Universality and Scaling in Shallow Bound States

Mario Gattobigio

Seattle, 13 May 2014

Outline

Efimov Physics
- Efimov Effect
- Discrete Scale Invariance

Finite-range Effect
- 3-Body Bound States
- Scattering Length
- Recombination
- Measured energies

N-body Universality
- N-Body States
- Universality

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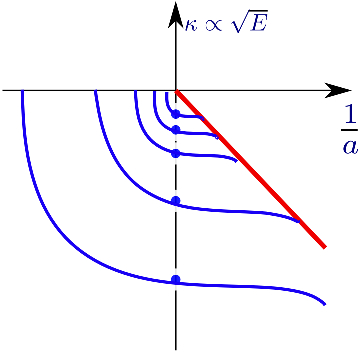
Efimov Effect

$$@1/a = 0 \quad \begin{cases} E_3^n \to 0 & n \to \infty \\ E_3^{n+1}/E_3^n \to 1/515 = 1/(22.7)^2 \end{cases}$$



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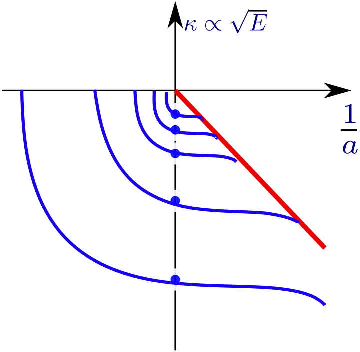
Discrete Scale Invariance

Sornette, Physics Reports 297, 239-270 (1998)



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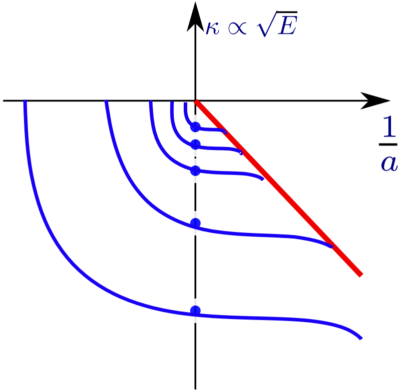

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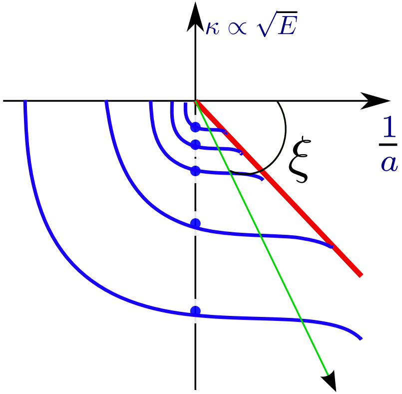

Polar coordinates

$$(H)^2 = (E_3 + E_2)/(\hbar^2/m)$$

$$\tan^2 \xi = E_3/E_2$$

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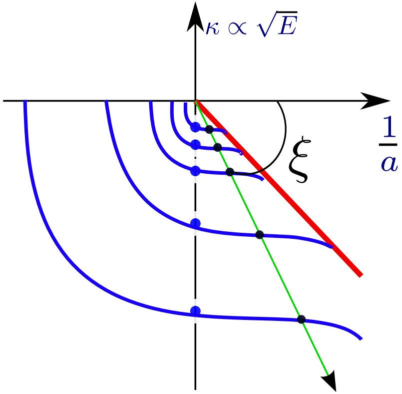

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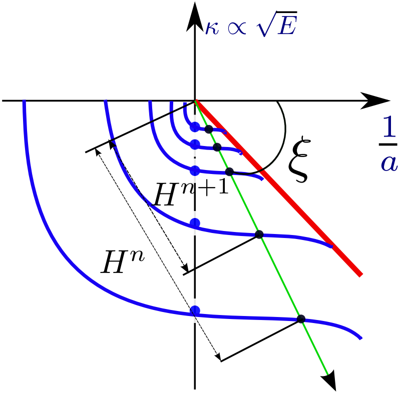

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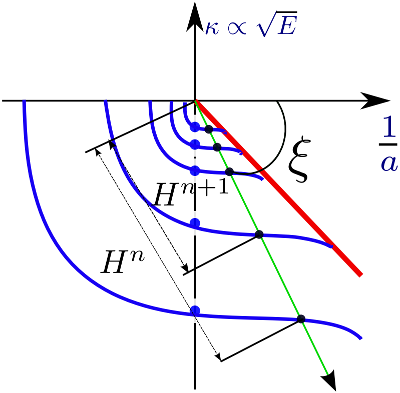

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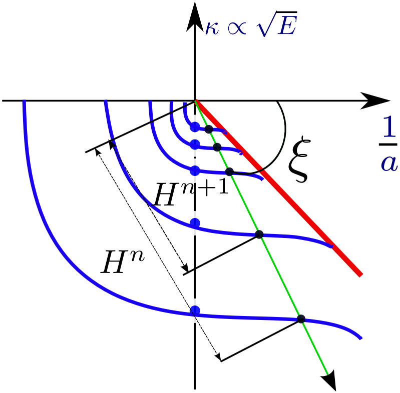

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Recombination Rate at the threshold

$$K_3 = \frac{128\pi^2(4\pi - 3\sqrt{3})}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\cot^2[s_0 \ln(\kappa_* a) + \gamma]} \frac{\hbar a^4}{m},$$

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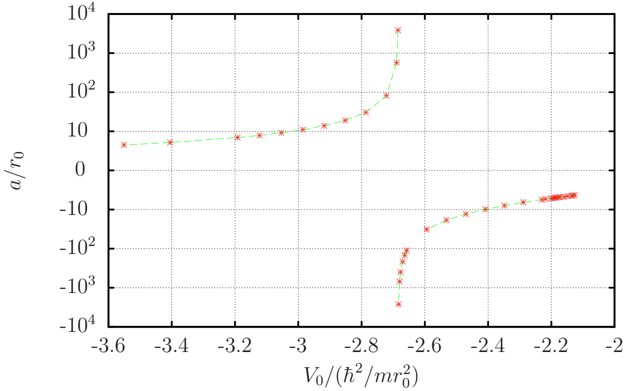
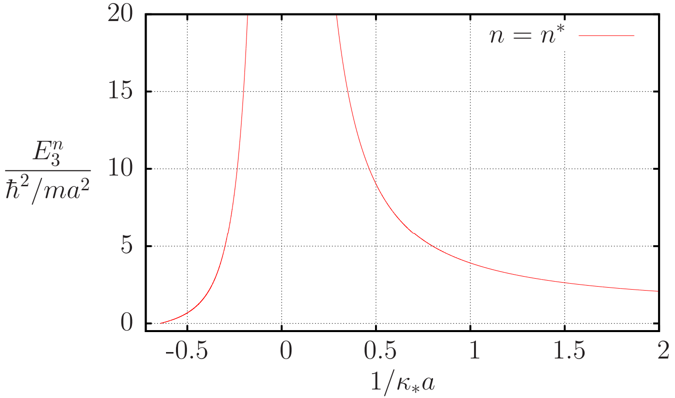


3-Body Bound States



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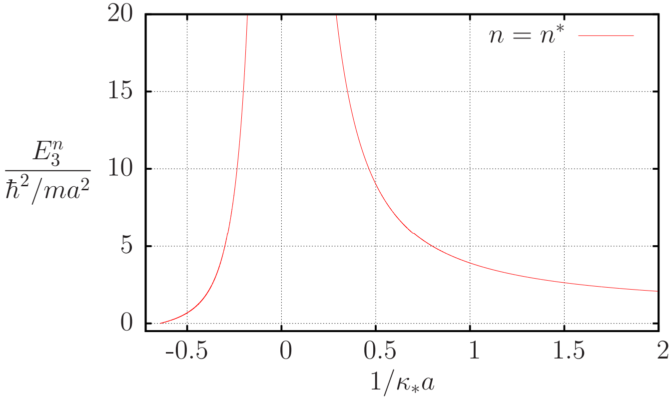

