The few-boson problem near unitarity: recent theory and experiments

Chris Greene, Purdue University

Revisiting the 3-body parameter for van der Waals two-body interactions

The N>3 boson problem near unitarity

•Thanks, NSF, AFOSR, DOE, BSF!

DAMOP thesis prize co-recipient in 2009

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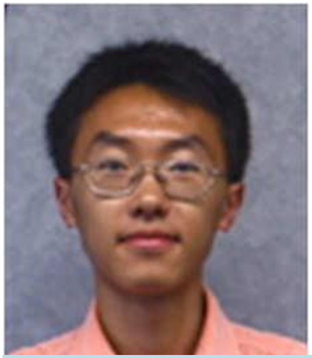
Collaborators on the ultra-cold few-body projects



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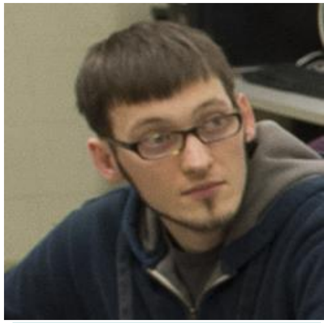


Kevin Daily
Research associate

Strategy of the adiabatic hyperspherical representation: FOR ANY NUMBER OF PARTICLES, convert the partial differential Schroedinger equation into an infinite set of coupled ordinary differential equations:

To solve:

$$\left[-\frac{1}{2\mu}\frac{\partial^2}{\partial R^2}+\frac{\Lambda^2}{2\mu R^2}+V(R,\theta,\varphi)\right]\psi_E=E\psi_E$$

First solve the fixed-R Schroedinger equation, for eigenvalues $U_n(R)$:

$$\left[\frac{\Lambda^2}{2\mu R^2}+\frac{15}{8\mu R^2}+V(R,\theta,\varphi)\right]\Phi_\nu(R;\Omega)=U_\nu(R)\Phi_\nu(R;\Omega)$$

Next expand the desired solution into the complete set of eigenfunctions with unknowns F(R)

$$\psi_E(R,\Omega)=\sum_\nu F_{\nu E}(R)\Phi_\nu(R;\Omega)$$

And the original T.I.S.Eqn. is transformed into the following set which can be truncated on physical grounds, with the eigenvalues interpretable as adiabatic potential curves, in the Born-Oppenheimer sense.

$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2}+U_\nu(R)\right]F_{\nu E}(R)-\frac{1}{2\mu}\sum_{\nu'}\left[2P_{\nu\nu'}(R)\frac{d}{dR}+Q_{\nu\nu'}(R)\right]F_{\nu' E}(R)=EF_{\nu E}(R)$$

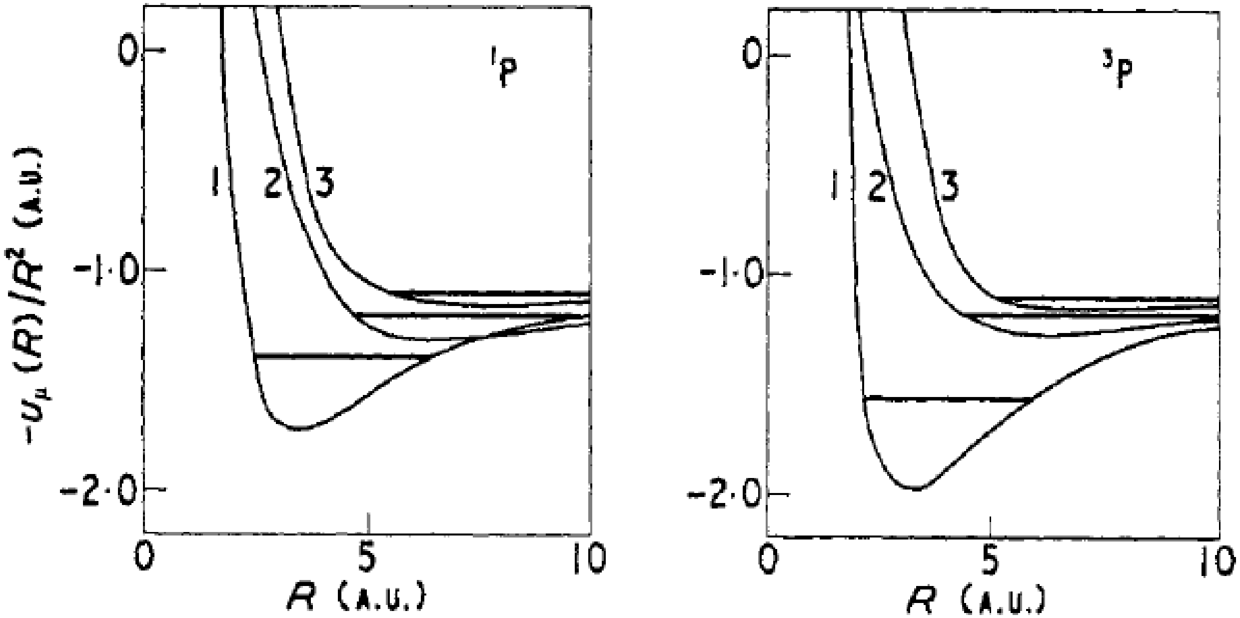
Typically, to solve this PDE, one expands in some basis set and diagonalizes:

$$\left[\frac{\Lambda^2}{2\mu R^2}+\frac{15}{8\mu R^2}+V(R,\theta,\varphi)\right]\Phi_\nu(R;\Omega)=U_\nu(R)\Phi_\nu(R;\Omega)$$

- **For three particles, we usually use a B-spline basis to directly solve the coupled PDEs in the two hyperangles → essentially exact**
- **For N>3 particles, the most efficient method we have found is the correlated Gaussian basis set, implemented for hyperspherical studies by Javier von Stecher, later extended by Doerte Blume**
- **Another method that works well for N>3 particles, especially at small or modest values of the hyperradius R, is the hyperspherical harmonic expansion, especially if augmented by a few basis functions designed to handle the two-body aymptotic channels**

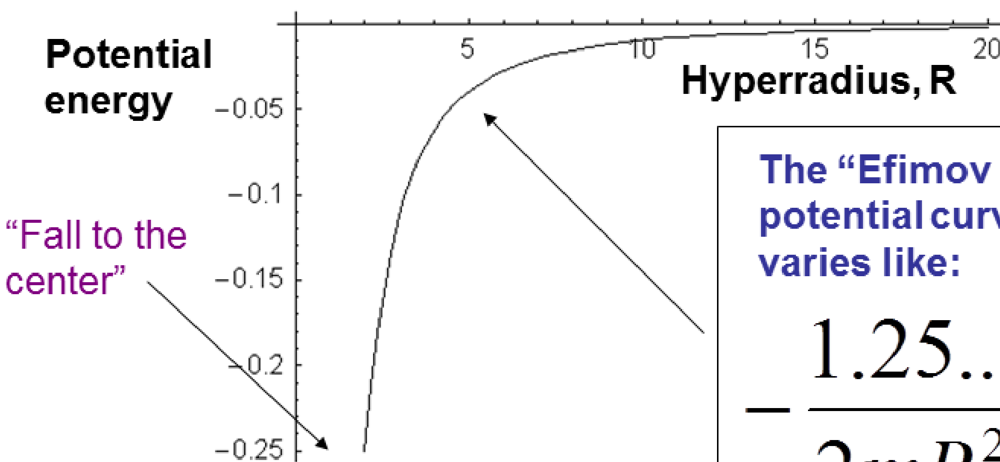
Examples

- **Macek, J. Phys. B 1, 831 (1968) ← first idea of adiabatic hyperspherical potential curves, for He two-electron excited states**



Effective potential energy versus hyperradius R for two 3-body systems

- **The Efimov effect for three particles with short range interactions and infinite scattering length. Efimov's → original paper can be viewed as an example of Macek's adiabatic theory in a limit where it becomes exact.**

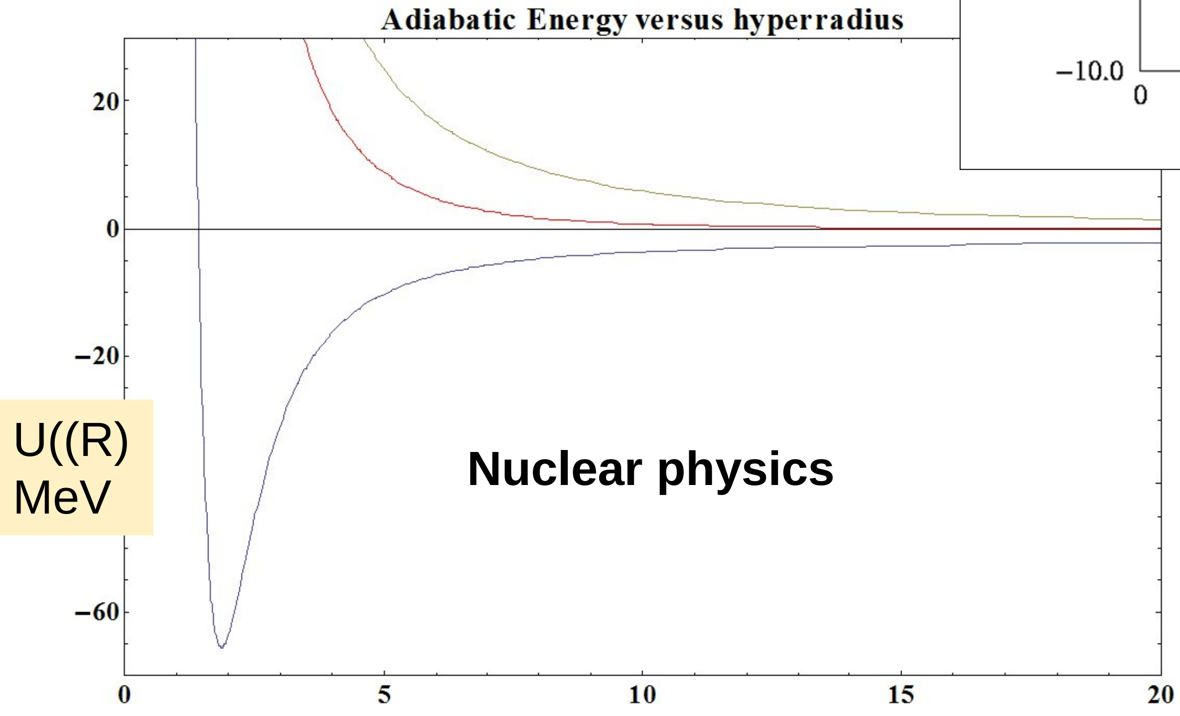


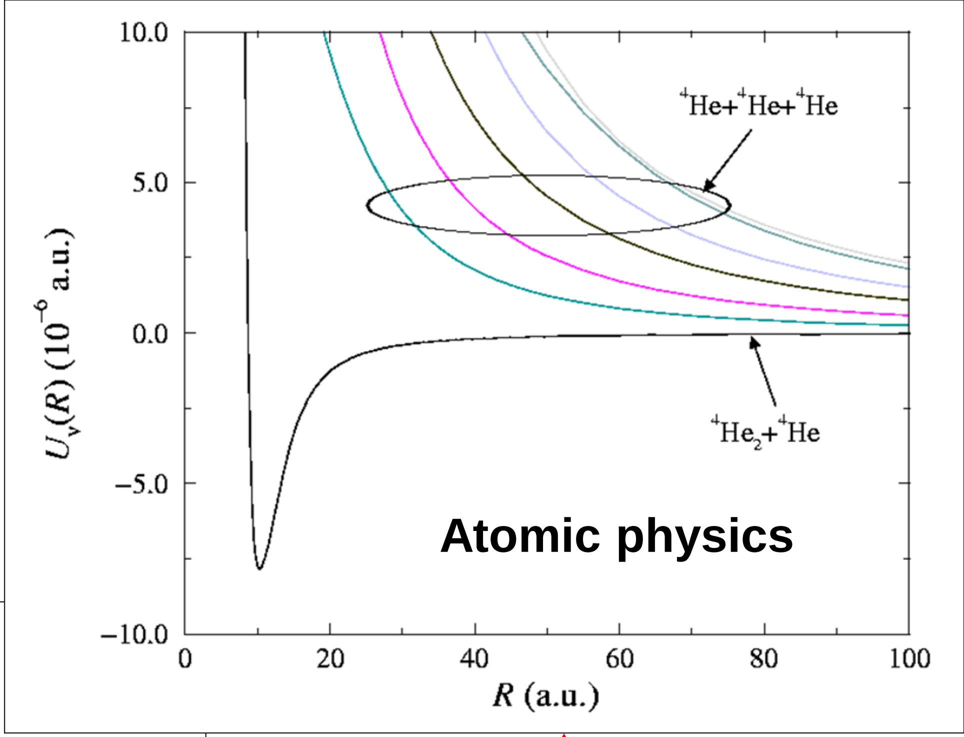
The "Efimov potential curve" varies like:

$$-\frac{1.25\ldots}{2mR^2}$$

Universality, from nuclear scale energies to the chemical

Preliminary results, adiabatic potential curves for n+n+p, in collaboration with Alejandro Kievsky and Kevin Daily, nuclear physics on 10^6 eV scale





3-atom hyperspherical potential curves for He+He+He on a 10^{-3} eV scale, looks very similar to the 3-nucleon potentials

Another example, a system of 2 positrons and 3 electrons, hyperspherical potential curves showing multiple fragmentation pathways.

Kevin Daily and CHG, 2014 Phys. Rev. A 89, 012503 (correlated gaussians)

e+ e+ e- e- e-
A 5-body problem

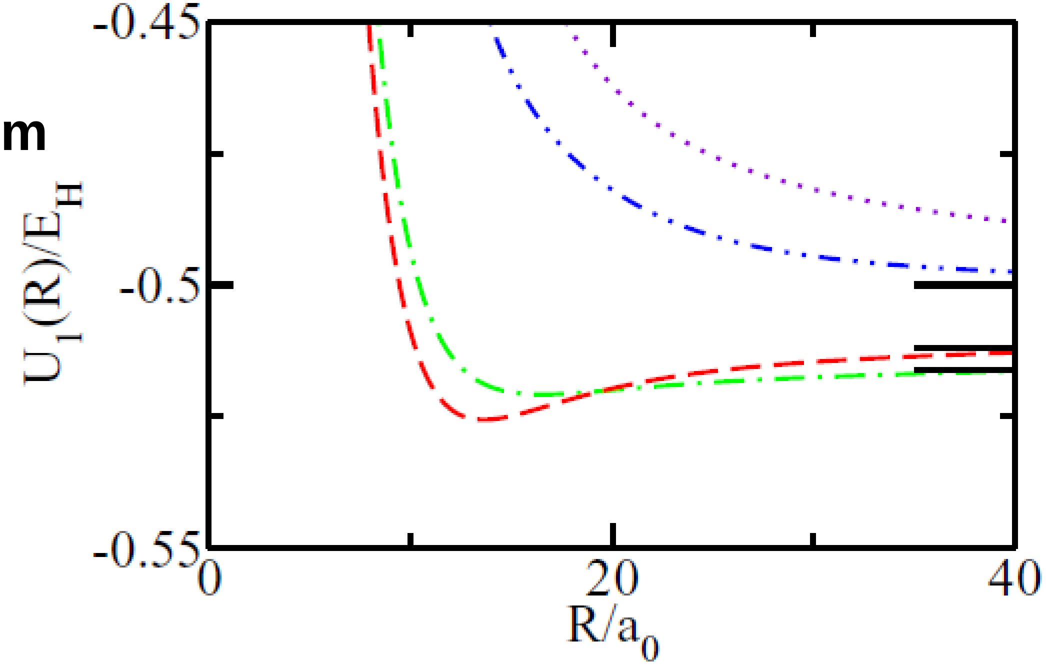


FIG. 4. (Color online) The lowest adiabatic potential curves with $L^\pi = 0^+$ symmetry for the $(+)_2(-)_3$ system in three dimensions. Dashed, dash-dotted, dash-dot-dotted, and dotted lines are for $(S_+, S_-) = (1, 1/2), (0, 1/2), (1, 3/2)$, and $(0, 3/2)$, respectively. From top to bottom, the solid lines indicate the asymptotic limits of break-up into $2\text{Ps} + e^-$, $\text{Ps} + \text{Ps}^-$, and $\text{Ps}_2 + e^-$, respectively.

Vitaly Efimov, 1970 - *A three-body system, whose dimers each have infinite scattering lengths and no bound states, must have an infinite number of trimer bound states.*

$E_{n+1} = E_n e^{-2\pi/s_0}$, where $s_0 = 1.00624\ldots$ is a universal constant.

