Spectroscopy for a few atoms harmonically-trapped in one dimension

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Symmetries for any* trap and any† interaction

$$\hat{H} = \sum_{i=1}^{N} \left(-\frac{1}{2m} \frac{\partial^2}{\partial x_i^{\,2}} + V_{\text{trap}}\left(|x_i|\right) \right) + \sum_{\langle i,j \rangle} V_{\text{int}}\left(|x_i - x_j|\right)$$

$$S_N \times O(1) \times O(1)$$

Permutation symmetry

Relative parity

COM parity

*one-dimensional, symmetric, spin-independent

†one-dimensional, Galilean-invariant, spin-independent

Symmetries for harmonic trap and contact interaction

$$\frac{\hat{H}}{\hbar\omega} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial x_i^{\ 2}} + x_i^2\right) + g\sum_{\langle i,j\rangle}\delta\left(x_i - x_j\right)$$

$$S_N \times O(1) \times U(1)$$

Permutation symmetry

Relative parity

COM harmonic oscillator

Additional symmetries:

- Contact symmetry $U(N)$ when $g = 0$
- Spectrum generating symmetry $SO(2,1)$ when $g = 0$ or $g \to \infty$

Outline

- Motivation
- Symmetries of Configuration Space
- Spectroscopic Classification of Spatial States
- Secret Motivation

Motivation, Part I

$$\frac{\hat{H}}{\hbar\omega} = \frac{1}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial x_i{}^2} + x_i^2\right) + g\sum_{\langle i,j\rangle}\delta\left(x_i - x_j\right)$$

- Mathematical physics
  - Symmetry and integrability
- Universality in few-body physics
  - Recent experiments
- Power of symmetry
  - Calculation and control: exact diagonalization, adiabatic evolution, quenches
- Limits of symmetry
  - Emergent complexity with increasing DOF

Harmonic Trap and Separation of Variables

$$\hat{H} = \frac{\hbar\omega}{2}\sum_{i=1}^{N}\left(-\frac{\partial^2}{\partial {x_i}^2} + x_i^2\right) + \sum_{\langle i,j\rangle} V_{\text{int}}\left(\left|x_i - x_j\right|\right)$$

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ R \end{pmatrix} = \mathbf{J}_N \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_2 \\ \frac{1}{\sqrt{6}} x_1 + \frac{1}{\sqrt{6}} x_2 - \sqrt{\frac{2}{3}} x_3 \\ \vdots \\ \frac{1}{\sqrt{N}} x_1 + \frac{1}{\sqrt{N}} x_2 + \cdots \frac{1}{\sqrt{N}} x_N \end{pmatrix}$$

$$\rho = \sqrt{{r_1}^2 + \ldots r_{N-1}^2}$$

$$\hat{H} = \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial R^2} + R^2\right) + \frac{\hbar\omega}{2}\sum_{i=1}^{N-1}\left(-\frac{\partial^2}{\partial {r_i}^2} + {r_i}^2\right) + \sum_{\langle i,j\rangle} V_{\text{int}}\left(\left|\vec{d}_{ij} \cdot \vec{r}\right|\right)$$

$$\left(\vec{d}_{ij}\right)_k = \left(\mathbf{J}_N\right)_{ki} - \left(\mathbf{J}_N\right)_{kj}$$

Symmetries for harmonic trap and any† interaction

$$\hat{H} = \tfrac{1}{2}\left(-\frac{\partial^2}{\partial R^2} + R^2\right) + \tfrac{1}{2}\sum_{i=1}^{N-1}\left(-\frac{\partial^2}{\partial r_i{}^2} + r_i{}^2\right) + \sum_{\langle i,j\rangle} V_{\text{int}}\left(\left|\vec{d}_{ij}\cdot\vec{r}\right|\right)$$

$$S_N \times O(1) \times U(1)$$

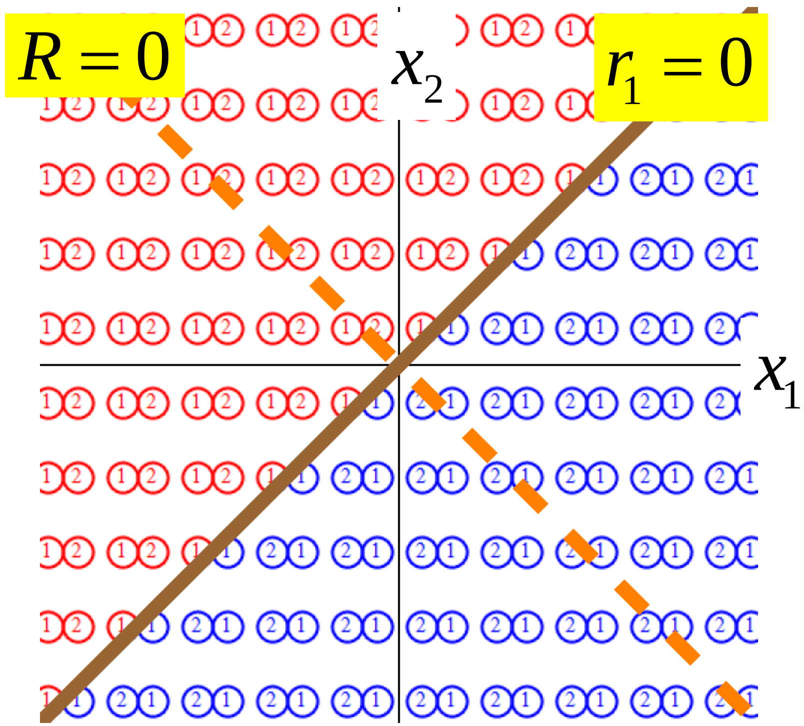
$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ R \end{pmatrix} \xrightarrow[p\in S_N]{} \begin{pmatrix} \sum_j D^{[N-1,1]}_{1j}(p) r_j \\ \sum_j D^{[N-1,1]}_{2j}(p) r_j \\ \vdots \\ R \end{pmatrix}$$

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ R \end{pmatrix} \xrightarrow[\pi\in O(1)]{} \begin{pmatrix} -r_1 \\ -r_2 \\ \vdots \\ R \end{pmatrix}$$

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ R \end{pmatrix} \xrightarrow[\pi\in U(1)]{} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ e^{i\theta} R \end{pmatrix}$$

