Universality in Four-Boson Systems

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INT Program: Universality in few-body systems: theoretical challenges and new directions
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Outline

Main Goal

- Very accurate solution for the 4B problem - scaling and universality

Motivation: why are weakly bound state problems interesting?

- Short-range interactions & large quantum systems
- Universality & model independence
- Reality: Cold atom physics close to a Feshbach resonance!

Different techniques for few-body bound state calculations

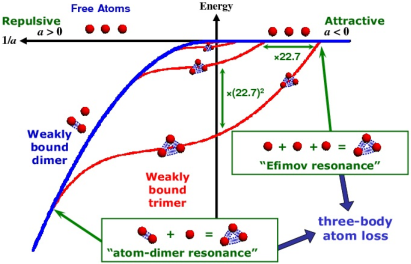
- NCSM (Navratil et al. PRC62),
- HH (Viviani et al. PRC71) and EIHH (Barnea et al. PRC67)
- GFMC (Viringa et al. PRC62),
- CRCGV (Hiyama et al. PRL85),
- SV (Usukura et al. PRB59),
- FY (Hadizadeh et al. FBS40, EPJA201, PTP120, PRC83)

Efimov Physics (1970): Nuclear Physics

Vitaly Efimov

- an infinite sequence of weakly bound 3-body states as $a \to \pm\infty$
- Unitary limit ($a \to \pm\infty$): $E_3^{n+1}/E_3^n \approx 1/22.7^2$ $\Longrightarrow$ discrete scaling with scaling factor 22.7
- For finite a: discrete scaling is exact when range$\to 0$

- Efimov effect: If two bosons interact in such a way that a two-body bound state is exactly on the verge of being formed, then in a three-boson system one should observe an infinite number of bound states. This phenomenon appears in a three-dimensional formalism for the three-body systems, and does not exist in one or two dimensions.
- This effect, predicted by Vitaly Efimov in 1970 [Phys. Lett. B 33 (1970) 563; Sov. J. Nucl. Phys. 12 (1971) 589], have been recently verified in ultracold atom laboratories, with the increasing number of three-body bound-state levels, as the two-body scattering length goes to infinity.



The observation of Efimov effect

The observation of this effect, first reported by Kraemer et al. in an ultracold gas of Cesium atoms [Nature 440 (2006) 315], was confirmed by several other atomic experimental groups, which are looking for the properties of such states.

- Zaccanti et al., Nature Phys. 5 (2009) 586
- Ferlaino et al, PRL 102 (2009) 140401
- Pollack et al., Science 326 (2009) 1683

Vol 440|16 March 2006|doi:10.1038/nature04626 nature

LETTERS

Evidence for Efimov quantum states in an ultracold gas of caesium atoms

T. Kraemer[1], M. Mark[1], P. Waldburger[1], J. G. Danzl[1], C. Chin[1,2], B. Engeser[1], A. D. Lange[1], K. Pilch[1], A. Jaakkola[1], H.-C. Nägerl[1] & R. Grimm[1,3]

nature physics LETTERS
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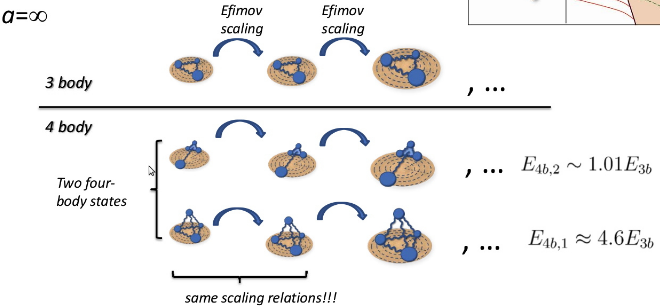
Observation of an Efimov-like trimer resonance in ultracold atom–dimer scattering

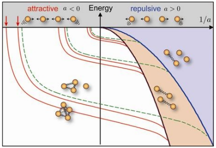
S. Knoop[1]*, F. Ferlaino[1], M. Mark[1], M. Berninger[1], H. Schöbel[1], H.-C. Nägerl[1] and R. Grimm[1,2]

Recent studies on four-boson system - Theory

The addition of one more particle to the quantum three-body system has long challenged the Efimov picture. The need of an independent four-body parameter is discussed, considering the observational possibilities in cold-atom experiments.

- L. Platter et al., *PRA*70 (2004); H.-W. Hammer and L. Platter, *EPJA*32 (2007)
  - Four-boson system with short-range interactions
  - Universal properties of the four-body system with large scattering length
- M. T. Yamashita et al., *EpL*75 (2006) Four-boson scale near a Feshbach resonance.
- R. Lazauskas and J. Carbonell, *PRA* 73 (2006) Description of ^{4}He tetramer bound and scattering states
- J. von Stecher et al., *Nature Physics* 5 (2009); J. von Stecher, *JPB*43 (2010)
  - Signatures of universal four-body phenomena and their relation to the Efimov physics.
  - Weakly bound cluster states of Efimov character
- Y. Wang, B. D. Esry, *PRL*102 (2009) Efimov trimer formation via ultracold four-body recombination
- A. Deltuva, *PRA*82 (2010) Efimov physics in bosonic atom-trimer scattering
- M. R. Hadizadeh et al., *PRL*107 (2011) Scaling properties of universal tetramers
- Gattobigio et al., PRA86 (2012) Energy spectra of small bosonic clusters having a large two-body scattering length
- Hiyama et al. PRA85 (2012) Linear correlations between ^{4}He trimer and tetramer energies calculated with various realistic ^{4}He potentials



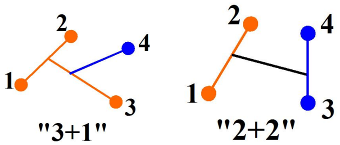


- *Hammer et al., EPJA32 (2007)* **[5, 1.01]**
- *Deltuva, PRA82 (2010)* **[4.61, 1.00227]**
- *Gattobigio et al., PRA86 (2012)* **[4.5, 1.020]**

No independent four-body parameter!

