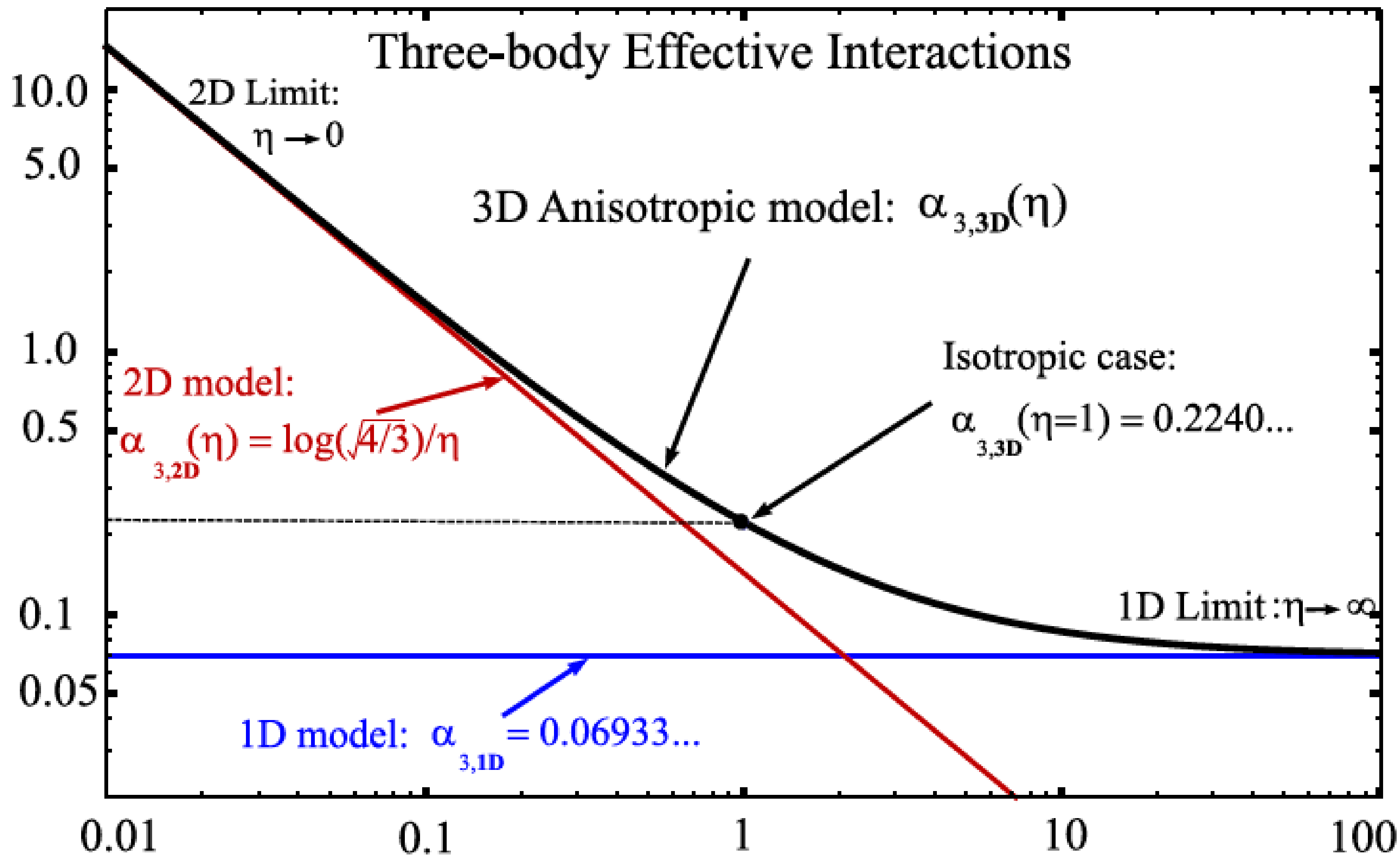
$$= -6\left(\sum_{\vec{n}=n_x n_y n_z} \frac{K_{000\vec{n}}K_{\vec{n}000}}{n_x + n_y + \eta^{-1} n_z}\right)\left(\frac{a_t}{\sigma_z}\right)^2$$

Can be reduced to infinite sum over single index.

