Three-body finite-volume formalism for lattice QCD

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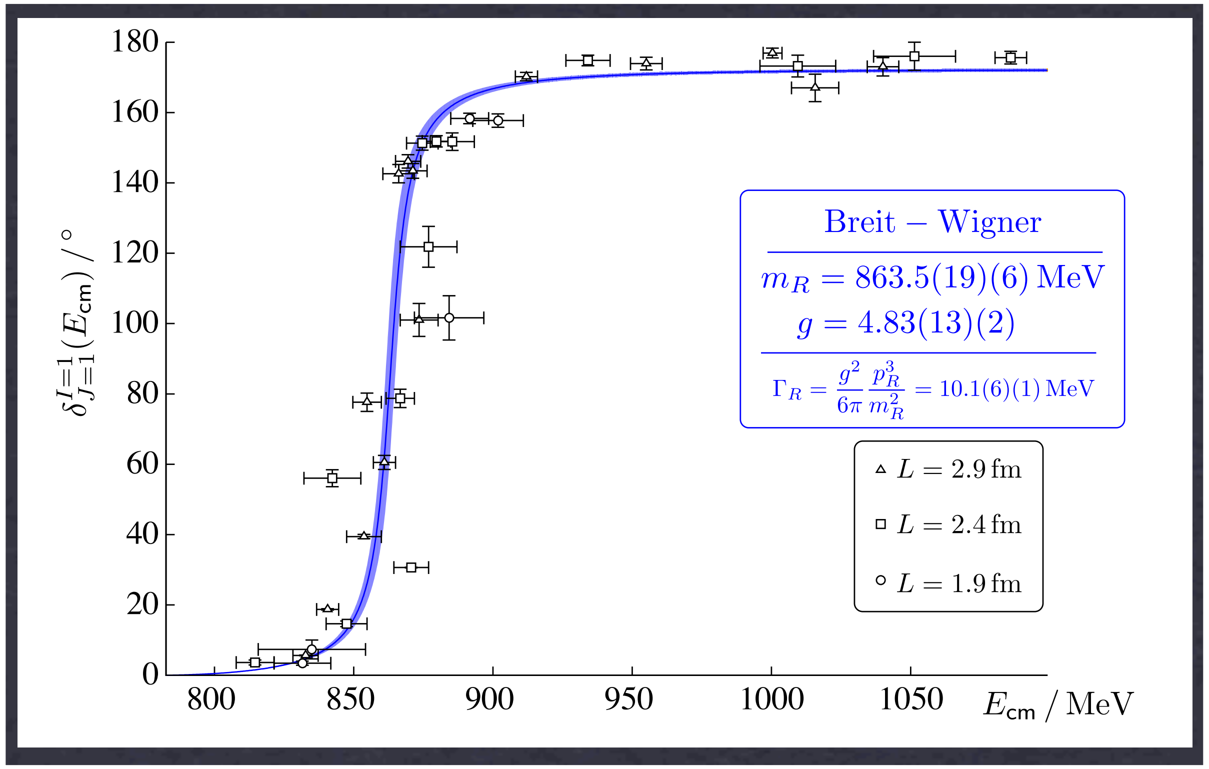
INT 14-1

NUCLEAR PHYSICS FROM LATTICE QCD

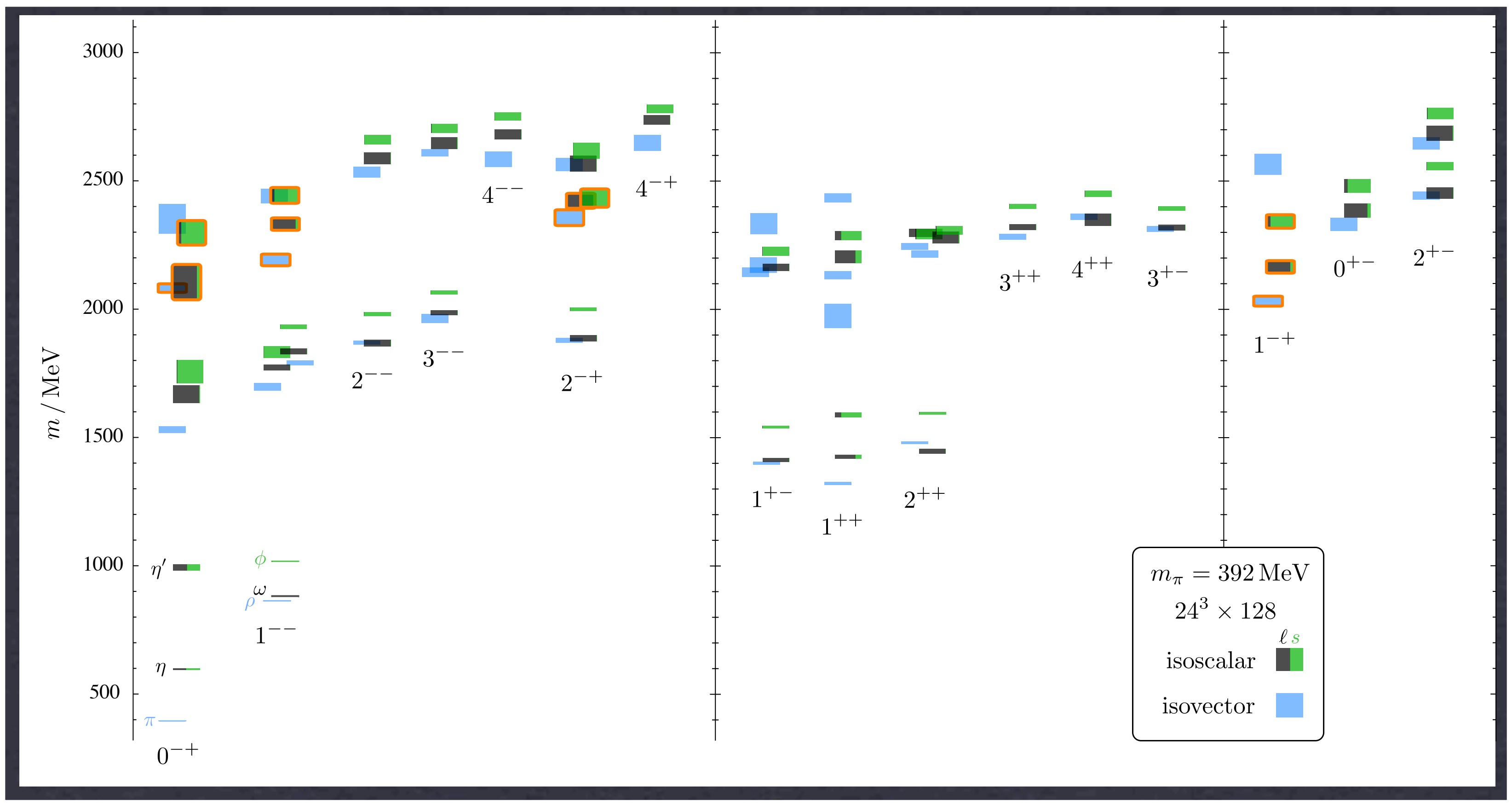
RECENT DEVELOPMENTS

- Hadronic interactions and resonances. Spectrum of QCD? Exotics and gluonic degrees of freedom, etc.



Dudek, et al, arXiv:13092608.

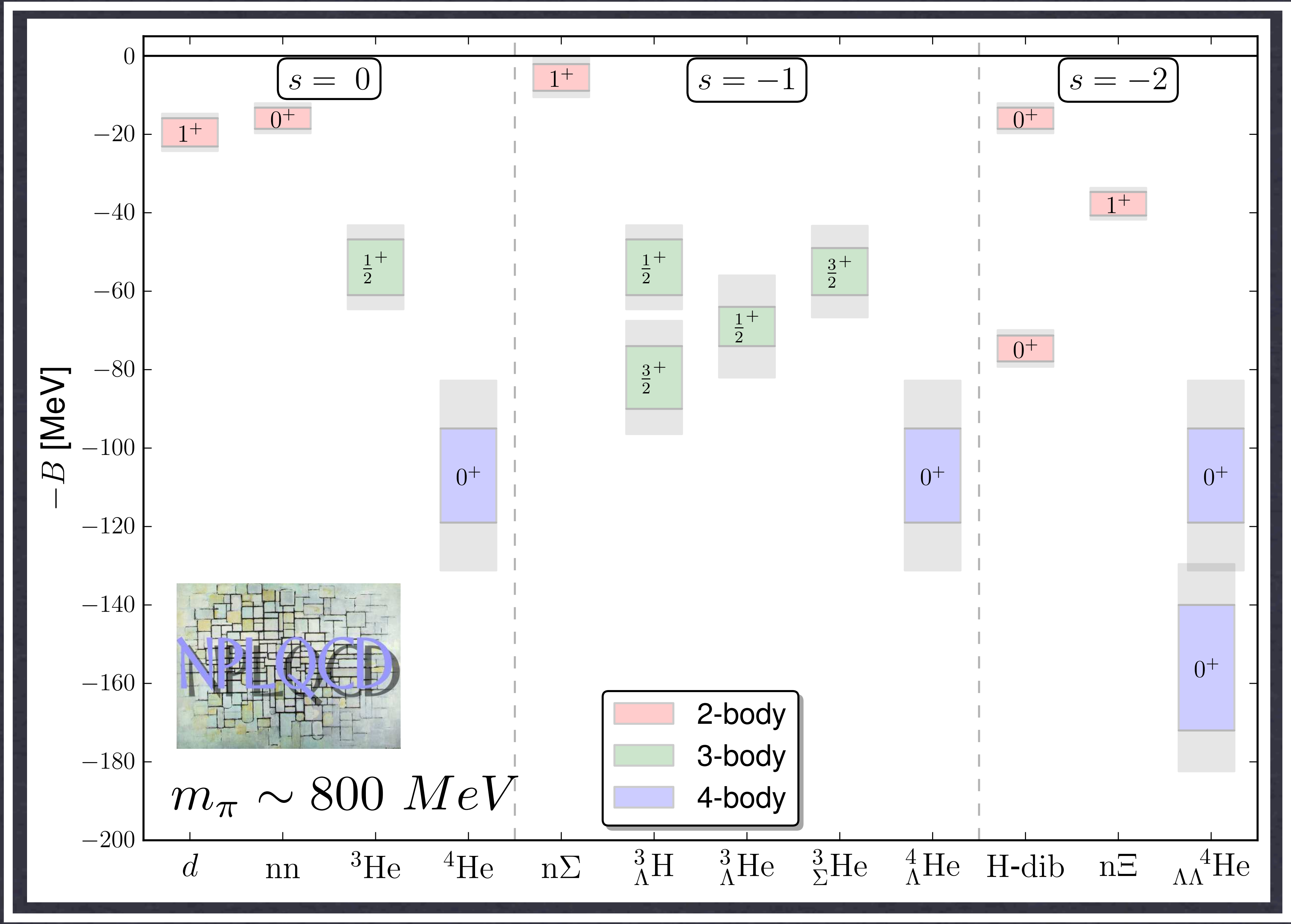
Dudek, Edwards, Thomas, et al, Phys.Rev. D87 (2013) 034505.