Two-Particle Configuration Space

$m_1 = m_2$

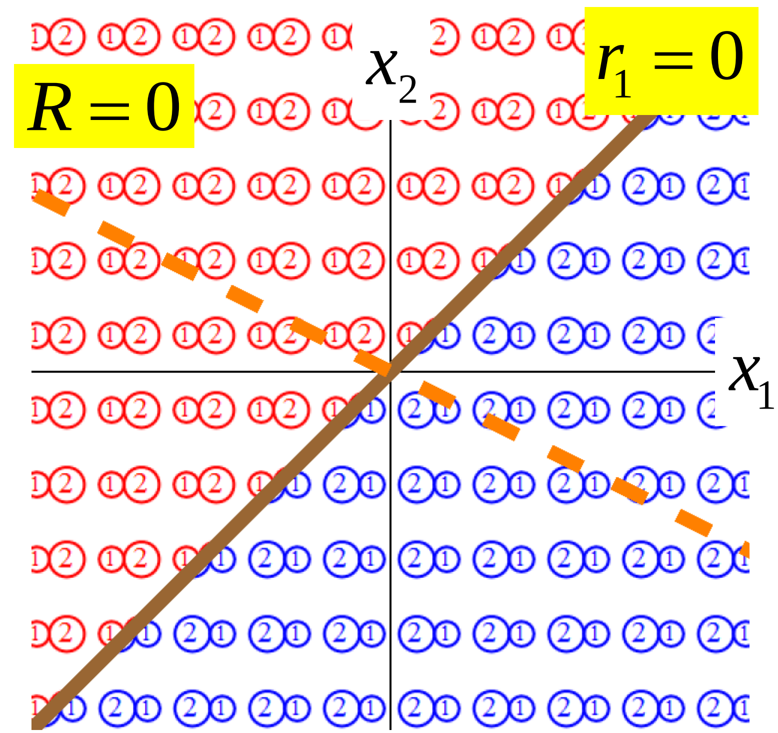


Symmetry group: $S_2 \times O(1)$

Transformations: $\pi, (12) \rightarrow \sigma_{r1}$ $\Pi \rightarrow C_2$

$\Pi(12) \rightarrow \sigma_R$

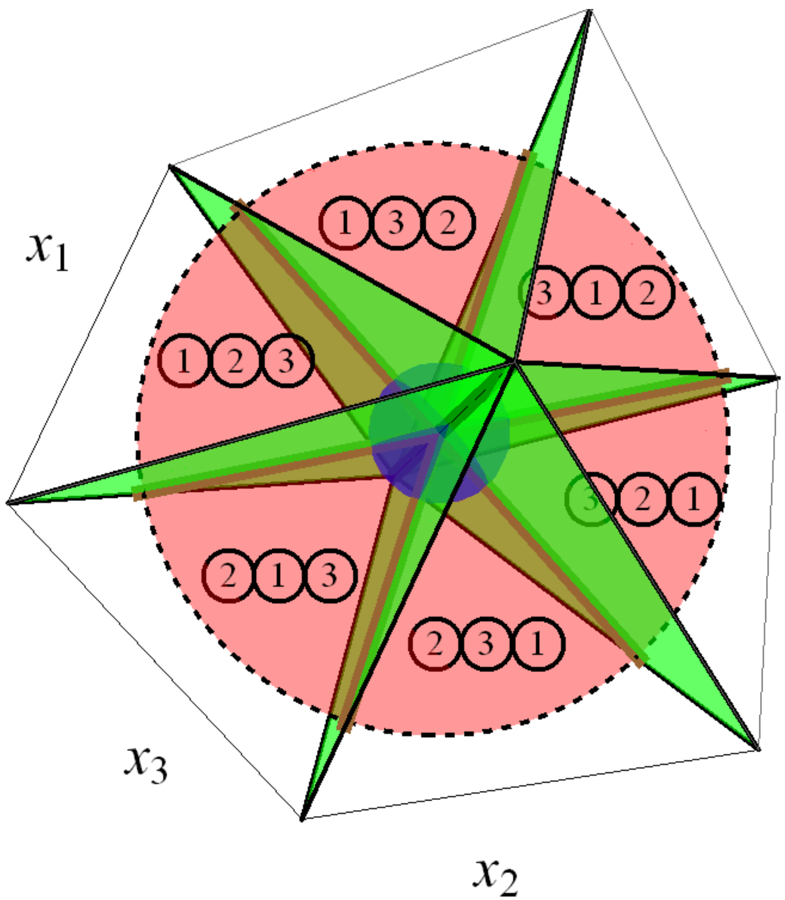
$2m_1 = m_2$

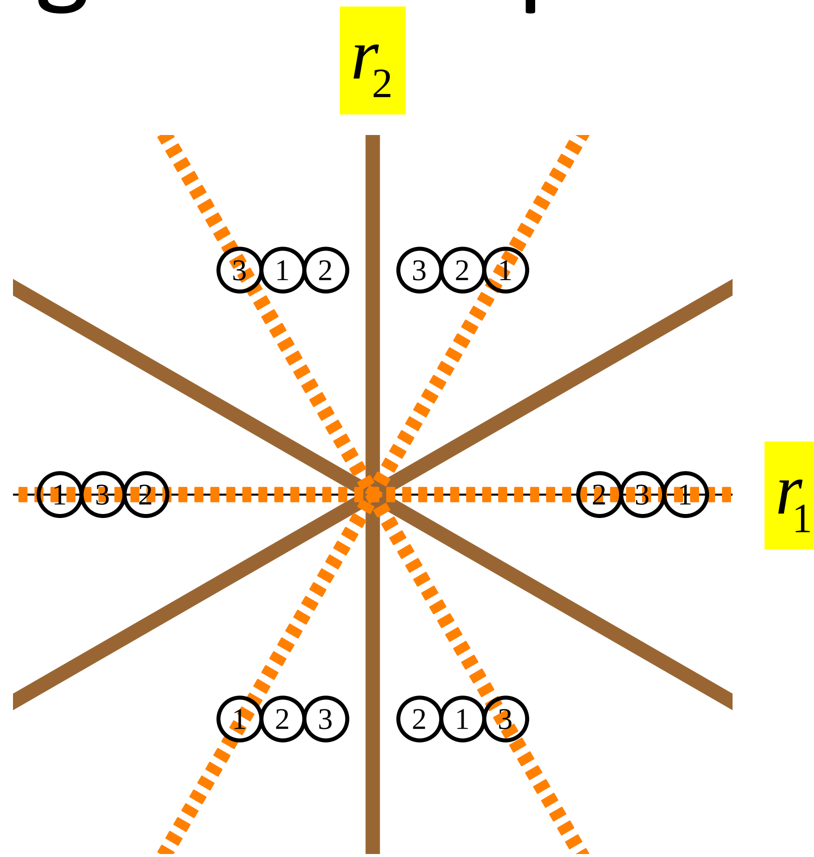


Symmetry group: $O(1)$

Transformations: $\Pi \rightarrow C_2$

Three-Particle Configuration Space





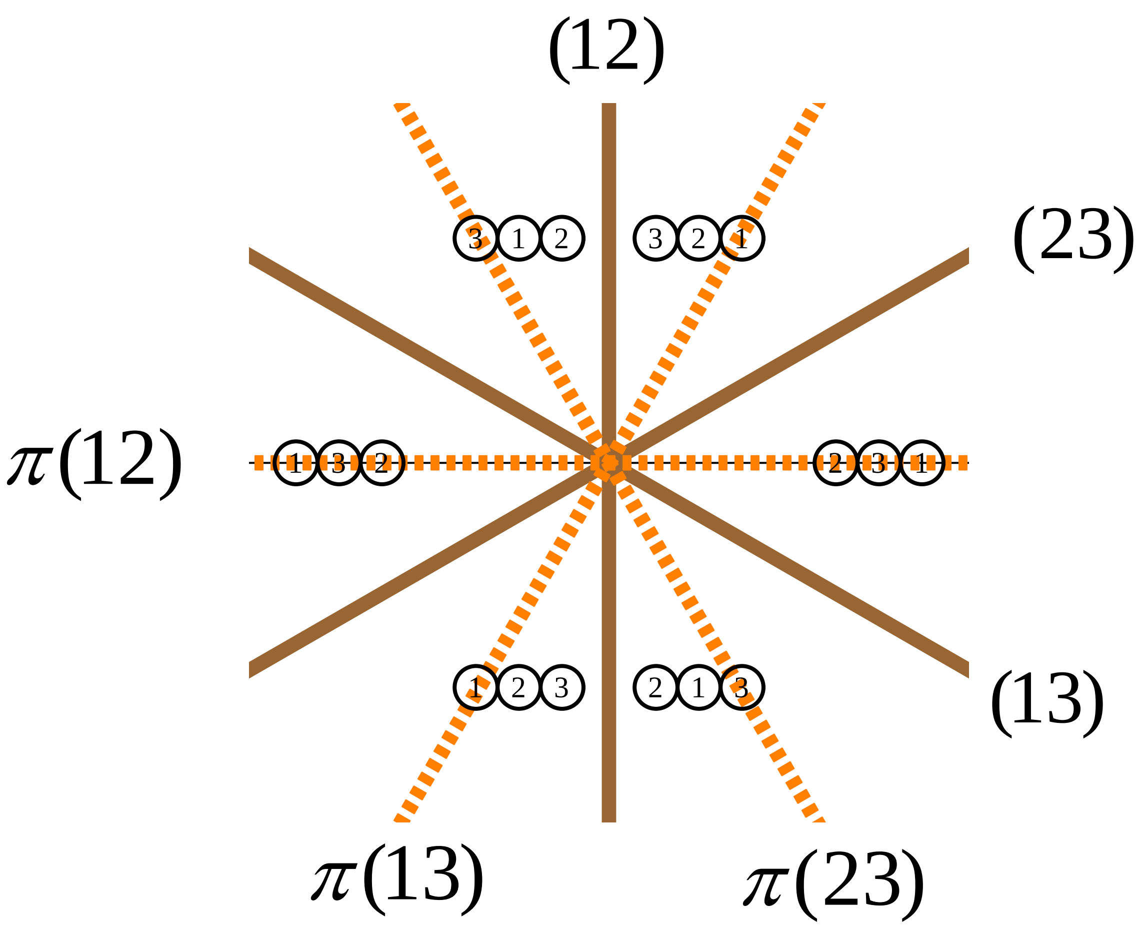
Symmetry group:

$$S_3 \times O(1) \times O(1) \square\ D_{6h}$$

Symmetry group:

$$S_3 \times O(1) \square\ C_{6v}$$

Three-Particle Configuration Space



Symmetry group:

$$S_3 \times O(1) \cong C_{6v}$$

Relative inversion:

$$\pi \to C_2$$

Reflections:

$$(12), (13), (23)$$

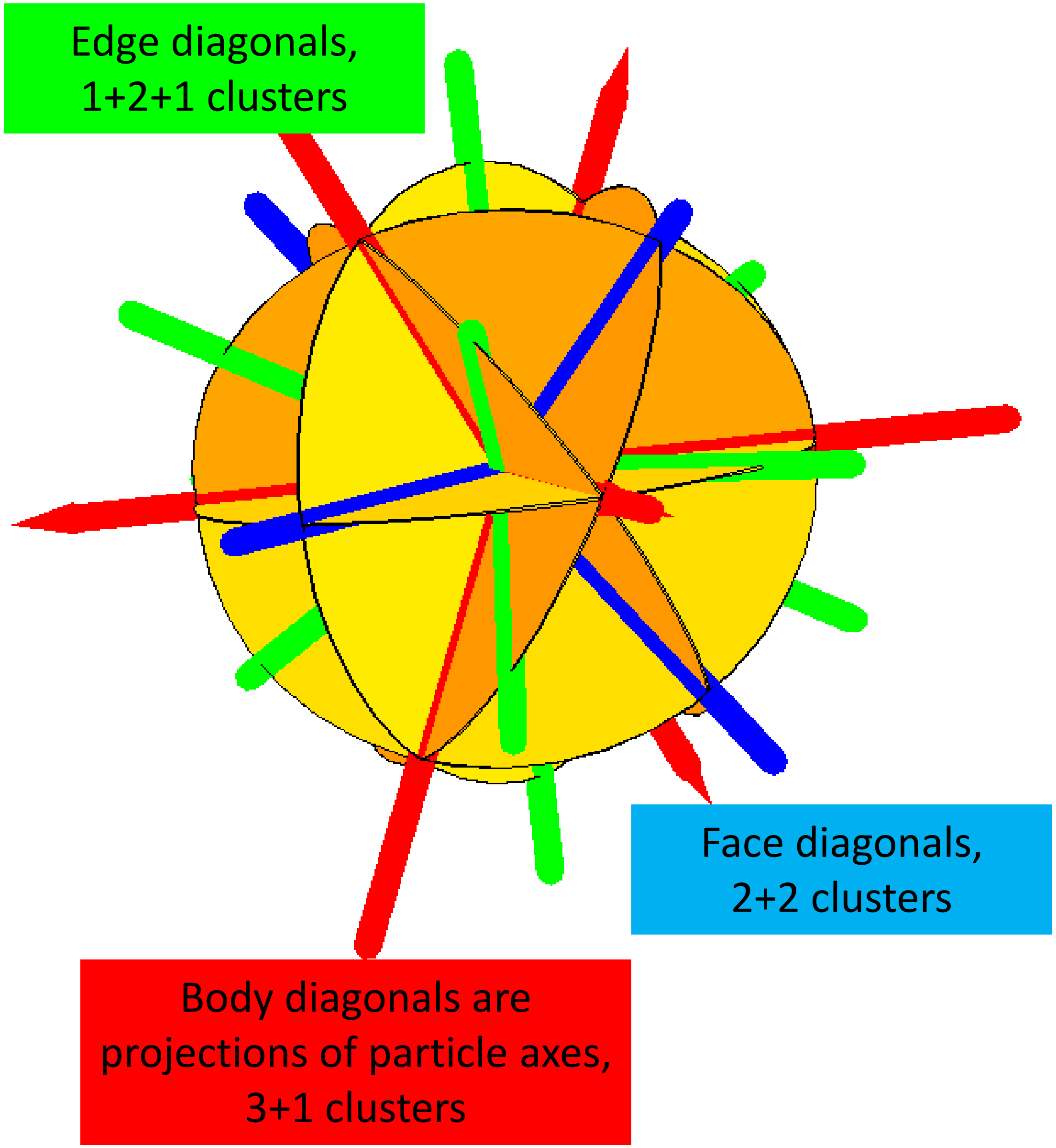
$$\pi(12), \pi(13), \pi(23)$$

Rotations:

$$(123), (132) \to 2C_3$$

$$\pi(123), \pi(132) \to 2C_6$$

Four-Particle Configuration Space



Symmetry group:

$$S_4 \times O(1) \square O_h$$

$$(12)\ldots \rightarrow 6\sigma_d$$

$$(123)\ldots \rightarrow 8C_3$$

$$(12)(34)\ldots \rightarrow 3C_2$$

$$(1234)\ldots \rightarrow 6S_4$$

$$\pi \rightarrow i$$

$$\pi(12)\ldots \rightarrow 6C_2'$$

$$\pi(123)\ldots \rightarrow 8S_6$$

$$\pi(12)(34)\ldots \rightarrow 3\sigma_h$$

$$\pi(1234)\ldots \rightarrow 6C_4$$

Five-Particle Configuration Space

$$S_5 \times O(1)$$

It is known, Khaleeshi.