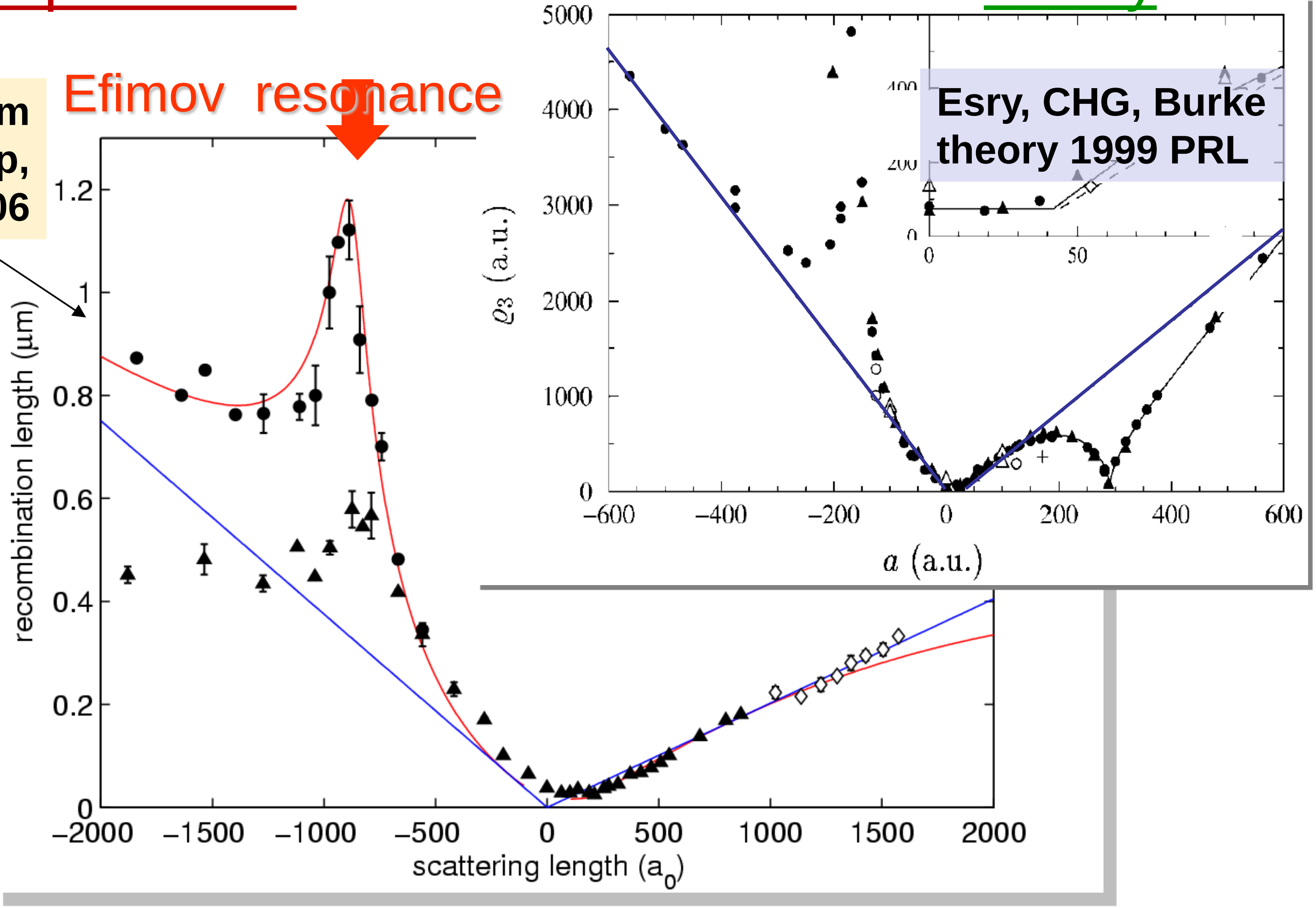
Three-body recombination ``length“ versus *a*

2006 exp. results

theory

Grimm group, Nature 2006

Efimov resonance

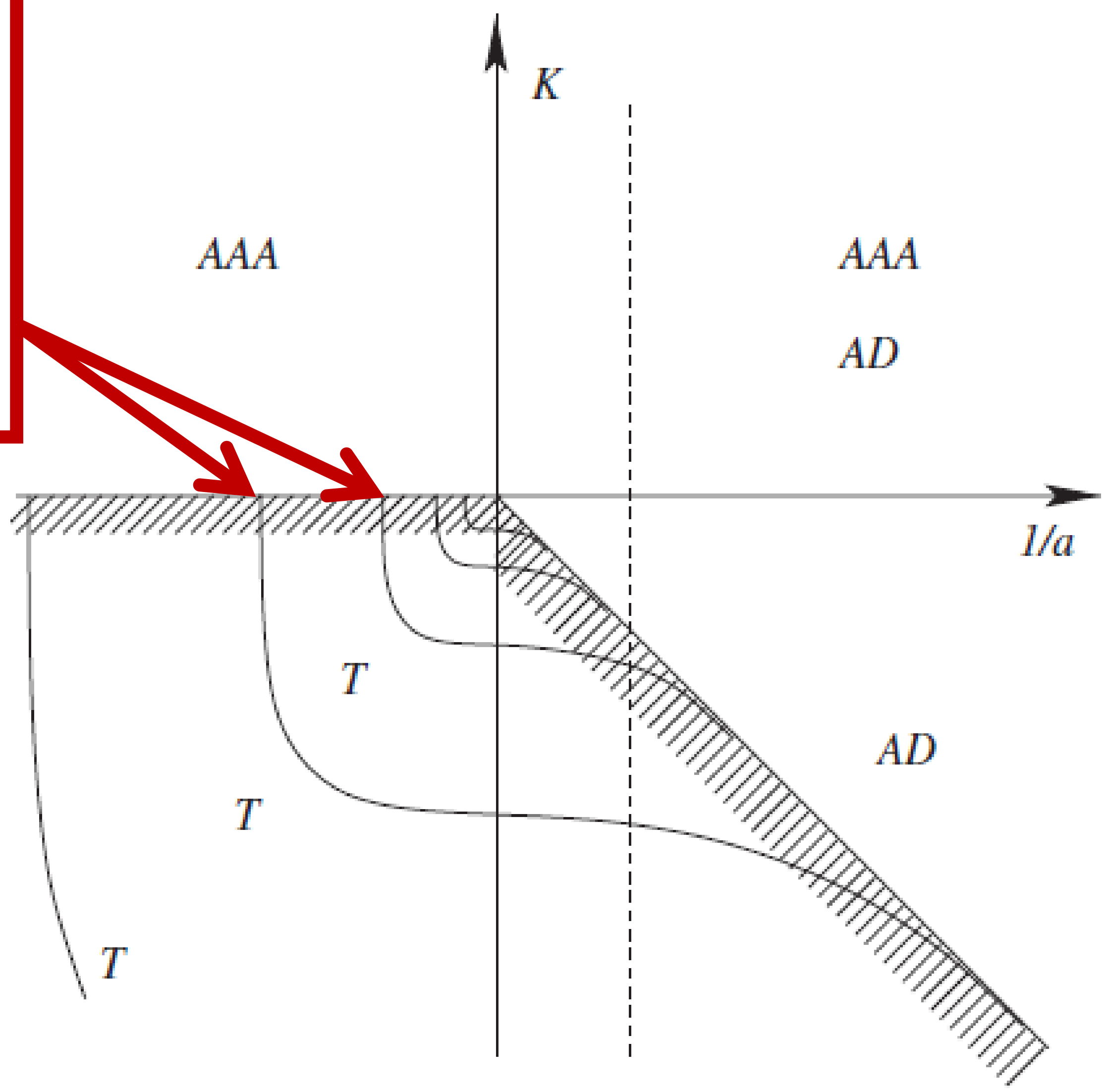


Esry, CHG, Burke theory 1999 PRL

Infinite ladder of 3-body energy levels (sqrt) versus 1/a

E. Braaten, H.-W. Hammer / Physics Reports 428 (2006) 259–390

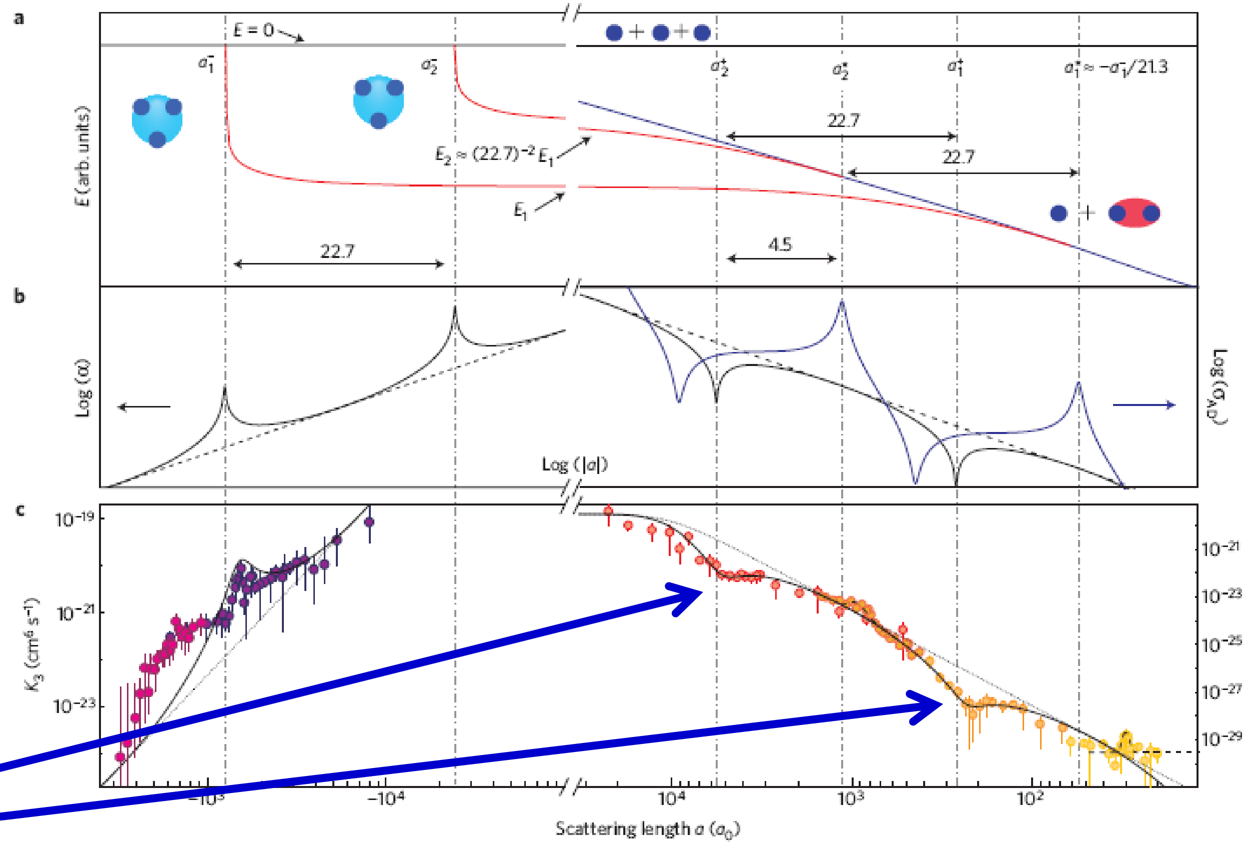
Efimov resonances observable in 3-body recombination in an ultracold gas



Observation of an Efimov spectrum in an atomic system

NATURE PHYSICS | VOL 5 | AUGUST 2009 586

M. Zaccanti[1]*, B. Deissler[1], C. D'Errico[1], M. Fattori[1,2], M. Jona-Lasinio[1], S. Müller[3], G. Roati[1], M. Inguscio[1] and G. Modugno[1]

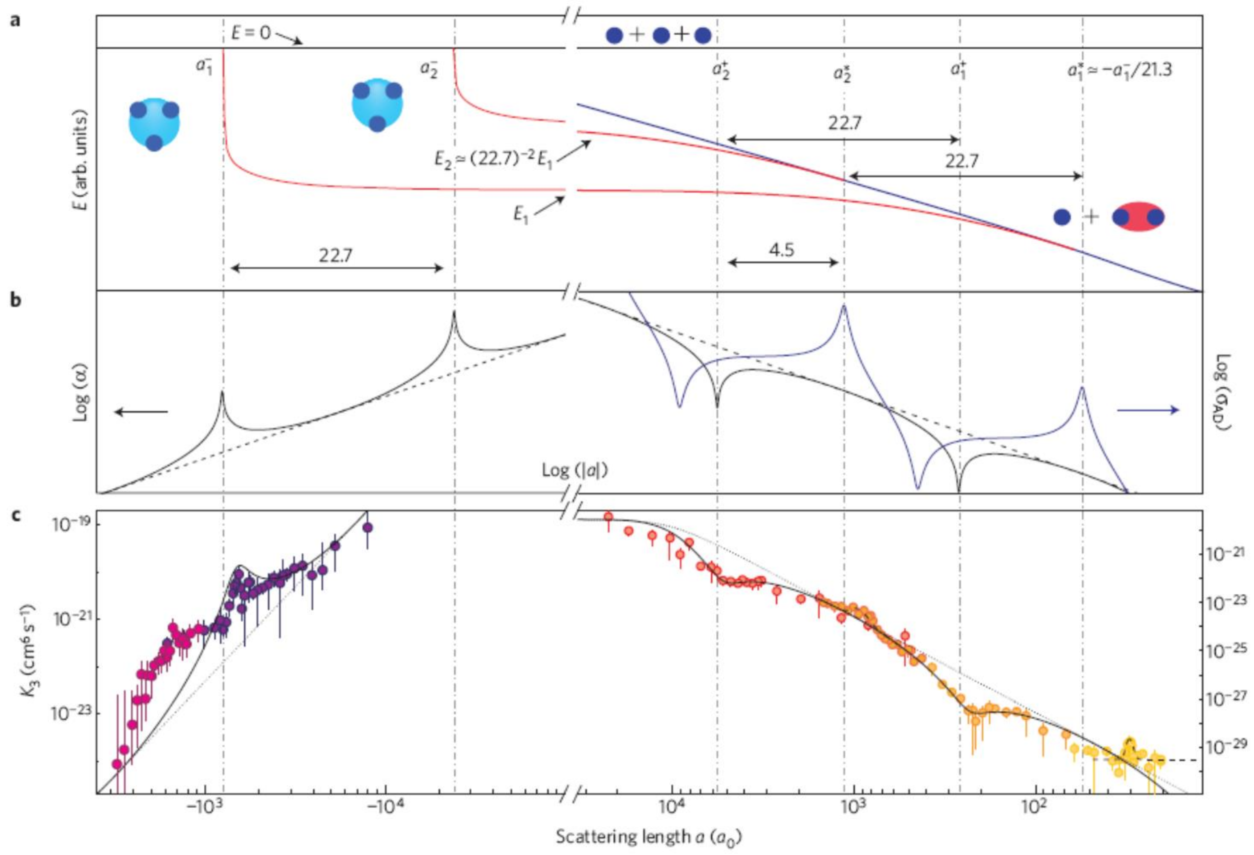


Experimental confirmation of predicted minima in the 3-body loss rate

Figure 1 | Efimov spectrum. a Theoretical binding energy of two consecutive Efimov states (red) and of the dimer state (blue) in the universal regime versus the scattering length *a*. **b**, Theoretical three-body recombination rate α (black) and atom–dimer elastic cross-section σ_{AD} (blue). The vertical dash–dotted lines indicate the position of the detectable maxima and minima in the three-body observables, for which the relevant scaling rules are summarized in **a**. The dashed lines indicate the a^4 behaviour of the three-body recombination rate expected in the absence of Efimov states. **c**, Measured recombination coefficient K_3 in an ultracold potassium gas (circles), featuring deviations from the bare a^4 trend (dotted line), and fitted behaviour assuming a local universal trend for K_3 in the vicinity of the two recombination minima at $a > 0$ and of the Efimov resonance at $a < 0$ (solid line), see text. The other two features due to the atom–dimer resonances a_1^* and a_2^*, not expected by theory, are locally fitted with a Gaussian profile superimposed to a constant background and to the universal behaviour, respectively (dashed lines). The various colours correspond to different data sets. For all data points, the error bars are the root sum squared of the standard error of the mean value resulting from the fit and of the uncertainty on the trap frequencies (see the Methods section).

