The Coulomb Problem in Momentum Space without Screening

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(The TORUS Collaboration)

Physics Problem:

Nuclear Reactions dominated by few degrees of freedom

Reactions: Elastic Scattering, Breakup & Transfer

$^{3}He(d,p)^{4}He$

Light nuclei

$^{140}Sn(d,p)^{141}Sn$

Heavy nuclei

?

Physics Problem:

Nuclear Reactions dominated by few degrees of freedom

Reactions: Elastic Scattering, Breakup & Transfer

Theory:

few-body techniques

Continuum
Discretized
Coupled
Channel

Faddeev

Three-particles

$^{3}He(d,p)^{4}He$

$^{140}Sn(d,p)^{141}Sn$

?

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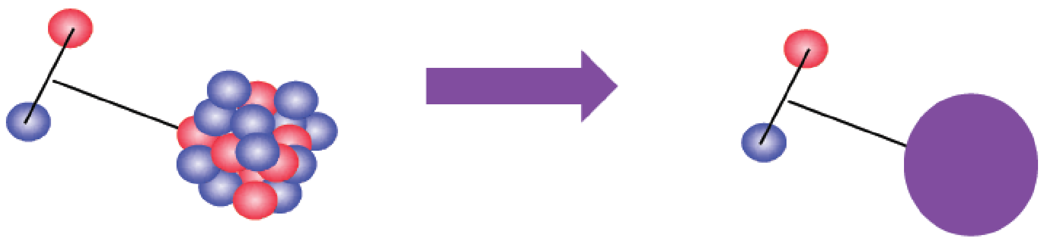
Reduce Many-Body to Few-Body Problem



Task:

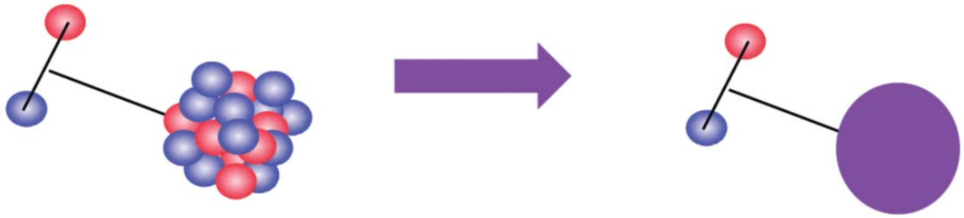
- Isolate important degrees of freedom in a reaction
- Keep track of important channels
- Connect back to the many-body problem

Hamiltonian for effective few-body poblem:

$$H = H_0 + \underbrace{V_{np}}_{\text{np interaction}} + \underbrace{V_{nA} + V_{pA}}_{\text{Optical potentials p+A and n+A}}$$

Three-Body Problem

(d,p) Reactions as three-body problem

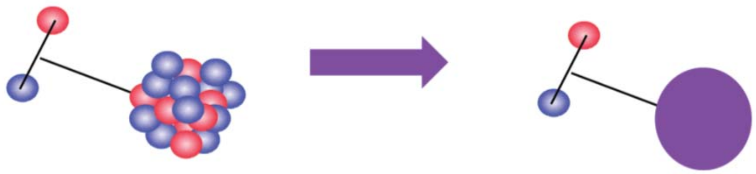


Deltuva and Fonseca, Phys. Rev. C79, 014606 (2009)

Applied Faddeev AGS Equations to $^{12}C(d,p)^{13}C$

Elastic, breakup, rearrangement channels are included and fully coupled
(compared to e.g. CDCC calculations)