Quantum numbers

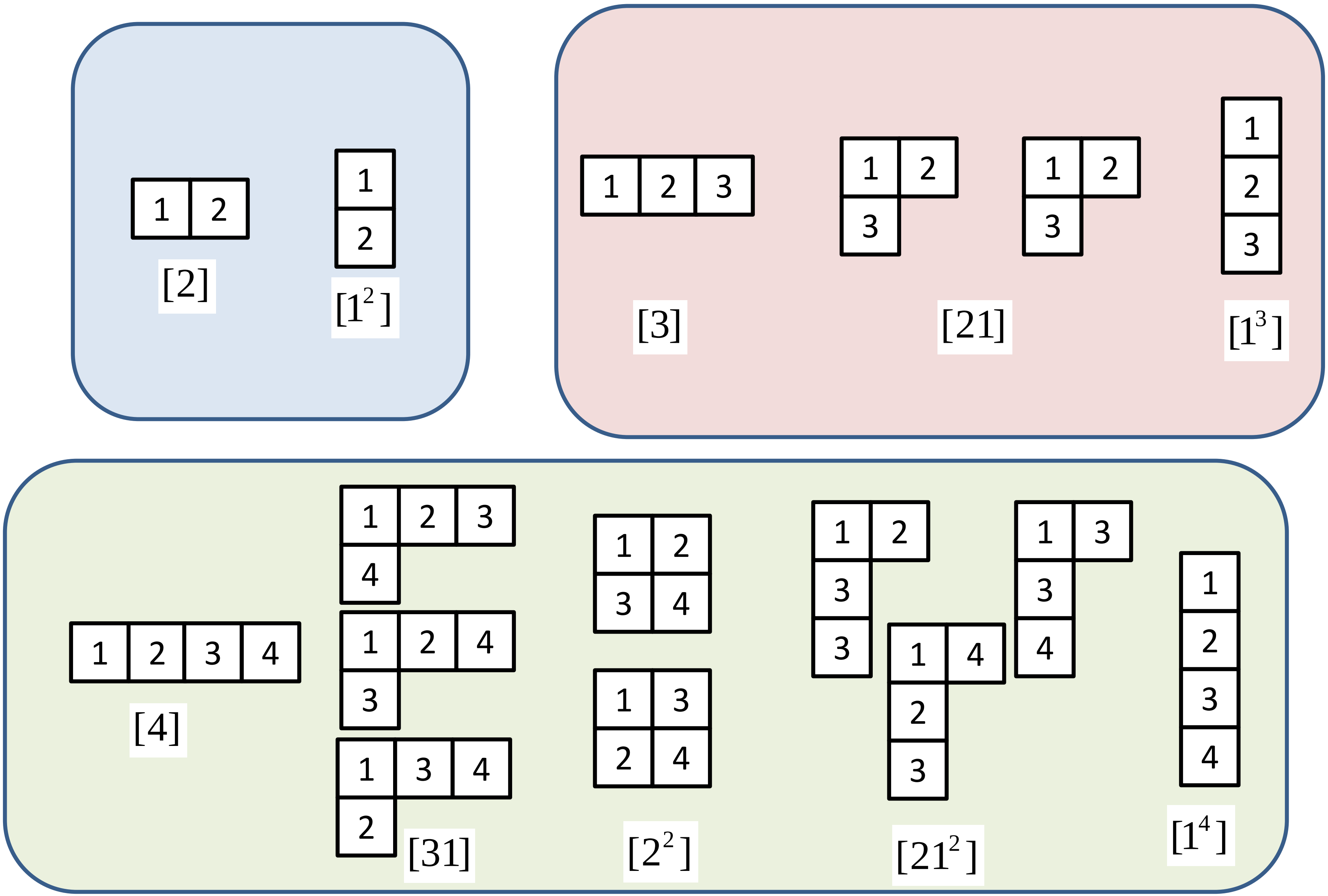
$$S_N \times O(1) \times U(1)$$

$$U(1), S_N{}^{O(1)} \longrightarrow \left| n_R [p]^{\pi} \tau; i \right]_g$$

COM excitation number

Symmetric group irrep

Symmetric Group Irreps



Quantum numbers

$$S_N \times O(1) \times U(1)$$

Relative parity

Index within irrep

$$U(1), S_N^{\;O(1)} \rightarrow \left| n_R [p]^\pi \tau; i \right]_g$$

COM excitation number

Symmetric group irrep

Degeneracy of irrep

$$\hat{H} \left| n_R [p]^\pi \tau; i \right]_g = E_{n_R [p]^\pi \tau} \left| n_R [p]^\pi \tau; i \right]_g$$

Each energy is distinct except at non-interacting and infinite repulsion limit.

$$E_{n_R [p]^\pi \tau} = \hbar \omega \left(n_R + X_{[p]^\pi \tau} + \frac{N}{2} \right)$$

Symmetries for contact interaction

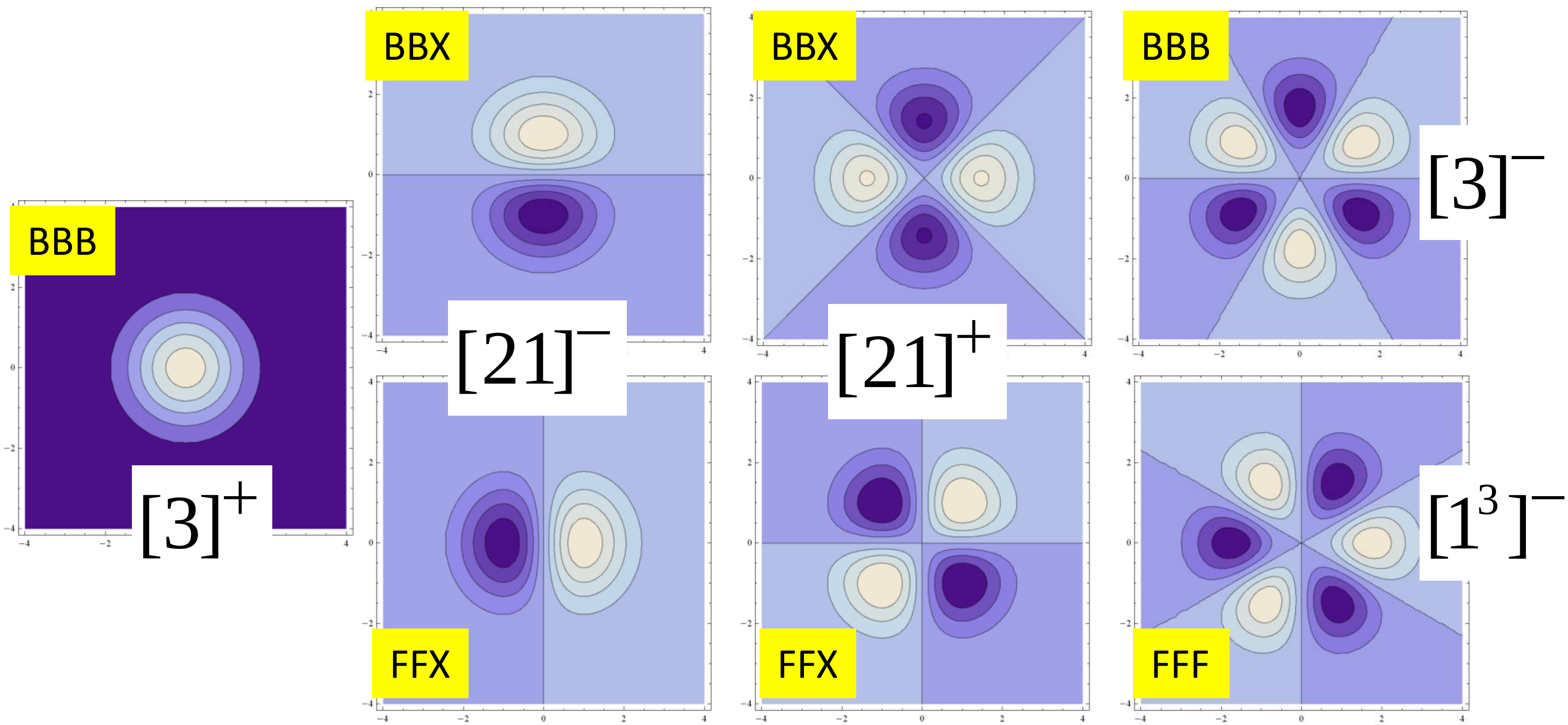
$$\hat{H} = \tfrac{1}{2}\left(-\frac{\partial^2}{\partial R^2} + R^2\right) + \tfrac{1}{2}\sum_{i=1}^{N-1}\left(-\frac{\partial^2}{\partial r_i^{\,2}} + r_i^{\,2}\right) + g\sum_{\langle i,j\rangle}\delta\left(\vec{d}_{ij}\cdot\vec{r}\right)$$

	Contact symmetry	Spectrum generating symmetry
For any interaction strength	$S_N \times O(1) \times U(1)$	NO
For zero interaction strength	$U(N)$	$SO(2,1)$
For infinite interaction strength	$S_N \times O(1) \times U(1)$ …and something else for $N > 3$…	$SO(2,1)$ $X = 2n_\rho + \lambda$

Three Particles: Harmonic Trap, Not Interacting

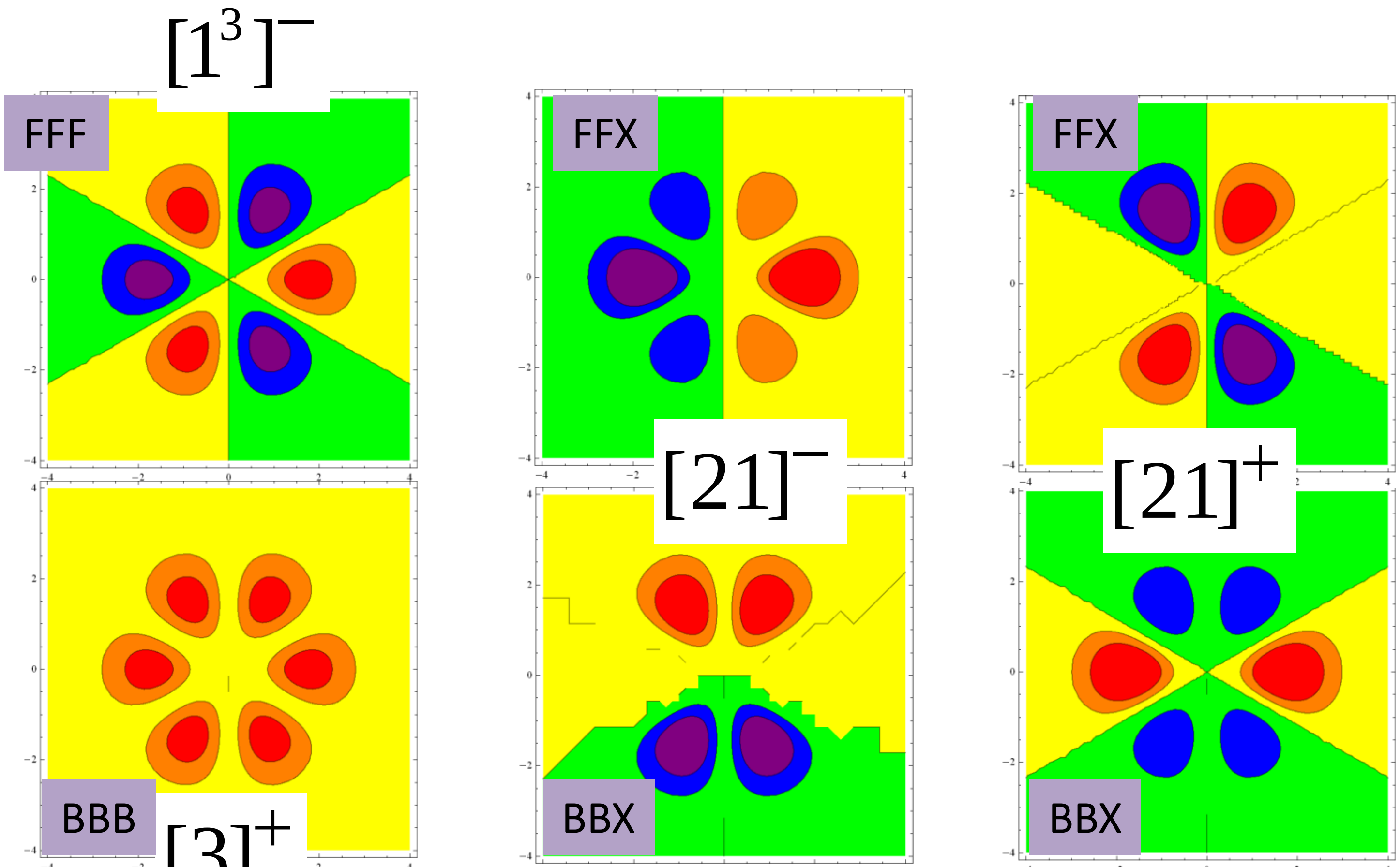
$X_{n_\rho \lambda} = 2n_\rho + \lambda$ $\quad d(X) = X + 1$ $\quad \varepsilon(\lambda > 0) = 2$ $\quad \pi = (-1)^\lambda$