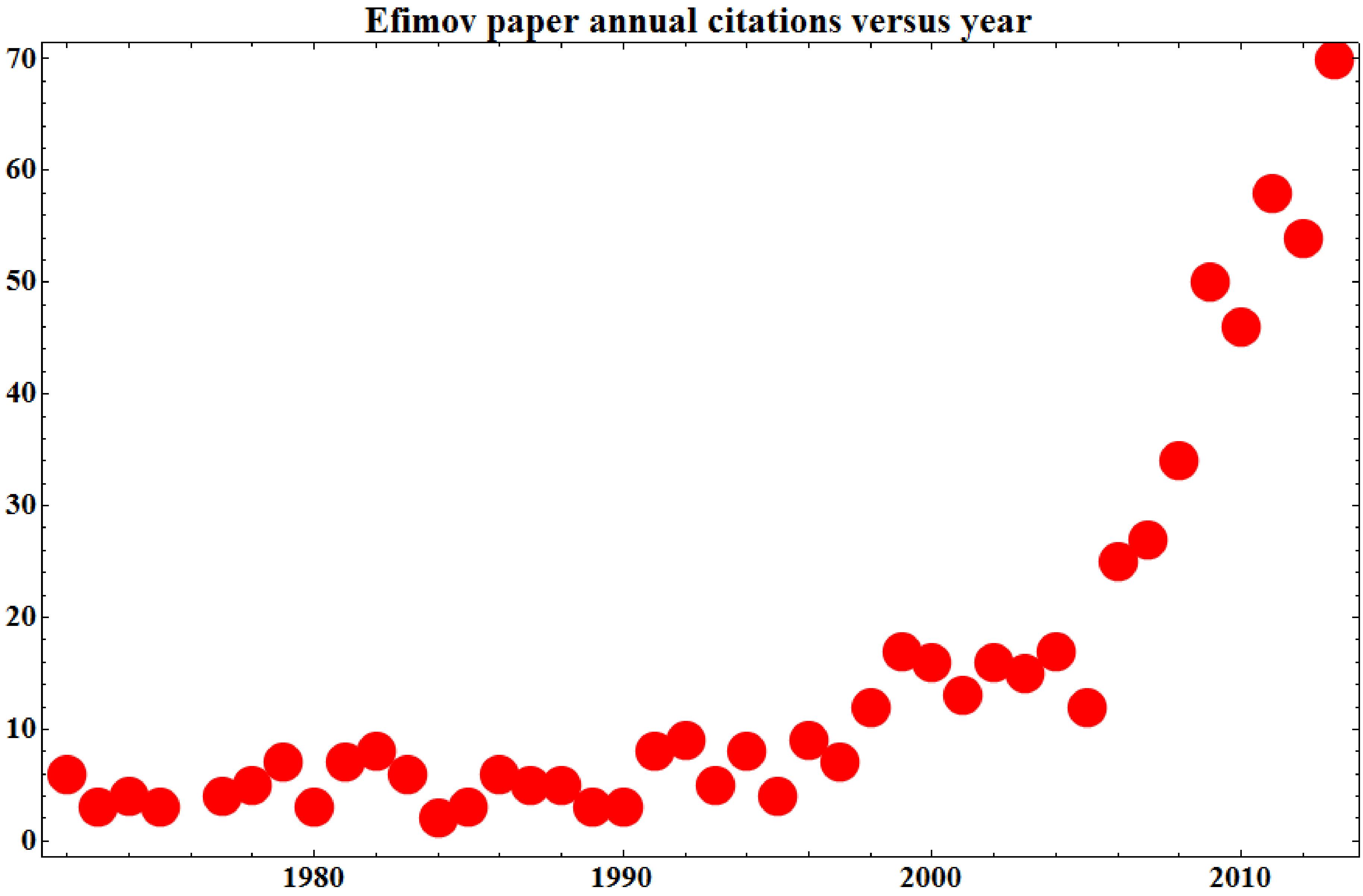
And experiments have confirmed (and sometimes led) a great deal of theory concerning 3-body recombination since the late 1990s, so that we now understand:

- **Efimov resonances occur at a<0 (an infinite number of these)**
- **Destructive interference minima occur at a>0**
- **The $K_3 \sim a^4$ scaling really is there and it makes it difficult to imagine exploring the unitary Bose gas where a→infinity**



Zaccanti et al. Nature Physics 2009 expt confirms the a^4 general scaling and also predicted resonance (a<0) and minima (a>0) features

An obvious conclusion: trying to make a BEC at a→ infinity would be bad news, explosive losses, etc…..



Total citations: 628 in early 2014

There has always been interest in understanding the three-body parameter, i.e. the value of the scattering length where the first Efimov state appears, which also tells one approximately the energy of the lowest Efimov state at unitarity. Brett Esry, Jose D'Incao, and I thought we understood this physics and summarized our state of knowledge in the following paper:

J. Phys. B: At. Mol. Opt. Phys. 42 (2009) 044016

The short-range three-body phase and other issues impacting the observation of Efimov physics in ultracold quantum gases

Finally, we emphasize the importance of the short-range three-body physics in determining the position of Efimov features and show that theoretically reproducing two-body physics is not generally sufficient to predict three-body properties quantitatively.

But in 2011, the "Bombshell PRL" by the Innsbruck group forced us to modify this understanding:

→ M. Berninger, A. Zenesini, B. Huang, W. Harm, H. C. Nägerl, F. Ferlaino, R. Grimm, P. S. Julienne, and J. M. Hutson, Phys. Rev. Lett. **107**, 120401 (2011).

The Innsbruck experiment generated a flurry of activity from theorists, attempting to understand this apparent near-constancy of the 3-body parameter observed experimentally

Origin of the Three-Body Parameter Universality in Efimov Physics

Jia Wang,[1] J. P. D'Incao,[1] B. D. Esry,[2] and Chris H. Greene[1]

PRL **108**, 263001 (2012)

Other relevant theoretical work to interpret this result:

Cheng Chin's toy model (arXiv 2011)

And detailed hyperspherical calculations by Naidon, Endo, & Ueda:

``Physical Origin of the Universal Three-body Parameter in Atomic Efimov Physics" *arXiv*:1208.3912 (largely confirms our interpretation)

See also ➔ R. Schmidt, S. P. Rath and W. Zwerger, Eur. Phys. J. B **85**, 386 (2012).

P. K. Sorensen, D. V. Fedorov, A. S. Jensen, N. T. Zinner, Phys. Rev. A **86**, 052516 (2012).

The "three-body parameter" controlling the first Efimov resonance location had been thought to be more or less "random", but the new experimental evidence strongly suggests that it must be approximately universal:

1) ^{133}Cs (Berninger et al.) PRL 107, 120401 (2011) :
   $|a_-|/L_{vdW}$ = 9.4, 11.1, 10.4, and 10.3

2) ^{7}Li (Hulet) Science 326, 1683 (2009) : $|a_-|/L_{vdW}$ = 10.0

3) ^{7}Li (Khaykovich) PRL 103, 163202 (2009) : $|a_-|/L_{vdW}$ = 8.9

4) ^{7}Li (Khaykovich) PRL 105, 103203 (2010) : $|a_-|/L_{vdW}$ = 9.0

5) ^{39}K (Modungno) Nat. Phys. 5, 586 (2009): $|a_-|/L_{vdW}$ = 11.0

6) ^{85}Rb(Cornell-Jin group at JILA) 2012 PRL: $|a_-|/L_{vdW}$ = 9.7(1)

3-body hyperspherical potential curves based on 2-body Lennard-Jones interaction potential with 10 s-wave bound states, around 100 total, including all angular momentum states

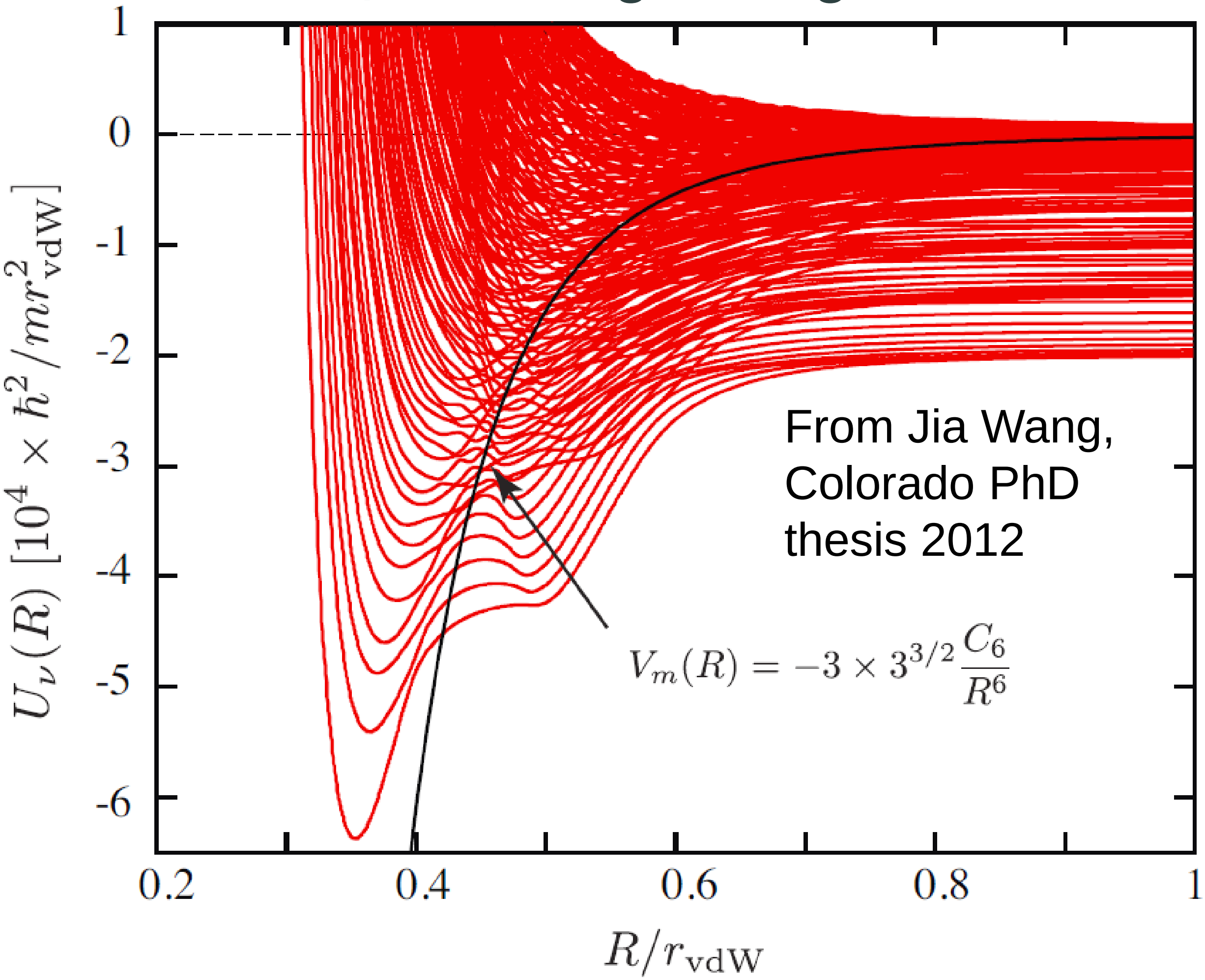


Figure 5.12: This figure shows the three-body potentials obtained using the $v^a_\lambda(\lambda = \lambda^*_{10})$ model supporting a total of 100 bound states. Roughly speaking, the potential of Eq. (5.18) [16] (black solid line) can be seen as a diabatic potential since it passes near one of the series of avoided crossings.

PRL 2012 Jia Wang, D'Incao, Esry, CHG

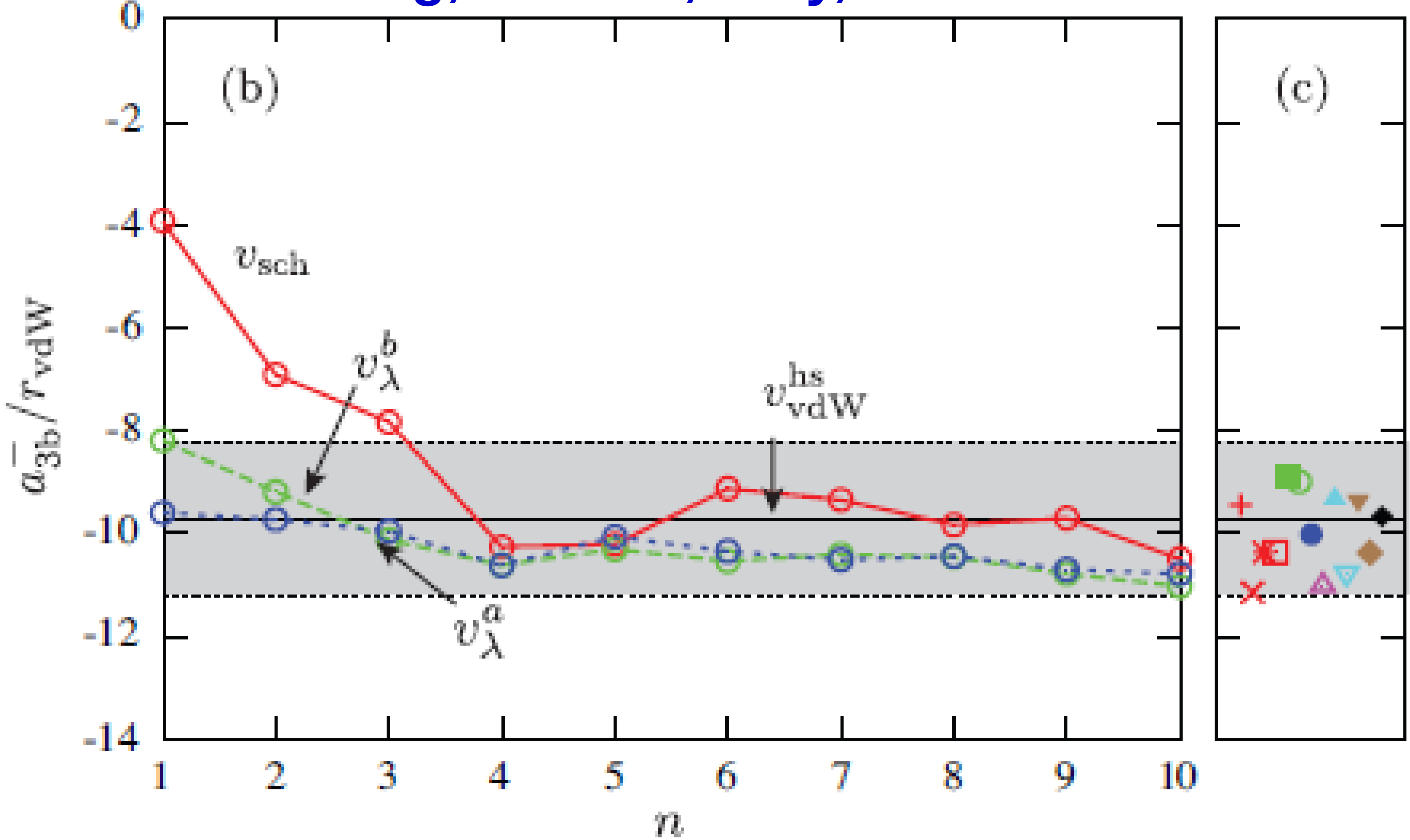
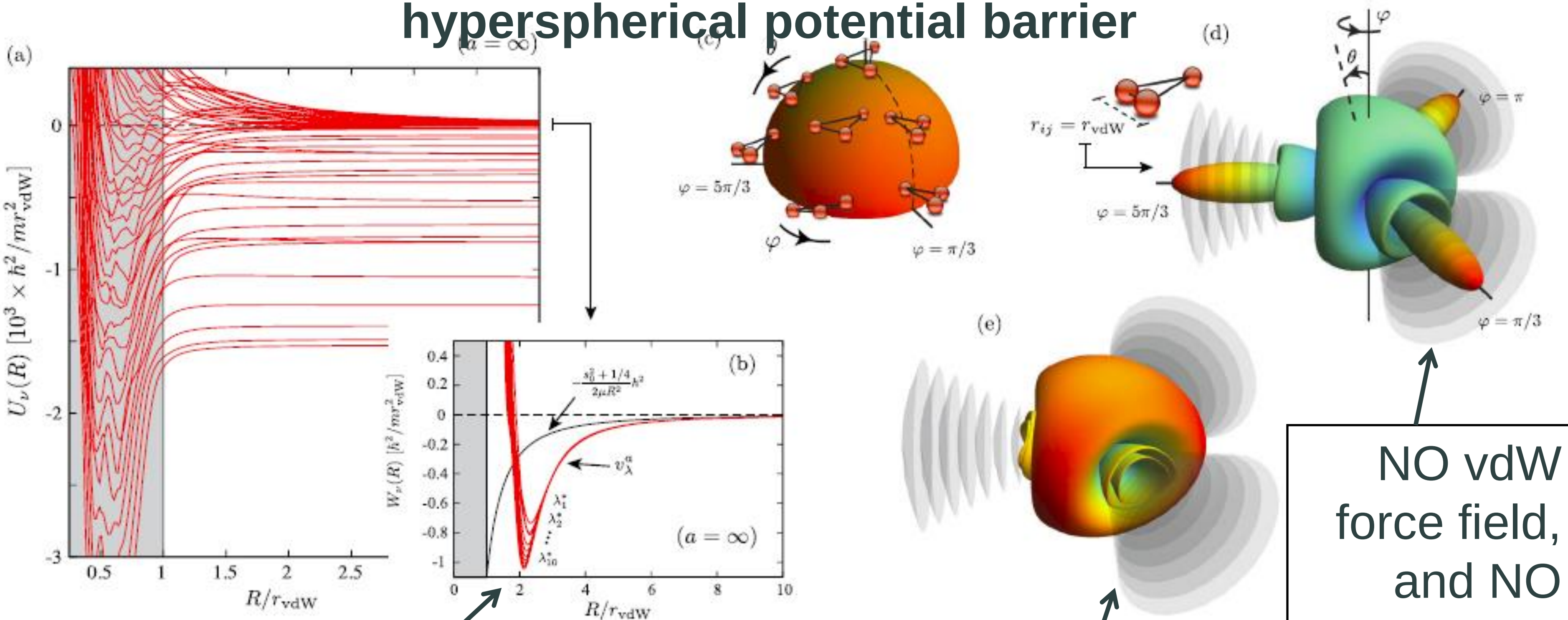


FIG. 4: Values for the three-body parameter (a) κ_* and (b) a_{3b}^- as functions of the number n of two-body s-wave bound states for each of the potential model studied here. (c) Experimental values for a_{3b}^- for ^{133}Cs [3] (red: ×, +, □, and ∗), ^{39}K [4] (magenta: △), ^{7}Li [5] (blue: •) and [6, 7] (green: ■ and ∘), ^{6}Li [8, 9] (cyan: ▲ and ▽) and [10, 11] (brown: ▼ and ◊), and ^{85}Rb [12] (black: ♦). The gray region specifies a band where there is a ±15% deviation from the $v_{\rm vdW}^{\rm hs}$ results. The inset of

Another finding: This property of 3-atom states is not expected to hold for nuclear systems, which have no van der Waals tail and few bound states.