We address this point with high-precision calculations using a zero-range model and searching the excited tetramer states and their dependences on short-range parameters.

Outline



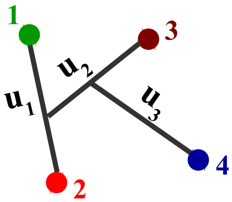
$$
\begin{aligned}
|K^{4}_{12,3}\rangle &= G_0 t_{12} P \left[\left(1 + P_{34}\right) |K^{4}_{12,3}\rangle + |H_{12,34}\rangle \right] \\
|H_{12,34}\rangle &= G_0 t_{12} \tilde{P} \left[\left(1 + P_{34}\right) |K^{4}_{12,3}\rangle + |H_{12,34}\rangle \right]
\end{aligned}
$$

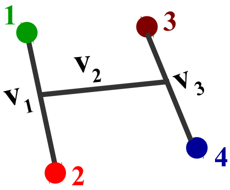
$$
|\Psi\rangle = \left(1 + P + P_{34} P + \tilde{P}\right) \left[\left(1 + P_{34}\right) |K^{4}_{12,3}\rangle + |H_{12,34}\rangle \right]
$$

$$
P = P_{12} P_{23} + P_{13} P_{23} \quad \tilde{P} = P_{13} P_{24}
$$

Kamada et al., NPA548 (1992)

Definition of 4B basis states

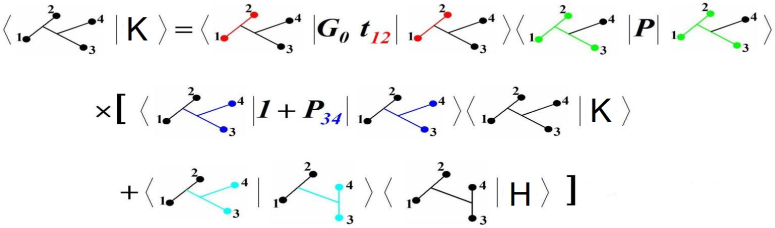


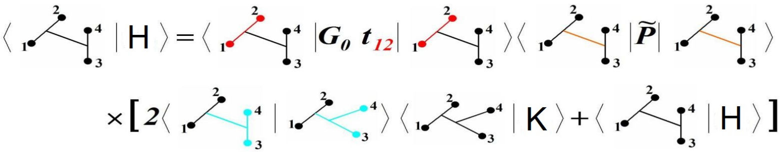


$$\begin{cases} \mathbf{u}_1 = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2) \\ \mathbf{u}_2 = \frac{2}{3}\left(\mathbf{k}_3 - (\mathbf{k}_1 + \mathbf{k}_2)\right) \\ \mathbf{u}_3 = \frac{3}{4}\left(\mathbf{k}_4 - \frac{1}{3}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\right) \end{cases}$$

$$\begin{cases} \mathbf{v}_1 = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2) \\ \mathbf{v}_2 = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) - \frac{1}{2}(\mathbf{k}_3 + \mathbf{k}_4) \\ \mathbf{v}_3 = \frac{1}{2}(\mathbf{k}_3 - \mathbf{k}_4) \end{cases}$$

Explicit representation of Yakubovsky equations





Hadizadeh and Bayegan, FBS40 (2007)

Two-body transition amplitude for zero-range interaction

$$V(\mathbf{r}) = (2\pi)^3\,\lambda\,\delta(\mathbf{r})$$

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = \lambda\langle \mathbf{p}|\chi\rangle\langle\chi|\mathbf{p}'\rangle; \quad \langle \mathbf{p}|\chi\rangle = \int d^3r\, e^{i\mathbf{p}.\mathbf{r}}\,\delta(\mathbf{r}) = 1$$

$$\tau(\epsilon) = \left[\lambda^{-1} - \int d^3p\,\frac{1}{\epsilon - p^2}\right]^{-1}$$

$$\lambda^{-1} = \int d^3p\,\frac{1}{B_2 - p^2}$$

$$\tau(\epsilon) = \frac{1}{2\pi^2}\left(\sqrt{-B_2} - \sqrt{-\epsilon}\right)^{-1} = \frac{1}{2\pi^2}\left(\frac{1}{a} - \sqrt{-\epsilon}\right)^{-1}$$