$n_\rho = 0; \lambda = 0$ $\quad n_\rho = 0; \lambda = 1$ $\quad n_\rho = 0; \lambda = 2$ $\quad n_\rho = 0; \lambda = 3$



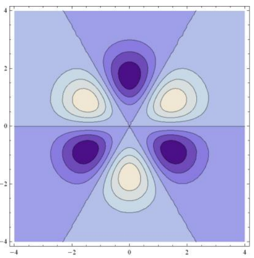
Three Particles: Harmonic Trap, Unitary Limit

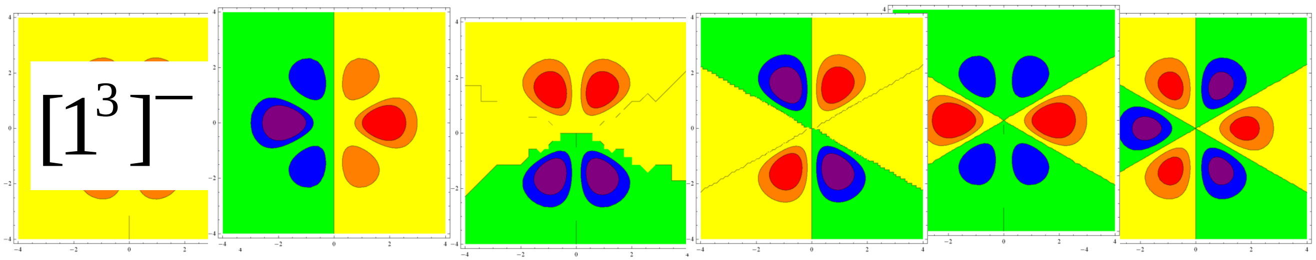
$X_{n_\rho \lambda} = 2n_\rho + \lambda \qquad d(X) = 3! \qquad \lambda \in 3,6,9,12,\ldots \quad \pi = (-1)^\lambda$



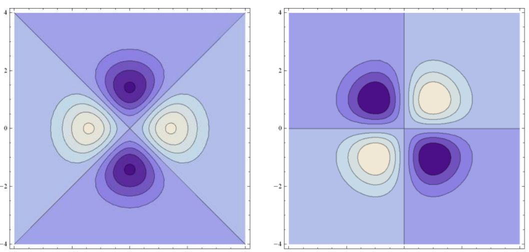
Girardeau

Three Particle: Adiabatic Mapping

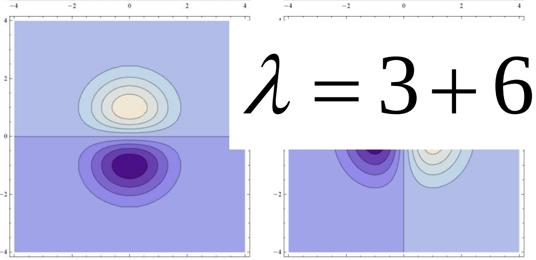




$[1^3]^-$

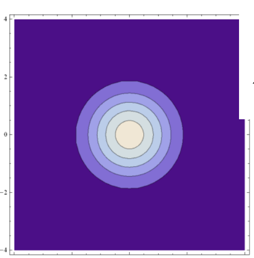


$[21]^+$



$$\lambda = 3 + 6k \rightarrow [3]^+ \oplus [21]^+ \oplus [21]^- \oplus [1^3]^-$$

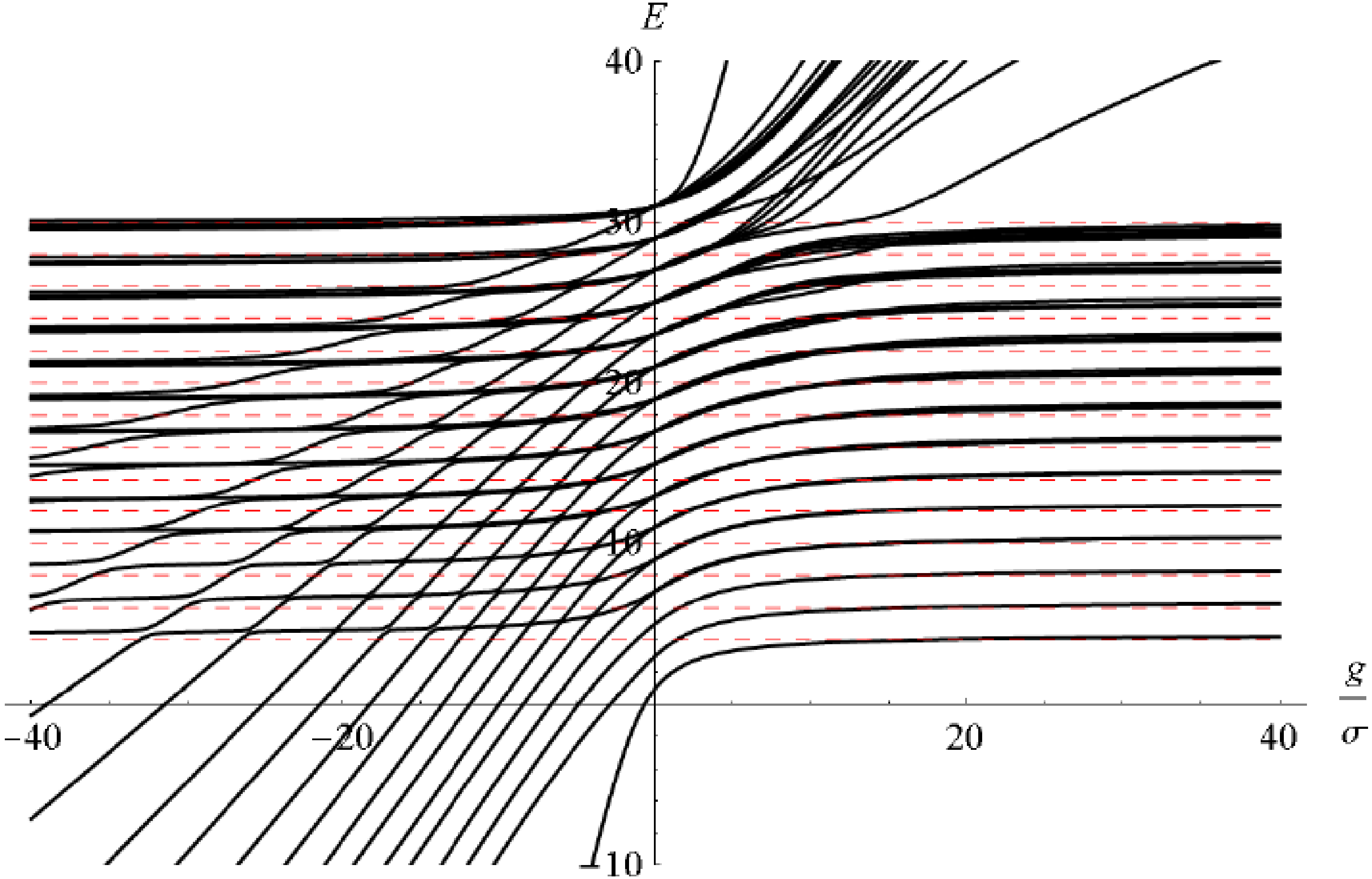
$[21]^-$



$$\lambda = 6 + 6k \rightarrow [3]^- \oplus [21]^- \oplus [21]^+ \oplus [1^3]^+$$