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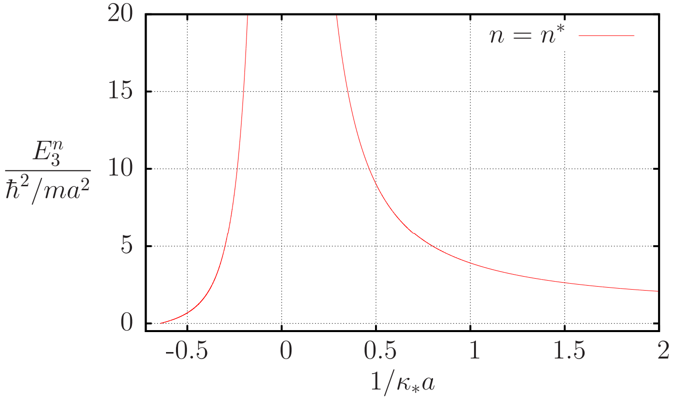

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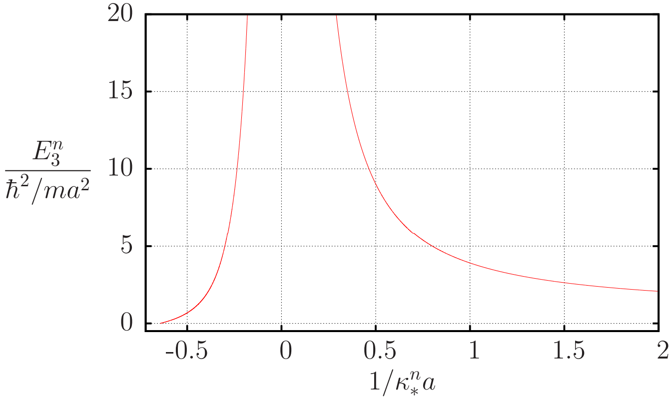

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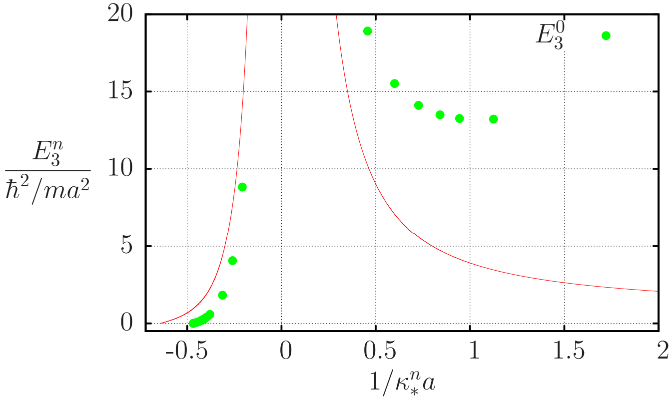

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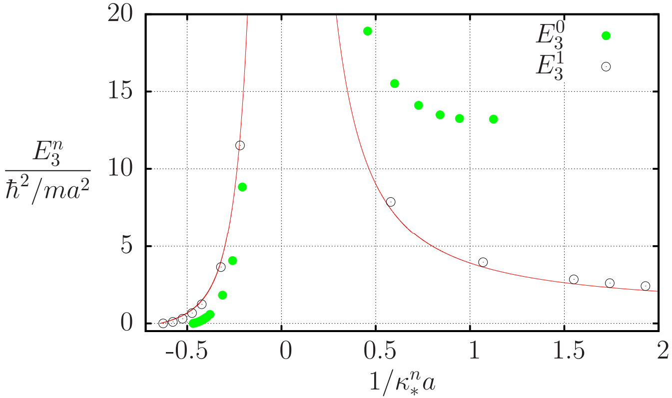

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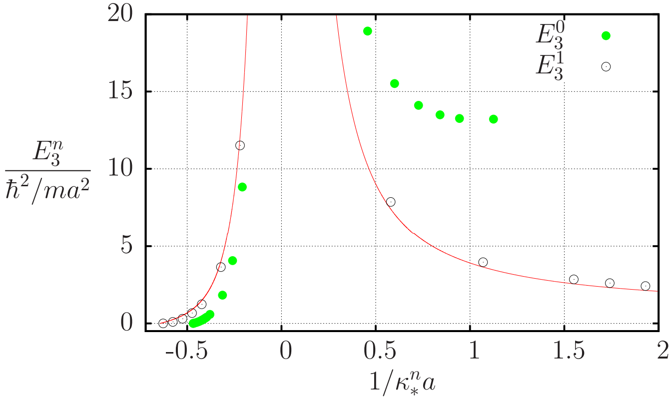

$$\begin{cases} E_3^n/(\hbar^2/m a_B{}^2) = \tan^2\xi \\ \kappa_*^n a_B = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} - \Gamma_n^3 \end{cases}$$

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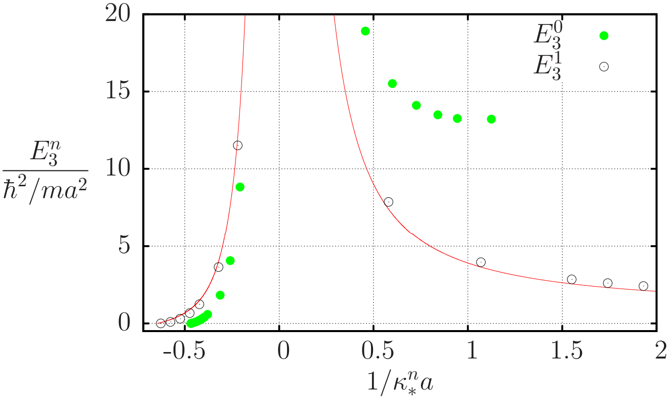

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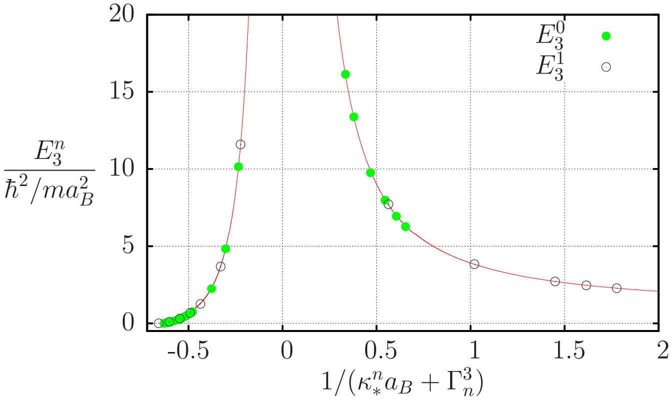

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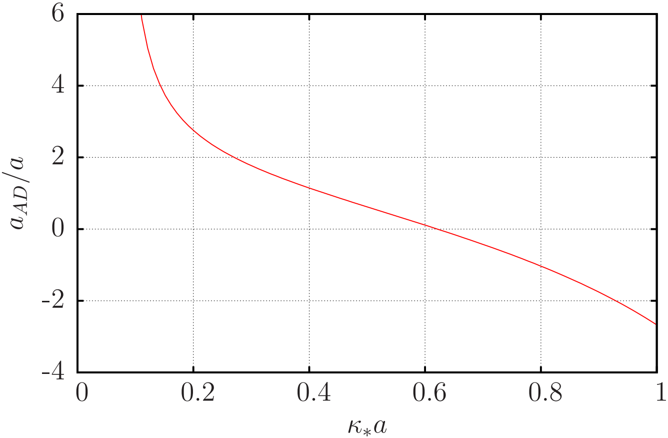