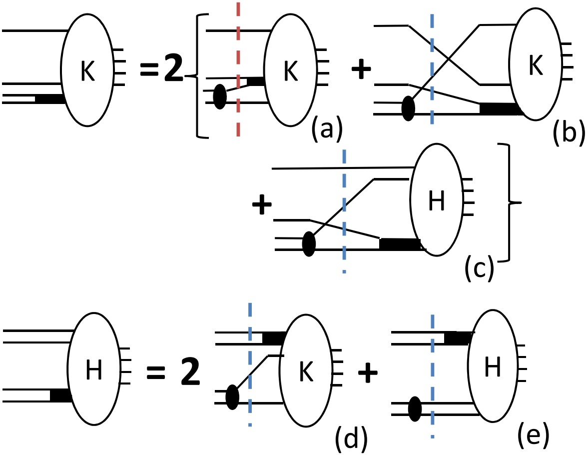
Amorim et al., PRC46 (1992)

Coupled Yakubovsky integral equations

$$\begin{aligned}
\mathcal{K}(u_2,u_3) &= 4\pi\,\tau(\epsilon)\int du'_2 u'^2_2 \int dx \\
&\times\Bigg[G_0^{(3)}\Big(\Pi(u'_2,u_2),u'_2,u_3\Big)\,\mathcal{K}\Big(u'_2,u_3\Big) \\
&+\frac{1}{2}\int dx'\,G_0^{(4)}\Big(\Pi(u'_2,u_2),\Pi_2(u'_2,u_3,x'),\Pi_3(u'_2,u_3,x')\Big)\,\mathcal{K}\Big(\Pi_2(u'_2,u_3,x'),\Pi_3(u'_2,u_3,x')\Big) \\
&+\frac{1}{2}\int dx'\,G_0^{(4)}\Big(\Pi(u'_2,u_2),\Pi_4(u'_2,u_3,x'),\Pi_5(u'_2,u_3,x')\Big)\,\mathcal{H}\Big(\Pi_4(u'_2,u_3,x'),\Pi_5(u'_2,u_3,x')\Big)\Bigg]
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}(v_2,v_3) &= 4\pi\,\tau(\epsilon^*)\int dv'_3 v'^2_3 \\
&\times\Bigg[\int dx\,G_0^{(4)}\Big(v_3,\Pi_6(v_2,v'_3,x),\Pi_7(v_2,v'_3,x)\Big)\,\mathcal{K}\Big(\Pi_6(v_2,v'_3,x),\Pi_7(v_2,v'_3,x)\Big) \\
&+G_0^{(4)}\Big(v_3,v_2,v'_3\Big)\,\mathcal{H}\Big(v_2,v'_3\Big)\Bigg]
\end{aligned}$$

Hadizadeh et al., PRA85 (2012)



Subtracted Green's Functions: $G_0^{(N)} = \frac{1}{E - H_0} - \frac{1}{-\mu_N^2 - H_0}$

with μ_3 (RED): 3B scale & μ_4 (BLUE): 4B scale

Outline

Numerical solution algorithm

$$
\begin{aligned}
|K\rangle &= G_0 t_{12} P\Big[\Big(1 + P_{34}\Big)|K\rangle + |H\rangle\Big] \\
|H\rangle &= G_0 t_{12} \tilde{P}\Big[\Big(1 + P_{34}\Big)|K\rangle + |H\rangle\Big]
\end{aligned}
$$

standard eigenvalue problem

$$
\lambda(E).\psi = K(E).\psi; \quad \psi = \begin{pmatrix} K \\ H \end{pmatrix}
$$

Searching E to get the solution of coupled Yakubovsky integral equations with $\lambda = 1$.

Integration: Gaussian quadrature

- Jacobi momenta: 80-160 mesh points; tricky mapping
- polar angles: 40 mesh points

Dimension of Kernel after discretization:
$120 \times 120 \times 120 \times 40 \times 40 \sim 10^9$

Eigenvalue problem: Lanczos type method

- Iterative orthogonal vectors (IOV) (Stadler PRC44 2319)
- ARPACK Fortran library (http://www.caam.rice.edu/software/ARPACK/)

Dimension of eigenvalue problem after using Lanczos technique:
# of iterations-1 $\sim$ 10!

Multidimensional interpolations: Cubic-Hermit Splines

- high computational speed and accuracy (Huber FBS22 107)

Tetramer ground and excited state binding energies for $B_2 = 0$

Within a renormalized zero-range model, where the relevant three- and four-body momentum scales (μ_3 and μ_4) are introduced in a subtractive regularization procedure, we have the following results:

μ_4/μ_3	$B_4^{(0)}/B_3$	$B_4^{(1)}/B_3$	$B_4^{(2)}/B_3$	$B_4^{(3)}/B_3$
1	3.10			
1.6	**4.70**	1.00071		
5	12.5	1.531		
10	24.6	2.44		
21	63.5	**4.62**	1.00032	
40	184	8.65	1.203	
$\Rightarrow$ 70	5.20×10^2	13.9	1.629	
100	1.04×10^3	21.5	2.17	
200	4.06×10^3	51.8	3.86	
240			≈ 4.6	≈ 1
300	9.11×10^3	103	5.53	
400	1.62×10^4	154	7.28	