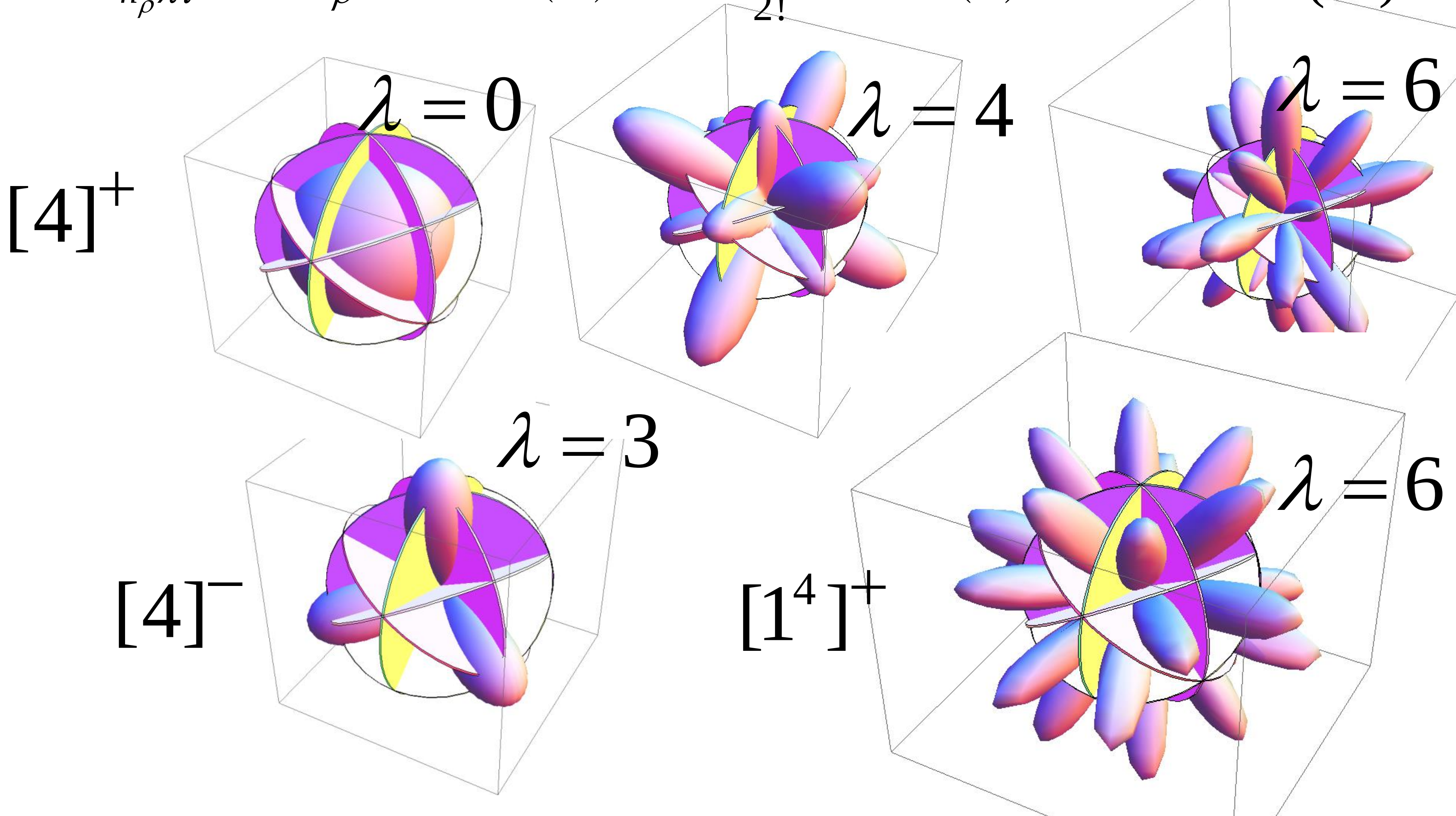
$[3]^+$

$[3]^{+}$ $\quad X_{\max} = 30 \quad$ 40 states

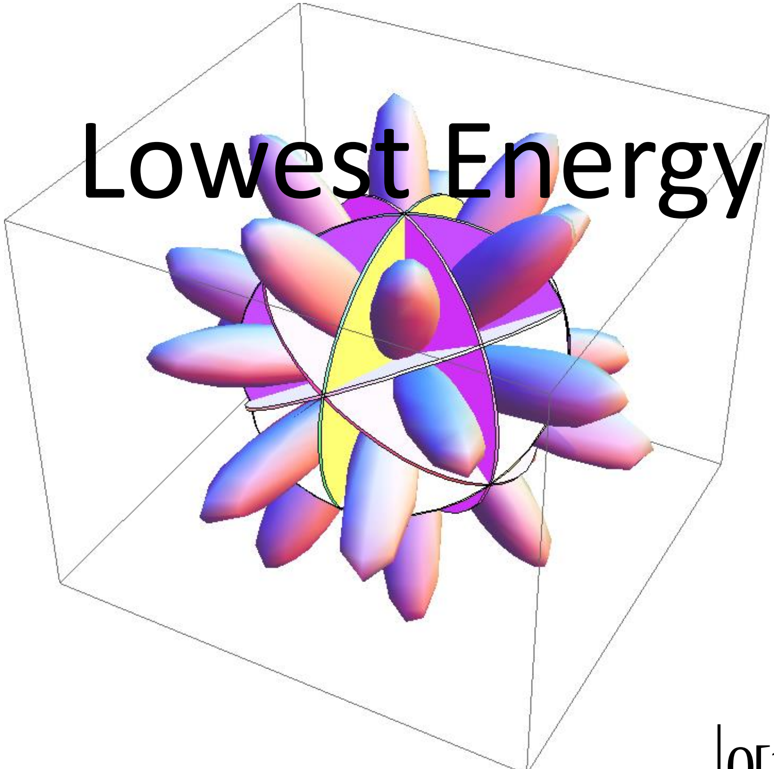


Four Particles:
Harmonic Trap, Not Interacting

$$X_{n_\rho \lambda \tau} = 2n_\rho + \lambda \quad d(X) = \frac{(X+1)(X+2)}{2!} \quad \varepsilon(\lambda) = 2\lambda + 1 \quad \pi = (-1)^\lambda$$



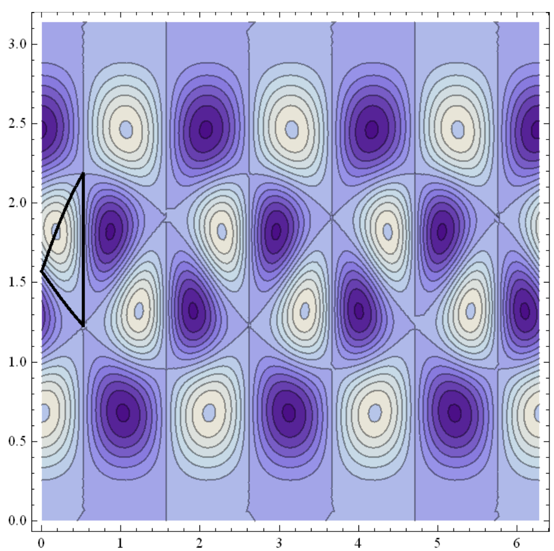
Lowest Energy Antisymmetric State



$$\left|n_R[p]^\pi \tau; i\right]_g \to \left|0[1^4]^+ 0\right]_g$$

Particle basis, Slater determinant:

$$\left|0[1^4]^+ 0\right]_g = \frac{1}{\sqrt{24}} \sum_{p\in S_4} \mathrm{sign}(p) \left|n_{p1} n_{p2} n_{p3} n_{p4}\right\rangle$$