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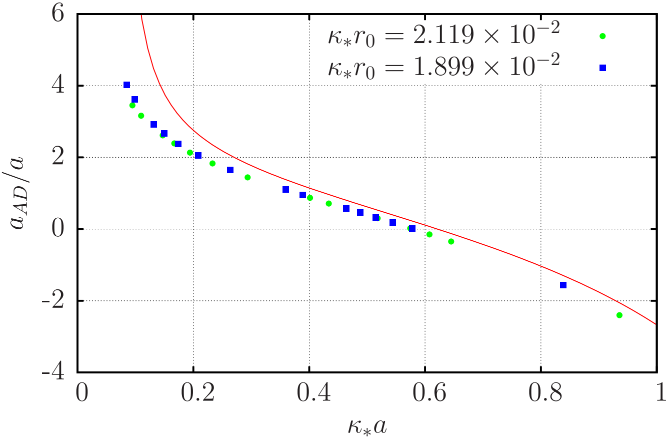

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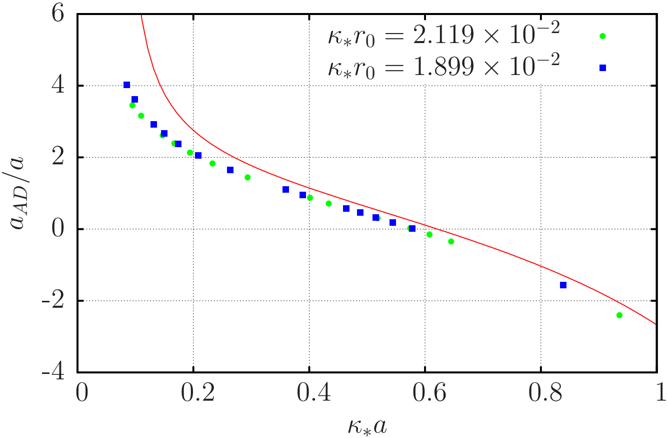

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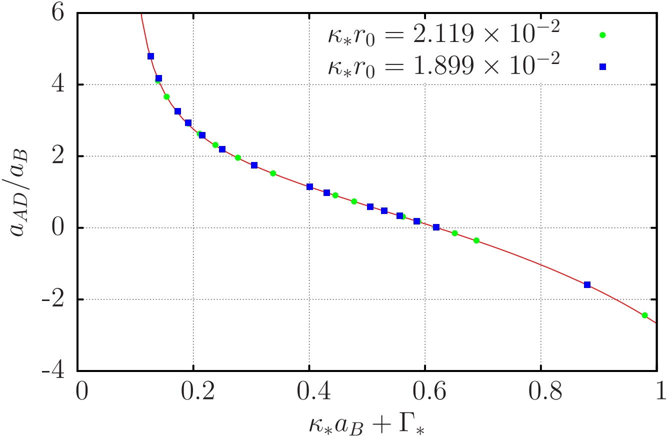

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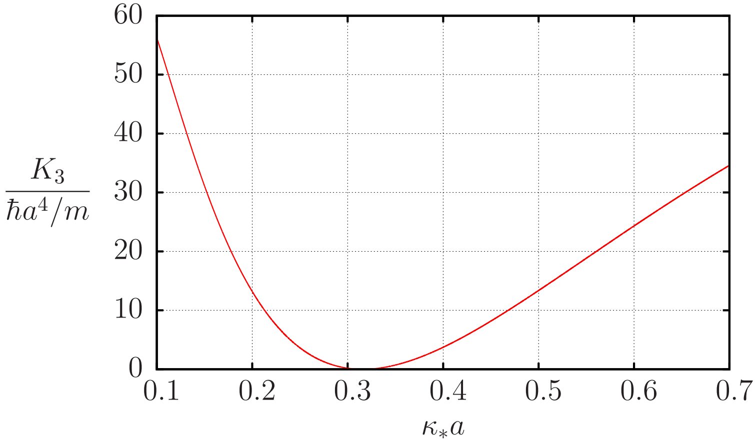

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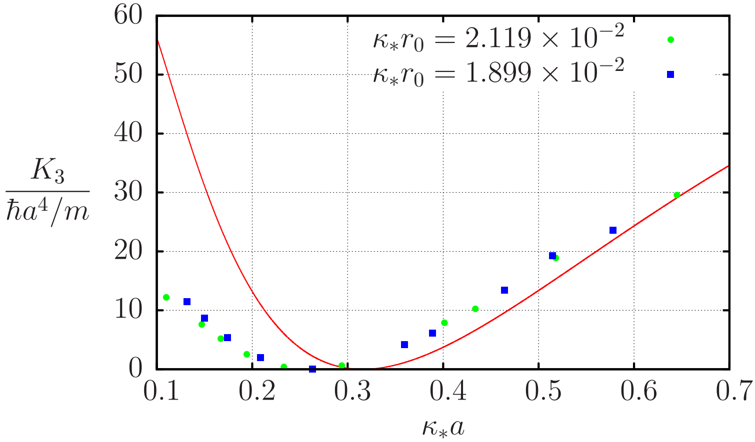

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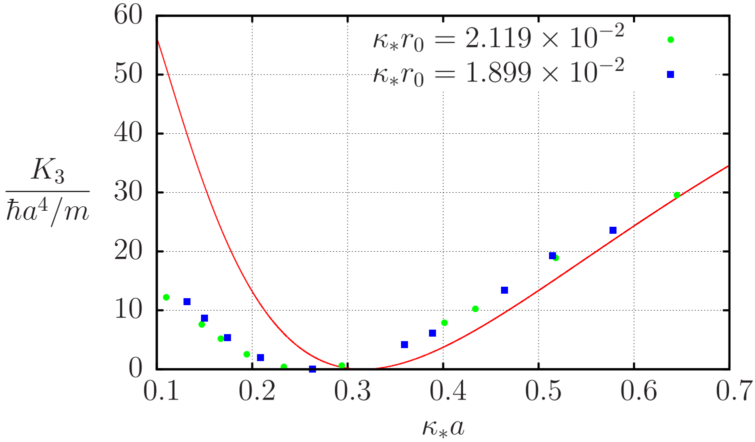

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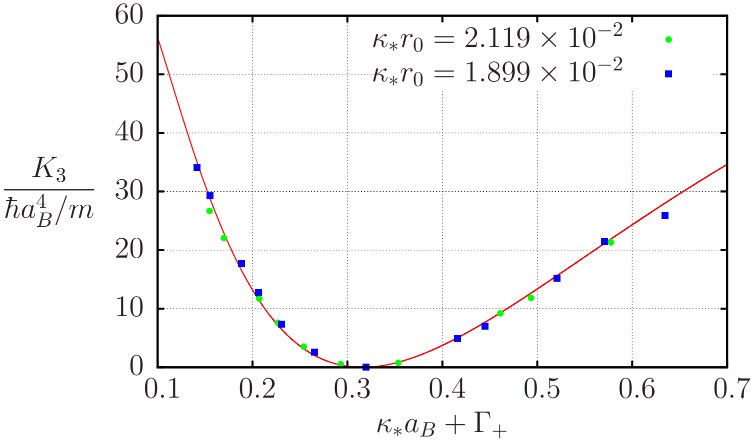