Our study of hyperspherical potentials in the bosonic A+A+A system, showing that any two atoms “go over the van der Waals cliff” when they approach within their vdW radius, and this rise in kinetic energy produces a repulsive hyperspherical potential barrier



Numerical evidence for the existence of a universal barrier when the two-body potential has a van der Waals tail

vdW force field, note wavefunction suppression in 2-

NO vdW force field, and NO suppression

Summary of our extensive numerical tests and analysis. There is a universal Efimov potential curve that includes a universal short range barrier that fixes the 3-body parameter, shown here:

PRL 108, 263001 (2012) PHYSICAL RE

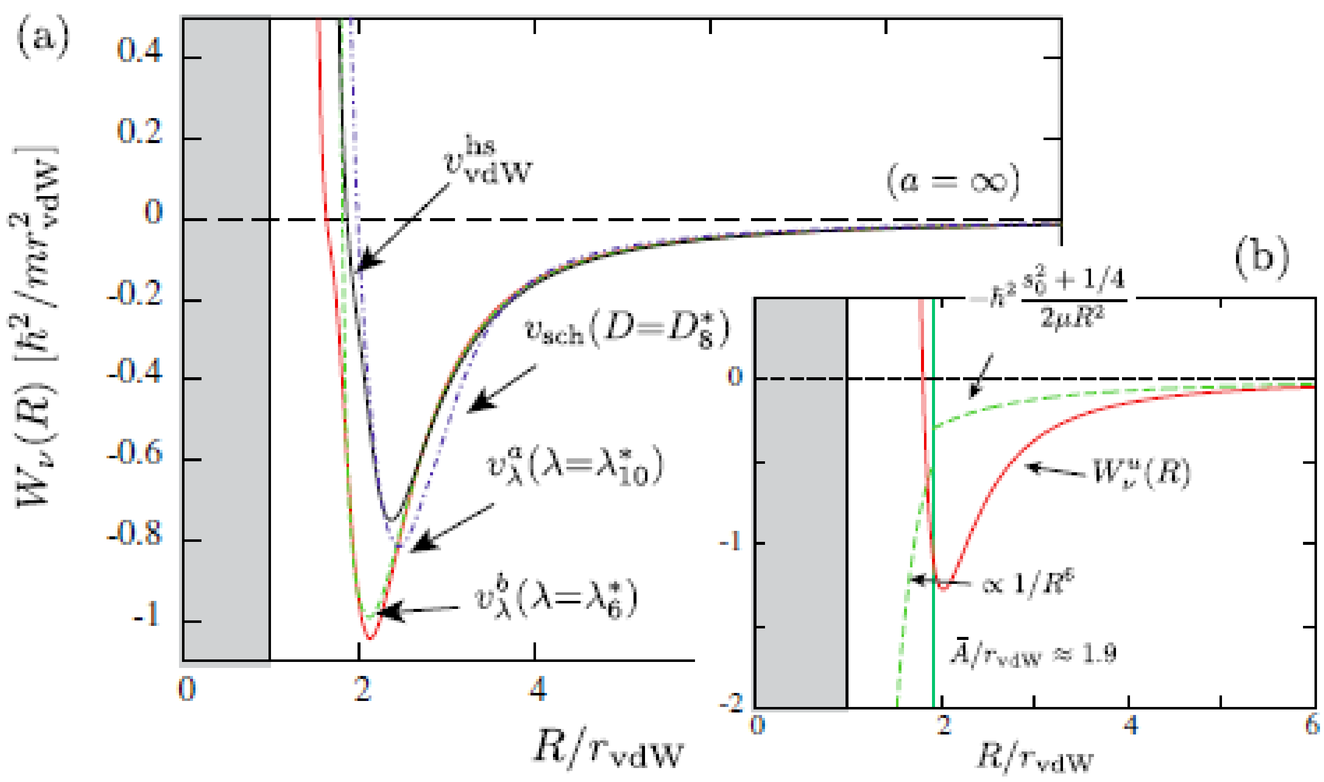


FIG. 3 (color online). (a) Efimov potential obtained from the different two-body potential models used here. The reasonably

Note that this barrier arises from a <u>classical</u> suppression of the wavefunction

Note that our detailed hyperspherical calculations have all assumed a single-channel interaction between each pair of atoms, which means that our conclusions are presumably valid for BROAD Fano-Feshbach resonances, but most likely inapplicable to NARROW resonances, but then along came another surprising experiment:

Universality of the three-body Efimov parameter at narrow Feshbach resonances

Sanjukta Roy[1], Manuele Landini[1], Andreas Trenkwalder[1], Giulia Semeghini[1], Giacomo Spagnolli[1], Andrea Simoni[3], Marco Fattori[1,2], Massimo Inguscio[1,2], and Giovanni Modugno[1,2]
[1] *LENS and Dipartimento di Fisica e Astronomia, Universitá di Firenze,*

This experiment by Sanjukta Roy, Giovanni Modugno, et al. has seen 7 Efimov resonances in ^{39}K, ranging from resonance width parameters: s_{res}=0.1 (narrow) up to s_{res}=2.8 (broad)

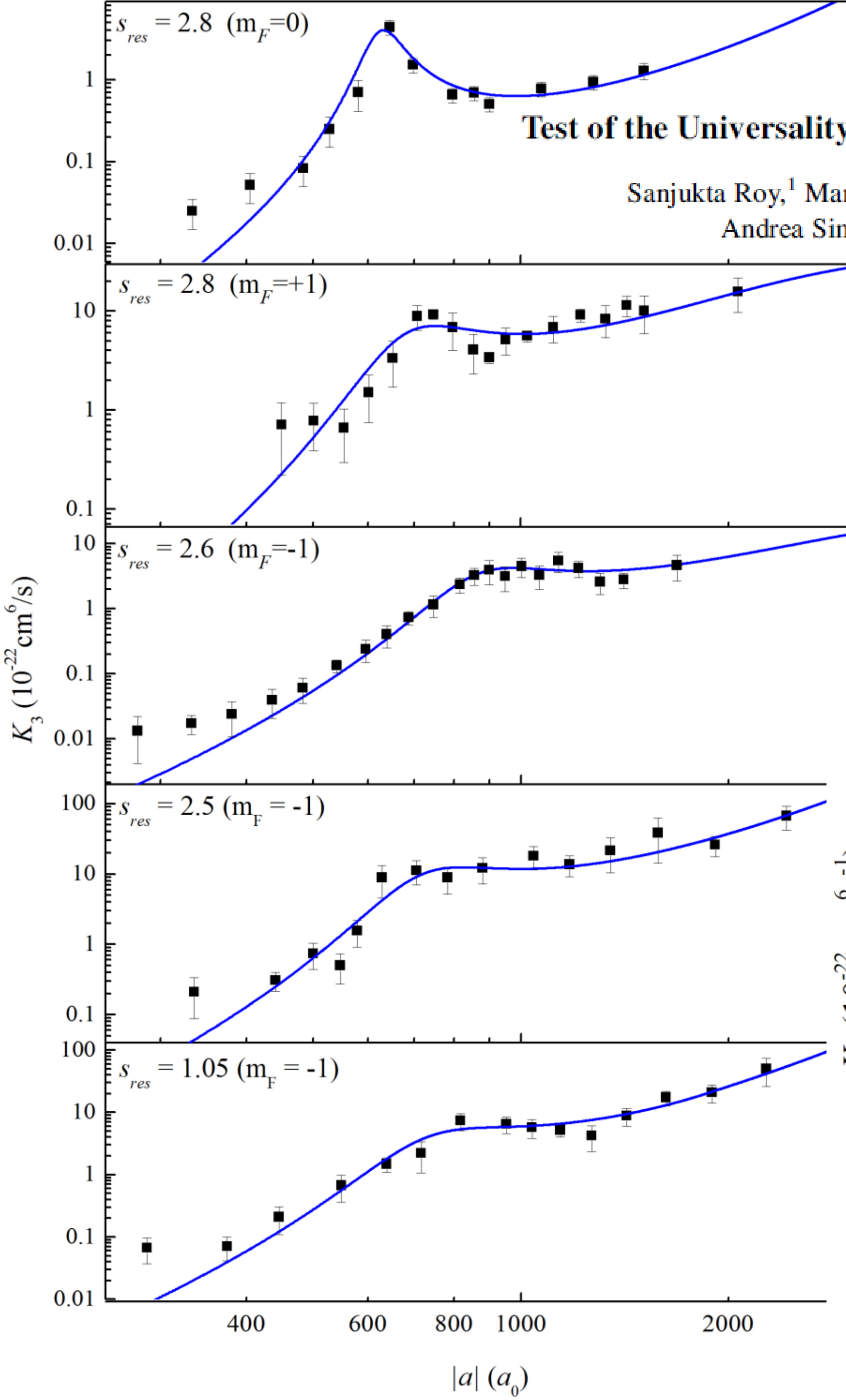
arXiv:1303.3843

PRL **111**, 053202 (2013)

PRL 111, 053202 (2013)

Test of the Universality of the Three-Body Efimov Parameter at Narrow Feshbach Resonances

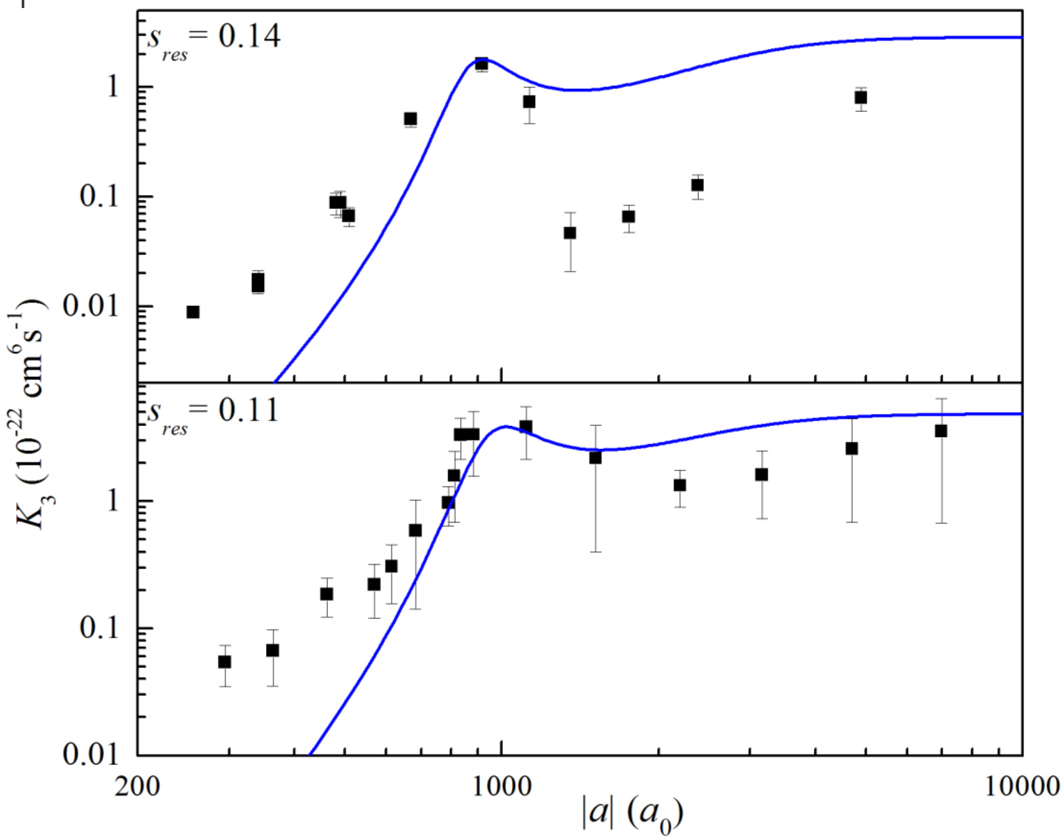
Sanjukta Roy,[1] Manuele Landini,[1] Andreas Trenkwalder,[1] Giulia Semeghini,[1] Giacomo Spagnolli,[1]
Andrea Simoni,[3] Marco Fattori,[1,2] Massimo Inguscio,[1,2] and Giovanni Modugno[1,2]



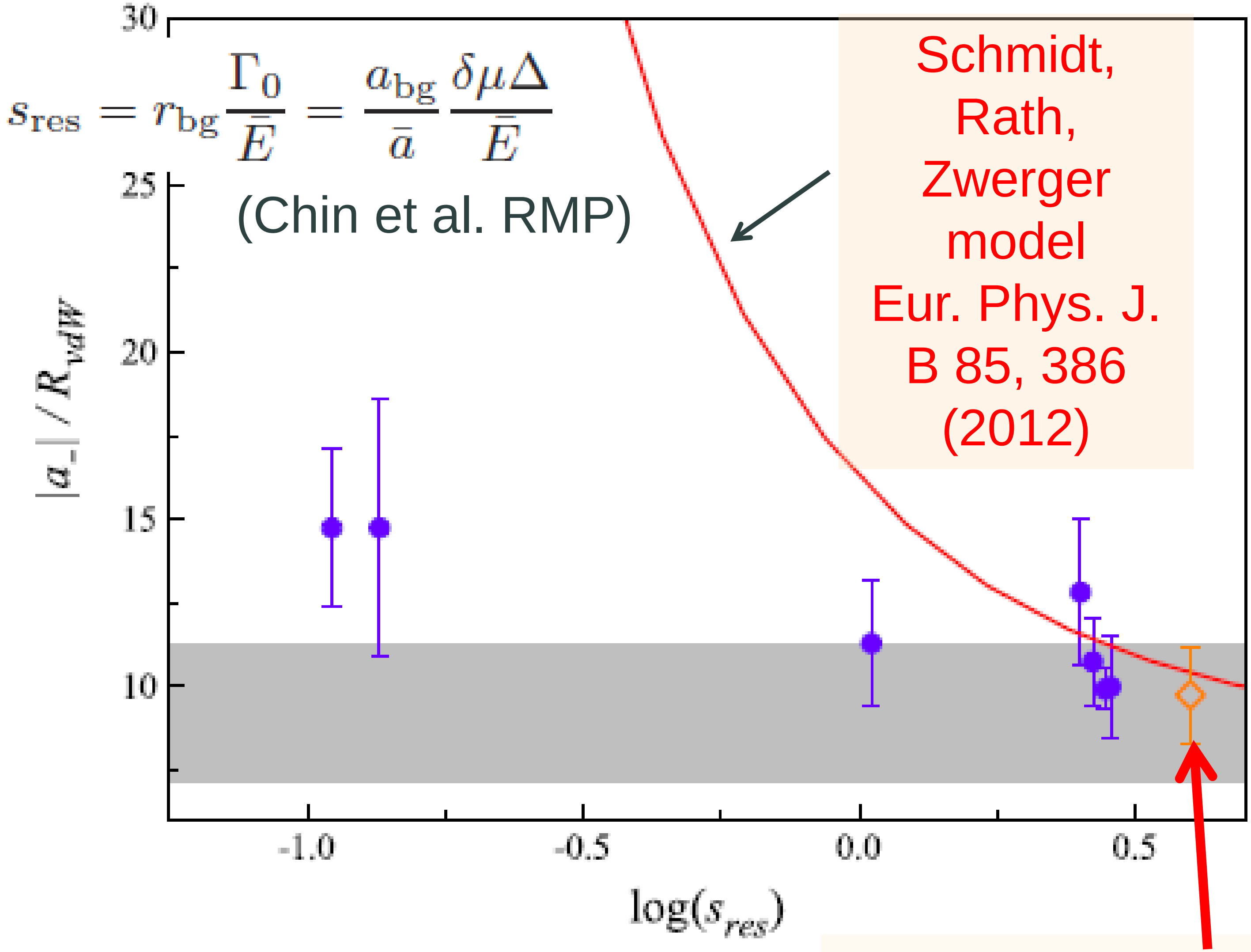
←**broad cases, s_{res}>1**

7 Efimov resonances in ^{39}K

Narrow cases s_{res}<0.15



^{39}K experimental 3-body parameter data, Roy, Modugno, et al. 2013 PRL

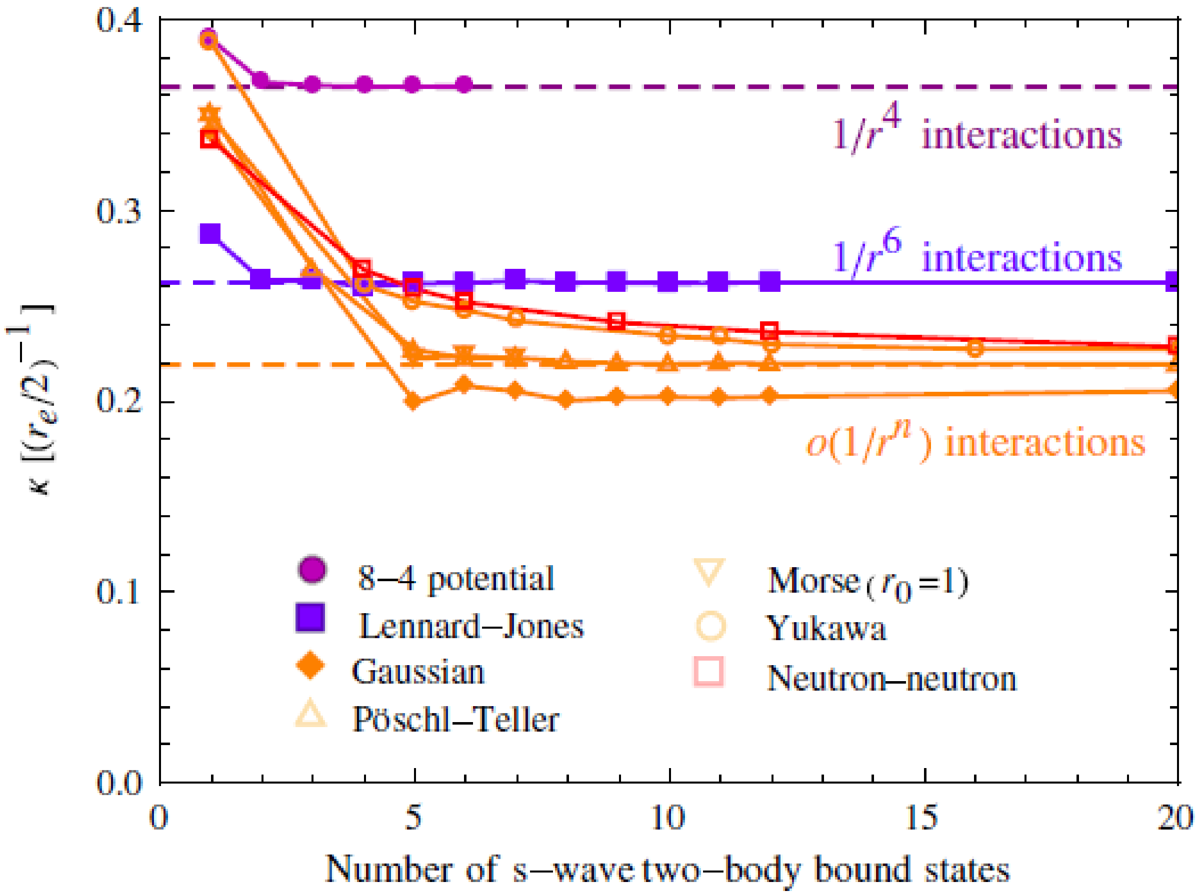


Expected theoretical range (Wang et al. 2012 PRL)

And now there is a very recent PRL (14 March) addressing again the broad resonance limit of the 3-body parameter, and its dependence on different 2-body interactions: PRL 112, 105301 (2014),

Microscopic Origin and Universality Classes of the Efimov Three-Body Parameter

Pascal Naidon,[1,*] Shimpei Endo,[2] and Masahito Ueda[2]



"…In the particular case of a van der Waals tail, we obtain a_=-10.86(1) r6, and kappa = 0.187(1)/r6 in good agreement with **Ref.25(Wang et al)** and experimental observations."