NUCLEAR PHYSICS FROM LATTICE QCD
RECENT DEVELOPMENTS

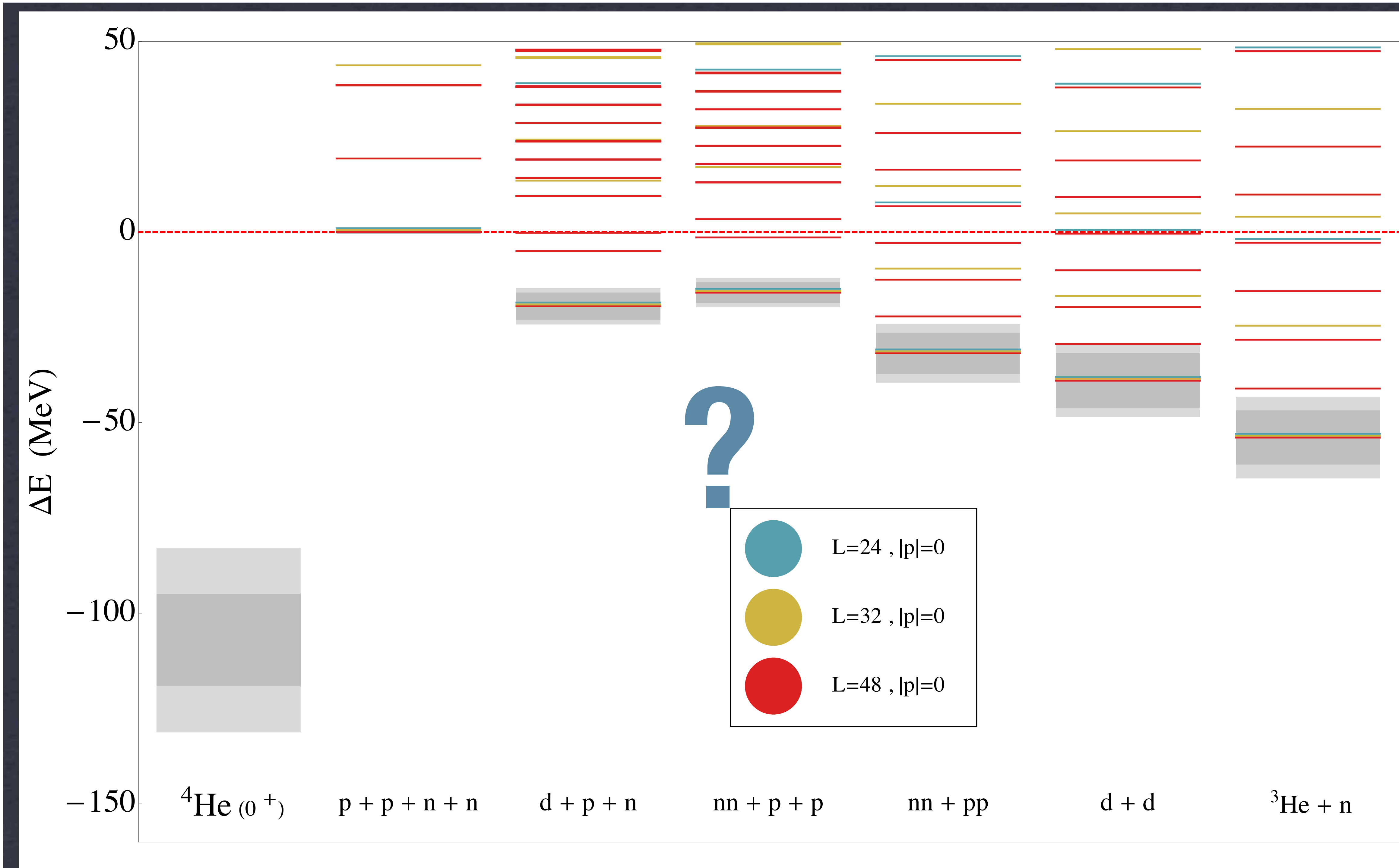
- Nuclei and hyper nuclei from first principles, nuclear structure.
  Nuclear landscape at unphysical pion masses.

Beane, et al, Phys.Rev. D87 (2013) 034506.
Yamazaki, et al, Phys.Rev. D86 (2012) 074514.
Barnea, et al, arXiv:1311.4966.



NUCLEAR PHYSICS FROM LATTICE QCD
CHALLENGES IN MULTI-PARTICLE SECTOR

Beane, et al, Phys.Rev. D87 (2013) 034506.



NUCLEAR PHYSICS FROM LATTICE QCD

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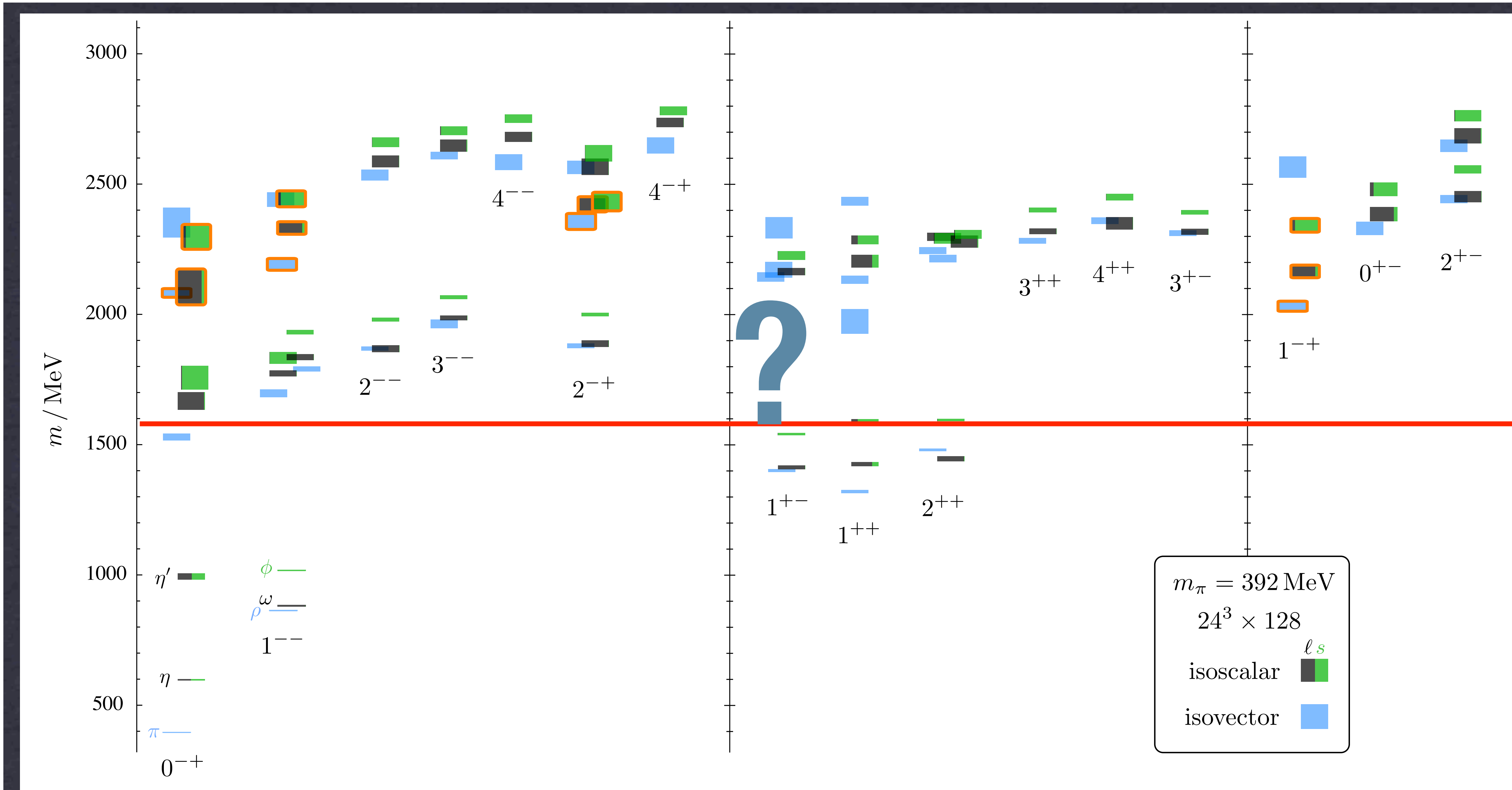

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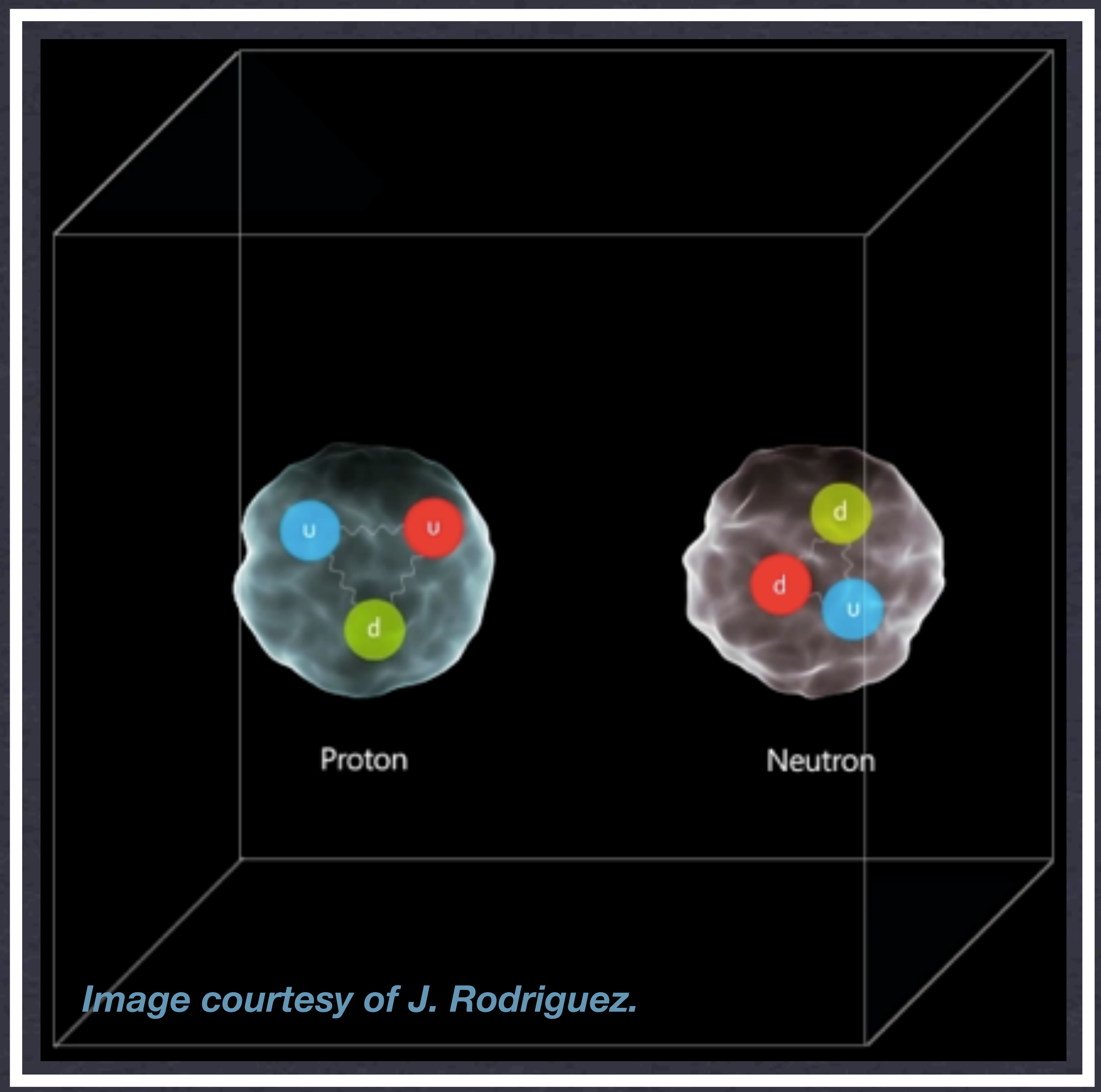
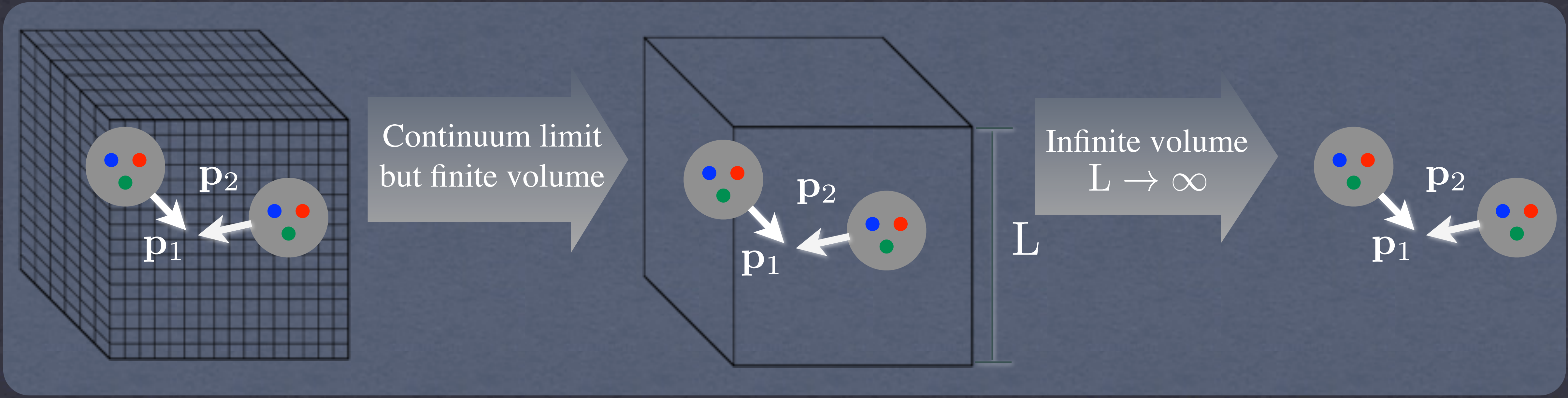
TWO-BODY SECTOR