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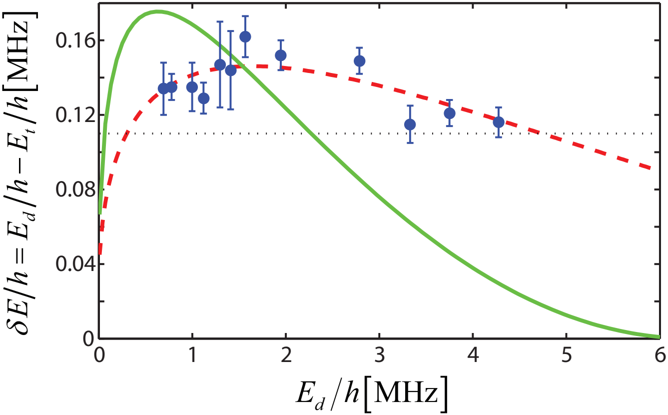

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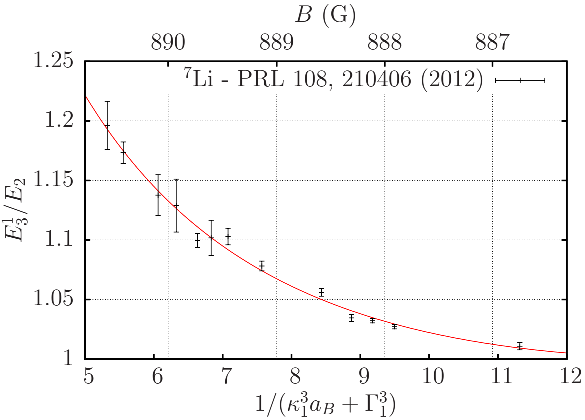

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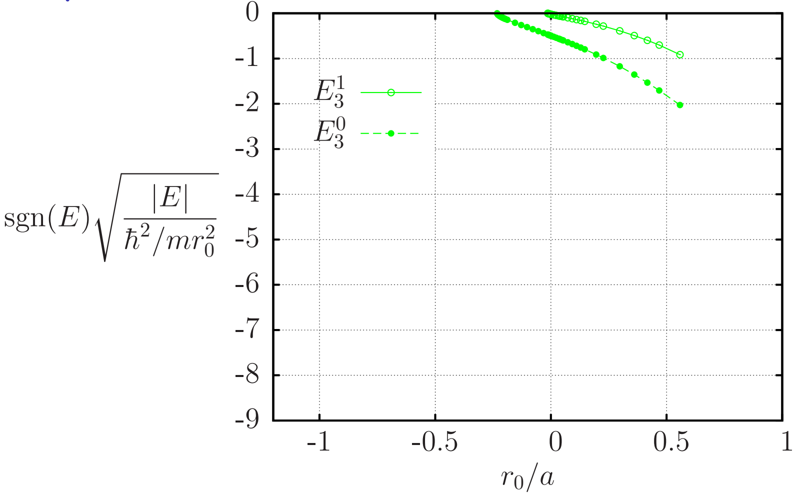

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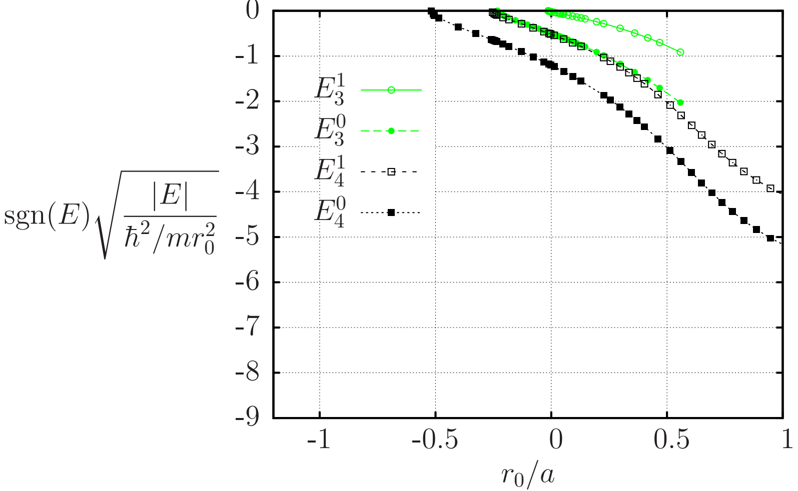

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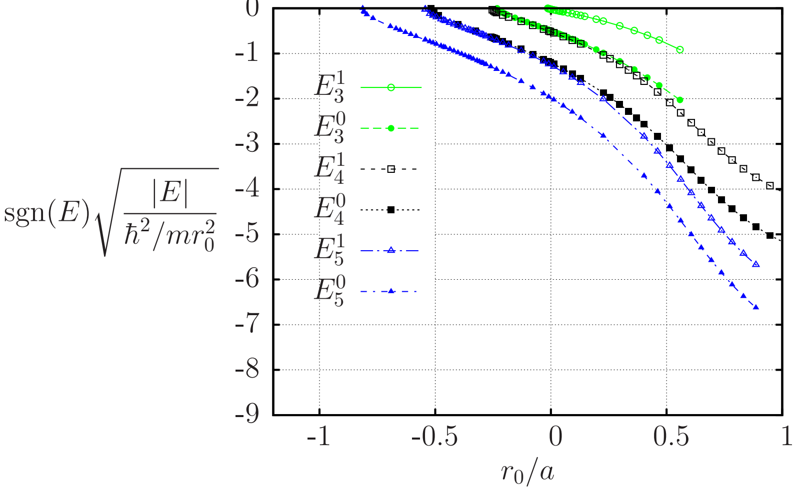

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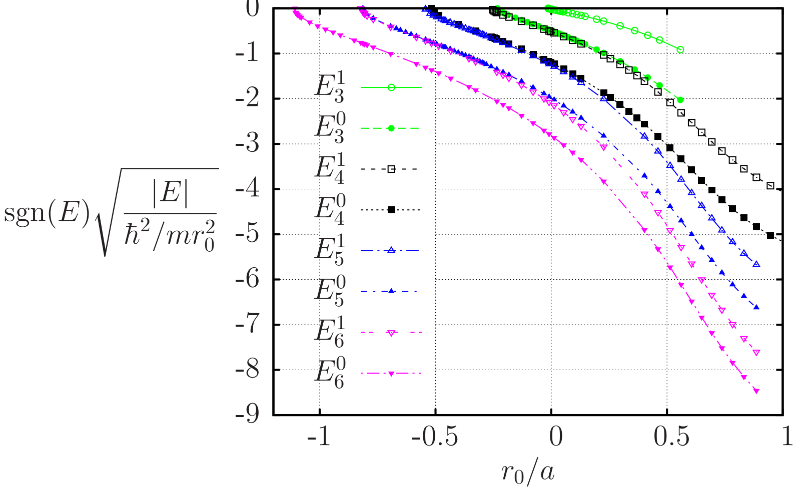

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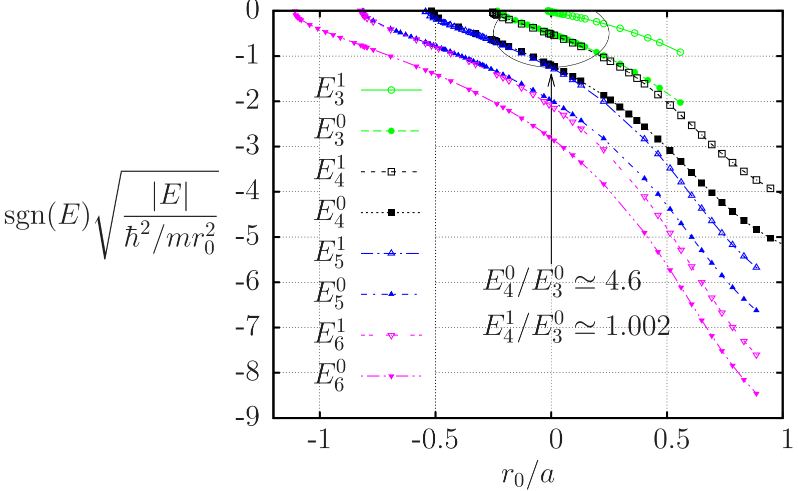