(d,p) Reactions as three-body problem



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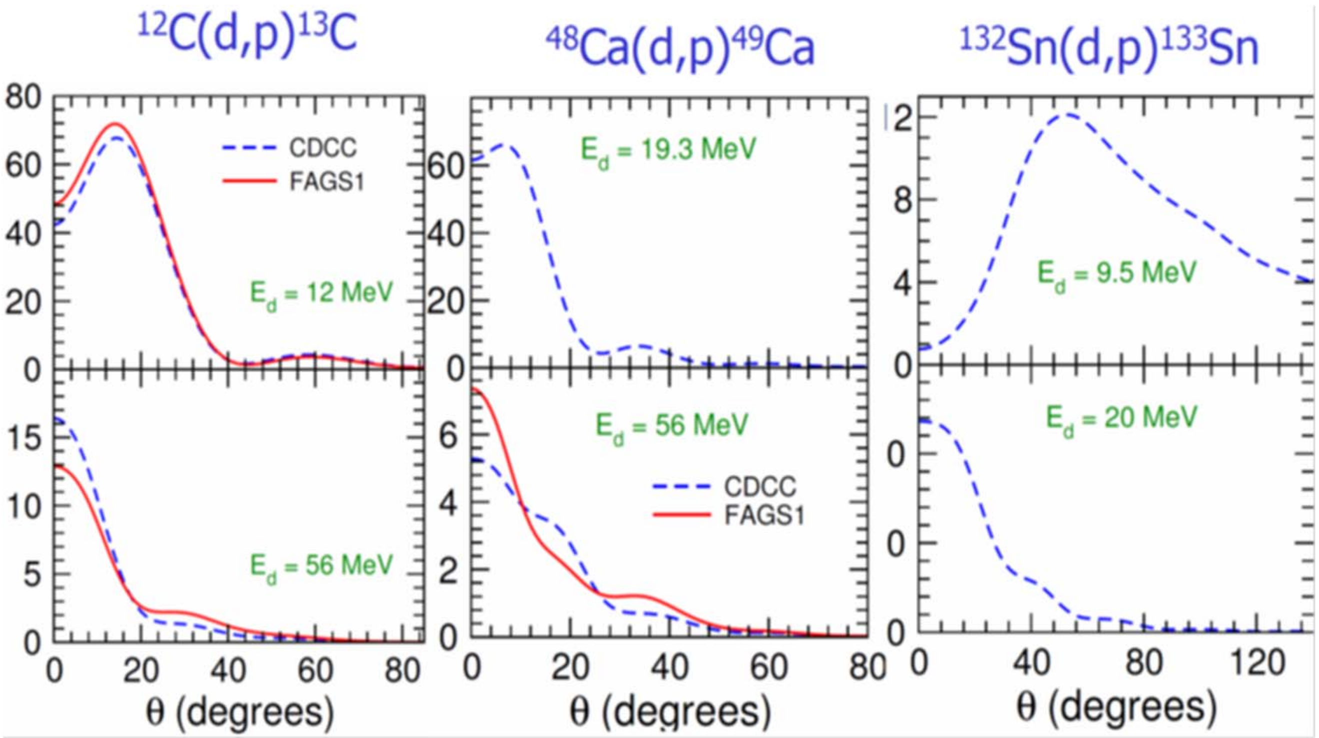

Issue: current momentum space implementation of Coulomb interaction (screening) does **not** converge for Z ≥ 20

Courtesy: F.M. Nunes

A.M. Mukhamedzhanov, V. Eremenko and A.I. Sattarov, Phys.Rev. C86 (2012) 034001

Solve Faddeev equations in Coulomb basis (no screening)

Scattering: Faddeev equations best solved in momentum space

Partial-wave Coulomb function in momentum space $\psi^{C}_{l,p}(q)$

Very nasty! *Oscillatory singular at p=q*

Matrix elements in Coulomb basis:

Example: plane wave basis: **V(p',p) ≡ <p' | V | p>**

Coulomb basis: 2 singularities, for p'=p: "pinch" singularity

Up to now

- *not directly solved*
- *Indirect:* Chinn, CE, Thaler, PRC44, 1569 (1991) for p+A scattering

Work with separable functions: **V(p',p) ≡ ∑ g(p') λ g(p)**

Can we handle this?

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First Test in Two-Body System

Separable t-matrix derived from p+A optical potential (generalized EST scheme)

$$t_l(E) = \sum_{i,j} u|f_{l,k_{E_i}}\rangle \tau_{ij}(E) \langle f^*_{l,k_{E_j}}|u$$

Nuclear matrix elements $\langle p|t_l(E)|p'\rangle$

$$\boxed{\begin{aligned} \langle p|u|f_{l,k_E}\rangle &= t_l(p, k_E; E_{k_E}) \equiv u_l(p) \\ \langle f^*_{l,k_E}|u|p'\rangle &= t_l(p', k_E; E_{k_E}) \equiv u_l(p') \end{aligned}}$$

Coulomb distorted nuclear matrix element

$$\boxed{\begin{aligned} \langle \psi^C_{l,p}|u|f_{l,k_E}\rangle &= \int_0^\infty \frac{dq\; q^2}{2\pi^2}\; u_l(q)\psi^C_{l,p}(q)^\star \equiv u^C_l(p) \\ \langle f^*_{l,k_E}|u|\psi^C_{l,p}\rangle &= \int_0^\infty \frac{dq\; q^2}{2\pi^2}\; u_l(q)\; \psi^C_{l,p}(q) \equiv u^C_l(p)^\dagger \end{aligned}}$$