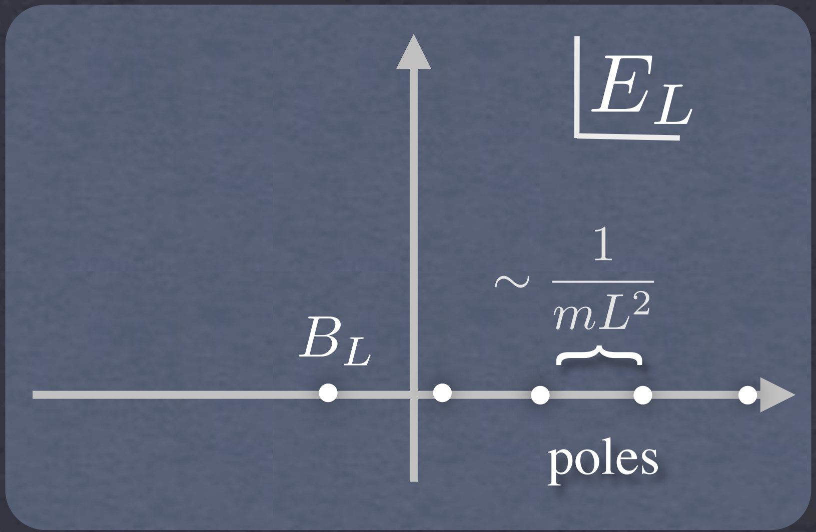


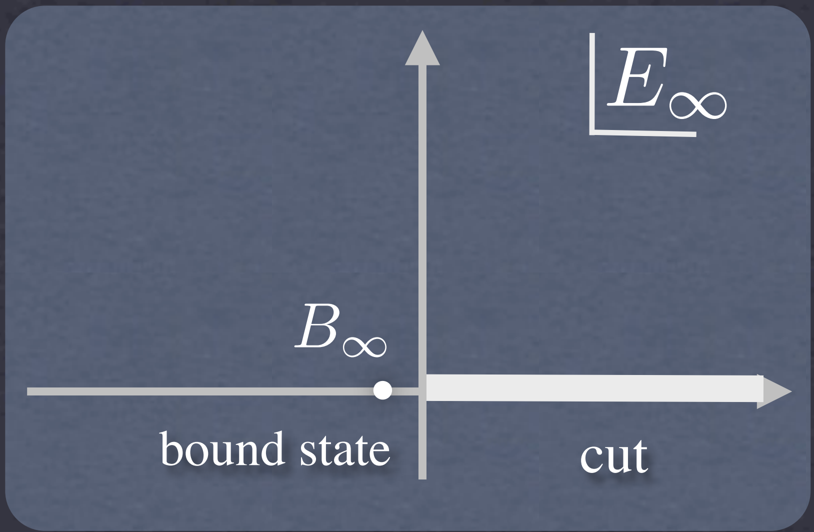
TWO HADRONS IN A FINITE VOLUME



* Cubic spatial volume with the PBCs

$$\mathbf{p}_i = \frac{2\pi \mathbf{n}_i}{L}$$





* Maiani-Testa no-go theorem

Maiani, Testa, Phys.Lett., B245, 585 (1990).

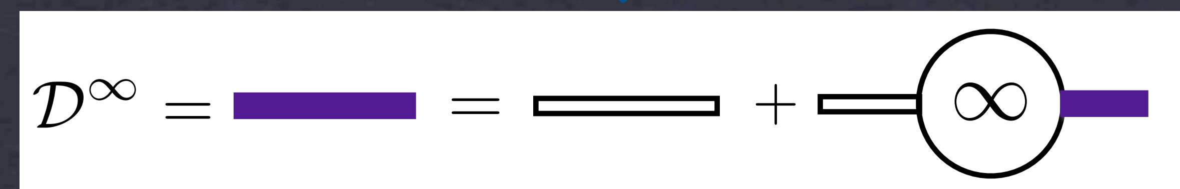
Luscher's formula

LUESCHER FORMULA

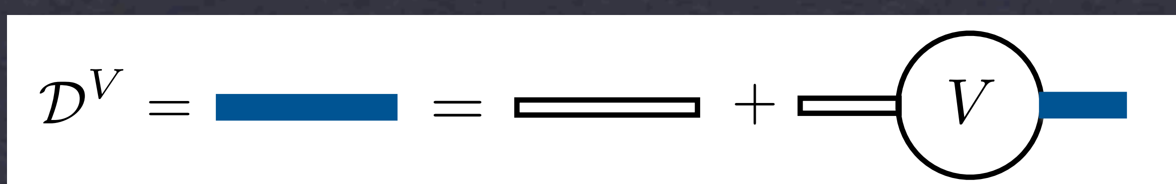
A DERIVATION BASED ON DIMER FORMALISM

A NR EFT $\mathcal{L} = \phi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \phi - d^\dagger \left(i\partial_0 + \frac{\nabla^2}{4m} - \Delta \right) d^\dagger - \frac{g_2}{2} (d\phi^2 + \text{h.c.}) + \ldots$

Eliminate in favor of physical observables: a, r



$$i\mathcal{D}^\infty(E, \mathbf{q}) = \frac{-imr/2}{\overline{q}\cot\delta_d - i\overline{q} + i\epsilon}$$



$$i\mathcal{D}^V(E, \mathbf{q}) = \frac{-imr/2}{\overline{q}\cot\delta_d - 4\pi\, c^q_{00}(\overline{q}^2 + i\epsilon) + i\epsilon}$$

S-wave quantization condition (QC)

The spectrum in FV can be written in a model-independent way

Luscher, Nucl.Phys. B354 (1991) 531-578.
Rummukainen, Gottlieb, Nucl.Phys. B450 (1995) 397-436.
Kim, Sachrajda, Sharpe, et al, Nucl.Phys. B727 (2005) 218-243.
Bour, et al, Phys.Rev. D84 (2011) 091503.
Davoudi, Savage, Phys.Rev. D84 (2012) 114502.

$$c^q_{00}(x) = \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \mathcal{P} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right] k^{*l} \frac{\sqrt{4\pi}\, Y_{00}(\hat{k}^*)}{k^{*2} - x} \qquad \mathbf{k}^* = \mathbf{k} - \mathbf{q}/2$$

TWO-NUCLEON SYSTEMS IN INFINITE VOLUME AND SYMMETRIES

Parity and total J — CONSERVED

Assuming isospin symmetry

Orbital angular momentum L — NOT CONSERVED

→ 4 channels

$$J^{\pi=\pm} \ (I=0,1)$$

e.g. deuteron

$$1^+ \ (I=0)$$

↓

$$L=0, L=2$$

$$\eta = -\tan\epsilon_1\big|_{k^*=i\kappa_d^\infty}$$

$$\approx 0.02713(6)$$

Biedenharn, Blatt, Phys.Rev. 93, 1387 (1954).
Mustafa, Phys. Rev. C47, 473 (1993).
de Swart, et al, 9509032 (1995).