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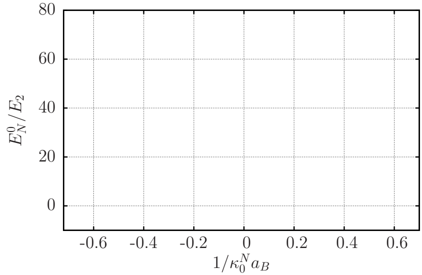

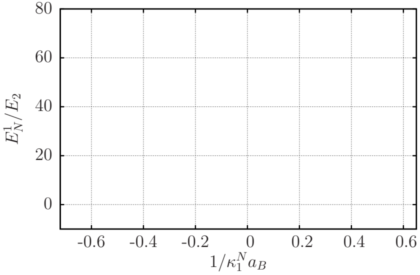

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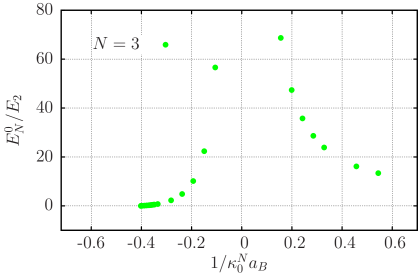

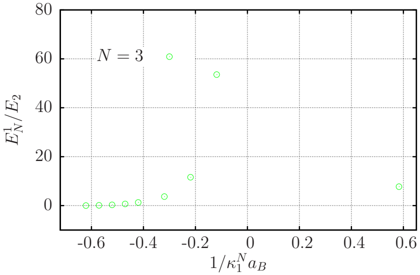

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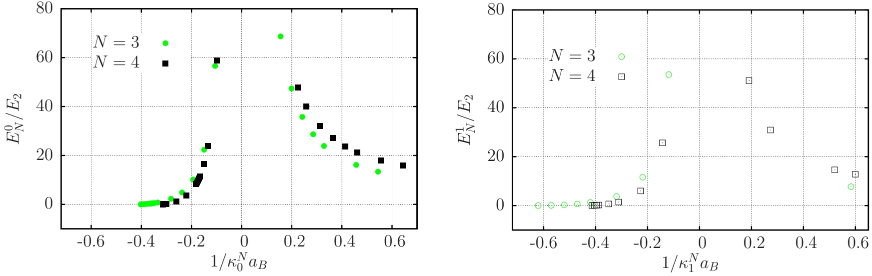

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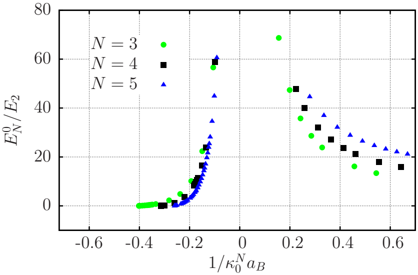

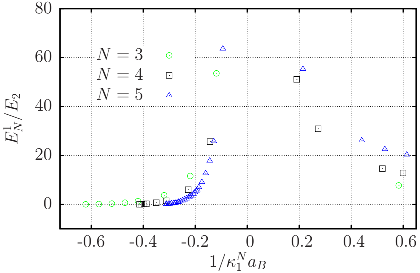

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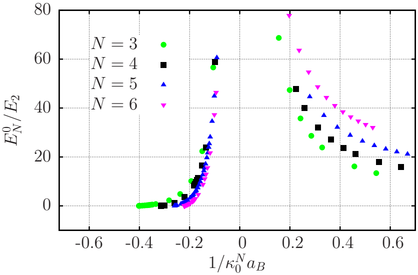

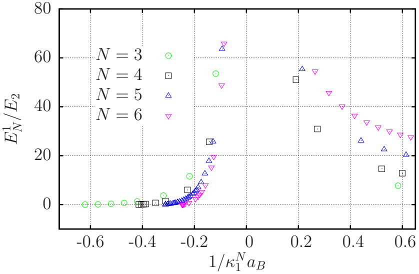

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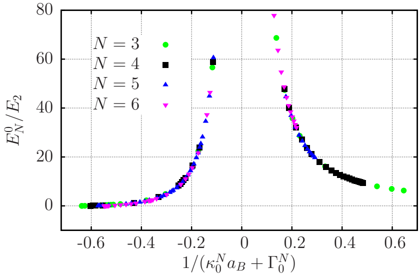

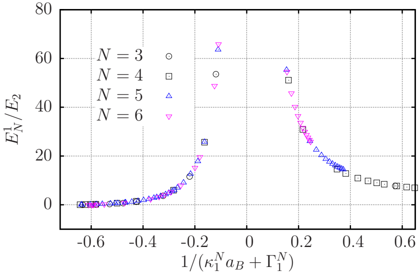

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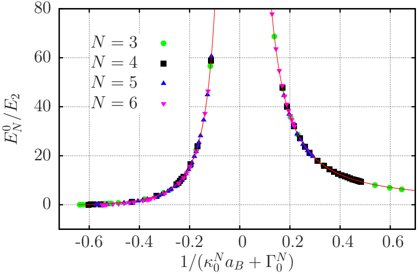

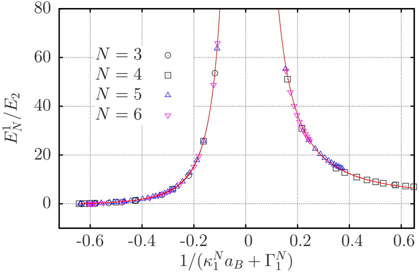

Universal Formula

$$E_N^n/E_2 = \tan^2 \xi$$

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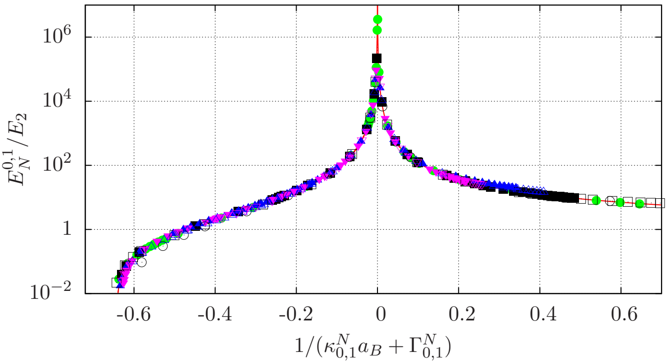


Universal Formula

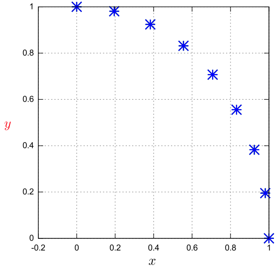
$$E_N^n / E_2 = \tan^2 \xi$$

$$\kappa_n^N a_B + \Gamma_n^N = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$