$\psi^C_{p_\alpha l}(p)$ is the Coulomb scattering wave function

Challenge I: momentum space Coulomb functions

General: $$\psi_{\mathbf{q},\eta}^{C(+)}(\mathbf{p}) = \lim_{\gamma\to+0}\int d^3\mathbf{r}\; e^{-i\mathbf{pr}-\gamma r}\;\psi_{\mathbf{q},\eta}^{C(+)}(r)$$

FT: A. Chan, MS thesis
U. Waterloo (2007)

$$= -4\pi e^{-\pi\eta/2}\Gamma(1+i\eta)\lim_{\gamma\to+0}\frac{d}{d\gamma}\left\{\frac{[p^2-(q+i\gamma)^2]^{i\eta}}{[|\mathbf{p}-\mathbf{q}|^2+\gamma^2]^{1+i\eta}}\right\}$$

Partial wave decomposition (Mukhamedzanov, Dolinskii) (1966)

$$\psi_{\mathbf{q},\eta}^{C(+)}(\mathbf{p}) \equiv \sum_{l=0}^{\infty}(2l+1)\psi_{l,q,\eta}^{C}(p)P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}),\qquad \psi_{l,q,\eta}^{C}(p) = \frac{1}{2}\int_{-1}^{1}dz\;P_l(z)\psi_{\mathbf{q},\eta}^{C(+)}(\mathbf{p}),$$

$$\frac{1}{2}\int_{-1}^{1}dz\;P_l(z)(\zeta-z)^{-1-i\eta} = \frac{e^{\pi\eta}}{\Gamma(1+i\eta)}(\zeta^2-1)^{-i\eta/2}Q_l^{i\eta}(\zeta)$$

$$\zeta \equiv \frac{p^2+q^2+\gamma^2}{2pq},$$

Essential:

$Q_l^{i\eta}(\zeta)$ has different representations depending on ζ

$Q_l^{i\eta}(\zeta)$ has different representations in terms of the hypergeometric function $_2F_1$ (a;b;c;z) depending on ζ

ζ large enough (p and q different) → **"regular" representation**

$$Q_l^{i\eta}(\zeta) = \frac{e^{-\pi\eta}\Gamma(l+i\eta+1)\Gamma(1/2)}{2^{l+1}\Gamma(l+3/2)}(\zeta^2-1)^{i\eta/2}\,\zeta^{-l-i\eta-1} \times {}_2F_1\left(\frac{l+i\eta+2}{2},\frac{l+i\eta+1}{2};l+\frac{3}{2};\frac{1}{\zeta^2}\right)$$

ζ ≈ 1 (p ≈ q) → **"pole-proximity" representation**

$$Q_l^{i\eta}(\zeta) = \frac{1}{2}e^{-\pi\eta}\left\{\Gamma(i\eta)\left(\frac{\zeta+1}{\zeta-1}\right)^{i\eta/2} {}_2F_1\left(-l,l+1;1-i\eta;\frac{1-\zeta}{2}\right) + \frac{\Gamma(-i\eta)\Gamma(l+i\eta+1)}{\Gamma(l-i\eta+1)}\left(\frac{\zeta-1}{\zeta+1}\right)^{i\eta/2} {}_2F_1\left(-l,l+1;1+i\eta;\frac{1-\zeta}{2}\right)\right\}$$

Partial-wave momentum space Coulomb functions

"regular" representation

$$\psi^C_{l,q}(p) = -\frac{4\pi\eta e^{-\pi\eta/2} q(pq)^l}{(p^2+q^2)^{1+l+i\eta}} \left[\frac{\Gamma(1+l+i\eta)}{(1/2)_{l+1}}\right] \times {}_2F_1\left(\frac{2+l+i\eta}{2}, \frac{1+l+i\eta}{2}; l+3/2; \frac{4q^2p^2}{(p^2+q^2)^2}\right) \times \lim_{\gamma\to 0}\left[p^2-(q+i\gamma)^2\right]^{-1+i\eta}$$