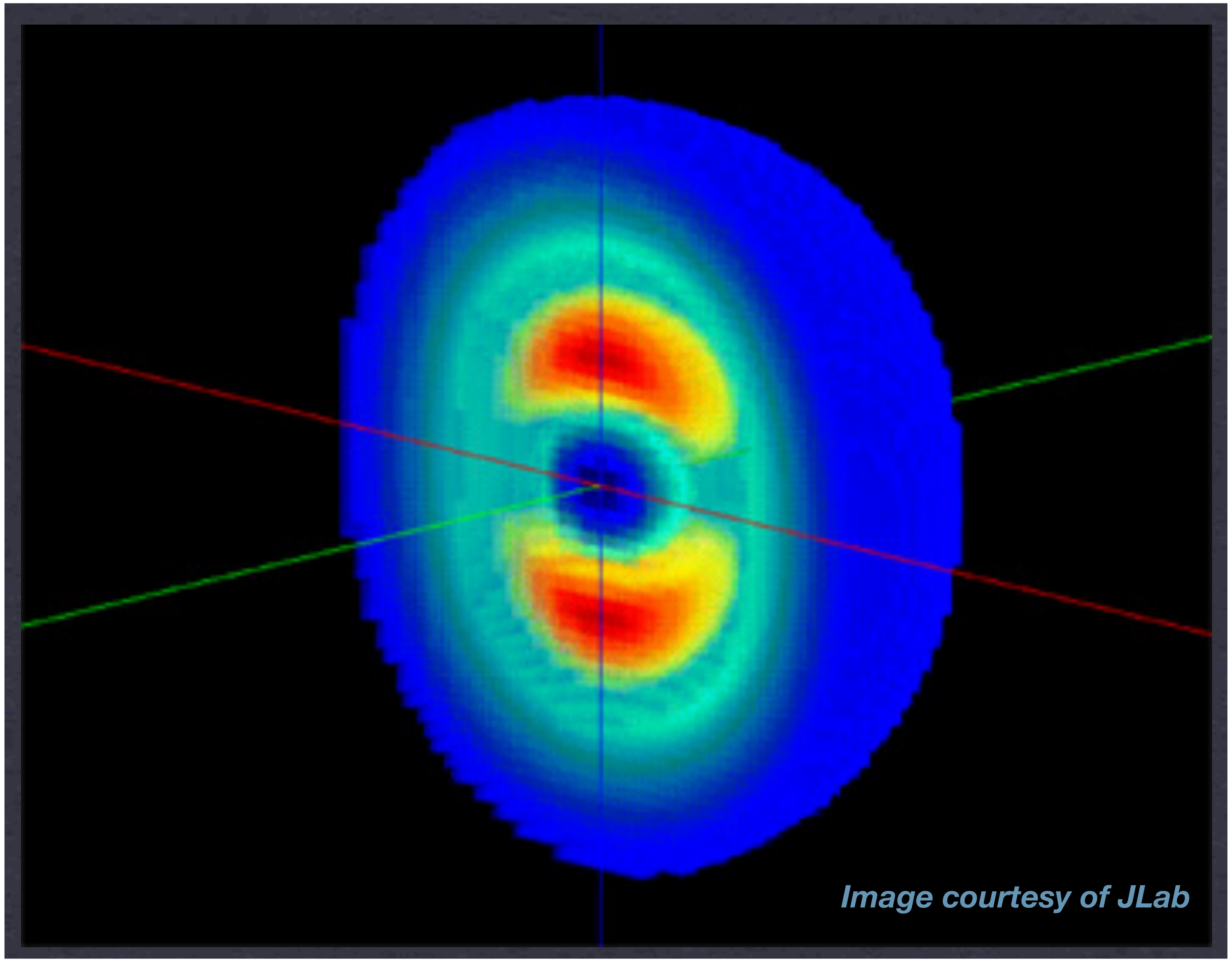


Image courtesy of JLab

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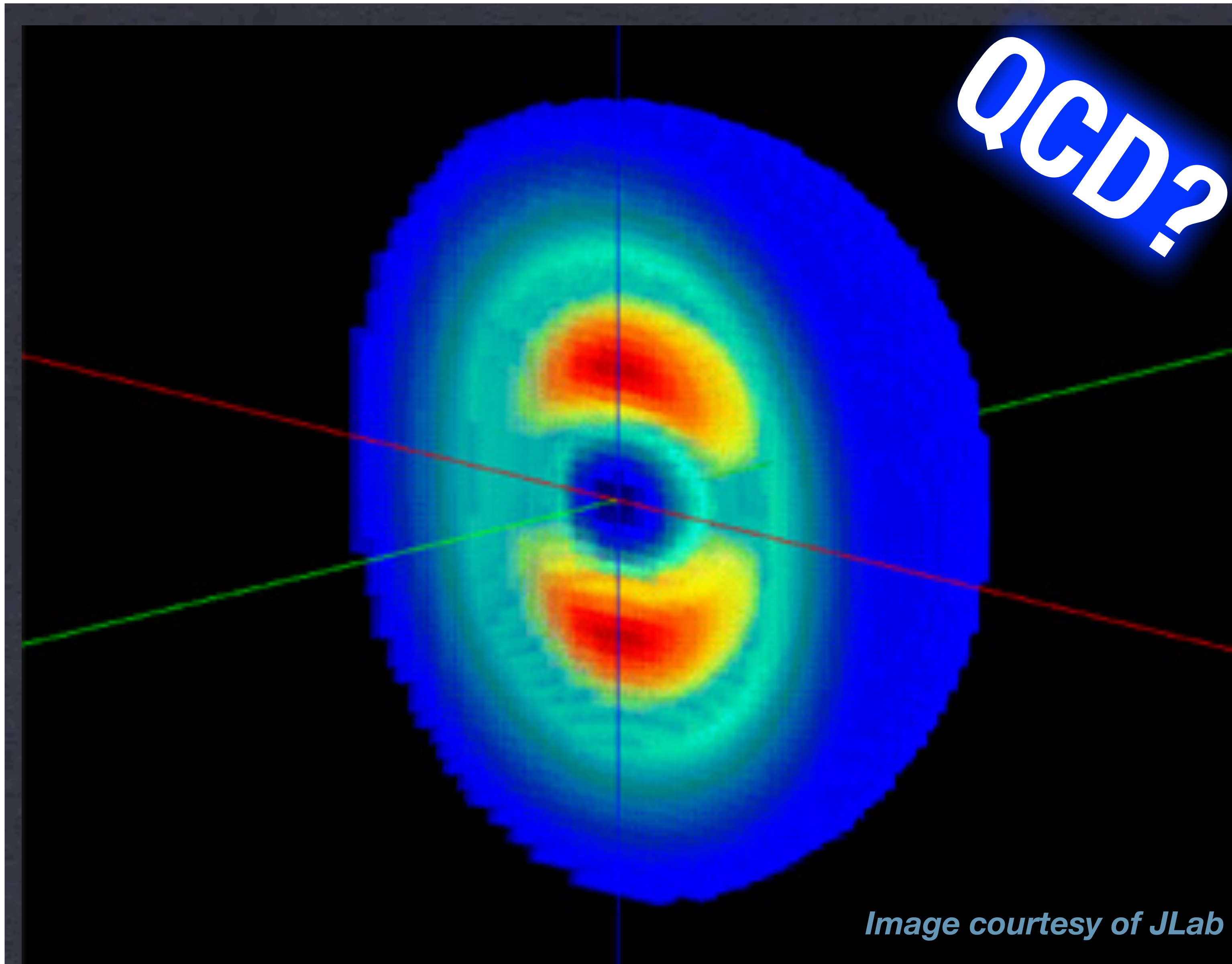


TWO-NUCLEON QUANTIZATION CONDITION

FINITE VOLUME SYMMETRY GROUPS

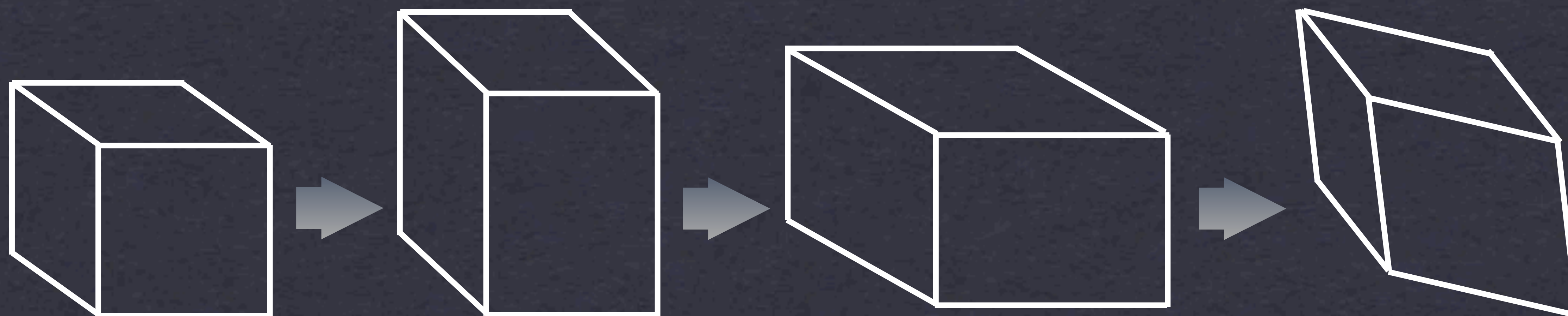
$$\det\left(\delta\mathcal{G}^V + \mathcal{M}^{-1}\right) = 0$$

Non-diagonal in both L and J basis

A function of energy and volume

Parametrized by phase shifts and mixing angles

Diagonal in J basis



$\mathbf{P} = \mathbf{0}$ $\quad$ $\mathbf{P} = \frac{2\pi}{L}(0,0,1)$ $\quad$ $\mathbf{P} = \frac{2\pi}{L}(1,1,0)$ $\quad$ $\mathbf{P} = \frac{2\pi}{L}(1,1,1)$

d	point group	classification	N_{elements}	irreps (dimension)
$(0,0,0)$	O	cubic	24	$\mathbb{A}_1(1), \mathbb{A}_2(1), \mathbb{E}(2), \mathbb{T}_1(3), \mathbb{T}_2(3)$
$(0,0,1)$	D_4	tetragonal	8	$\mathbb{A}_1(1), \mathbb{A}_2(1), \mathbb{E}(2), \mathbb{B}_1(1), \mathbb{B}_2(1)$
$(1,1,0)$	D_2	orthorhombic	4	$\mathbb{A}(1), \mathbb{B}_1(1), \mathbb{B}_2(1), \mathbb{B}_3(1)$
$(1,1,1)$	D_3	trigonal	6	$\mathbb{A}_1(1), \mathbb{A}_2(1), \mathbb{E}(2)$

e.g.

$1^+ \ (I=0)$

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{1,S} & \mathcal{M}_{1,SD} & 0 & 0 \\ \mathcal{M}_{1,SD} & \mathcal{M}_{1,D} & 0 & 0 \\ 0 & 0 & \mathcal{M}_{2,D} & 0 \\ 0 & 0 & 0 & \mathcal{M}_{3,D} \end{pmatrix}$$

Group theory decompositions

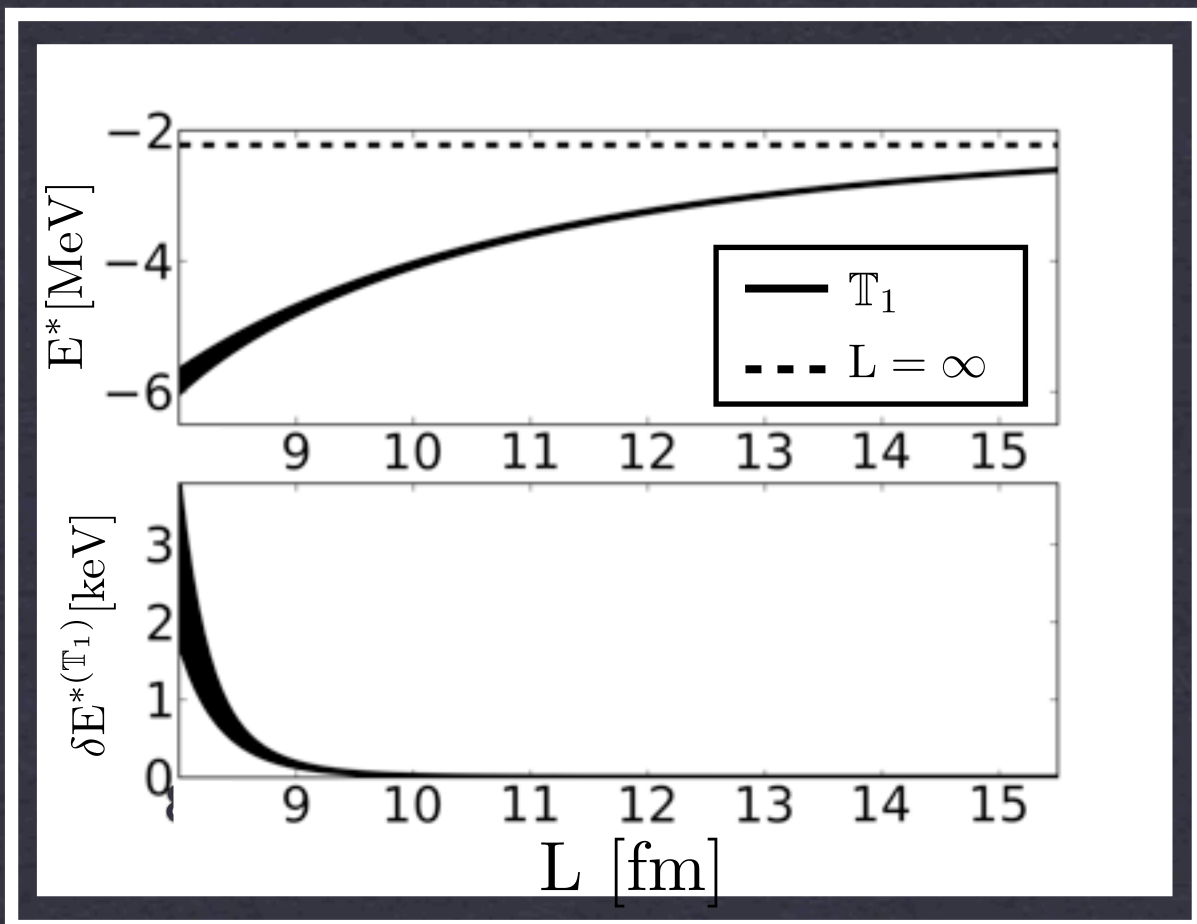
Neglecting scattering with $l > 3$

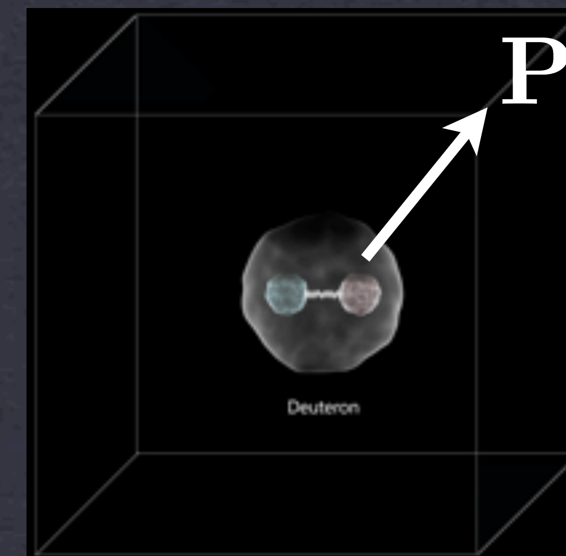
49 quantization conditions for 16 scattering parameters