Efimov Straighteners

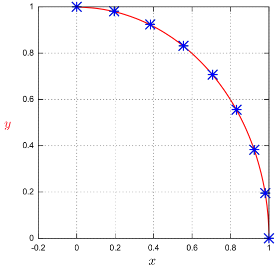
Efimov Straighteners

Data on a Circle



Efimov Straighteners

Data on a Circle

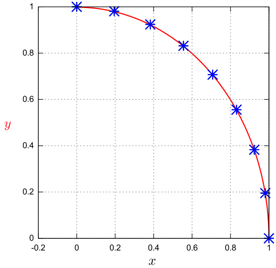


$$y = \sin\xi$$

$$x = \cos\xi$$

Efimov Straighteners

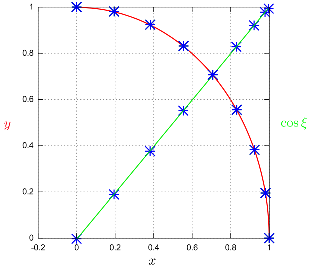
Data on a Circle



$$\begin{matrix} y = \sin\xi \\ x = \cos\xi \end{matrix} \Leftrightarrow \begin{matrix} y/x = \tan\xi \\ x = \cos\xi(x, y) \end{matrix}$$

Efimov Straighteners

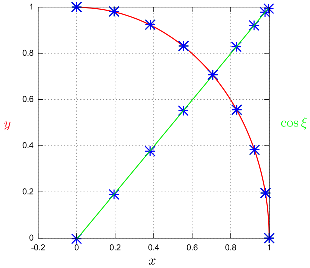
Data on a Circle



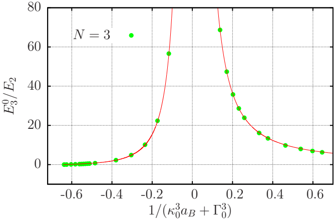
$$\begin{array}{c} y = \sin\xi \\ x = \cos\xi \end{array} \Leftrightarrow \begin{array}{c} y/x = \tan\xi \\ x = \cos\xi(x,y) \end{array}$$

Efimov Straighteners

Data on a Circle



Data on Efimov curve



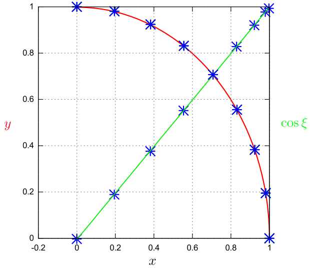
$$y = \sin\xi \quad \Leftrightarrow \quad y/x = \tan\xi$$
$$x = \cos\xi \qquad\quad x = \cos\xi(x,y)$$

$$E_3^0/E_2 = \tan^2\xi$$

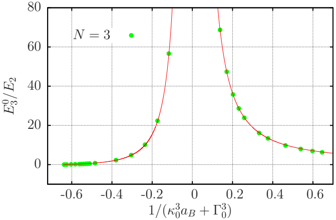
$$\kappa_0^3 a_B + \Gamma_0^3 = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Efimov Straighteners

Data on a Circle



Data on Efimov curve



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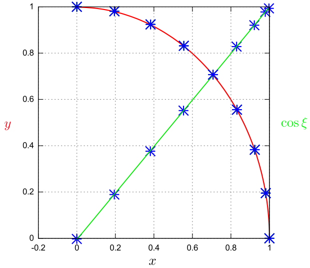
$$E_3^0/E_2 = \tan^2\xi$$

$$\kappa_0^3 a_B + \Gamma_0^3 = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

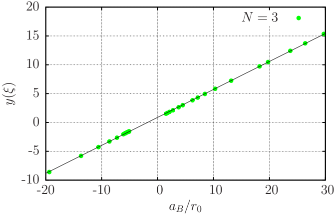
$$y(\xi) \stackrel{\text{def}}{=} \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Efimov Straighteners

Data on a Circle



Data on Efimov curve



$$y = \sin\xi \quad \Leftrightarrow \quad y/x = \tan\xi$$
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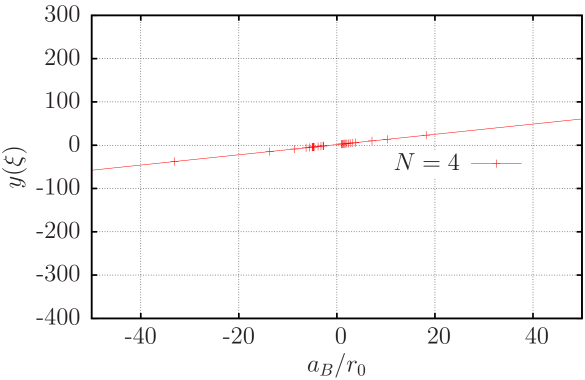
Universality up to $N = 16$