Separable potential model

associated with that tail. The authors of Ref. [25] remarked that this three-body repulsion is not explained by quantum reflection, as originally suggested in Ref. [29], but attributed it to an increase in kinetic energy due to the squeezing of the hyperangular wave function into a smaller volume caused by the suppression of two-body probability inside the well or the repulsive core of the two-body potential. This point was confirmed and clarified in Ref. [28] where the kinetic energy was shown to originate from an abrupt change of the geometry of the three-particle system caused by the two-particle exclusion in the van der Waals region. At large separation, the system has indeed an elongated geometry due to its Efimov nature, but it must deform to an equilateral configuration to accommodate for the mutual exclusion between the particles. Reference [28] showed that this deformation causes a nonadiabatic increase in kinetic energy that manifests itself as a three-body repulsive barrier. This phenomenon could be reproduced by simple models involving only the knowledge of the pair correlation causing the mutual exclusion between two particles at short separations.

The interpretation by Naidon et al., 2014 PRL, which claims to revise our (Wang et al. 2012) interpretation, but in fact appears to be almost identical

?

Some of the most exciting recent news concerns the observation of MULTIPLE Efimov resonances near a single Fano-Feshbach resonance, in two preprints this month:

Observation of the Second Triatomic Resonance in Efimov's Scenario

arXiv:1402.6161

Bo Huang (黃博)[1], Leonid A. Sidorenkov[1,2], and Rudolf Grimm[1,2]

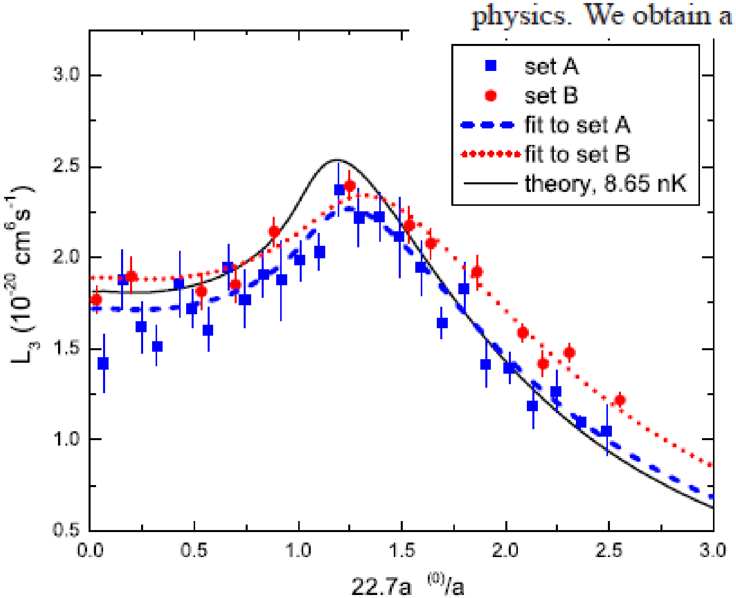
[1]*Institut für Experimentalphysik, Universität Innsbruck, 6020 Innsbruck, Austria and*
[2]*Institut für Quantenoptik und Quanteninformation (IQOQI), Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria*

Jeremy M. Hutson

Joint Quantum Centre (JQC) Durham/Newcastle, Department of Chemistry, Durham University, South Road, Durham, DH1 3LE, United Kingdom

(Dated: February 26, 2014)

We report on the observation of a three-body recombination resonance in an ultracold gas of cesium atoms at a very large negative value of the s-wave scattering length. The resonance is identified as the second triatomic Efimov resonance, which corresponds to the situation where the first excited Efimov state appears at the threshold of three free atoms. The observation, together with a finite temperature analysis and the known first resonance, allows the most accurate demonstration to date of the discrete scaling behavior at the heart of Efimov physics. We obtain a scaling factor of 21.1(1.4), close to the ideal value of 22.7.



Set	T/nK	$a_{-}^{(1)}/a_0$	$\eta_{-}^{(1)}$	λ
A	9.8(4)	-21000(400)	0.13(2)	1*
B	10.5(4)	-19800(600)	0.18(2)	1*
A	7.7*	-20600(400)	0.17(3)	0.52(5)
B	9.6*	-19700(400)	0.19(4)	0.80(7)

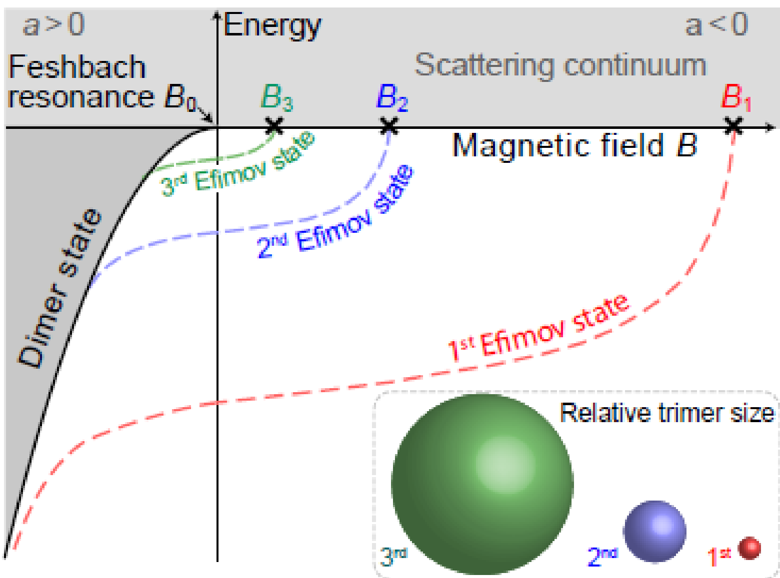
TABLE I. Fitted parameters for the second Efimov resonance. The upper part of the table shows the fitting results when no amplitude scaling of L_3 is applied and temperature T is a free parameter, while the lower part corresponds to fixed-temperature fitting with λ being a free amplitude scaling factor (see text). The uncertainties indicate 1σ errors from fitting. The symbol * indicates that the corresponding parameter is kept fixed.

And the other big piece of excitement comes from the 6Li-Cs-Cs experiment of Cheng Chin, Shi-Kuang Tung, and collaborators at the University of Chicago, who have observed 3 Efimov trimers with approximately the expected Efimov 4.87 geometric scaling factor between them:

arXiv:1402.5943v1

Observation of geometric scaling of Efimov states in a Fermi-Bose Li-Cs mixture

Shih-Kuang Tung, Karina Jiménez-García, Jacob Johansen, Colin Parker, and Cheng Chin*



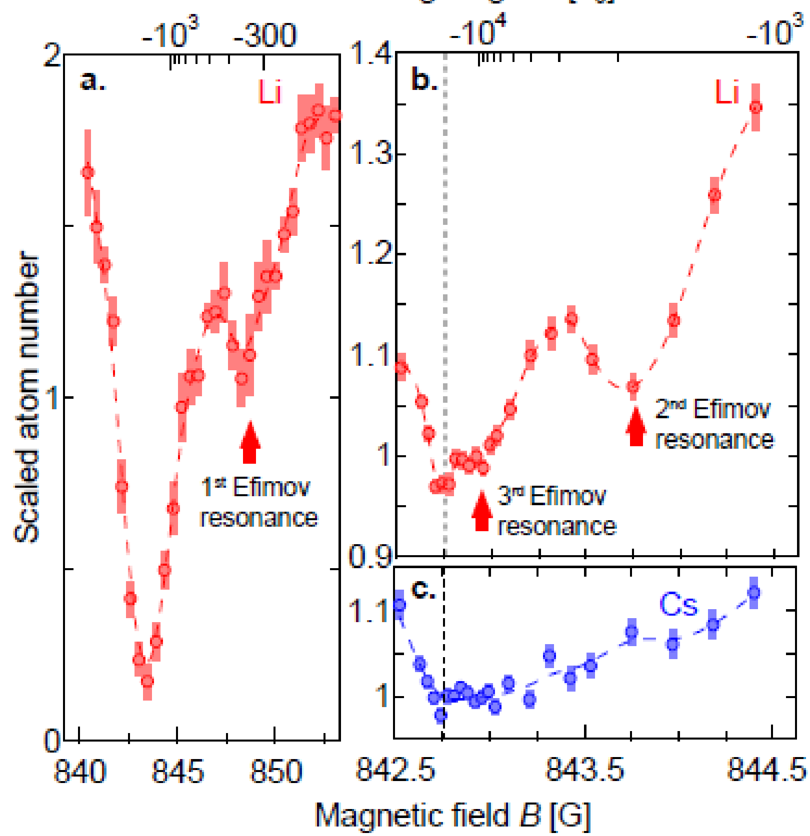


FIG. 3. **Observation of three Li-Cs-Cs Efimov resonances. a.** Scaled Li number versus magnetic field showing

Next, what can theory PREDICT for the heteronuclear Efimov effect?

Universal three-body parameter in heteronuclear atomic systems

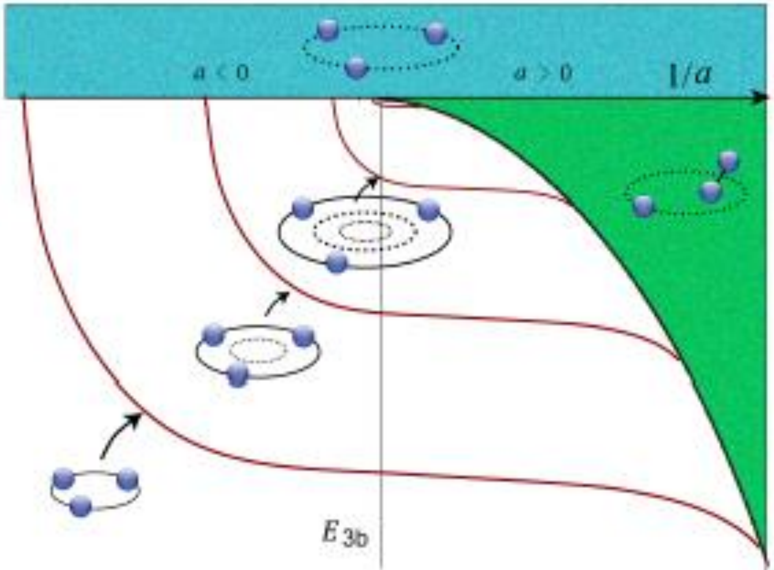
Yujun Wang, Jia Wang, J. P. D'Incao, & CHG

PRL 109, 243201 (2012)

Main result: we see that the Efimov physics is also universal for the case of 2 identical bosonic atoms (AA) and 1 distinguishable atom (X), but the parameter space is larger and more complicated. This is because the universality values predicted depend on the mass ratio, M_A/M_X, and on the background A-A scattering length, and on TWO different vdW radii (A-X and A-A).

arXiv:1207.6439

The Efimov effect: universality



Efimov spectrum

For three particles with two or three resonant interactions (scattering length $a \to \infty$), an infinite series of three-body bound states emerge with $E_n = E_0 e^{-2n\pi/s_0}$ [1].

Heteronuclear system AAX:
Efimov-favored when $m_A/m_X \gg 1$ such that $s_0 > 1$;
Efimov-unfavored when $m_A/m_X \lesssim 1$ such that $s_0 < 1$.

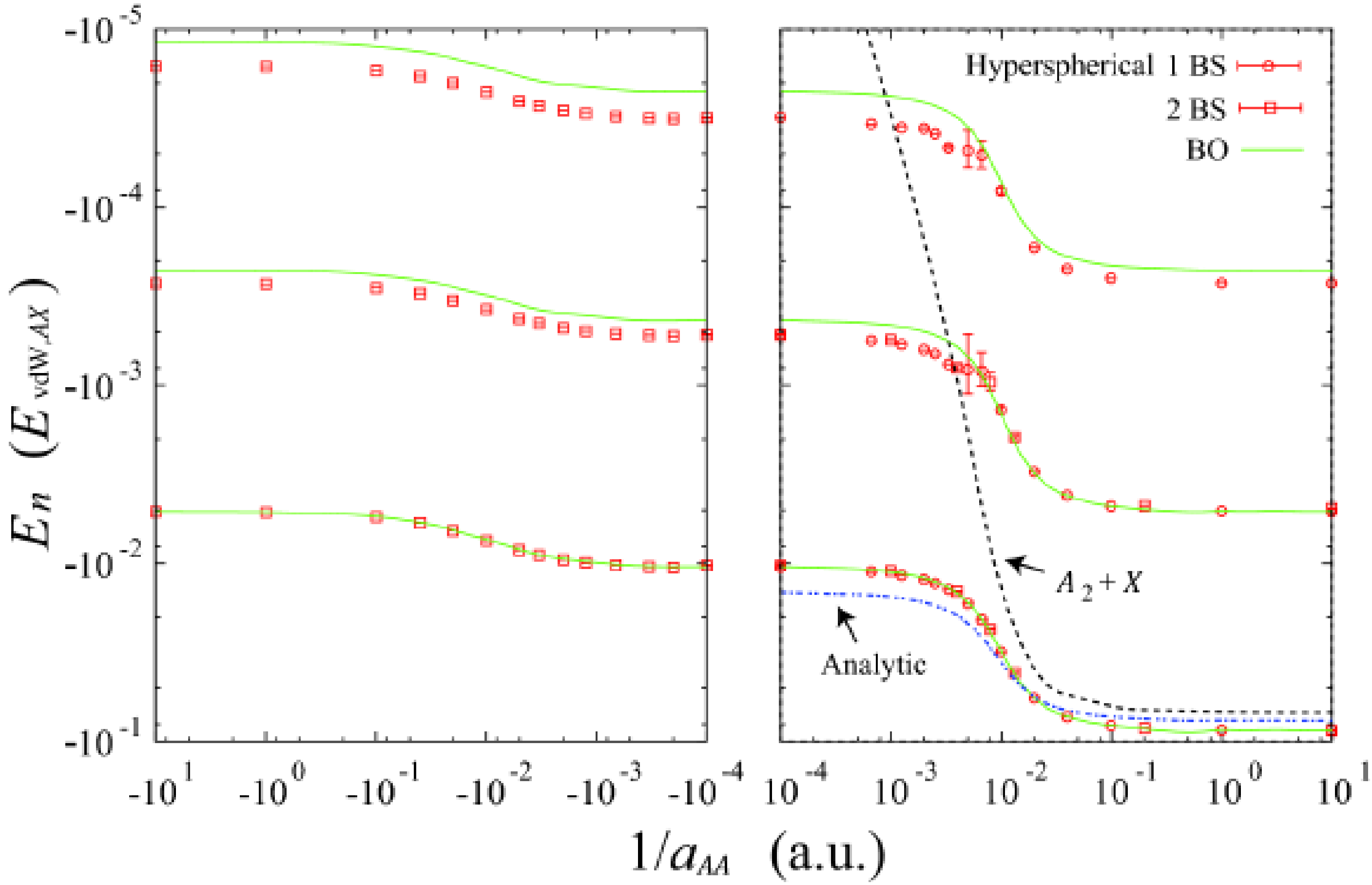
Three-body parameter can be expressed in three-body recombination observables a_-^* (first Efimov resonance) or a_0^* (first interference minimum).