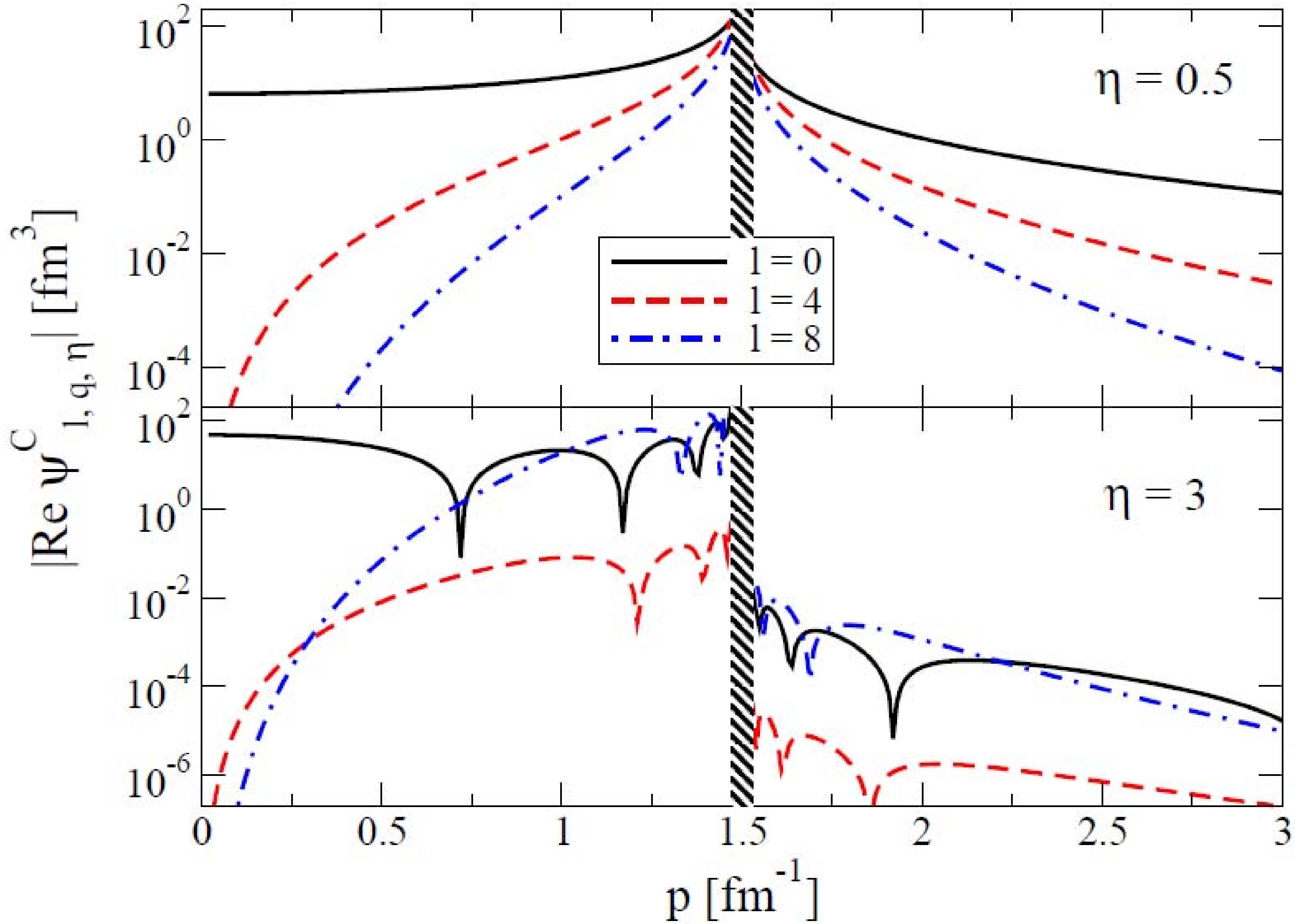
"pole-proximity" representation:

$$\psi^C_{l,q}(p) = -\frac{2\pi}{p}\exp(-\pi\eta/2+i\sigma_l)\left[\frac{(p+q)^2}{4pq}\right]^l \lim_{\gamma\to 0} 2\,\Im\mathrm{m}\,\mathcal{D}.$$

$$\mathcal{D} \equiv \frac{\Gamma(1+i\eta)e^{-i\sigma_l}(p+q)^{-1+i\eta}}{(p-q+i\gamma)^{1+i\eta}}\,{}_2F_1\left(-l, -l-i\eta; 1-i\eta; \frac{(p-q)^2}{(p+q)^2}\right)$$