$$y(\xi) = \kappa_N a_B + \Gamma_N$$

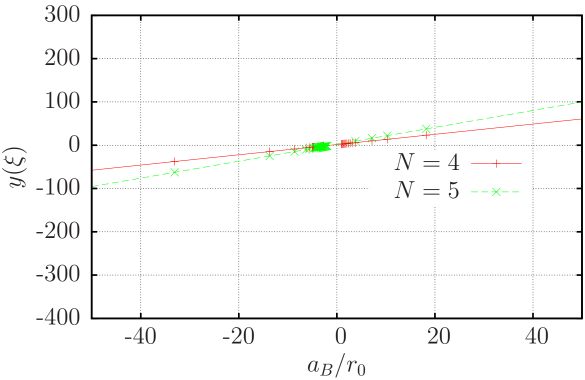


N	$\kappa_N r_0$	Γ_N
4	1.185	1.475

$$V(r) = V_0\ e^{-r^2/r_0^2}$$

Universality up to $N = 16$

$$y(\xi) = \kappa_N a_B + \Gamma_N$$

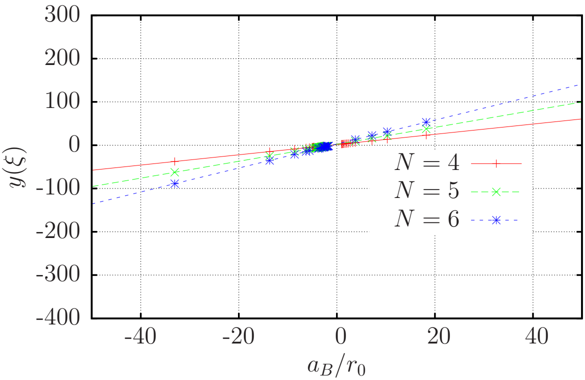


N	$\kappa_N r_0$	Γ_N
4	1.185	1.475
5	1.955	2.128

$$V(r) = V_0 \, e^{-r^2/r_0^2}$$

Universality up to $N = 16$

$$y(\xi) = \kappa_N a_B + \Gamma_N$$

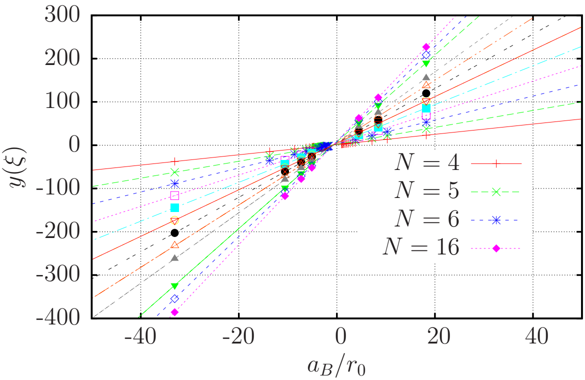


N	$\kappa_N r_0$	Γ_N
4	1.185	1.475
5	1.955	2.128
6	2.770	2.752

$$V(r) = V_0 \; e^{-r^2/r_0^2}$$

Universality up to $N = 16$

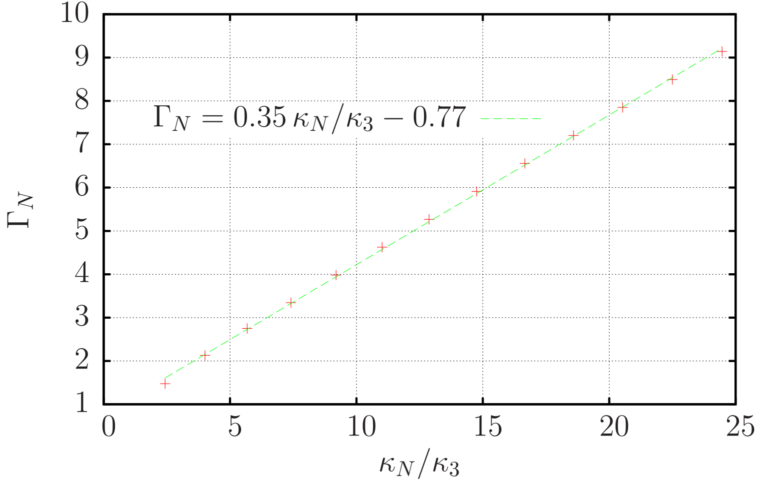
$$y(\xi) = \kappa_N a_B + \Gamma_N$$



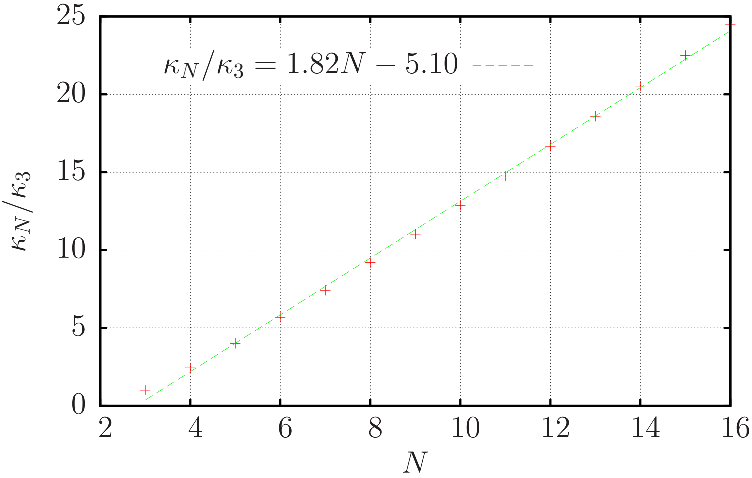
$$V(r) = V_0 \; e^{-r^2/r_0^2}$$

N	$\kappa_N r_0$	Γ_N
4	1.185	1.475
5	1.955	2.128
6	2.770	2.752
7	3.617	3.344
8	4.487	3.983
9	5.377	4.625
10	6.282	5.268
11	7.201	5.912
12	8.131	6.557
13	9.071	7.202
14	10.02	7.848
15	10.98	8.494
16	11.94	9.141

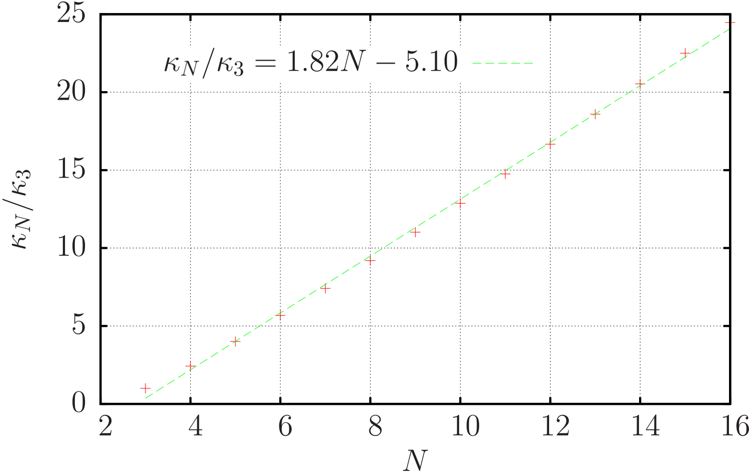
Universality up to $N = 16$



Universality up to $N = 16$

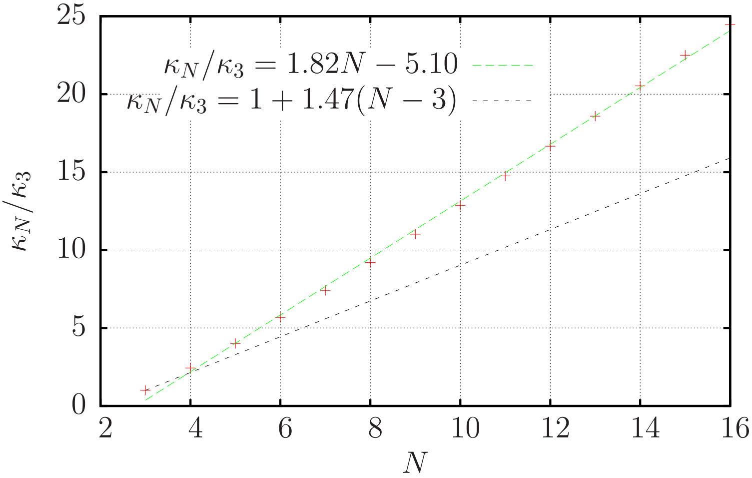


Universality up to $N = 16$



$$\kappa_N/\kappa_3 = 1 + (N - 3)(\kappa_4/\kappa_3 - 1)$$

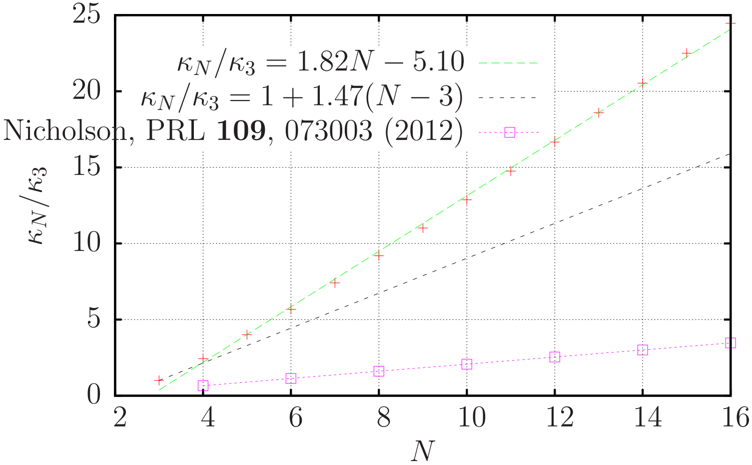
Universality up to $N = 16$



$$\kappa_N/\kappa_3 = 1 + (N-3)(\kappa_4/\kappa_3 - 1)$$
$$= 1 + 1.147(N-3)$$

$\kappa_4 = 2.147\kappa_3$ - Deltuva, Few-Body Syst 54, 569 (2013)

Universality up to $N = 16$



$$\kappa_N/\kappa_3 = 1 + (N - 3)(\kappa_4/\kappa_3 - 1)$$
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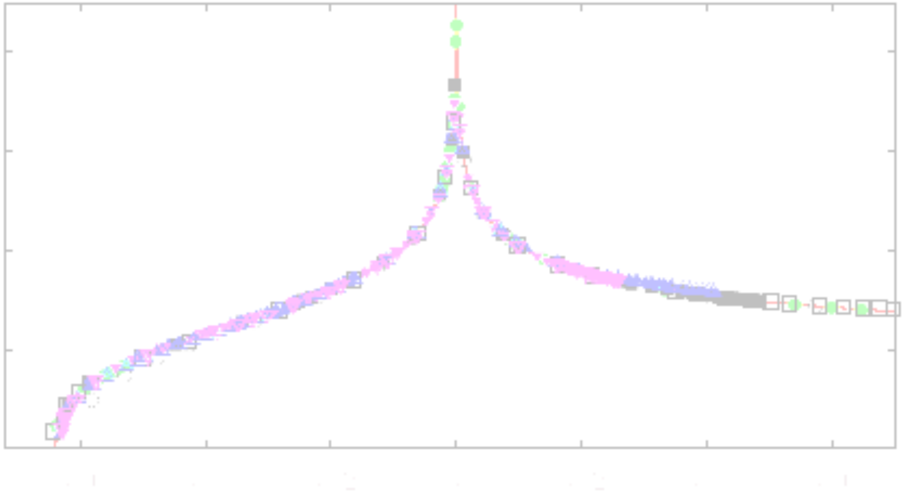