For identical bosonic atoms, $a_-^* \approx -9.1 r_{\mathrm{vdW}}$ [$r_{\mathrm{vdW}} = (2\mu_2 C_6)^{1/4}/2$].

Universal three-body parameter for AAX?

Key finding: Our numerical evidence suggests that the 3-body parameter is UNIVERSAL for heteronuclear AAX systems also, but this universality depends on the AA scattering length, the mass ratio, the two van der Waals lengths, etc, and must be mapped out

Efimov-favored *AAX* systems — universal three-body parameter



Universal Efimov spectrum for YbYbLi [1]

Predictions of first Efimov resonance (negative a) and destructive interference Stueckelberg minimum (positive a)

	s_0	s_0^*	$a_{AA,bg}$ (a.u.)	a_0^* (a.u.)	a_-^* (a.u.)
$^{174}Yb_2{}^6Li$	2.246	2.382	104 [32, 33]	1.3×10^3	-8.4×10^2
$^{133}Cs_2{}^6Li$	1.983	2.155	2000 [34]	9.6×10^2	-1.4×10^3
$^{87}Rb_2{}^6Li$	1.633	1.860	100 [35]	3.8×10^2	-1.6×10^3
$^{41}K_2{}^6Li$	1.154	1.477	62 [36]	3.7×10^2	-2.4×10^3
$^{23}Na_2{}^6Li$	0.875	1.269	100 [37]	1.5×10^3	-1.3×10^4
$^{87}Rb_2{}^{40}K$	0.653	1.125	100	2.8×10^3	$< -3 \times 10^4$
$^{133}Cs_2{}^{87}Rb$	0.535	1.060	2000	2.3×10^3	$< -4 \times 10^4$
$^{41}K_2{}^{87}Rb$	0.246	0.961	62	$> 7 \times 10^3$	$< -1 \times 10^6$

TABLE I: The universal Efimov scaling constants s_0, s_0^* and the 3BPs $a_{AX} = a_0^*$ and $a_{AX} = a_-^*$ obtained by keeping a_{AA} fixed at its background value ($a_{AA,bg}$).

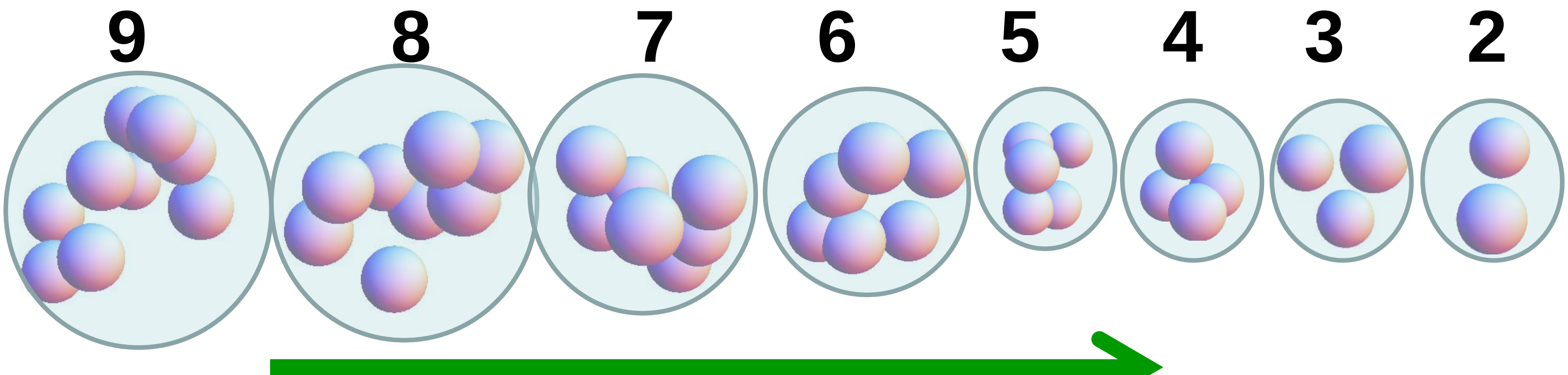
Our prediction from this 2012 PRL was that the first Cs-Cs-Li resonance should appear at either a= -1400 or else -1400/4.88 = -287 a.u. The new Chicago experiment observes a_(expt)= -337(9) a.u.

Extensions of Universal Efimov Physics to N>3 Bosons in 3D

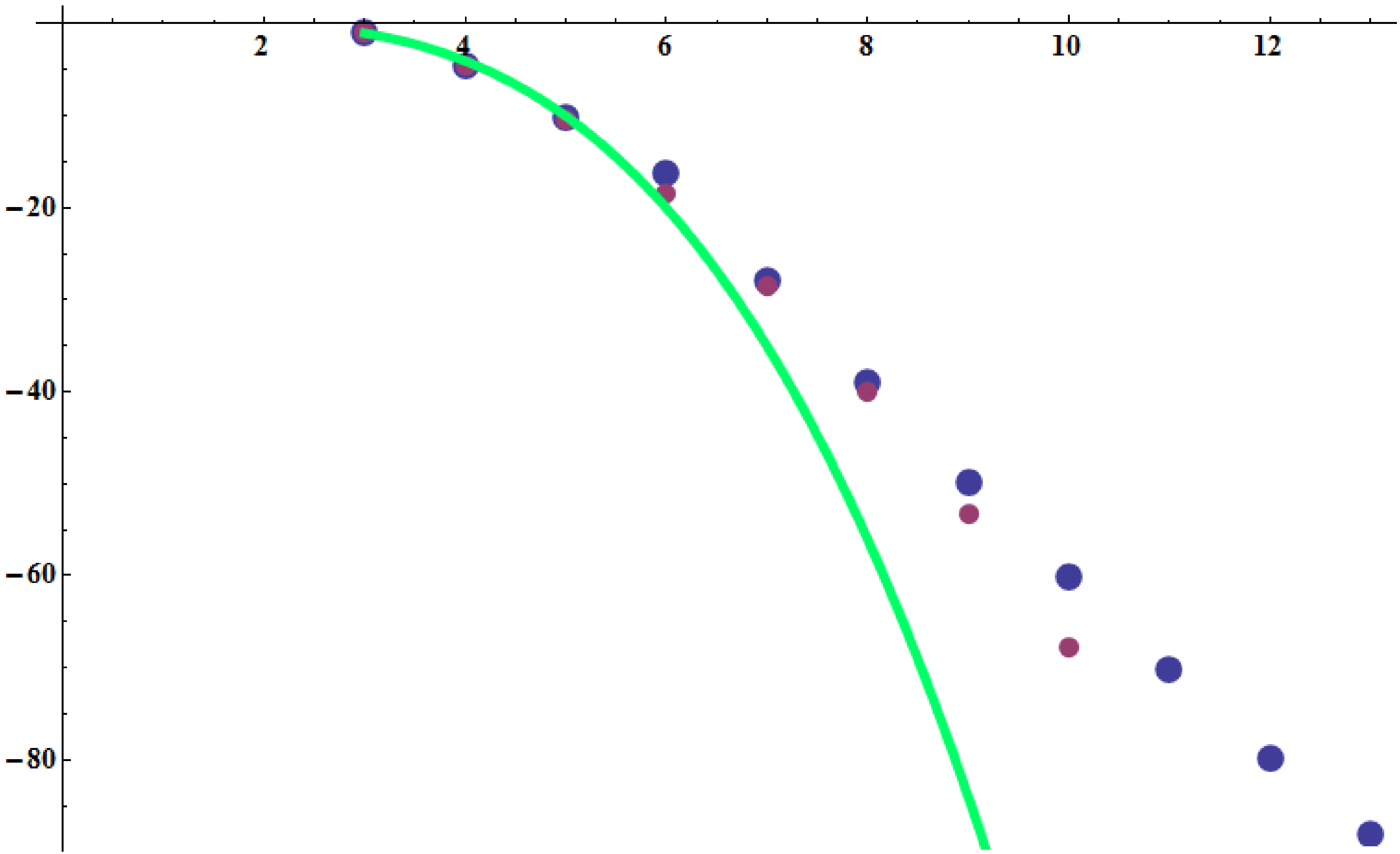
$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i<j} v(r_{ij})$$

N repulsive terms > 0

↑ $\frac{N(N-1)}{2}$ attractive terms < 0



INCREASING ATTRACTION (a gets more negative) →



See also Hanna and
Blume, 2006 PRA, and Gattobigio et al 2011 PRA

J. Phys. B: At. Mol. Opt. Phys. **43** (2010) 101002 (5pp) doi:10.1088/0953-4075/43/10/101002

FAST TRACK COMMUNICATION

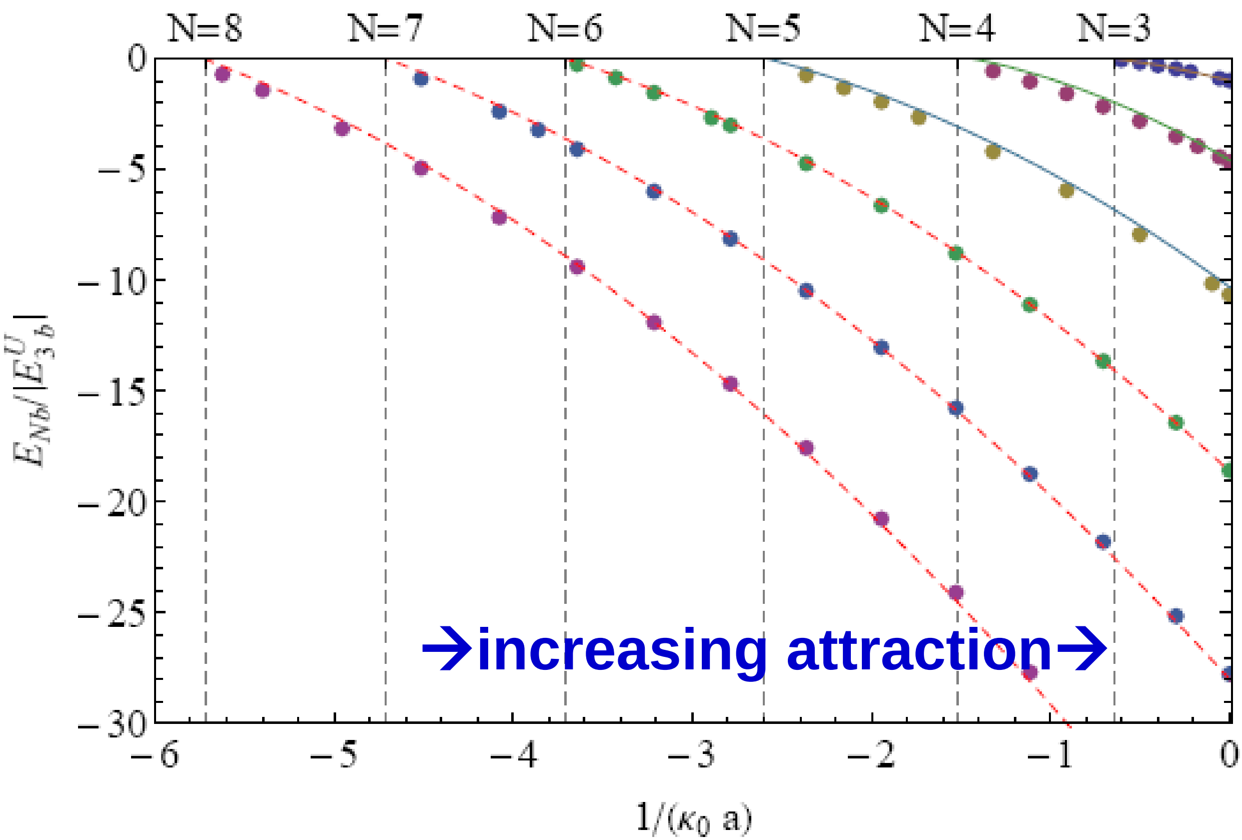
Javier von Stecher (former PhD student)

Weakly bound cluster states of Efimov character

Merging points into N-free atoms

Javier von Stecher

Clusters predicted up to N=13.



How Efimov physics extends to more than 3 particles. This figure shows the schematic entrance channel **potential curve expected for N particles at negative 2-body scattering length**, from Mehta et al., 2009 PRL

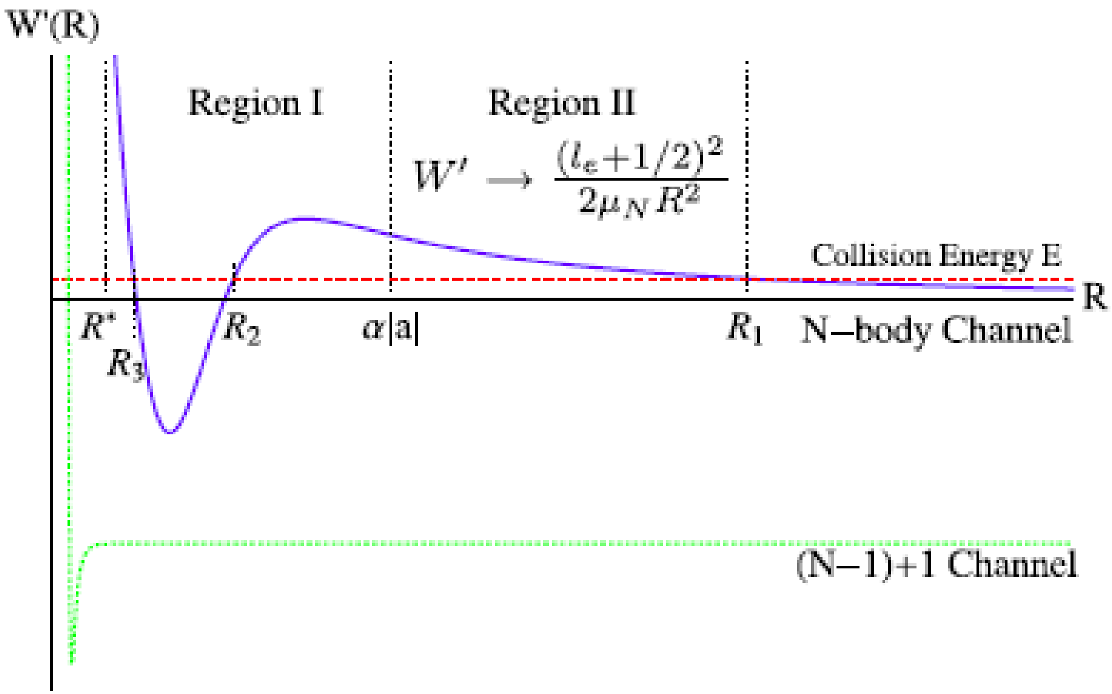


FIG. 1 (color online). A schematic representation of the N-boson hyperradial potential curves is shown. When a metastable N-boson state crosses the collision energy threshold at $E = 0$, N-body recombination into a lower channel with $N - 1$ atoms bound plus one free atom is resonantly enhanced.