Oscillatory singularity for p→q

q = 1.5 fm^{-1}



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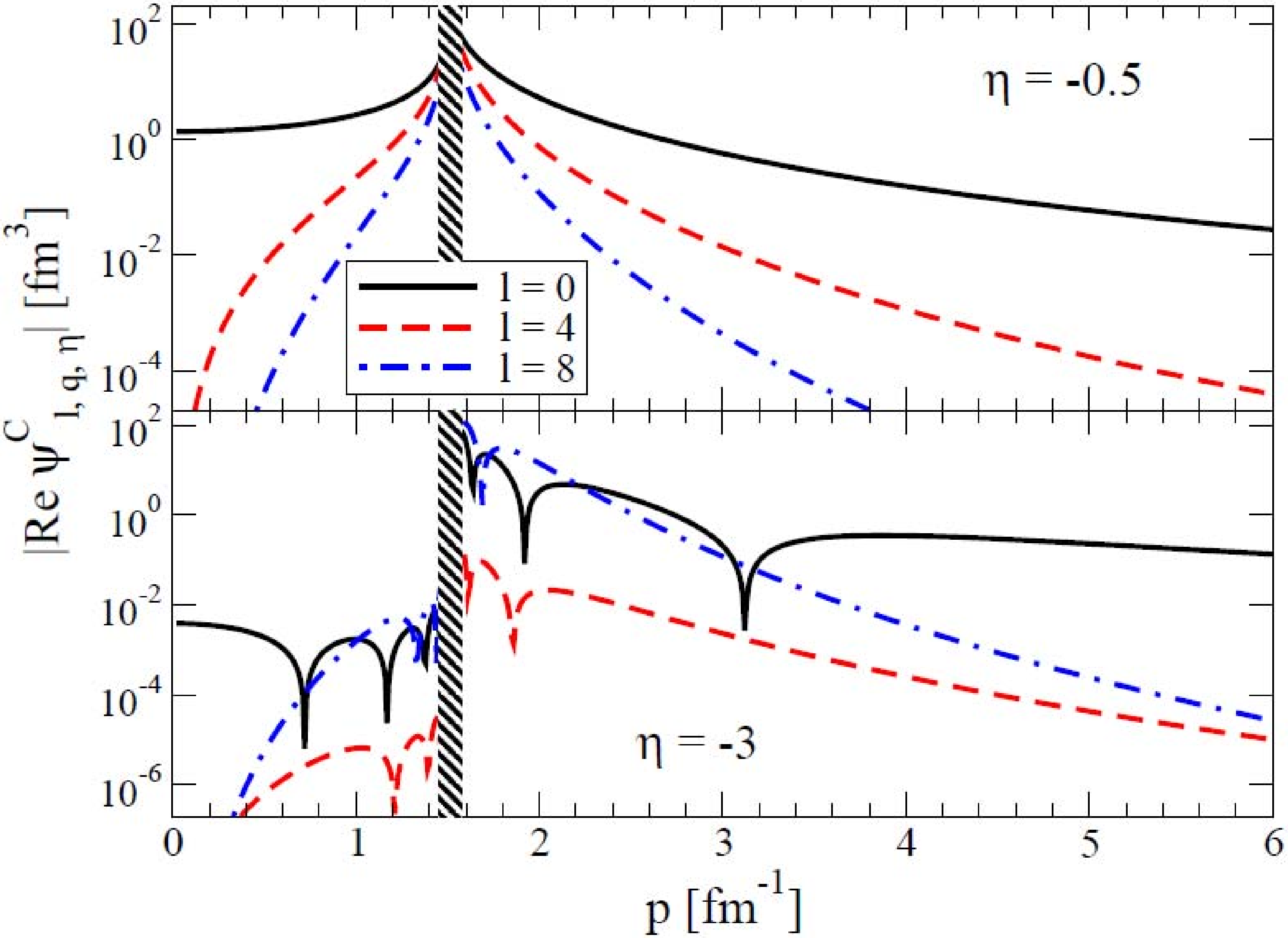

Work in progress: publish code in CPC

Challenge II:
Matrix elements with Coulomb basis functions

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"oscillatory" singularity at $q = p$: $\lim_{\gamma\to+0} \frac{1}{(q-p+i\gamma)^{1+i\eta}}$

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Gel'fand-Shilov Regularization:

Generalization of Principal value regularization
Idea: reduce value of integrand near singularity

$$\int_{-\Delta}^{\Delta} dx \frac{\varphi(x)}{x^{1+i\eta}} = \int_{-\Delta}^{\Delta} dx \frac{\varphi(x) - \varphi(0) - \varphi'(0)x}{x^{1+i\eta}}$$