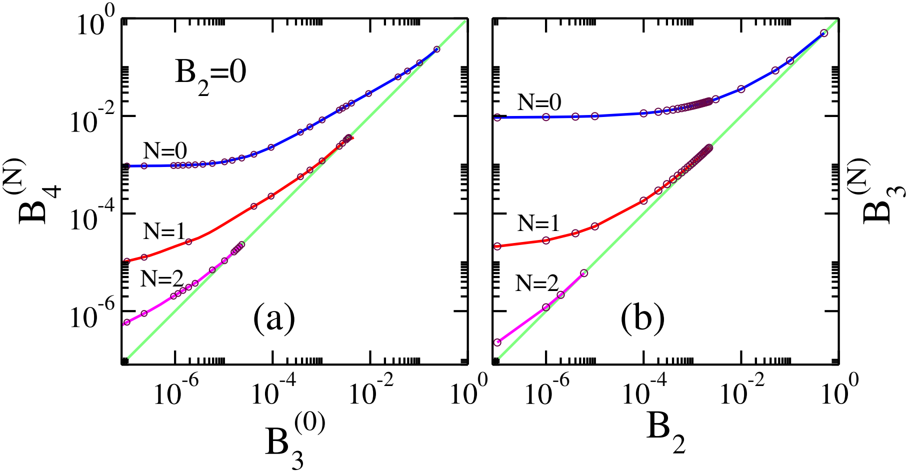
Tetramer ground and excited state binding energies for $B_2 = 0.02\ B_3$

μ_4/μ_3	$B_4^{(0)}/B_3$	$B_4^{(1)}/B_3 - 1$
1	2.66	
1.76	4.24	9.8×10^{-4}
5	10.0	0.421
20	45.9	2.77
40	139	6.10
80	506	13.0
200	2.86×10^{3}	39.5
300	6.00×10^{3}	69.3
400	9.81×10^{3}	104

Tetramer ground and excited state binding energies for $B_2^{virtual} = 0.02\ B_3$

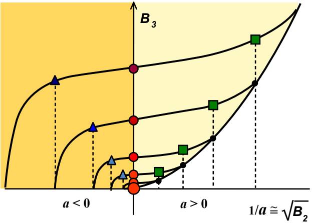
μ_4/μ_3	$B_4^{(0)}/B_3$	$B_4^{(1)}/B_3 - 1$
1	3.62	
1.7	5.91	0.014
5	15.4	0.658
20	74.8	4.18
40	236	9.46
80	873	20.6
200	5.02×10^3	64.5
300	1.06×10^4	115
400	1.73×10^4	174

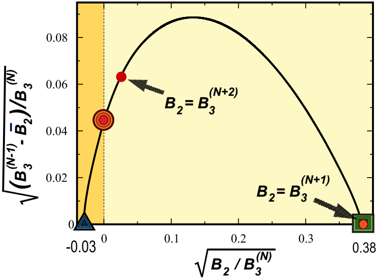
Three- and Four-body Binding Energies



3B Scaling Plot

Yamashita et al., PRA66 (2002)

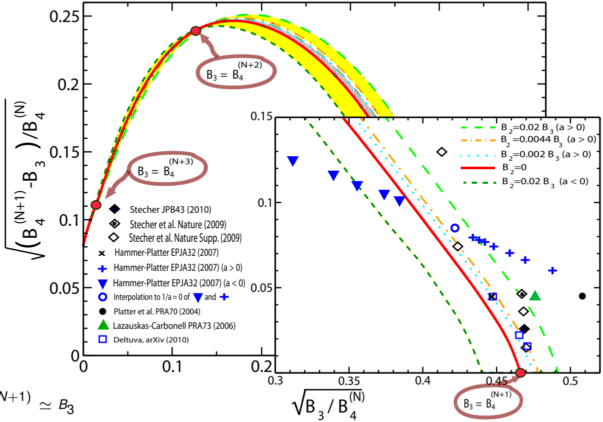




- $\frac{B_3^{(N)}}{B_2} \simeq 6.61 \Rightarrow B_3^{(N+1)} \simeq B_2$
- $\frac{B_3^{(N)}}{B_2} \approx 1385 \Rightarrow \frac{B_3^{(N+1)}}{B_2} \simeq 6.61 \Rightarrow B_3^{(N+2)} \simeq B_2$
- $B_2 = 0 \Rightarrow \frac{B_3^{(N)}}{B_3^{(N+1)}} = 515$
- $\frac{B_3^{(N)}}{B_2^{virtual}} \approx 1084 \Rightarrow B_3^{(N+1)} = 0 \quad \left[a_-^{(N)}/a_-^{(N+1)} = 22.7\right]$

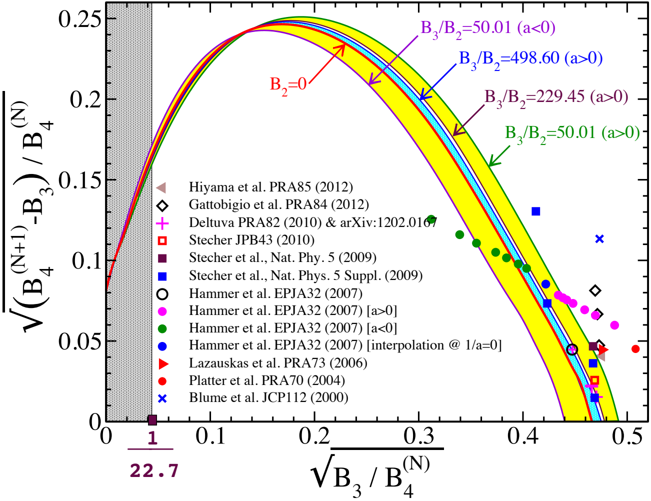
4B Scaling Plot

Hadizadeh et al., PRL 107, 135304 (2011)