But before we could actually calculate the rate of 4-body recombination in an ultracold gas, we had to develop some scattering theory:

PRL 103, 153201 (2009)

A general theoretical description of N-body recombination

N. P. Mehta,[1,2] Seth T. Rittenhouse,[1] J. P. D'Incao,[1] J. von Stecher,[1] and Chris H. Greene[1]

[1] Department of Physics and JILA, University of Colorado, Boulder, CO 80309

[2] Grinnell College, Department of Physics, Grinnell, IA 50112*

(Dated: March 24, 2009)

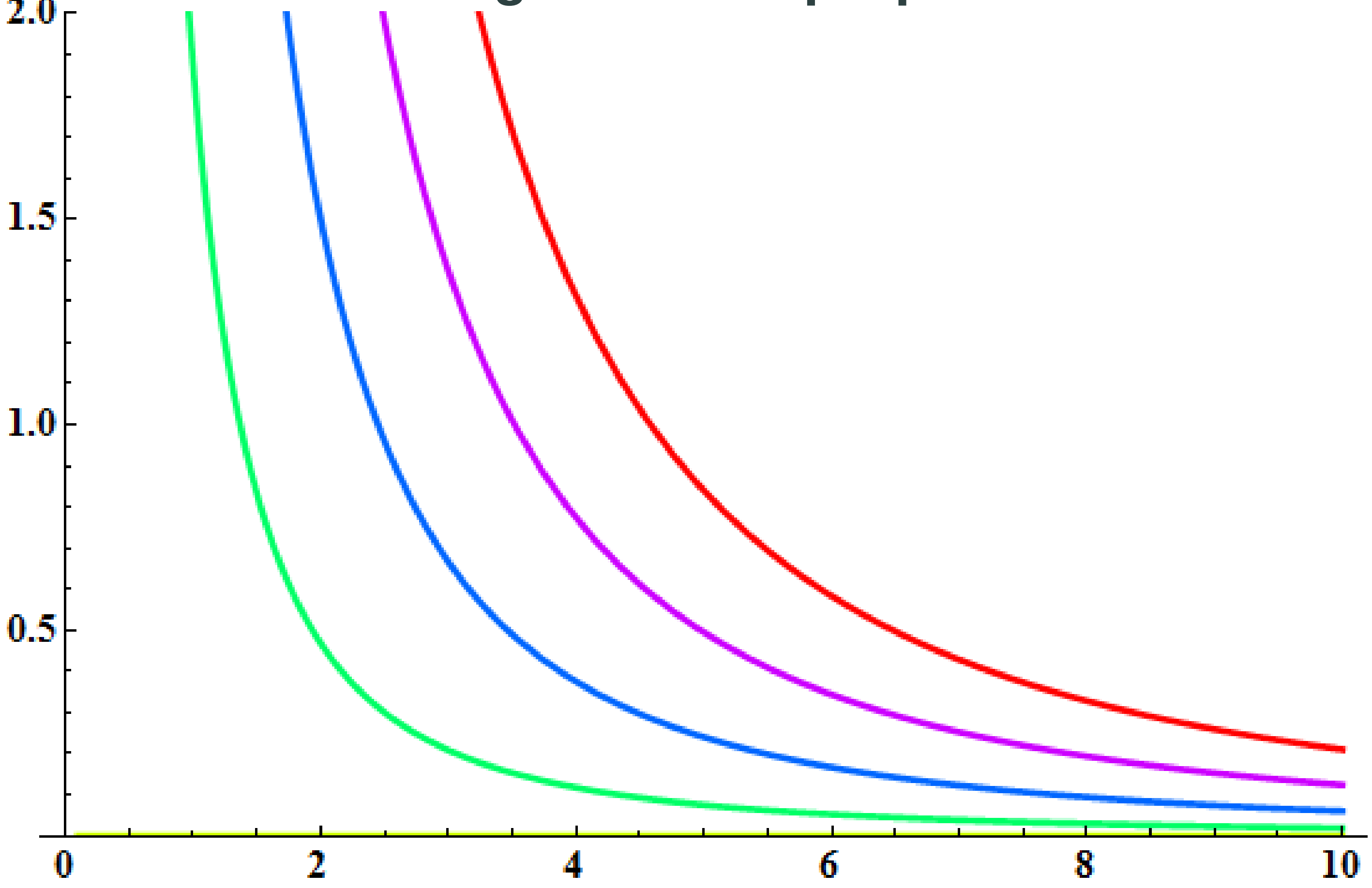
And here it is, THE FORMULA for N-body recombination, i.e. for the process: A+A+A+….+..A → A_{N-1}+A or A_{N-2}+A+A +…etc.

$$K_N^{0^+} = \frac{2\pi\hbar}{\mu_N} N! \left(\frac{2\pi}{k}\right)^{(3N-5)} \frac{\Gamma\left((3N-3)/2\right)}{2\pi^{(3N-3)/2}} \left|S_{f0}^{0^+}\right|^2$$

Semiclassical parameterization:

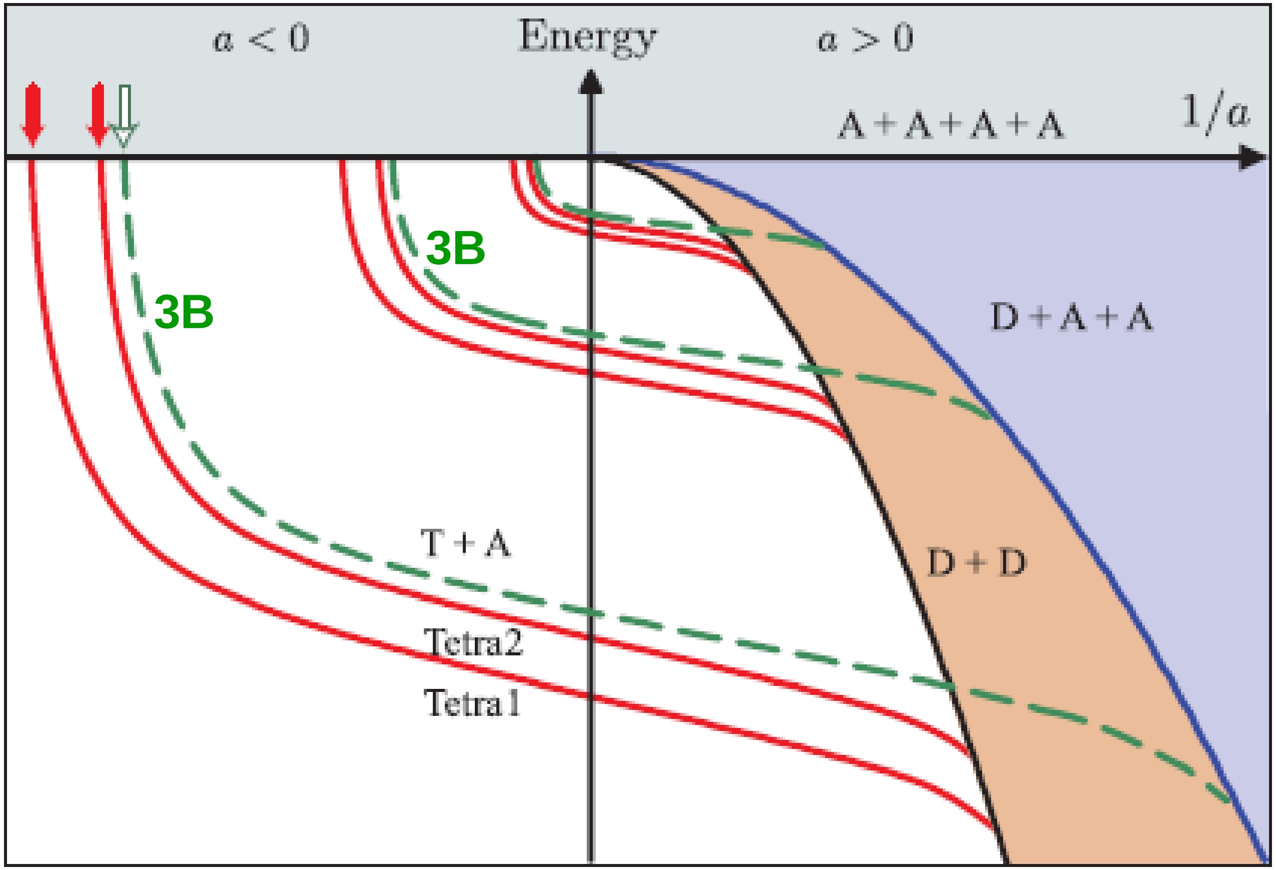
$$K_N^{0^+} = \frac{\pi\hbar N!}{\mu_N \Omega(3N-3)} \left(\frac{4\pi|a|}{3N-5}\right)^{3N-5} \frac{C(a)\sinh(2\eta)}{\cos^2\phi + \sinh^2\eta}.$$

Lowest Continuum Potential Curves for an N-body system with no interactions, showing the mock-centrifugal barrier proportional to $1/R^2$



Key finding:

Two four-body states are found to lie between each successive pair of Efimov trimers - von Stecher et al. Nature Phys. 2009 – which confirms insightful work by Platter and Hammer (Eur. Phys. J. A. 2007), and extends it



Extended Efimov plot showing universal dimer, trimer, and tetramer states of four identical bosons with short-range interactions.

Tetramer states predicted to hit zero energy at

a=0.43 a(Efimov)

and

a=0.90 a(Efimov)

See also Ferlaino et al., PRL 102, 140401 (2009) for experimental confirmation, and recent theoretical extensions by Deltuva with higher accuracy.

EPL, 95 (2011) 43002
doi: 10.1209/0295-5075/95/43002 www.epljournal.org

Shallow Efimov tetramer as inelastic virtual state and resonant enhancement of the atom-trimer relaxation

A. Deltuva[a]

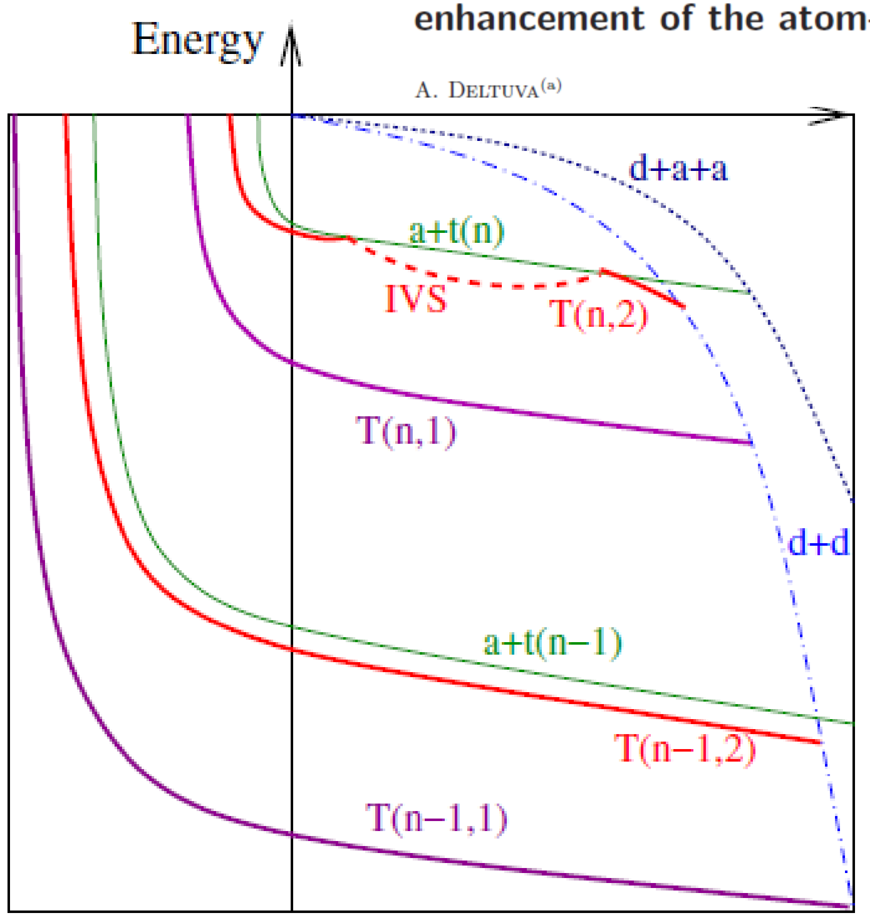


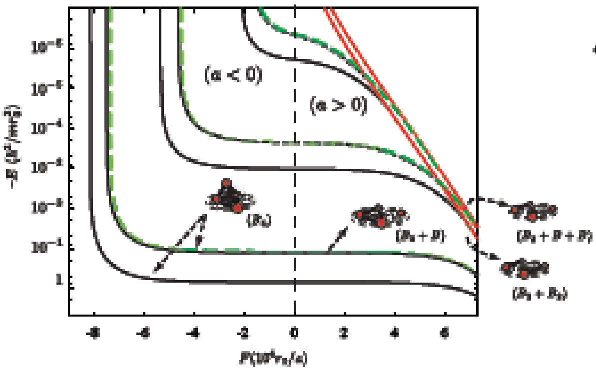
Fig. 1: (Colour on-line) Schematic representation of the four-boson energy spectrum as a function of the two-boson scattering length. The tetramer (T) energies, atom-trimer (a + t), dimer-dimer (d + d), and dimer-atom-atom (d + a + a) thresholds are shown as thick solid, thin solid, dash-dotted, and dotted curves, respectively. For a better visualization only

Deltuva's more accurate 4-body spectrum, showing that the excited tetramer can become unbound at positive a, for a range close to the intersection of dimer-dimer channels with atom-trimer channels

Four-boson Spectrum

JILA University of Colorado at Boulder

2, 3, and 4-body energy levels



$(a = \infty)$

Consistent with work of Platter and Hammer, we agree that there are:

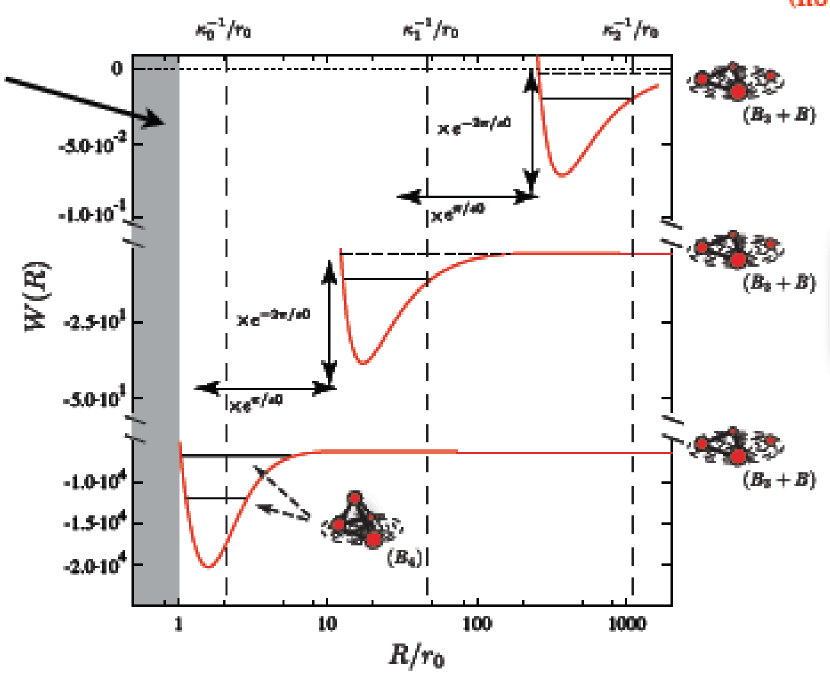
Two four-body states per Efimov trimer !!!

$$E_{4b}^{(n,m)} = c_m E_{3b}^{(n)} \qquad m = 1, 2 \qquad n = 1, 2, \ldots, \infty$$

$$(c_1 \approx 4.58,\ c_2 \approx 1.01)$$

(no four-body parameter)

details !

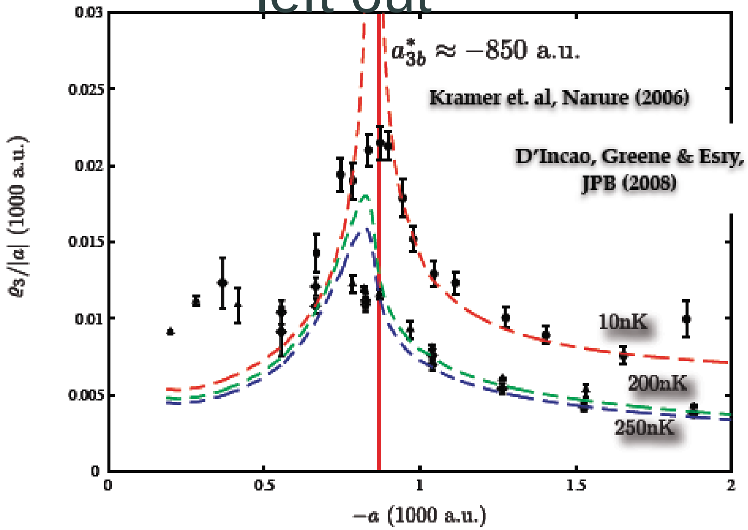


Hyperspherical 4-body potential curves converging asymptotically to Efimov trimer levels

Note that $E(He_4)/E(He_3) = 4.44$ (Blume & CHG 2000 JCP)

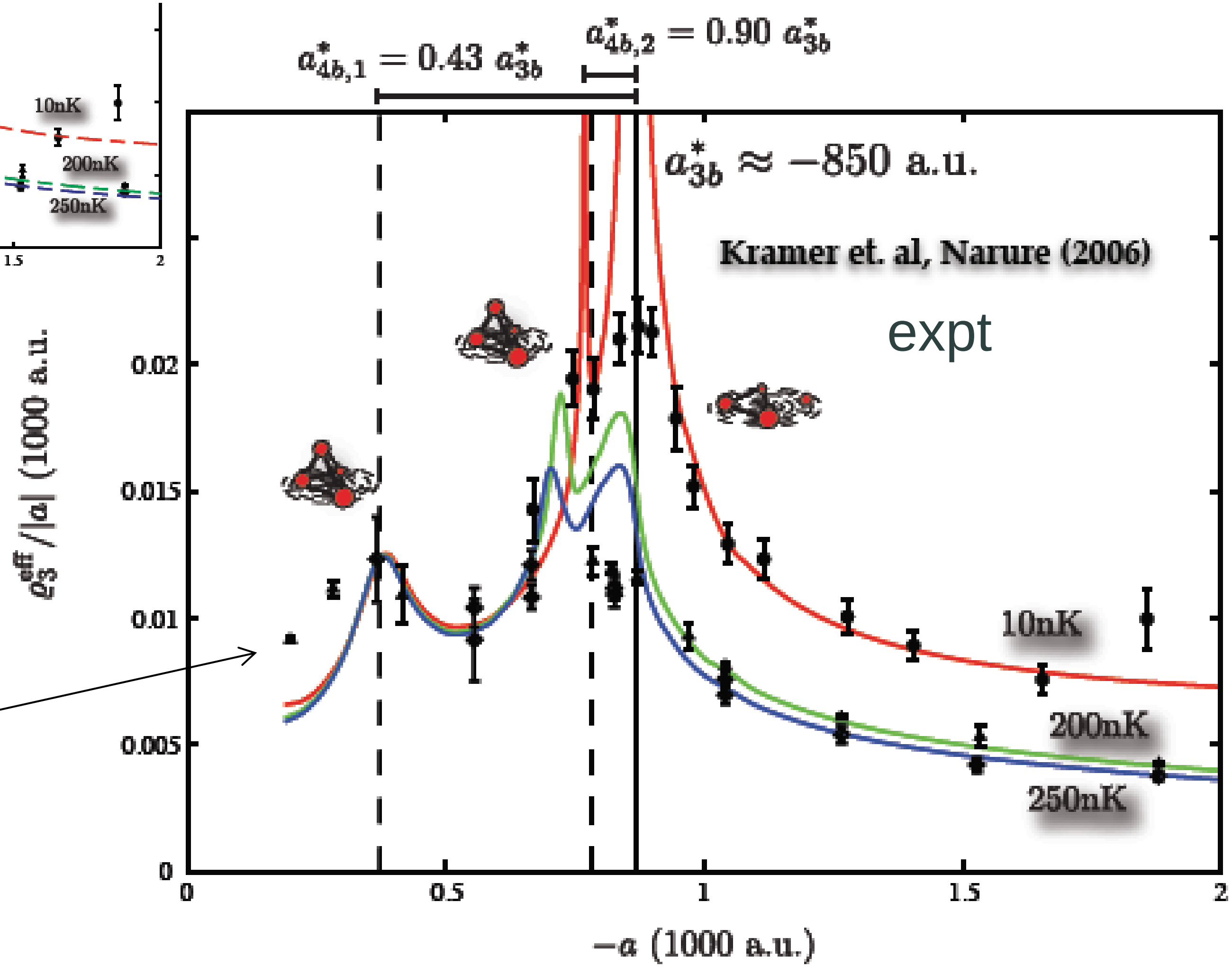
Considering only three-body recombination ...