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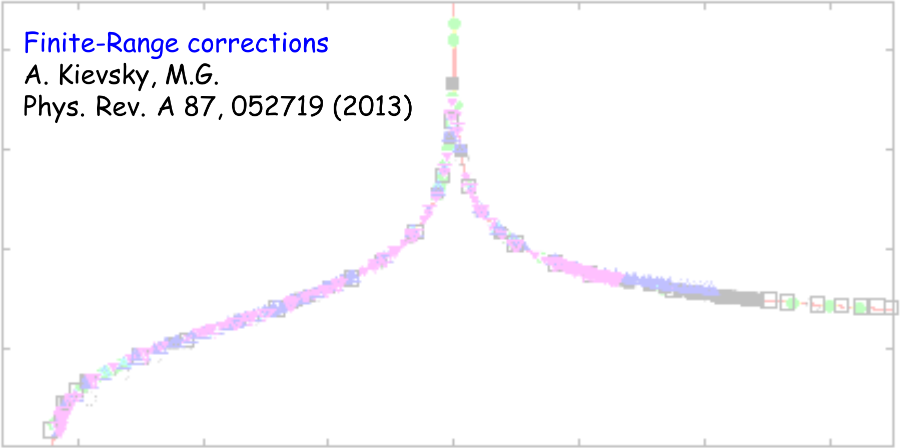

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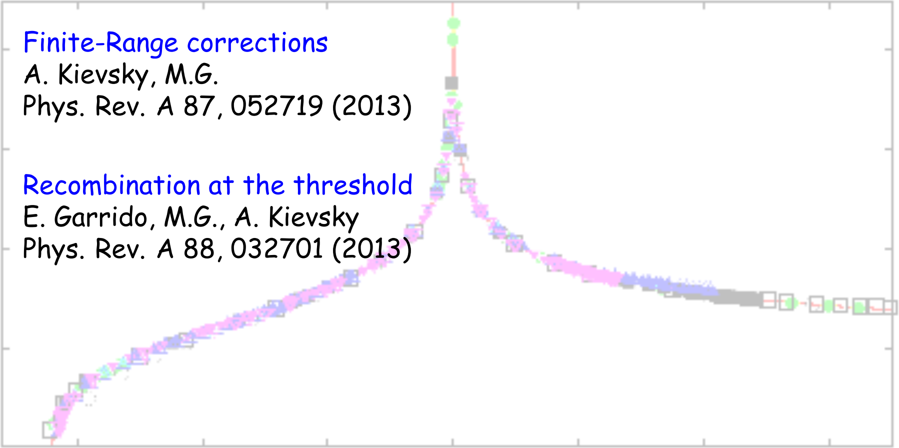

Finite-Range corrections
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Phys. Rev. A 87, 052719 (2013)

Recombination at the threshold
E. Garrido, M.G., A. Kievsky
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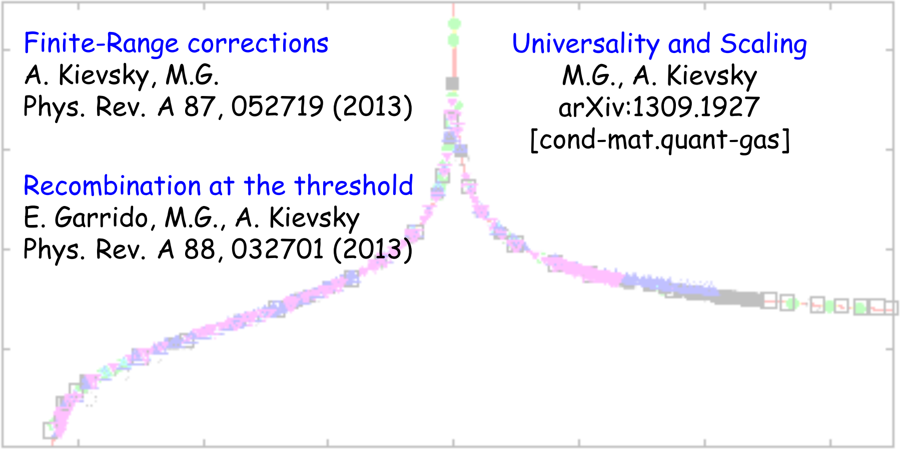

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[cond-mat.quant-gas]

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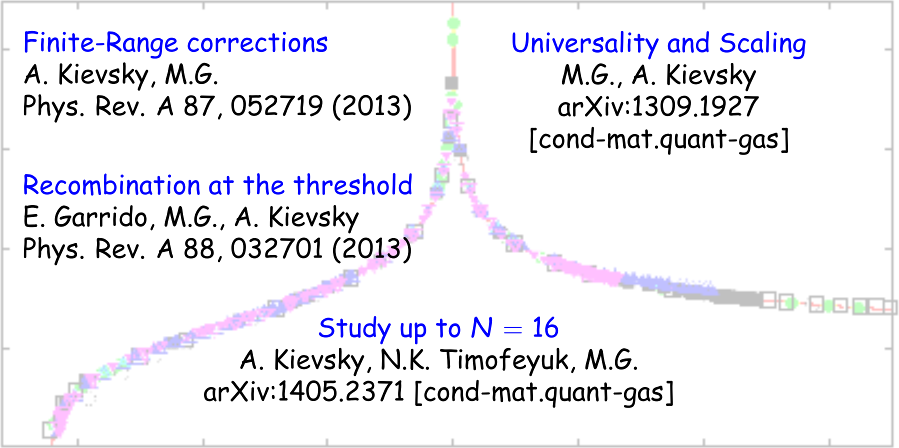

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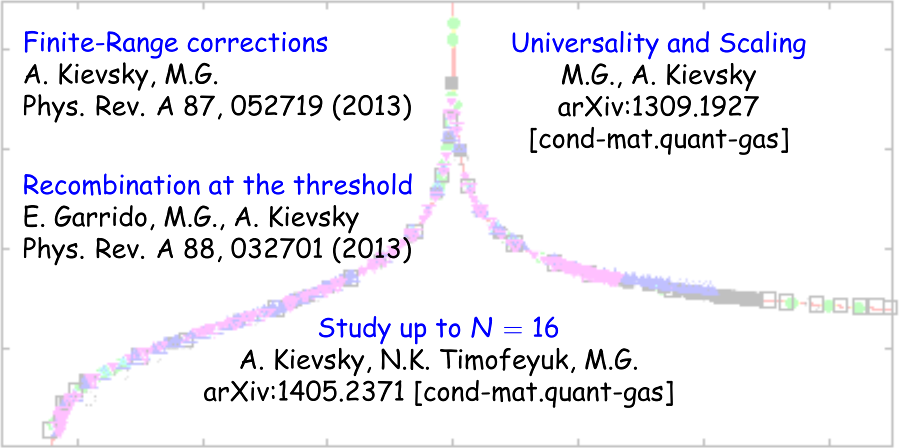

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Thanks!