- $\frac{B_4^{(N)}}{B_3} \simeq 4.60 \Rightarrow B_4^{(N+1)} \simeq B_3$
- $\frac{B_4^{(N)}}{B_4^{(N+1)}} \simeq 13.8 \Rightarrow \frac{B_4^{(N+1)}}{B_3} \simeq 4.60 \Rightarrow B_4^{(N+2)} \simeq B_3$
- $\frac{B_4^{(N)}}{B_4^{(N+1)}} \approx 79.89 \Rightarrow \frac{B_4^{(N+1)}}{B_4^{(N+2)}} \approx 13.88 \Rightarrow \frac{B_4^{(N+2)}}{B_3} \simeq 4.60 \Rightarrow B_4^{(N+3)} \simeq B_3$
- For ground trimer: $\frac{B_3}{B_4^{(N)}} \to 0$ (with $B_2 = 0$): an infinite tetramer levels with $\sqrt{B_4^{(N+1)}} \approx 0.085\sqrt{B_4^{(N)}}$.

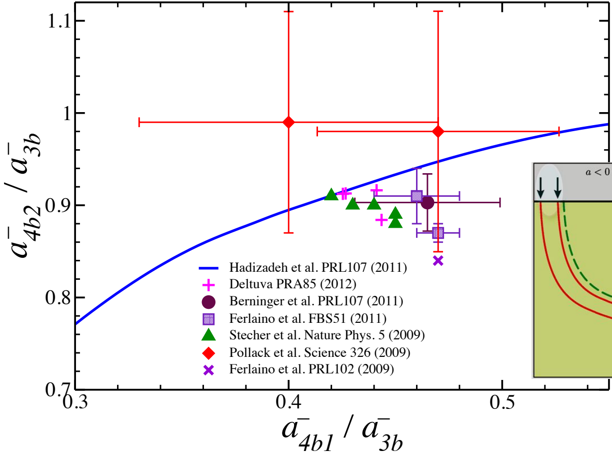
4B Scaling Plot

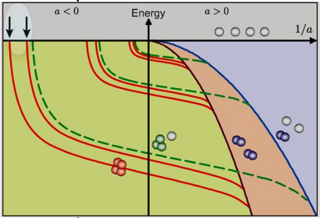


For $\sqrt{\frac{B_3}{B_4^{(N)}}} > \frac{1}{22.7} = 0.044$, at most three tetramers fit between two consecutive Efimov trimers.

4-atom recombination losses

Locations of four-atom loss features ($a^-_{4b1}, a^-_{4b2} < 0$) where two successive tetramers become unbound (blue-solid line for $r_0 \to 0$).





Outline

Our goal:

- Derive range corrections to the tetramer scaling functions calculated with the zero-range model
- Compare with potential models results
- Extract the effective range from cold atom experiments for the four-body recombination data close to a Feshbach resonance.

LO range correction in the two-body scattering amplitude:

- Two-body scattering amplitude:

$$\tau(\epsilon) = -\frac{1/(2\pi^2)}{k\cot(\delta_0) - \mathrm{i}k}$$

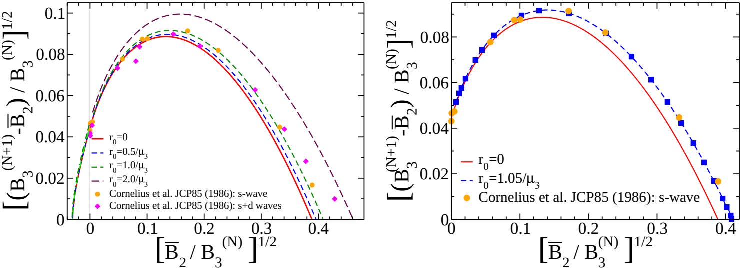
- effective range expansion @ low energies:
$k\cot(\delta_0) = -\frac{1}{a} + \frac{r_0}{2}k^2 + \mathcal{O}(r_0^3 k^4)$
- pole @ the bound (+) or virtual states (-): $\frac{1}{a} = \pm\sqrt{|\epsilon_2|} - \frac{r_0}{2}|\epsilon_2|$
- Up to first order in r_0,

$$\tau(\epsilon) \approx \frac{(2\pi^2)^{-1}}{-\sqrt{-\epsilon} \pm \sqrt{-\epsilon_2}}\left[1 + \frac{r_0}{2}\left(\sqrt{-\epsilon} \pm \sqrt{-\epsilon_2}\right)\right] .$$

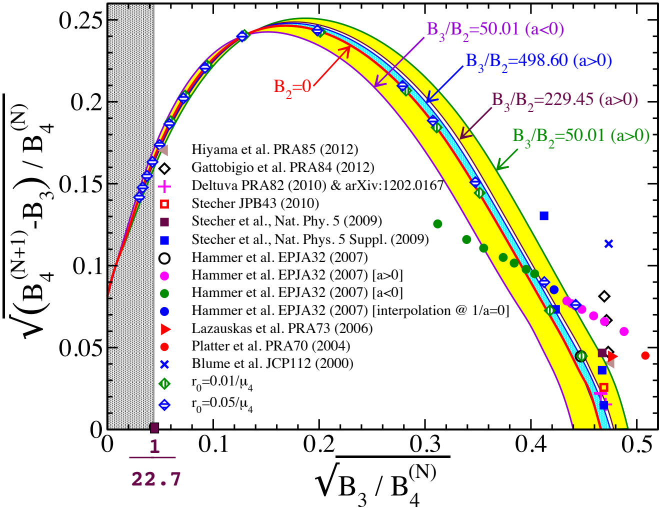
- The subtracted form of the STM and FY are able to consider the higher momentum power in the effective range expansion.