4-body recombination left out



Considering three- and four-body recombination ...

$$K_3^{\text{eff}}(a,t) = K_3(a) + n(t)K_4(a)$$



3-body recombination included AND 4-body losses too

von Stecher - arXiv:0909.4056
and 2010 JPB

TABLE I: Energies at unitarity and scattering-length ratios that characterize weakly bound cluster states. The scattering length ratios can be transformed to an absolute scale using $1/(\kappa_0 a_{3b}) \approx 0.64$.

N	E_N^U/E_3^U	$a^*_{Nb}/a^*_{(N-1)b}$	N	E_N^U/E_3^U
4	4.66(4)	0.42(1)	9	49.9(6)
5	10.64(4)	0.60(1)	10	60.2(6)
6	18.59(5)	0.71(1)	11	70.1(7)
7	27.9(2)	0.78(1)	12	79.9(3)
8	38.9(3)	0.82(1)	13	88.0(7)

0.46(1) 0.65(2) 0.73(1) are latest revised/improved values from von Stecher,

Five- and Six-Body Resonances Tied to an Efimov Trimer

Javier von Stecher

PRL **107**, 200402 (2011)

Remarkable prediction, that **all** larger cluster resonances are determined once the 3-body parameter is known!

How to tackle 5-body recombination for 5 free bosonic atoms with pairwise additive forces?

i.e. the reaction A+A+A+A+A→ A_3+A_2 or A_4+A or…

Start with the time-independent Schroedinger equation:

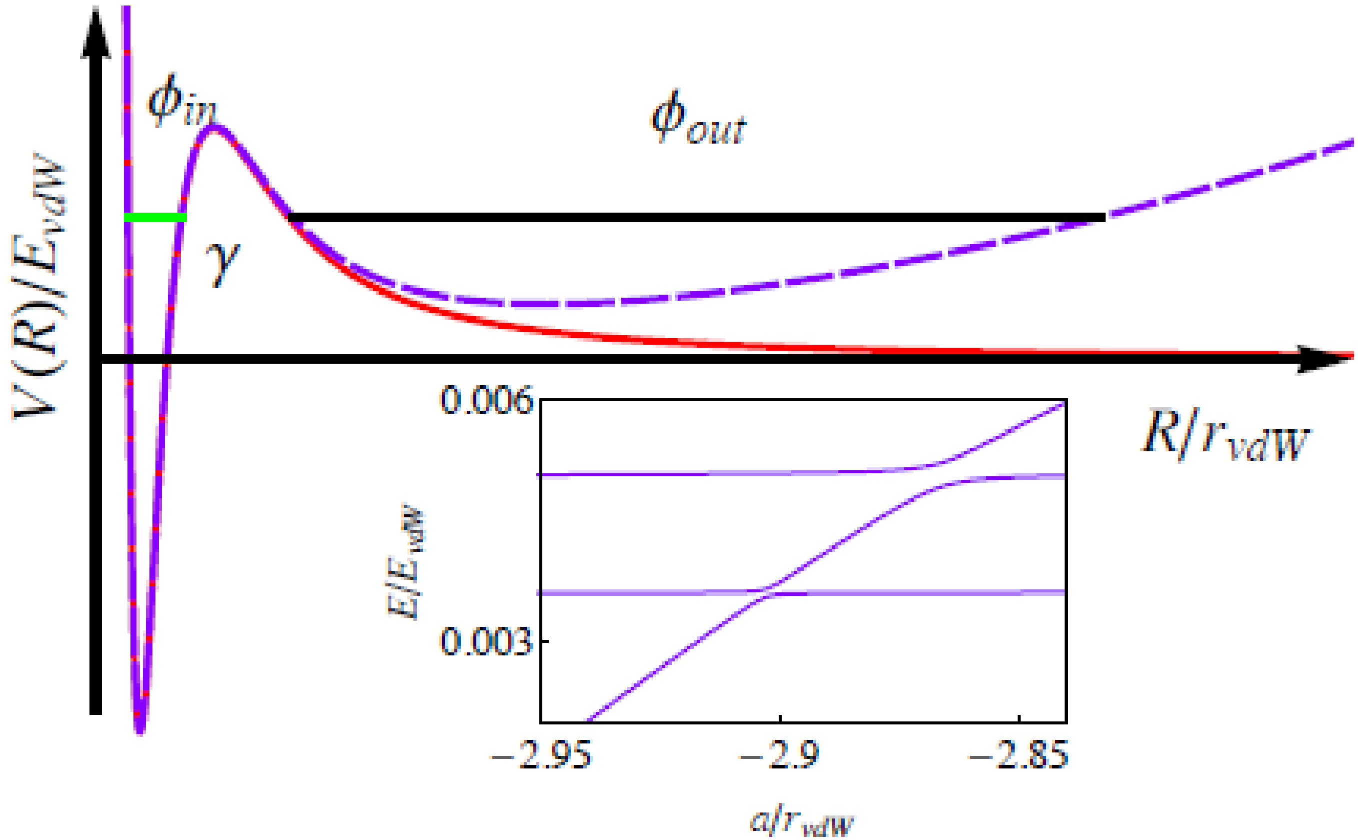
$$H=\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2}+\frac{p_3^2}{2m_3}+\frac{p_4^2}{2m_4}+\frac{p_5^2}{2m_5}$$

$$+V(r_{12})+V(r_{13})+V(r_{14})+V(r_{15})+V(r_{23})+V(r_{24})+V(r_{25})+V(r_{34})+V(r_{35})+V(r_{45})$$

After eliminating the center-of-mass degree of freedom, we're left with a 12-dimensional PDE to solve, which can be reduced to **a mere 9 dimensions** for J=0 states after going to the body frame.

Resonant five-body recombination in an ultracold gas of bosonic atoms

Mulliken-style potential energy versus hyperradius R for 5 free Cs atoms (solid red) or harmonically trapped (dashed)



New Journal of Physics
The open access journal for physics

Resonant five-body recombination in an ultracold gas of bosonic atoms

Alessandro Zenesini[1,5,7], Bo Huang[1], Martin Berninger[1], Stefan Besler[1], Hanns-Christoph Nägerl[1], Francesca Ferlaino[1], Rudolf Grimm[1,2], Chris H Greene[3,6] and Javier von Stecher[3,4]

Our recent paper with the Innsbruck group: New Journal of Physics 2013

Resonant Five-Body Recombination in an Ultracold Gas

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Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, 6020 Innsbruck, Austria

Chris H. Greene and J. von Stecher*
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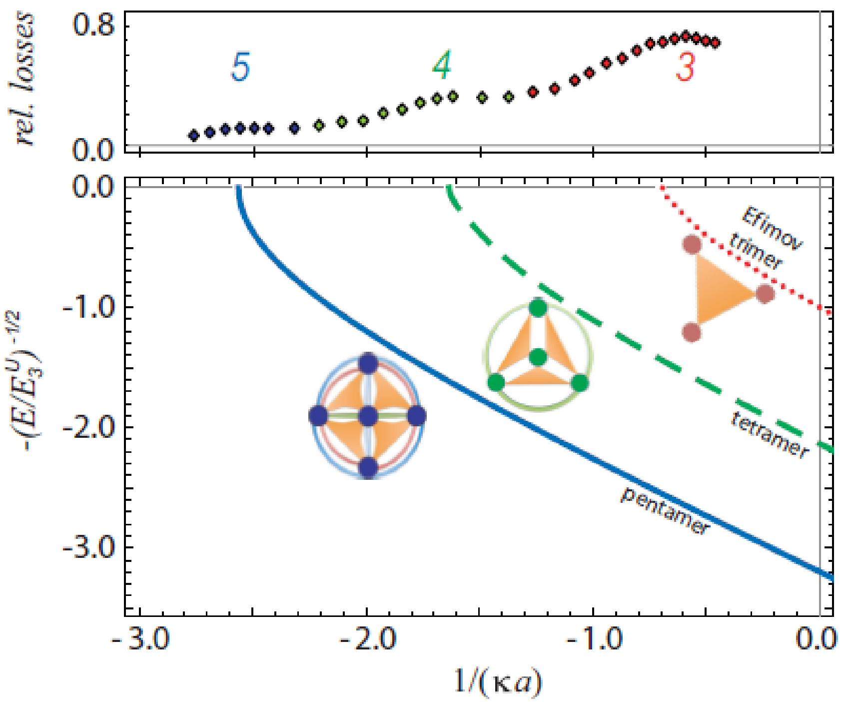
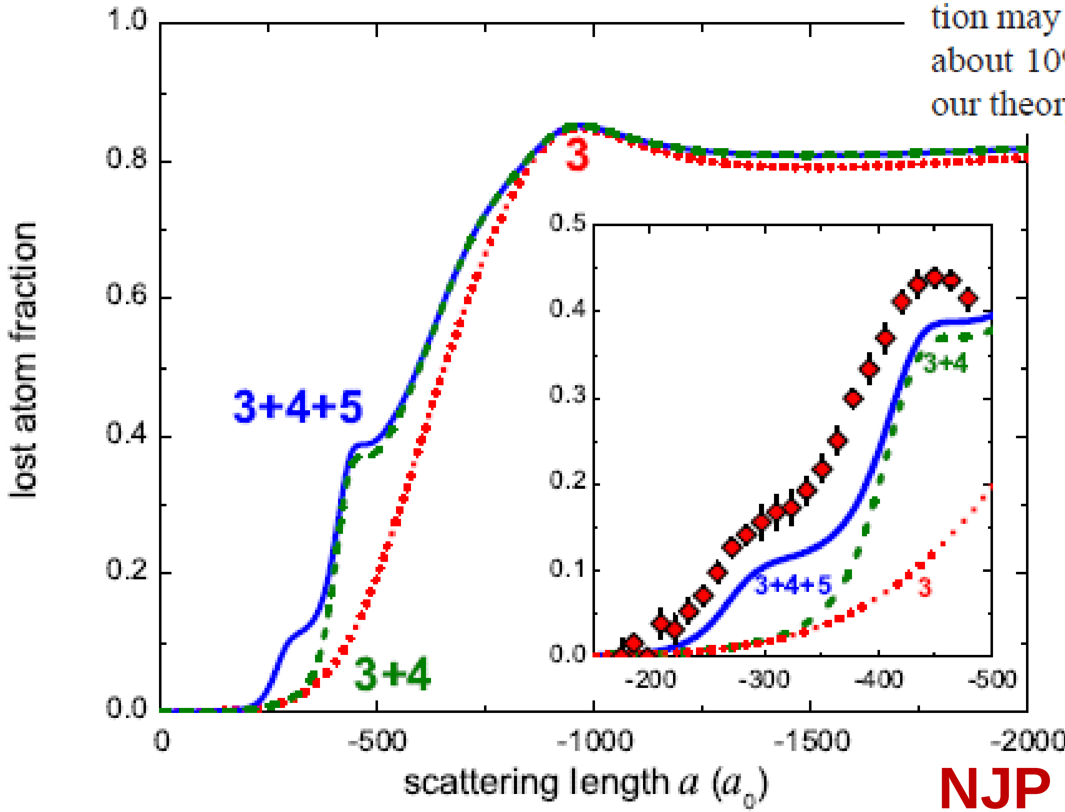


FIG. 1. (color online) N-body scenario in the region of negative two-body scattering length a. The lower panel shows the N-body binding energies as a function of the inverse scattering length. $E_3^U = (\hbar\kappa)^2/m$ is the trimer binding energy for resonant interaction. The dotted,

arXiv:1205.1921 and 2013 NJP , Zenesini et al.

count for the experimental observations. Remarkably, the resonance position $a_{5,-} = 0.64(2)\,a_{4,-}$ is in agreement with the theoretical predictions $0.65(1)\,a_{4,-}$ [37, 38]. However, quantitatively, the experimental values for L_5 are about 15 times larger than the calculated ones. To account for this, we introduce a corresponding scaling factor. We find that this deviation may be explained by a small error in the WKB phase γ of about 10%, which remains in a realistic uncertainty range of our theory.



NJP 2013 Zenesini et al.

FIG. 4. (color online) Calculated and measured fraction of loss atoms from an atomic sample of initially 5×10^4 atoms at a temperature of 80 nK after a hold time of 100 ms. The red dotted line corresponds to the losses predicted for three-body recombination only, while the dashed green line and the blue solid line include also contributions from four- and five body recombination, as quantified in this work. A

4-body recomb only

5-body recomb only

Zenesini et al., NJP 15, 043040 (2013)

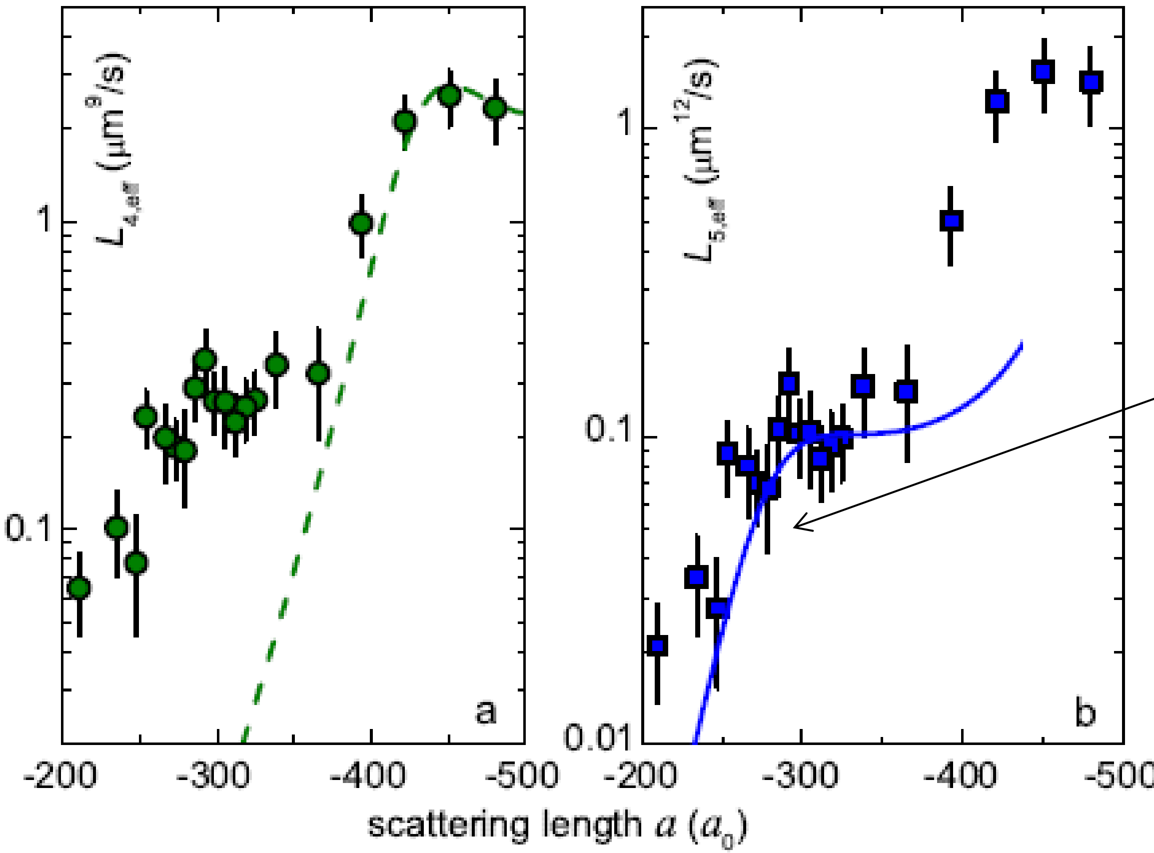


FIG. 3. (color online) Effective four- (a) and five-body recombination rates (b). The green dashed curve and the blue solid line follow the theoretical model for L_4 and L_5, respectively, with additional scaling factor for L_5; see text. The error bars include the statistical uncertainties from the fitting routine, the temperature and the trap frequencies.

Position of the predicted 4-body resonance and now the 5-body resonance is in agreement with experiment! Kewl!

Innsbruck group experiment, theory by von Stecher & CHG

Conclusions:

- Systematics of three-body recombination are getting gradually better known, but much work in theory and experiment still remains to make the treatments as quantitative as possible

- For more than 3 particles, results in experiment as well as in theory are very few in number and it is desirable to encourage further developments on both fronts