Jacobi hypercylindrical:

$$\left|0[1^4]^+ 0\right]_g = \sum_{\mu=-6}^{6} c_\mu \left|n_R n_\rho 6\mu\right\}$$

Four Particles: Harmonic Trap, Non-interacting Including Spin

$$\mathcal{H} = \mathcal{S} \otimes \mathcal{K}$$

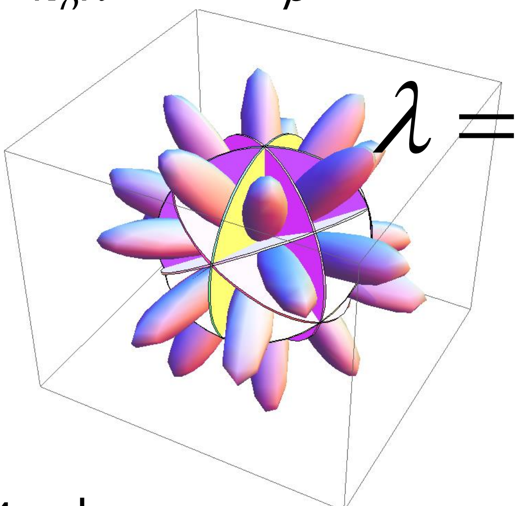
number of components
1
2
2
3
1
2
2
3
4

component pattern	λ 0	1	2	3	4	5	6	7	8	9	10	11	12
$(4)_B$	1	0	0	1	1	0	1	1	1	1	1	1	2
$(31)_B$	1	1	1	2	2	2	3	3	3	4	4	4	5
$(22)_B$	1	1	2	2	3	3	4	4	5	5	6	6	7
$(211)_B$	1	2	3	4	5	6	7	8	9	10	11	12	13
$(4)_F$	0	0	0	0	0	0	1	0	0	1	1	0	1
$(31)_F$	0	0	0	1	1	1	2	3	3	3	3	3	4
$(22)_F$	0	0	1	1	2	2	3	3	4	4	5	5	6
$(211)_F$	0	1	2	3	4	5	6	7	8	9	10	11	12
(1111)	1	3	5	7	9	11	13	15	17	19	21	23	25

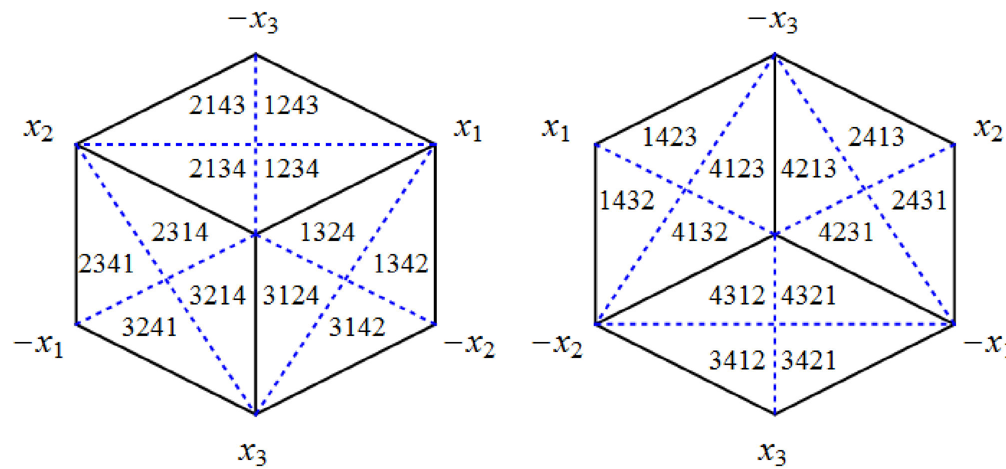
$s = 0$	Total
	1
	4
↑↑ / ↓↓	6
	4
	1
$2D^0$	

Four Particles: Harmonic Trap, Unitary Limit

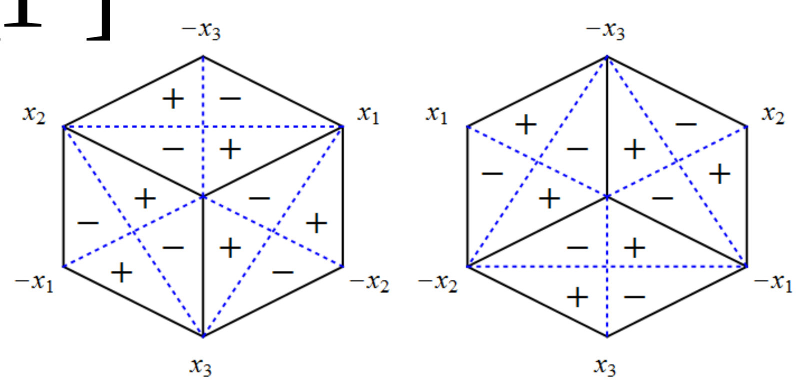
$$X_{n_\rho \lambda} = 2n_\rho + \lambda \quad d(X) = 4! \ \lambda \in 6, 8, 9, 10, 12, 13, \ldots \quad \pi = (-1)^\lambda$$