simplified

$$- \frac{i\varphi(0)}{\eta}[\Delta^{-i\eta} - (\Delta)^{-i\eta}] + \dots$$

- **Reduce integrand around pole by subtracting 2 terms of the Laurent series**

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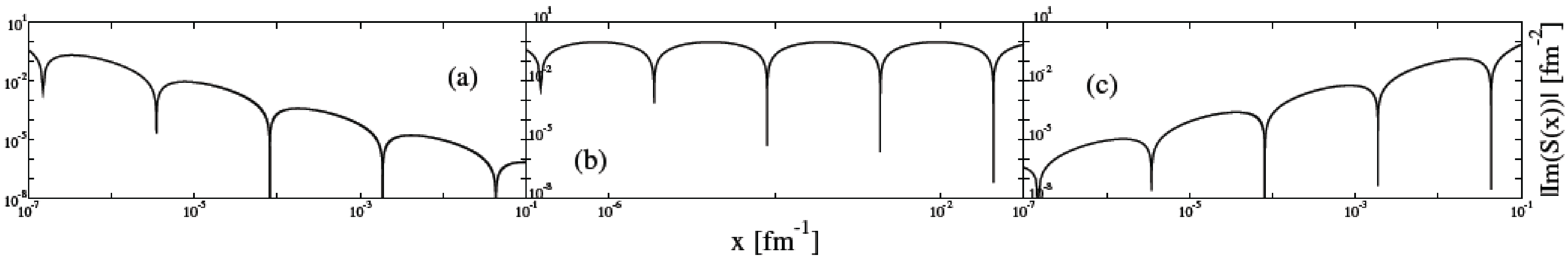