Range Correction to 3B Scaling Plot

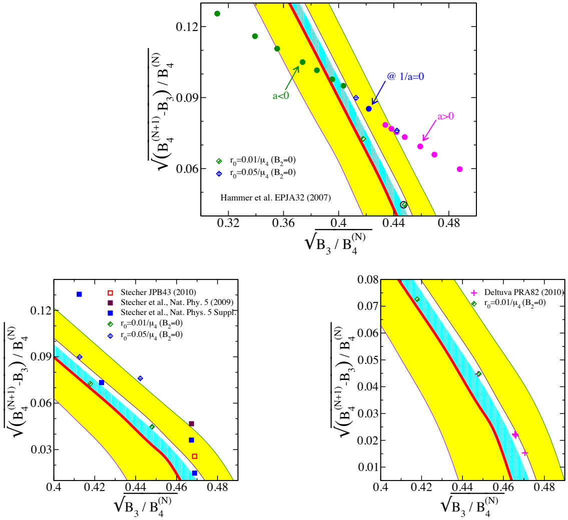
- Cornelius and Glöckle, JCP**85** (1986) - realistic HFDHE2 interatomic potential of Aziz et al.



Range correction to 4B Scaling Plot



Comparison to other calculations



Even at unitary limit, a very small value of range parameter can shift the scaling plot to right.

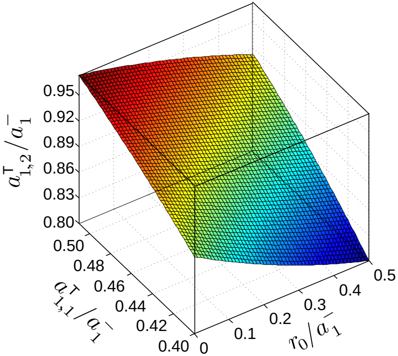
Range correction to the scaling function for the positions of successive 4-atom resonances:

$$\frac{a^T_{1,2}}{a^-_1} = \mathcal{A}(x,y) = \sum_{0\le m+n\le 2} c_{mn}\,(x-0.45)^m y^n;\quad m,n\ge 0$$

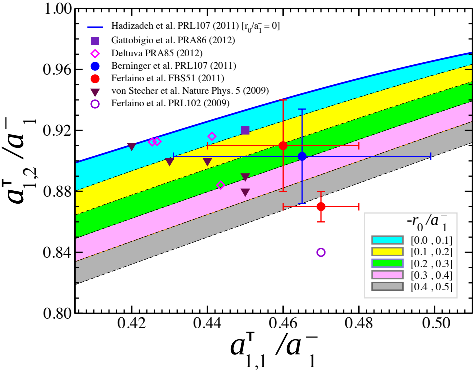
where $x \equiv a^T_{1,1}/a^-_1$, $y \equiv r_0/|a^-_1|$.

Coefficients of the parametrization

c_{00}	c_{10}	c_{01}
0.932	0.724	-0.144
c_{11}	c_{20}	c_{02}
0.347	-0.645	0.001



Range correction to the position of 4-atom resonance;



Hadizadeh et al PRA 87, 013620 (2013)

- von Stecher (priv. comm.) 0.38 vs. ∼0.36
- Deltuva (priv. comm.) 0.33 vs. ∼0.29

r_0 from the shift of the peaks of the four-atom losses

Ref.	$a^T_{1,1}/a^-_1$	$a^T_{1,2}/a^-_1$	a^-_1 [$R_{\rm vdW}$]	r_0 [$R_{\rm vdW}$]
Ferlaino et al PRL'09	0.47	0.84	-8.7(1)	> 5
Berniger et al PRL'11	0.465(34)	0.903(31)	-9.54(28)	2.5 ± 1.7
Ferlaino et al FBS'11	0.47(1)	0.87(1)	-8.71	4.8 ± 1.0
Ferlaino et al FBS'11	0.46(2)	0.91(3)	-9.64	2 ± 2

- $R^{Cs_2}_{\rm vdW} = 101.0\, a_0$ [Chin et al RMP82(2010)]
- $\bar{a}^{Cs_2} \simeq 0.955978\, R^{Cs_2}_{\rm vdW} = 96.5\, a_0$.
- $3.5 < r_0 < 4.3\, R_{\rm vdW}$
- Weighted average for the fitted r_0 values: $3.9 \pm 0.8\, R_{\rm vdW}$

$$r_0 \simeq 2.9179\, \bar{a} \left[\left(\frac{\bar{a}}{a} \right)^2 + \left(\frac{\bar{a}}{a} - 1 \right)^2 \right]$$