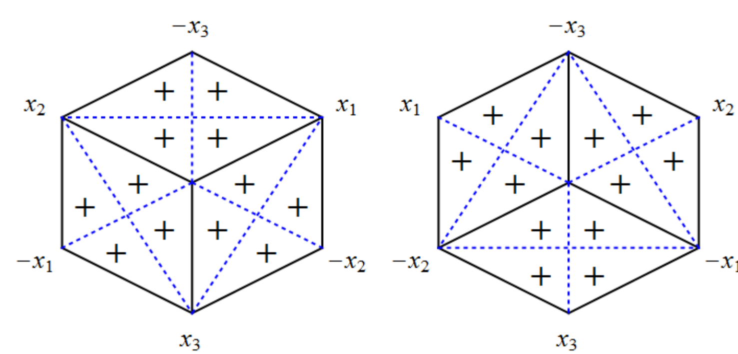
$\lambda = 6$



$[1^4]^+$

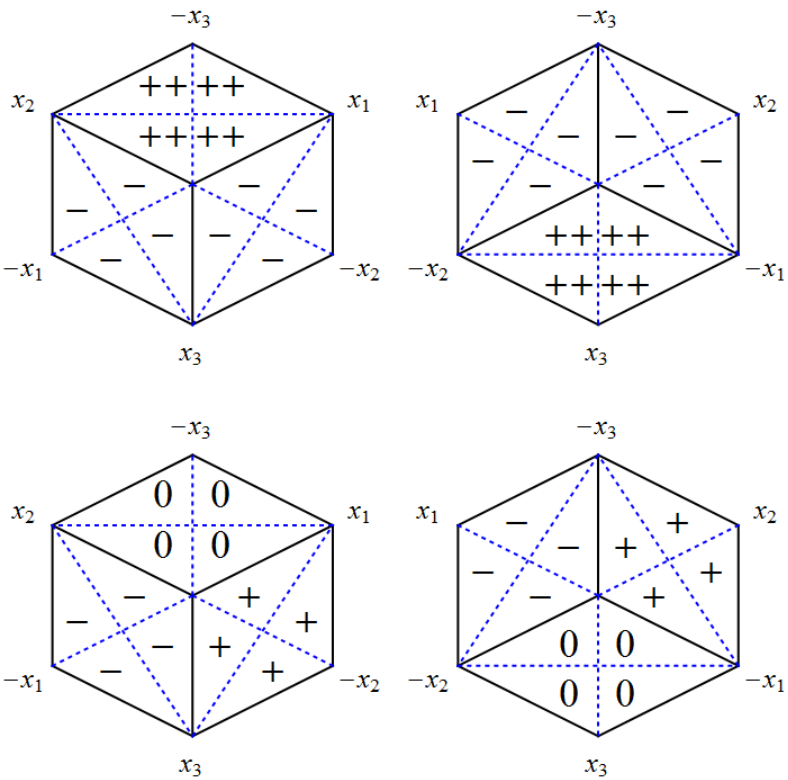


$[4]^+$

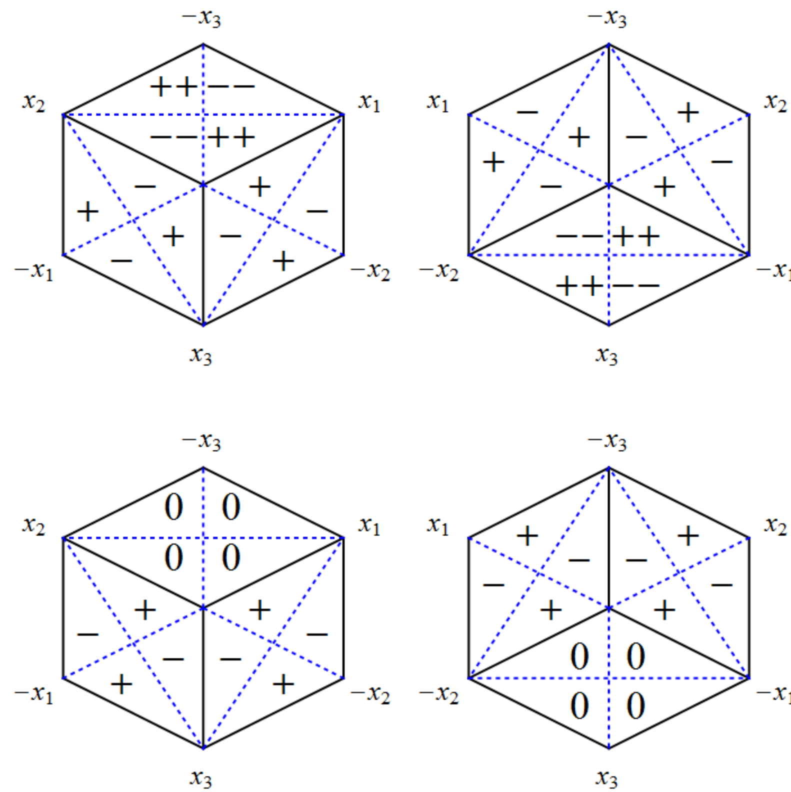


$$\lambda = \text{even} \rightarrow [4]^{+} \oplus [31]^{+} \oplus 2[2^{2}]^{+} \oplus [21^{2}]^{+} \oplus [1^{4}]^{+} \oplus 2[31]^{-} \oplus 2[21^{2}]^{-}$$

$$\lambda = \text{odd} \rightarrow 2[31]^{+} \oplus 2[21^{2}]^{+} \oplus [4]^{-} \oplus [31]^{-} \oplus 2[2^{2}]^{-} \oplus [21^{2}]^{-} \oplus [1^{4}]^{-}$$

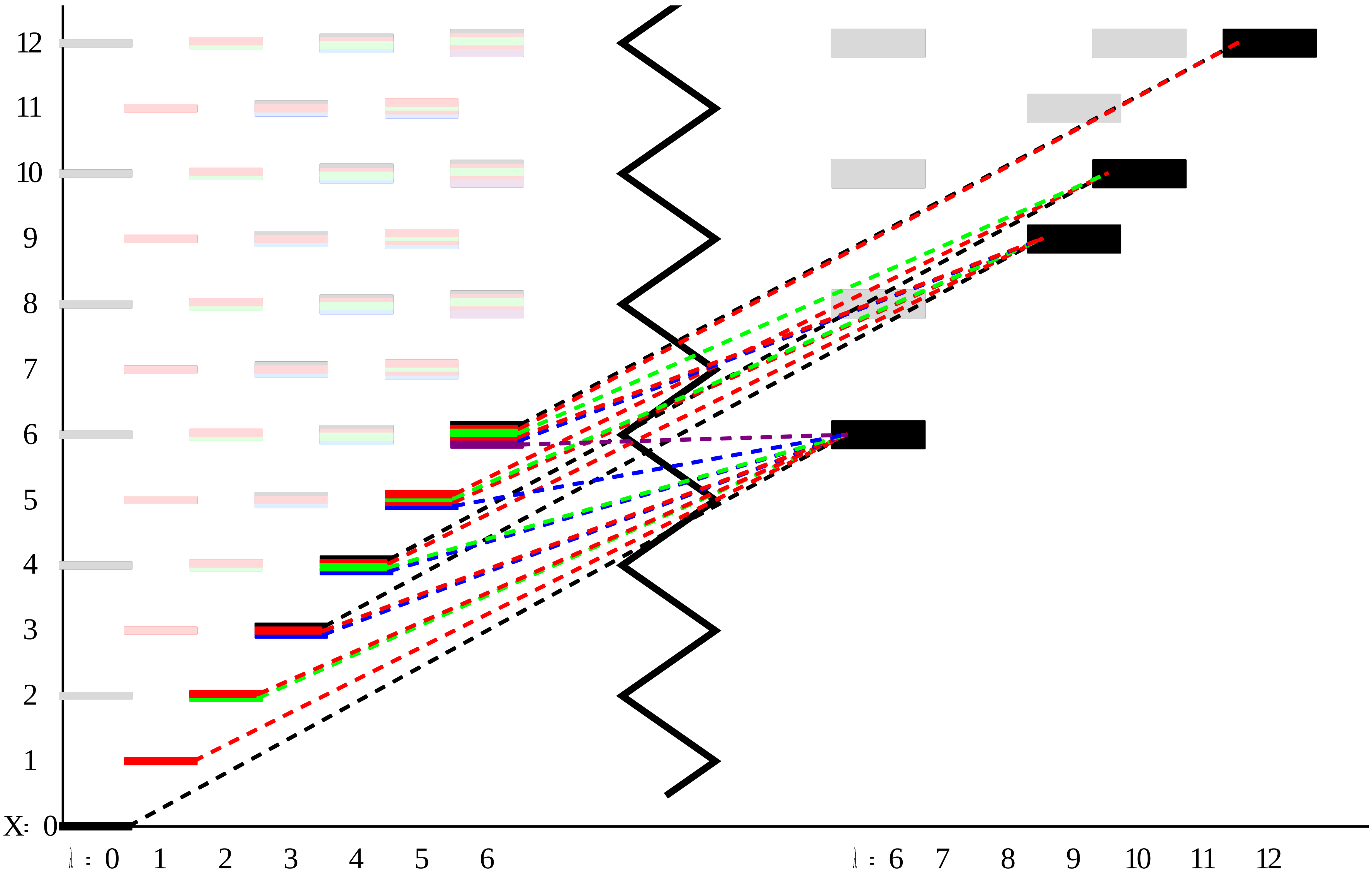


[2^2]^+ A



[2^2]^+ B

MYSTERY SYMMETRY



Caveat emptor

Irrep Reduction for Five Particles

Non-interacting

λ	$\lfloor 5 \rfloor$
0	1
1	0
2	0
3	1
4	1
5	1
6	1
7	1
8	2
9	2
10	2
11	2
12	3
13	3

no. of comps.	component pattern	$\lfloor p \rfloor$						
		$\lfloor 5 \rfloor$	$\lfloor 41 \rfloor$	$\lfloor 3^2 \rfloor$	$\lfloor 31^2 \rfloor$	$\lfloor 2^2 1 \rfloor$	$\lfloor 21^3 \rfloor$	$\lfloor 1^4 \rfloor$
1	$(5)_B$	1	0	0	0	0	0	0
2	$(41)_B$	1	1	0	0	0	0	0
2	$(32)_B$	1	1	1	0	0	0	0
2	$(311)_B$	1	2	1	1	0	0	0
3	$(221)_B$	1	2	2	1	1	0	0
4	$(2111)_B$	1	3	3	3	2	1	0
1	$(5)_F$	0	0	0	0	0	0	1
2	$(41)_F$	0	0	0	0	0	1	1
2	$(32)_F$	0	0	0	0	1	1	1
2	$(311)_F$	0	0	0	1	2	1	1
3	$(221)_F$	0	0	1	1	2	2	1
4	$(2111)_F$	0	1	2	3	3	3	1
5	(11111)	1	4	5	6	5	4	1

Unitary limit

		odd λ
		0
		2
		2
		4
		2
		2
		0
		1
		2
		3
		2
		3
		2
$\lfloor 1^5 \rfloor^-$	0	1

Working on more efficient state construction method

Secret Motivation

Integrability, separability and entanglement

- Abstractly: Particles and Tailored Observables
- Directly: Few body systems as a resource for quantum information processing

THE FUTURE

- Identify mystery symmetry
- Basis transformation coefficients, matrix elements
- More particles
  - Five is different
  - Heterogeneous particle mixtures
- More dimensions
  - Symmetry is less constraining
- More traps, more interactions
  - Lose spectrum generating group and $U(N)$ symmetry

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