I. M. Gel’fand and G. E. Shilov. “Generalized Functions”. Vol. 1. Academic Press, New York and London. 1964.

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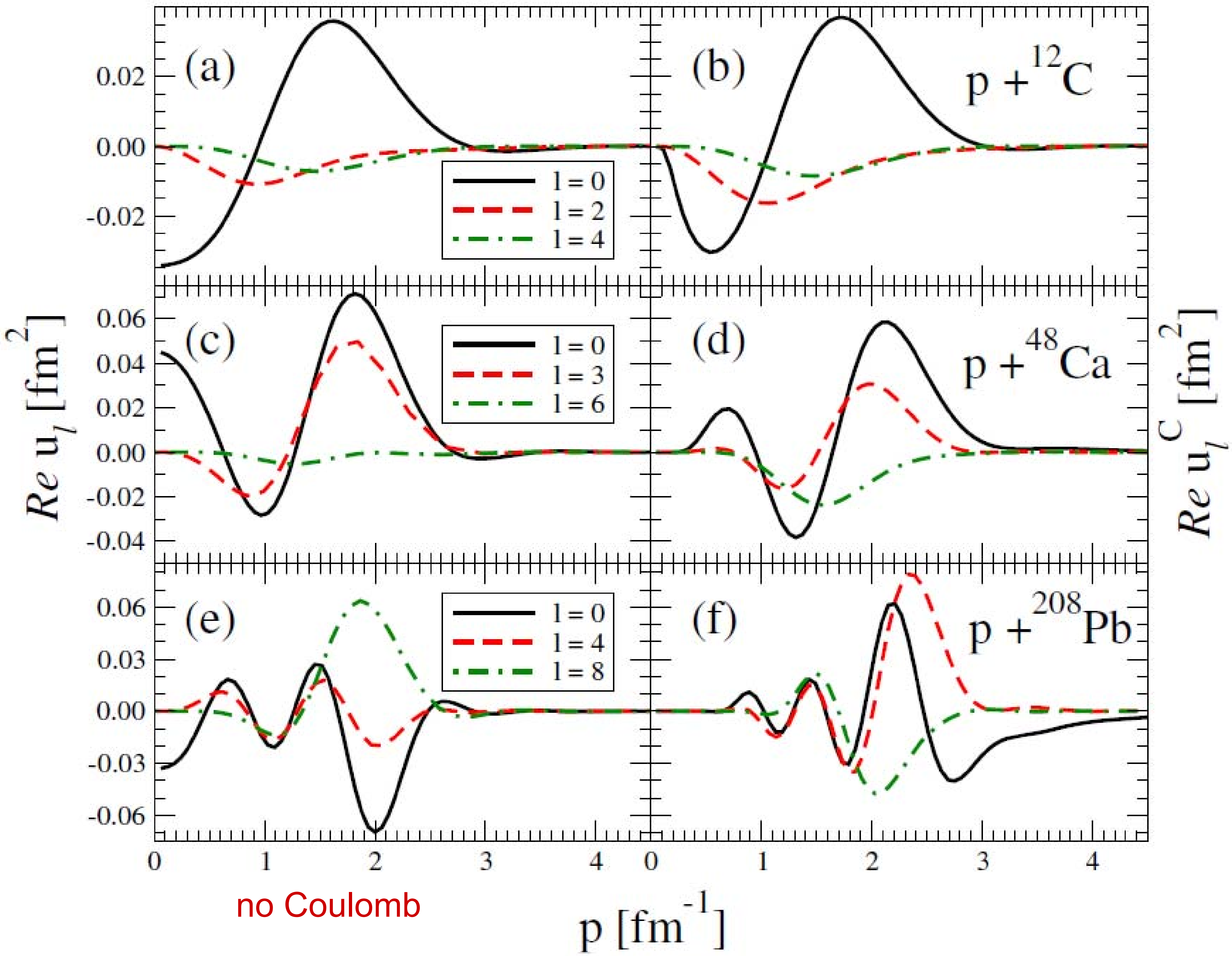
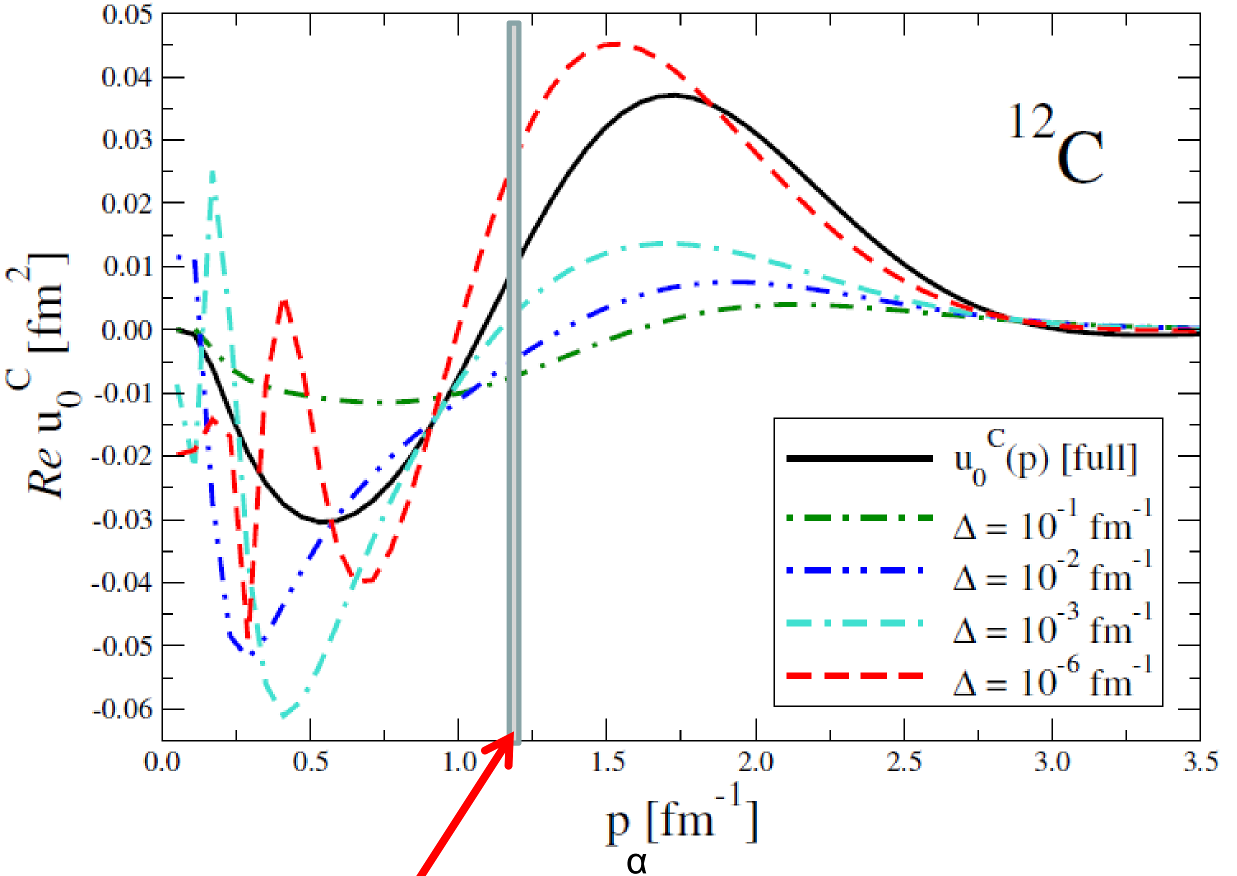