BOUND STATE SPECTRUM

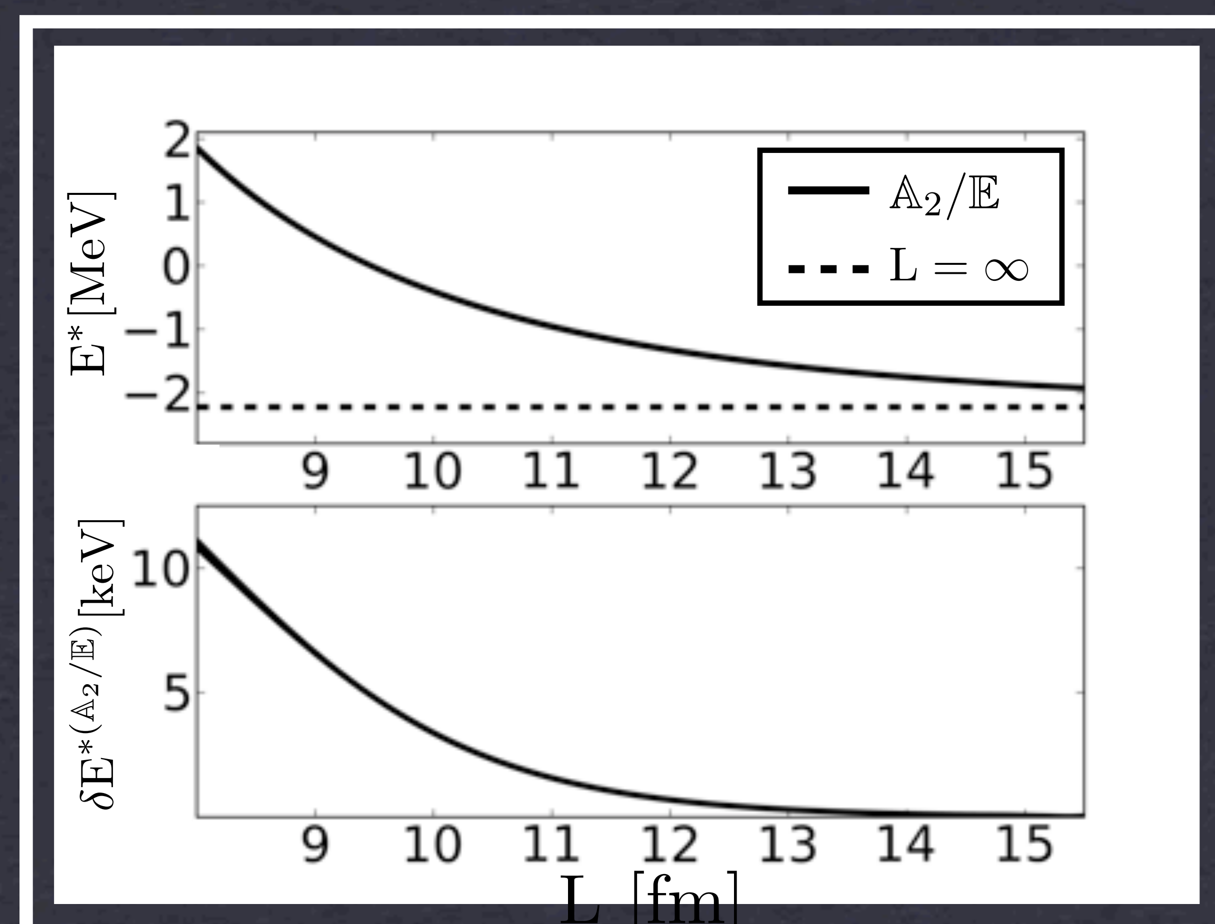
IS THE DEUTERON BINDING ENERGY SENSITIVE TO THE MIXING BETWEEN S AND D WAVE?

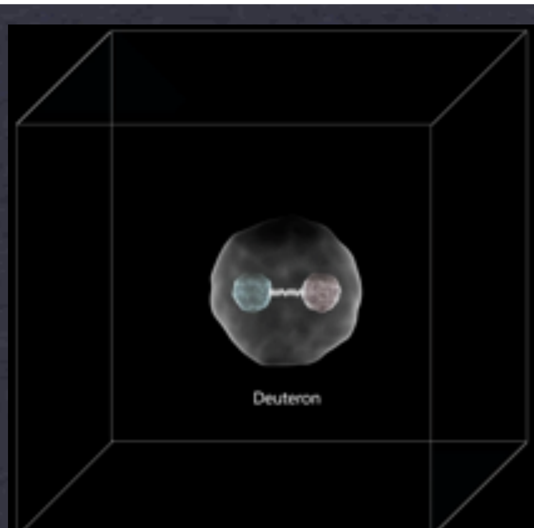
$$\delta \mathrm{E}^{*(\Gamma)} = \mathrm{E}^{*(\Gamma)} - \mathrm{E}^{*(\Gamma)}(\epsilon_1 = 0)$$





$$\mathbf{P} = \frac{2\pi}{L}(1,1,1)$$





$$\mathbf{P} = 0$$

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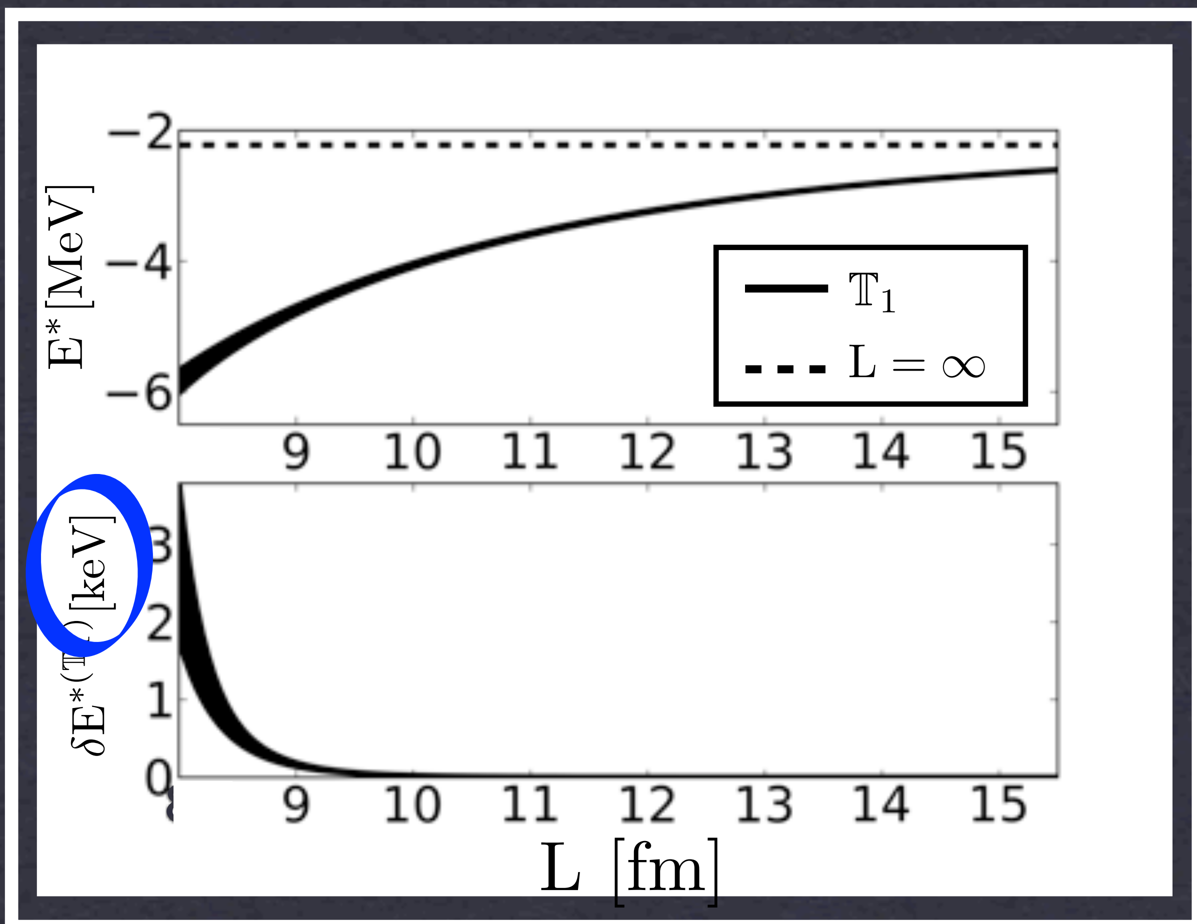