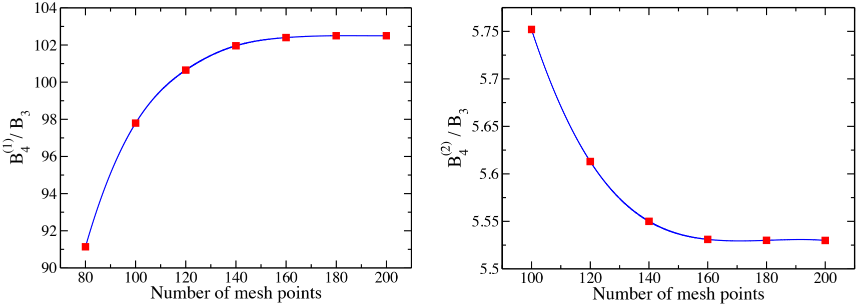
Outline

Example

Convergence of 1st and 2nd exited tetramer energies for $\frac{\mu_4}{\mu_3} = 300$:

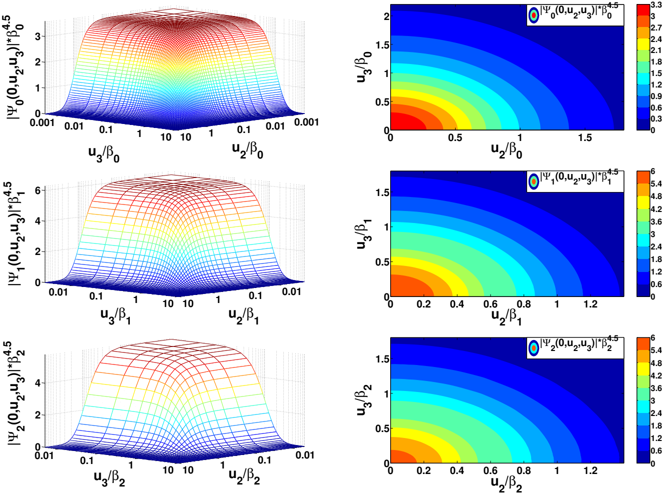
$$u_i = \frac{1 + x_i}{c_1(1 - x_i) + c_2 x_i}; \quad c_1 \equiv \frac{\mu_4}{\mu_3}, c_2 = 0.4$$

$$x_i \in [-1, +1] \quad \Longrightarrow \quad u_i \in [0, 0.003] + [0.003, 5]$$



Example

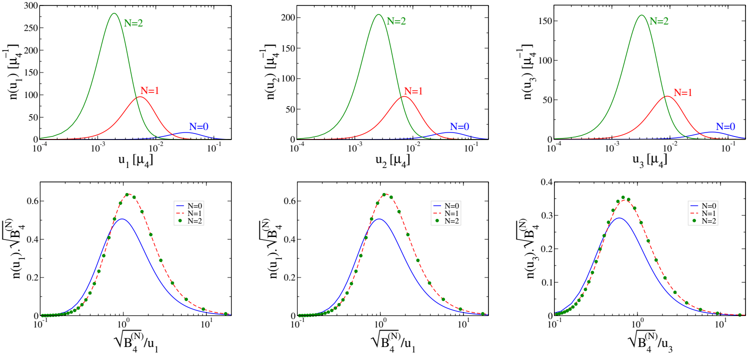
$\Psi(0, u_2, u_3)$ for $\frac{\mu_4}{\mu_3} = 50$: where $\beta_N = \sqrt{E_4^{(N)}}$, $N = 0, 1, 2$



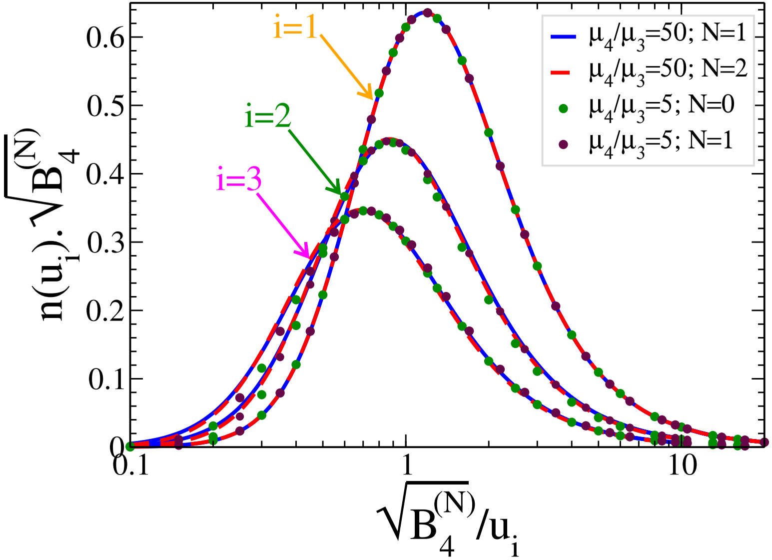
$$n(u_i) = u_i^2 \int du_j\, u_j^2 \int du_k\, u_k^2\, \Psi^2(u_i, u_j, u_k); \quad (i,j,k) \equiv (1,2,3)$$

Example

$n(u_1), n(u_2)$ and $n(u_3)$ for $\frac{\mu_4}{\mu_3} = 50$:



Universal momentum distribution functions at unitary ($B_2 = 0$)



Outline

Conclusion & Summary

- In our study on a general four-boson problem, within a renormalized zero-range model, we verify that it is not enough only two parameters (which determines trimer properties) to describe the the four-boson system.
- Four-body scale moving two successive tetramers below a given trimer
- Model independence of the limit cycle - comparison with other models
- No more than 3 tetramers between successive Efimov trimers for $a \to \pm\infty$
- range correction $r_0 > 0$: widens the 3B and 4B scaling functions
- range correction shifts the position of the four-atom losses
- Shift of the four-atom loss peaks found in a trapped cold Cesium gas close to a FB resonance can be associated with range effects
- Interwoven trimer-tetramer states?
- Scattering atom-trimer, dimer-dimer ...?
- More bosons?