(a)
(b)
$p + ^{12}C$
l = 0
l = 2
l = 4
(c)
(d)
$p + ^{48}Ca$
l = 0
l = 3
l = 6
(e)
(f)
$p + ^{208}Pb$
l = 0
l = 4
l = 8
$Re\ u_l\ [fm^2]$
$Re\ u_l^C\ [fm^2]$
$p\ [fm^{-1}]$

no Coulomb

p + ^{12}C

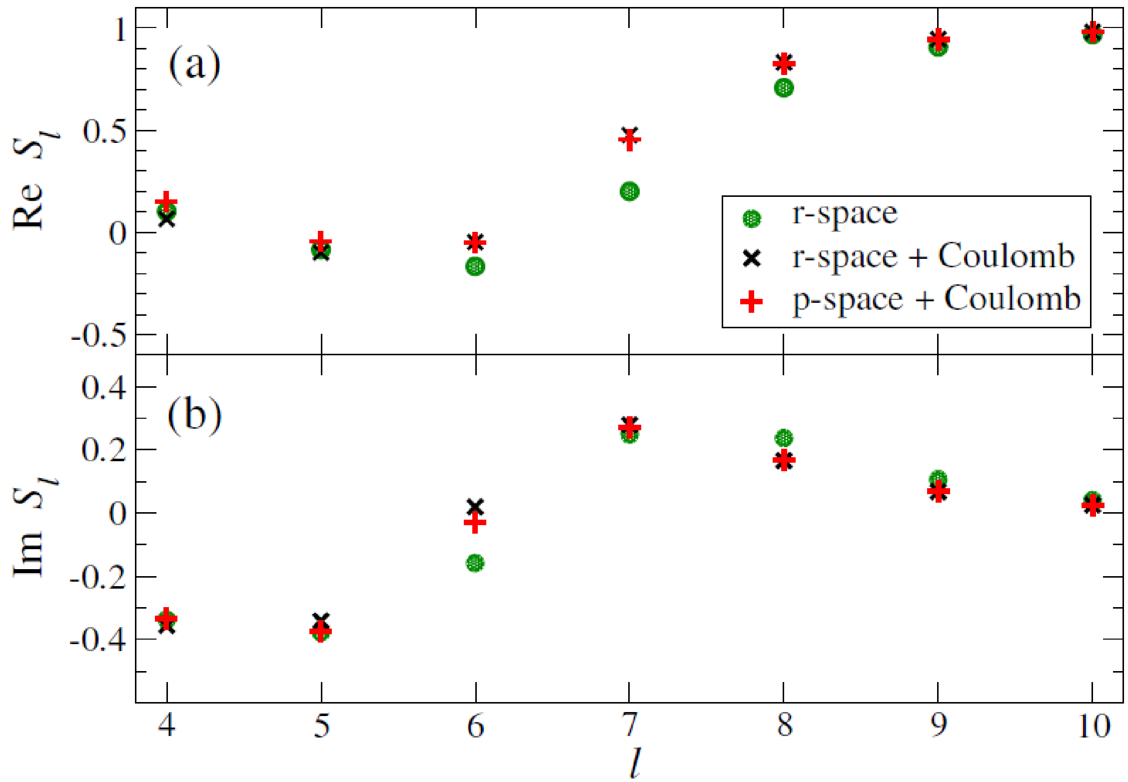
$$u_l^C(p_\alpha) = \overbrace{\int_0^{p_\alpha-\Delta} \frac{dp}{2\pi^2}\, p^2 u_l(p)\, \psi^C_{p_\alpha l}(p)}^{I} + \overbrace{\int_{p_\alpha-\Delta}^{p_\alpha+\Delta} \ldots}^{II} + \overbrace{\int_{p_\alpha+\Delta}^{\infty} \ldots}^{III}$$



Fixed p_α

First Physics Check:

p + ^{48}Ca



Selected partial wave S-matrix elements S_{l+1} for p+^{48}Ca (CH89 optical potential) with Coulomb distorted n+^{48}Ca formfactors

Method not designed for two-body scattering!

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Summary & Outlook

Faddeev-AGS framework in Coulomb basis passed first test!

- **Momentum space nuclear form factors obtained in a Coulomb distorted basis for high charges for the first time.**
- **"Oscillatory singularity" of $\psi_{q,l}{}^{c}(p)$ at p →q successfully regularized.**
- **Algorithms to compute $\psi_{q,l}{}^{c}(p)$ and overlap integrals successfully implemented**

Near Future:

Implementation of Faddeev-AGS equations in Coulomb basis

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TORUS: Theory of Reactions for Ustable iSotopes

A Topical Collaboration for Nuclear Theory

http://www.reactiontheory.org/

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