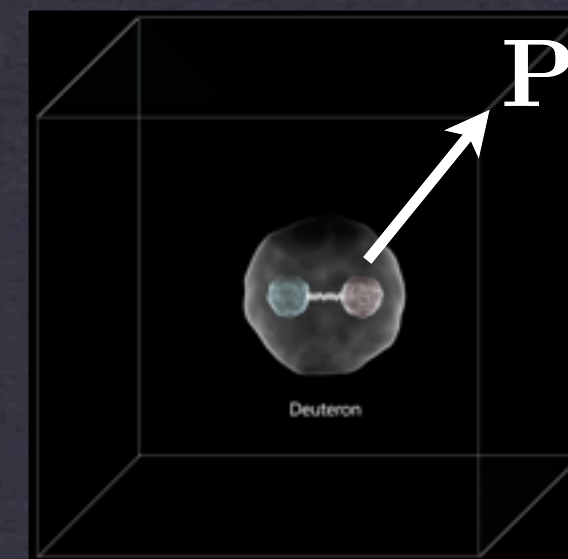

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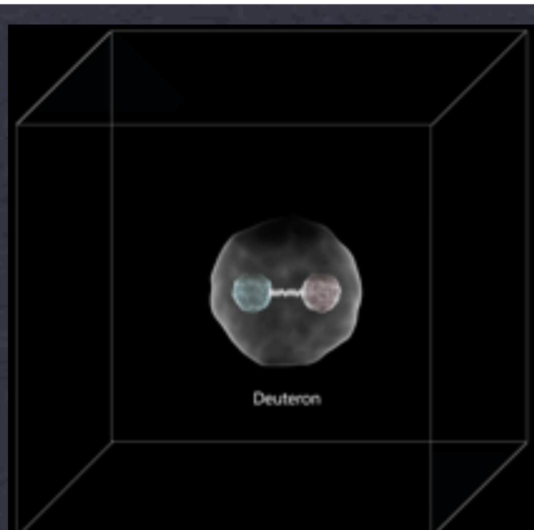
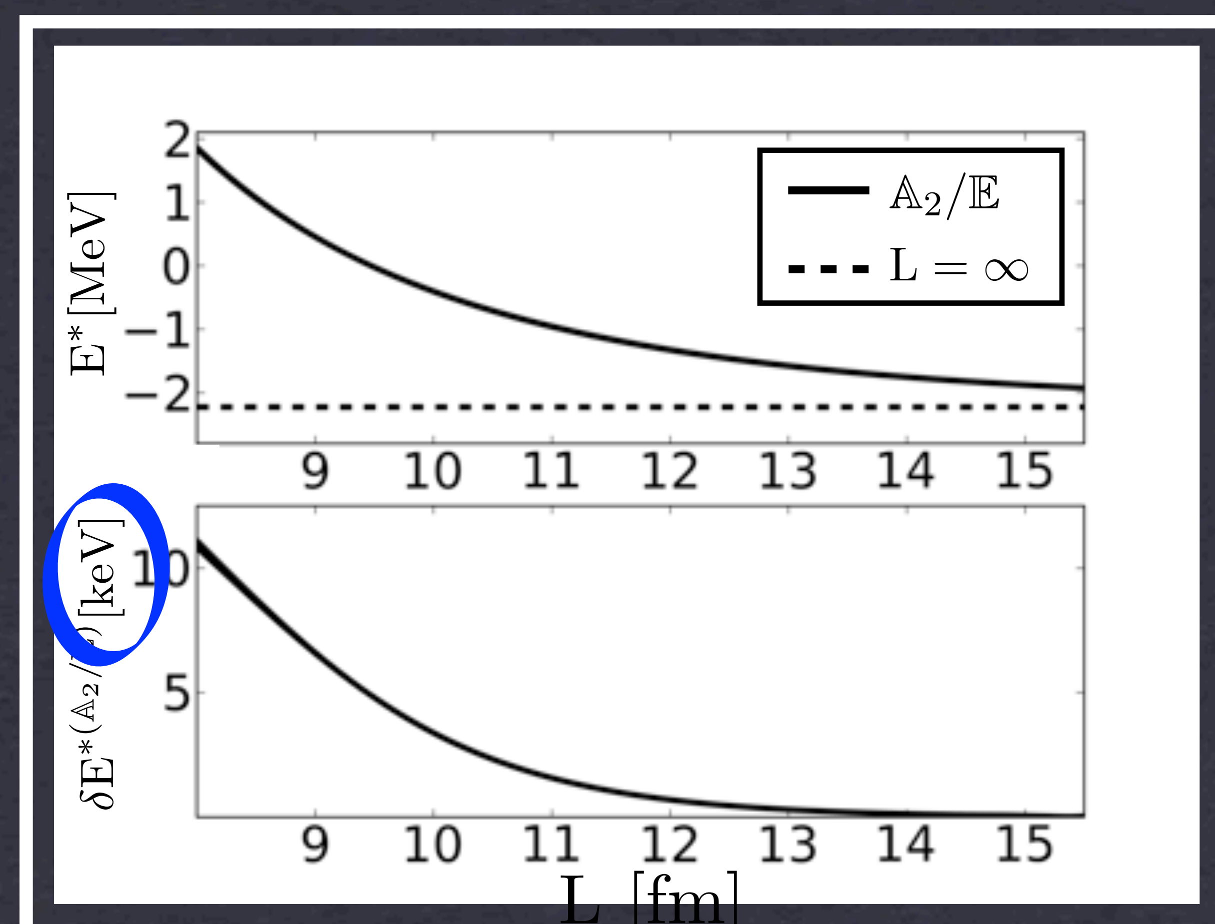
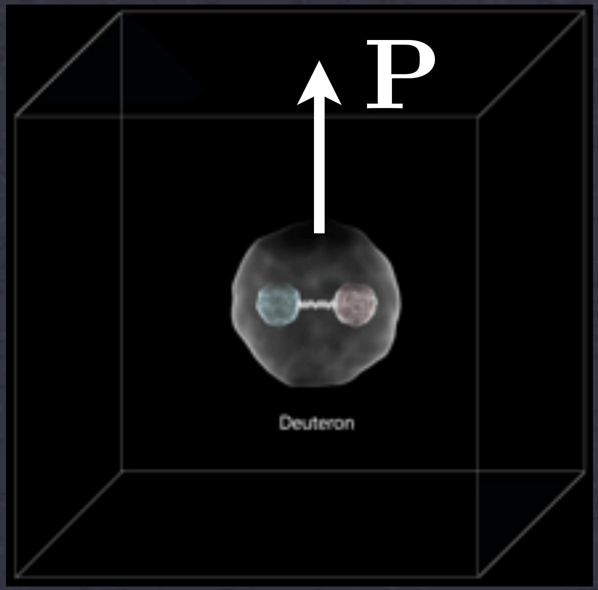


$$\mathbf{P} = 0$$



BOUND-STATE SPECTRUM

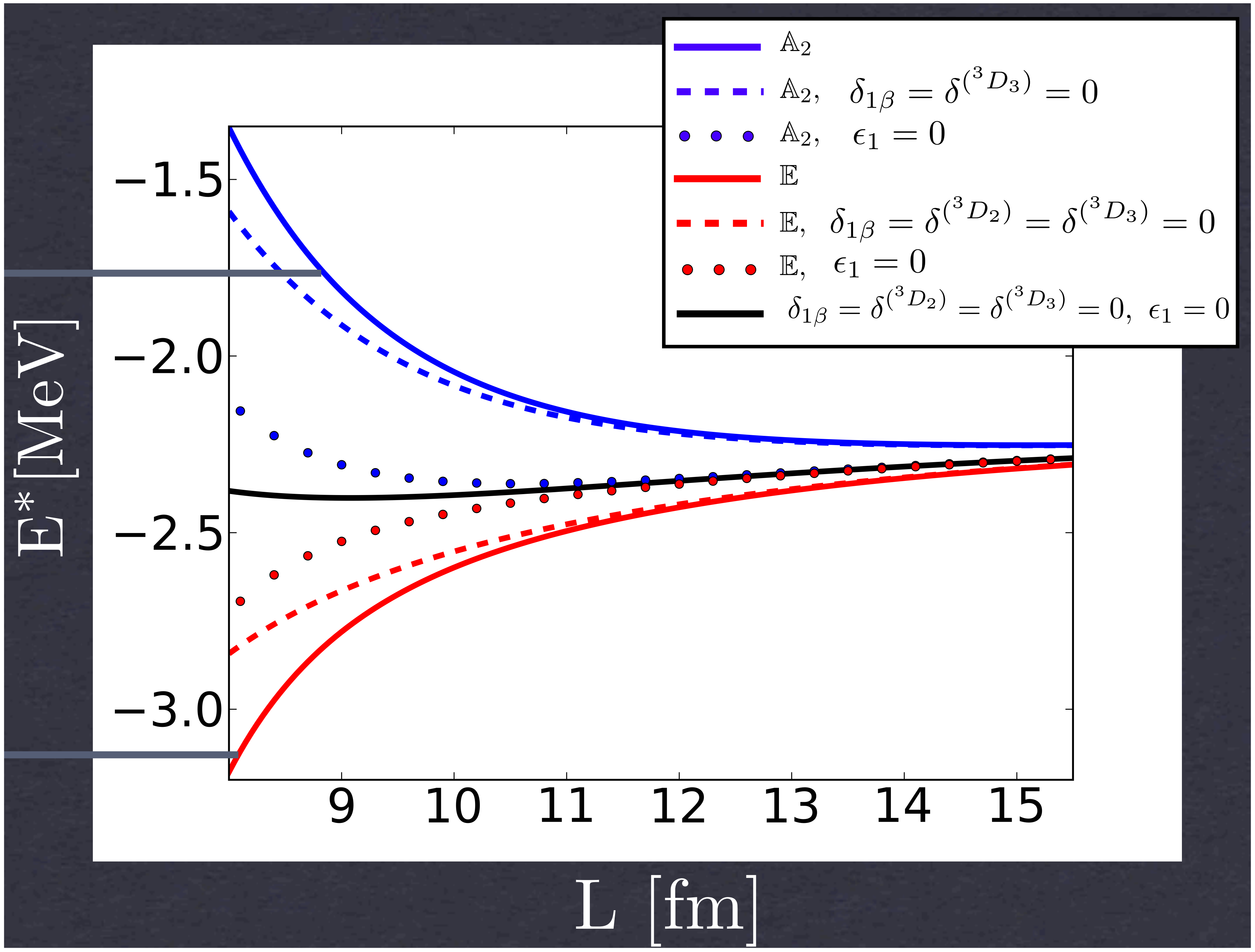
HOW ABOUT OTHER BOOST VECTORS?



$$\mathbf{P} = \frac{2\pi}{L}(0,0,1)$$

$M_J = 0$
one dimensional irrep

$M_J = \pm 1$
two dimensional irrep



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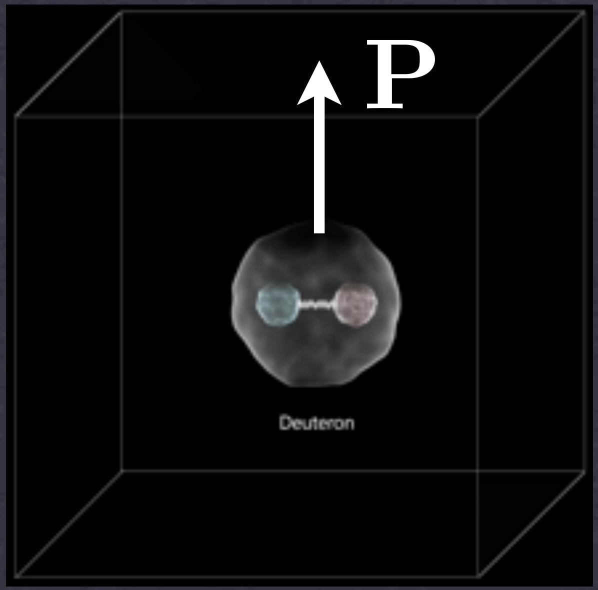