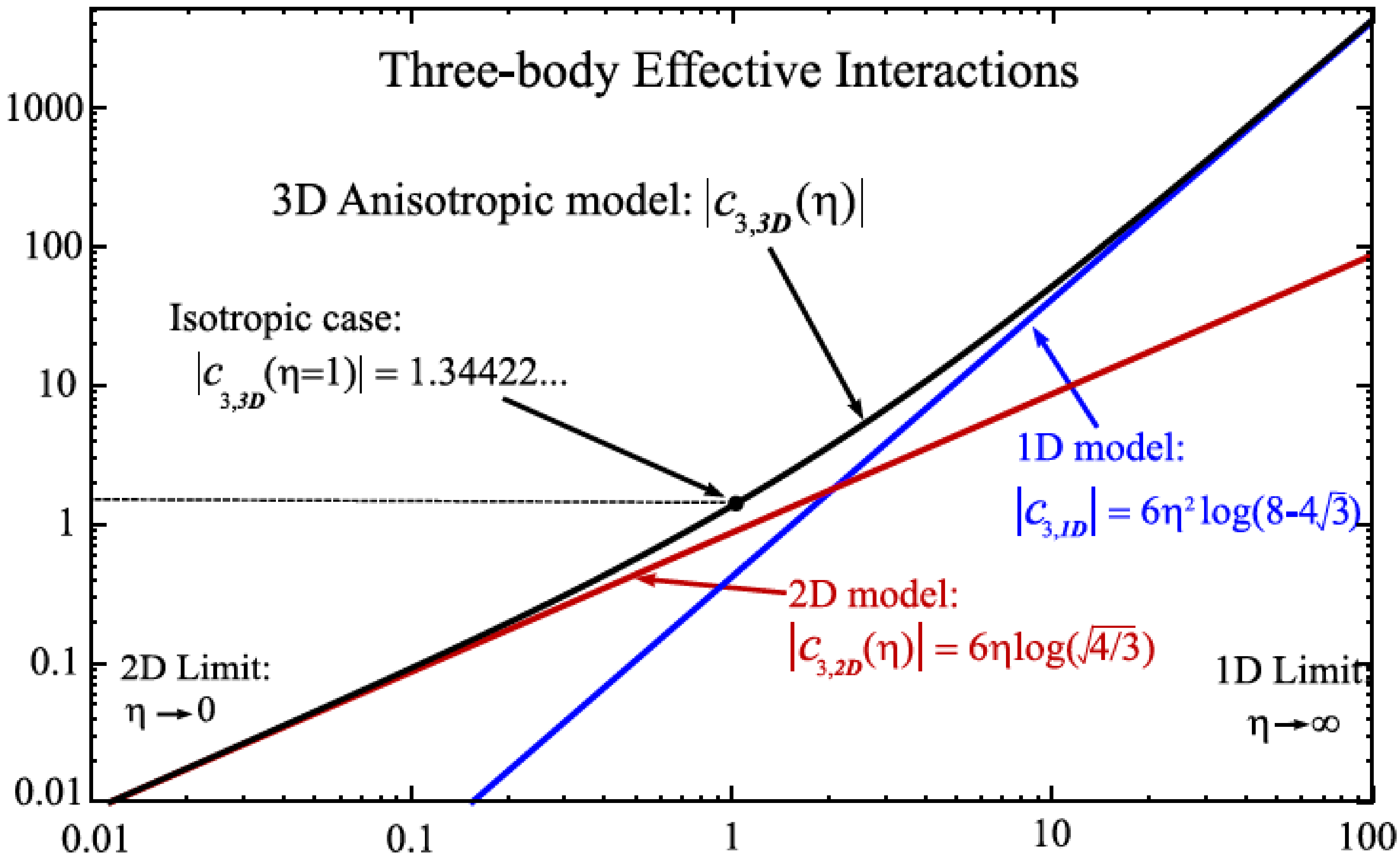
$$c_3^{(2)}(\eta) = -6\eta^2 \alpha_3^{(2)}(\eta).$$

$$\alpha_3^{(2)}(\eta) = \log(8 - 4\sqrt{3}) + \frac{1}{2}\sum_{\rho=1}^{\infty} 2^{2\rho(\eta-1)} B_{1/4}(\eta\rho, 1/2).$$

Cylindrically symmetric trap



Cylindrically symmetric trap



Cylindrical trap: Two-body energy

$$U_2^{(2)}(\eta) = c_2^{(2)}(\omega_z, \eta; \omega_z', \eta')\gamma_{\mathrm{t}}^2,$$

$$c_2^{(2)}(\omega_z, \eta; \omega_z', \eta') = \eta\left(\eta'\sqrt{\frac{\omega_z'}{\omega_z}} - \eta\right)\log 2 + \frac{\eta}{2}\left(\sqrt{\frac{\omega_z'}{\omega_z}}F(\eta') - F(\eta)\right).$$

$$F(\eta) \equiv \eta\sqrt{\pi}\sum_{\rho=1}^{\infty}\left(\frac{\Gamma(\eta\rho)}{\Gamma(\eta\rho+1/2)} - \frac{1}{\sqrt{\eta\rho}}\right) + \sqrt{\eta\pi}\zeta(1/2),$$

Agrees perturbatively with result of Idziaszek and Calarco

Z. Idziaszek and T. Calarco, Phys. Rev. A, 74, 022712 (2006).

Summary of Results

- Our new 3D results for anisotropic 2-body energy agree (perturbatively) with previous exact results for isotropic and anisotropic harmonic traps.

- Our new 3D results for anisotropic 3-body agree with our previous 3-body results in isotropic limit.

- The quasi-1D limit of our 3D, 3-body anisotropic results agree with true 1D model prediction. The 1D model also agrees perturbatively with Busch et al predictions for for 2-body energy in 1D.

- The quasi-2D limit of our 3D, 3-body anisotropic results agree with the true 2D model prediction.