Six-body problem in Yakubovsky scheme

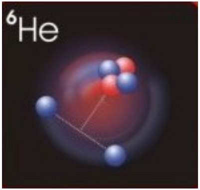
Few-Body Systems
July 2011, Volume 51, Issue 1, pp 27-44

The Six-Nucleon Yakubovsky Equations for ^{6}He

W. Glöckle, H. Witała

- 5 coupled Yakubovsky integral equations for 6 identical particles
- 5 Jacobi momenta for each Yakubovsky component

First attempt for solution of 6B Yakubovsky equations: two-neutron Halo nucleus ^{6}He



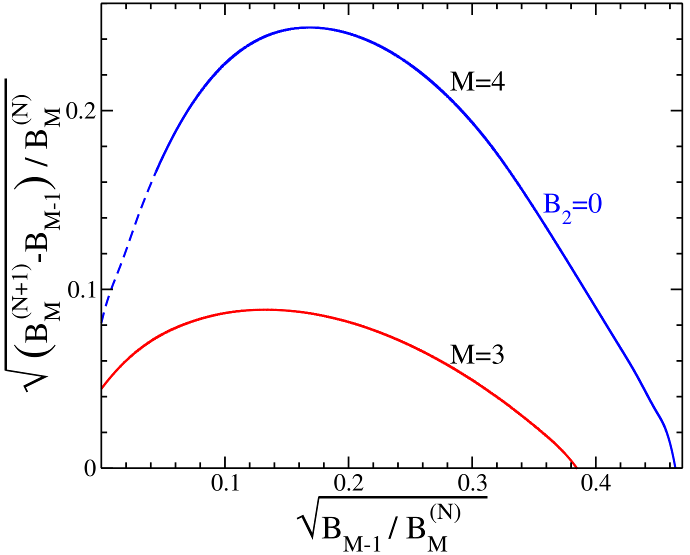
- Two coupled equations for the halo structure of the two loosely bound neutrons with respect to the core nucleons
- separable $s-$wave nucleon-nucleon interaction

Work in progress in collaboration with E. Ahmadi Pouya, M. Harzchi and S. Bayegan (Tehran few-body group)

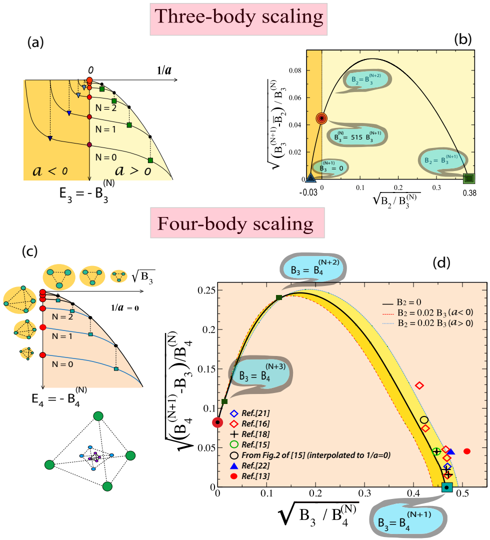
Thank you :)[1]

[1] Work supported by the Brazilian agencies FAPESP and CNPq

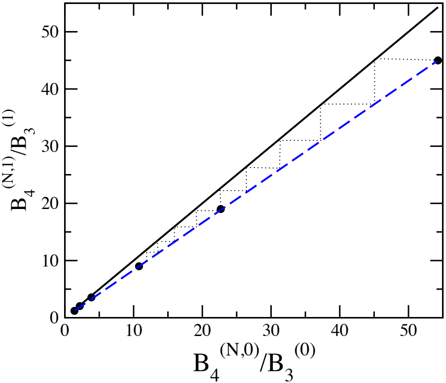
Three- and Four-body scaling plots



4B Limit Cycle: Scaling Plot



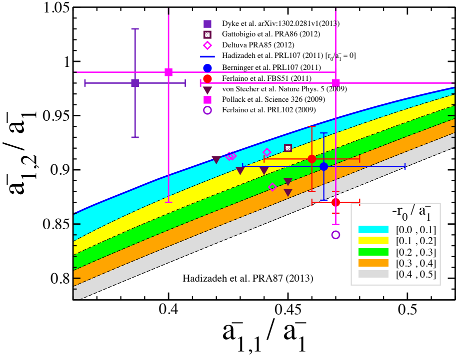
3B and 4B Interwoven Cycles



The fate of tetramers: cross the trimers thresholds...

New analysis of the peaks of the four-atom losses for ^{7}Li

Dike, Pollack, Hulet arxiv:1302.0281; Pollack, Dries, Hulet, Science (2009)



- ^{7}Li resonance is not open-channel dominated;
- Effective range expansion in lowest order fails for coupled-channels
- Relative change of the three- and four-body short-range scales (induced few-body forces from the coupled-channel potential).