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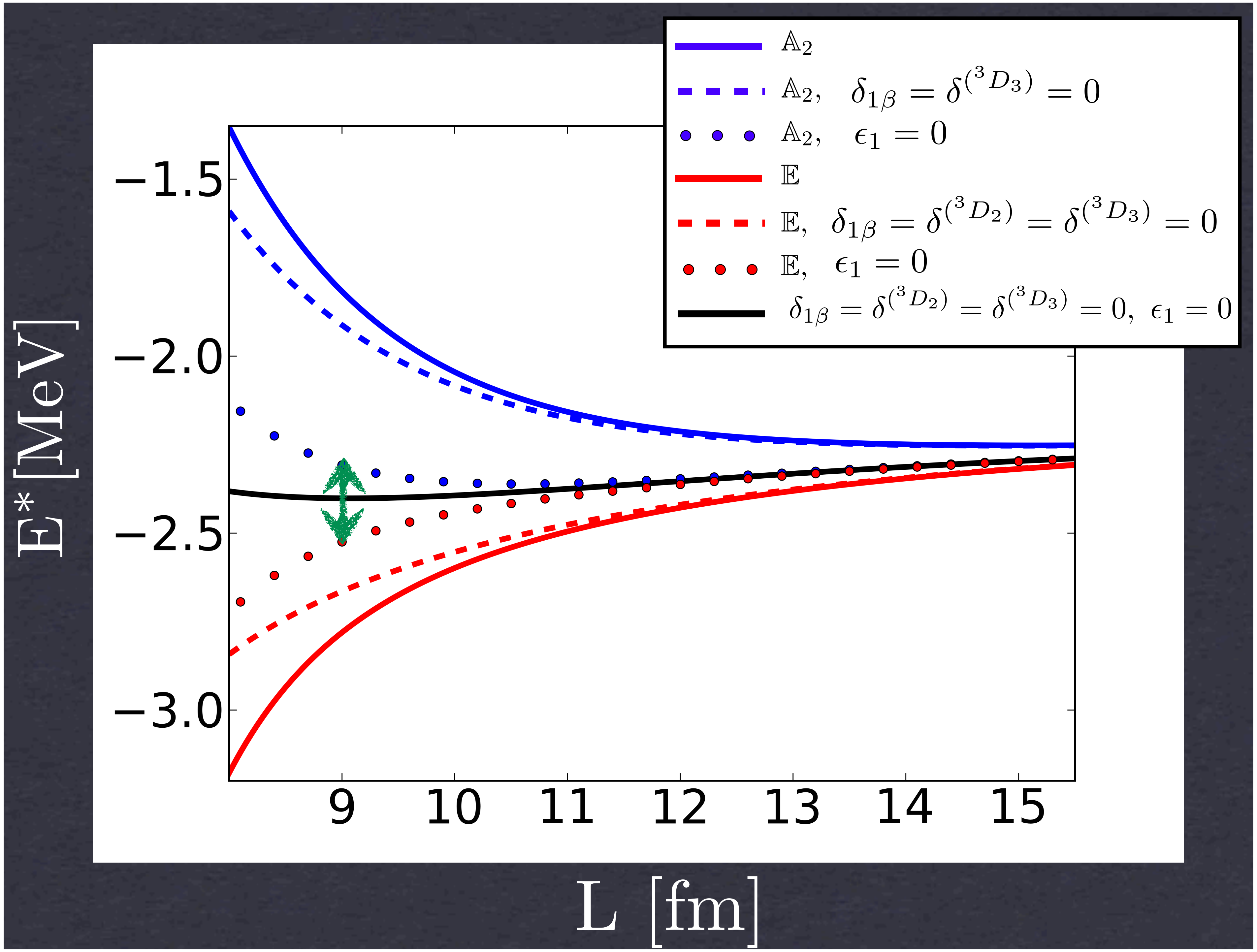

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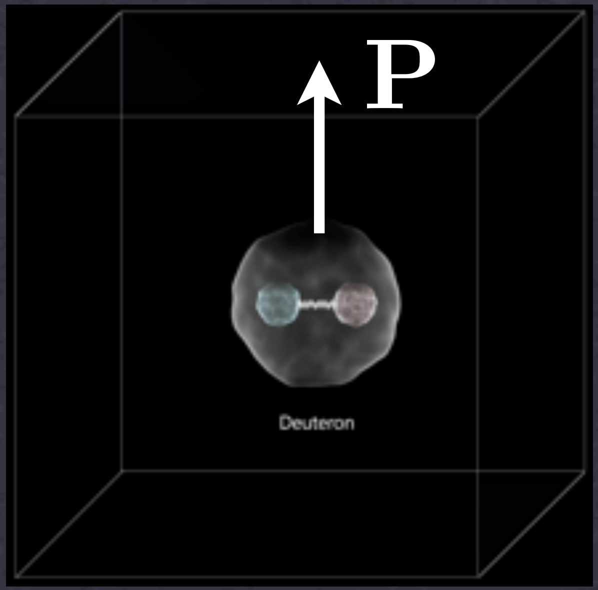

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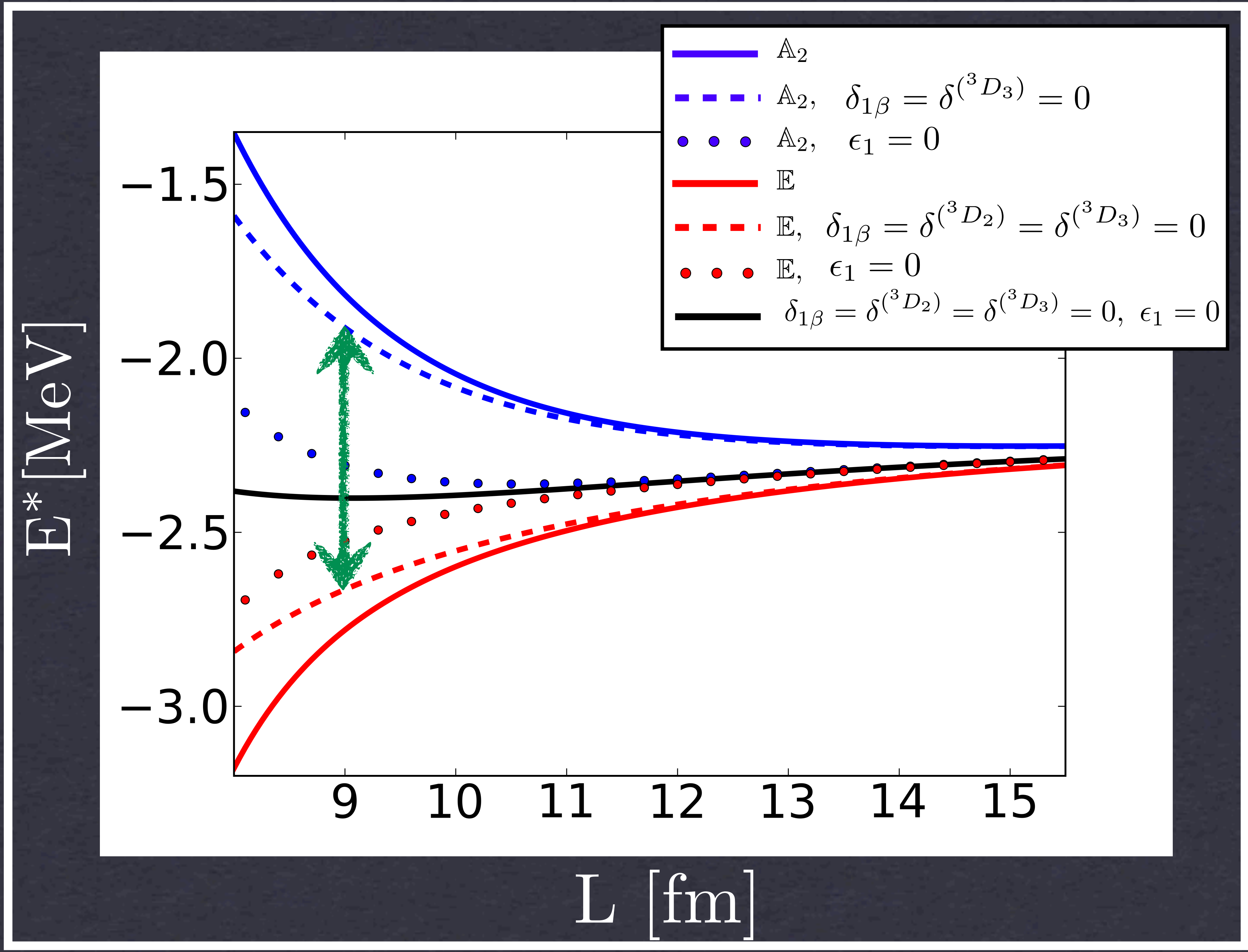

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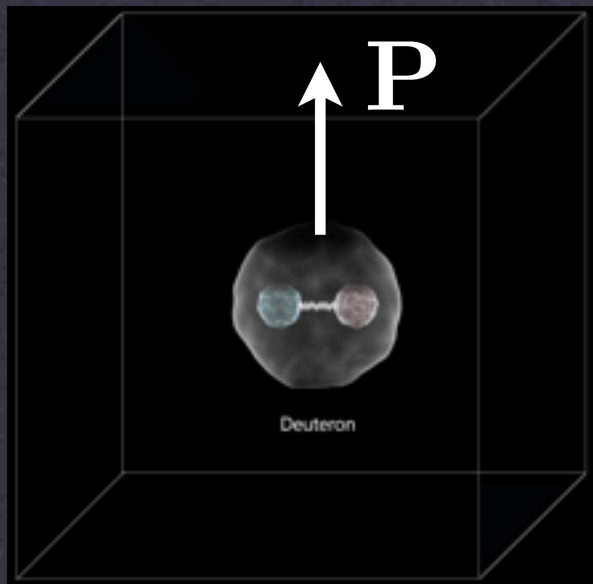

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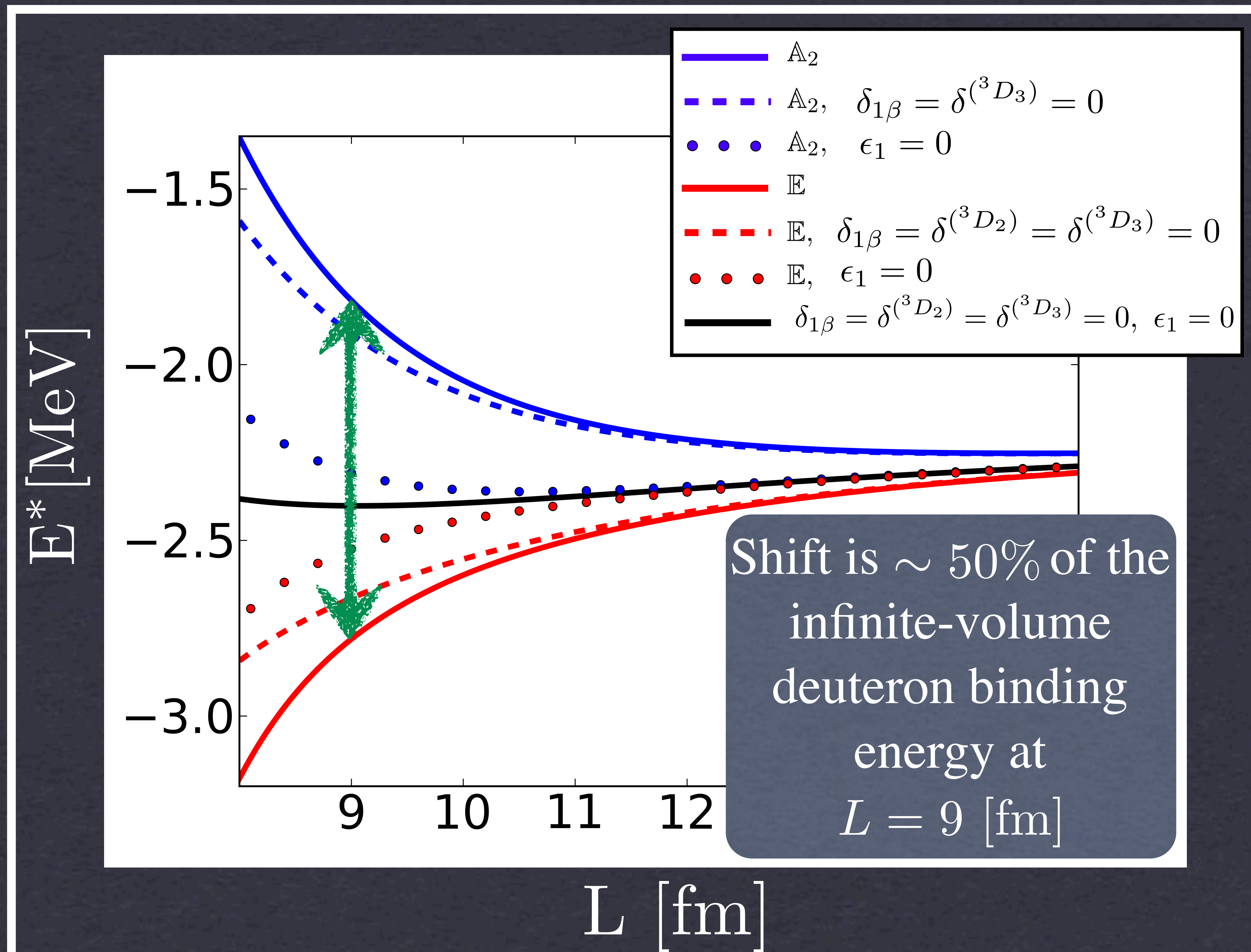
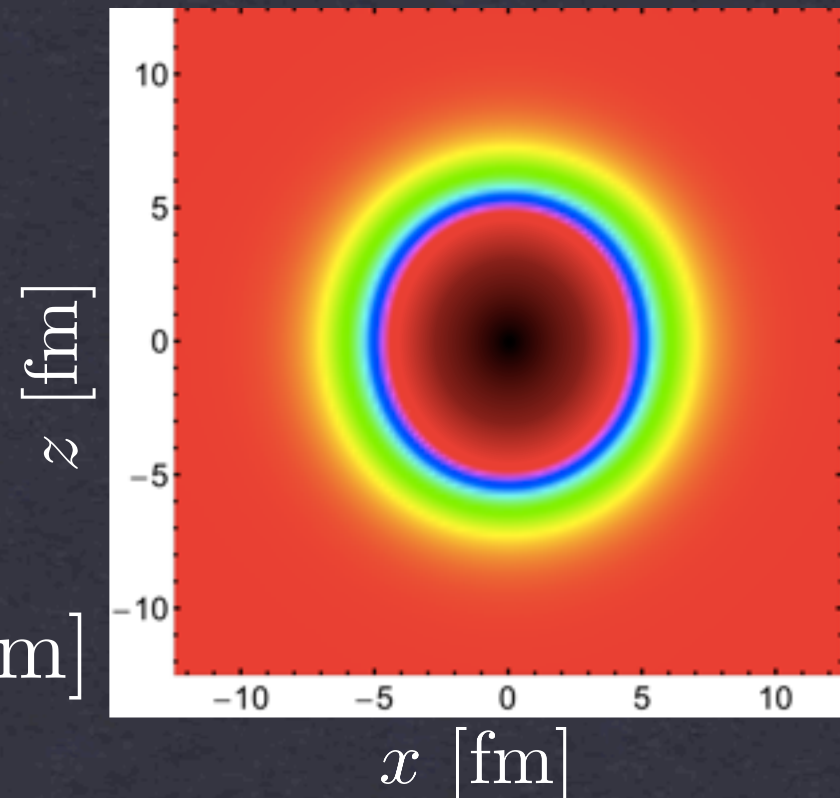


DEUTERON WAVEFUNCTION

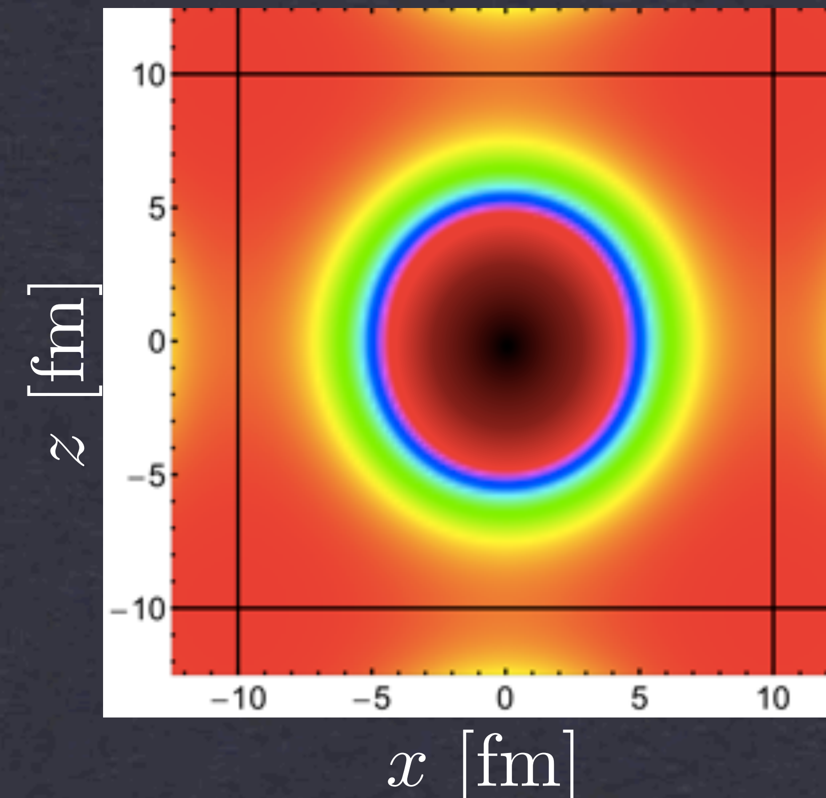
HOW DOES DEUTERON EVOLVE AS A FUNCTION OF VOLUME?

$\mathbf{P} = \mathbf{0}$

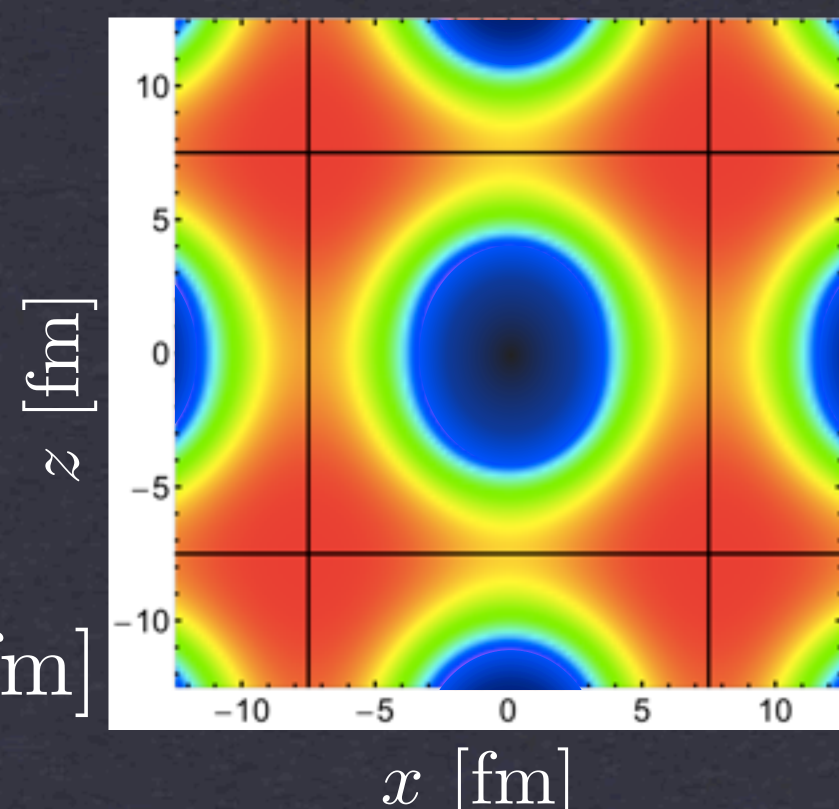
$\mathbb{T}_1$ irrep



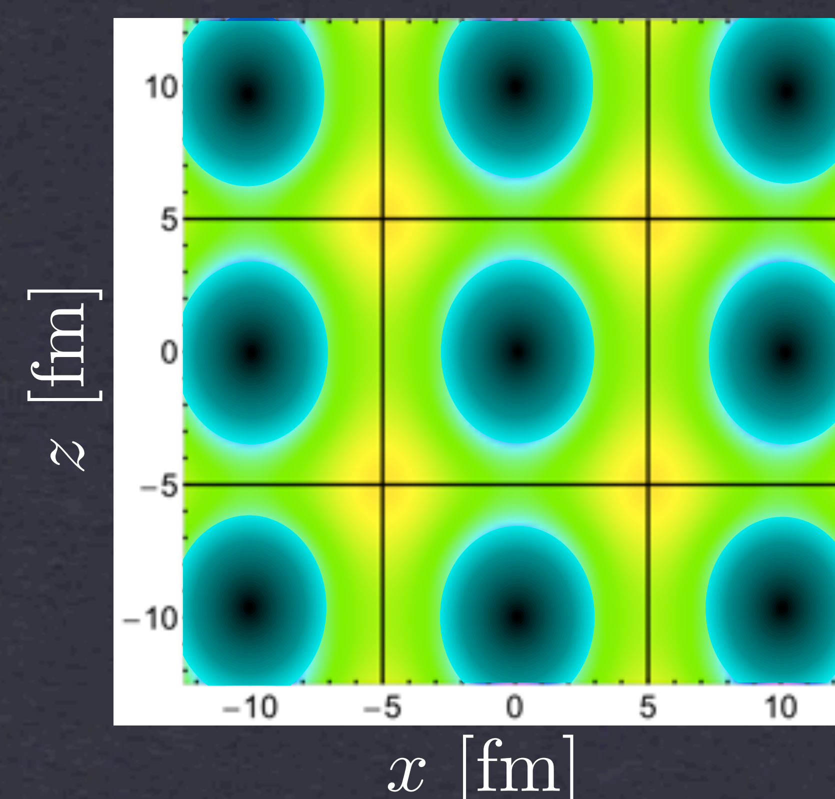
L = 30 [fm]



L = 20 [fm]



L = 15 [fm]



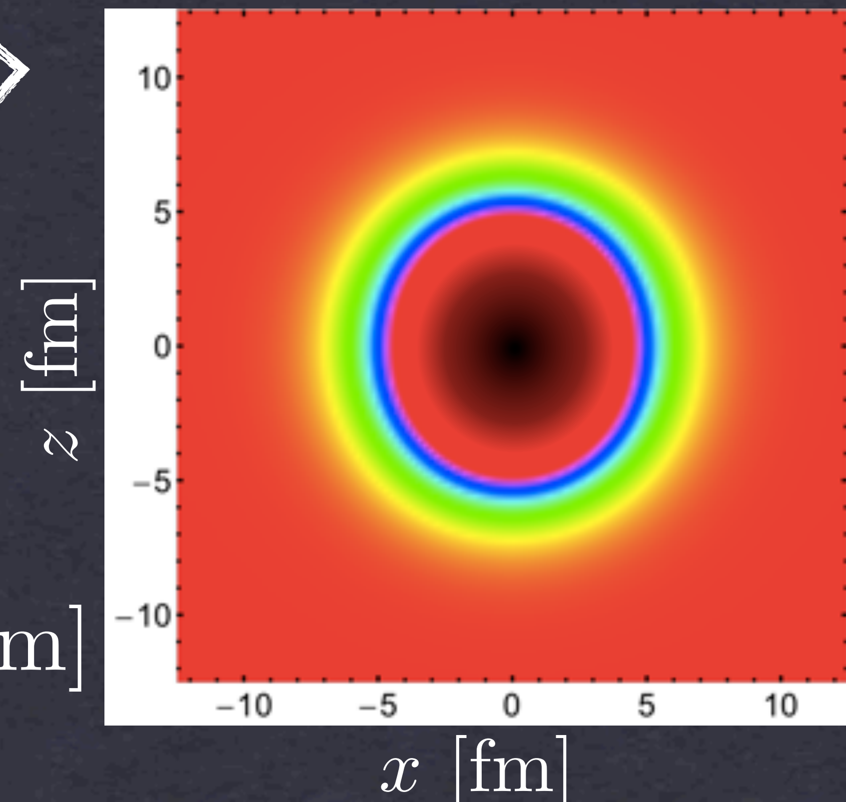
L = 10 [fm]

DEUTERON WAVEFUNCTION

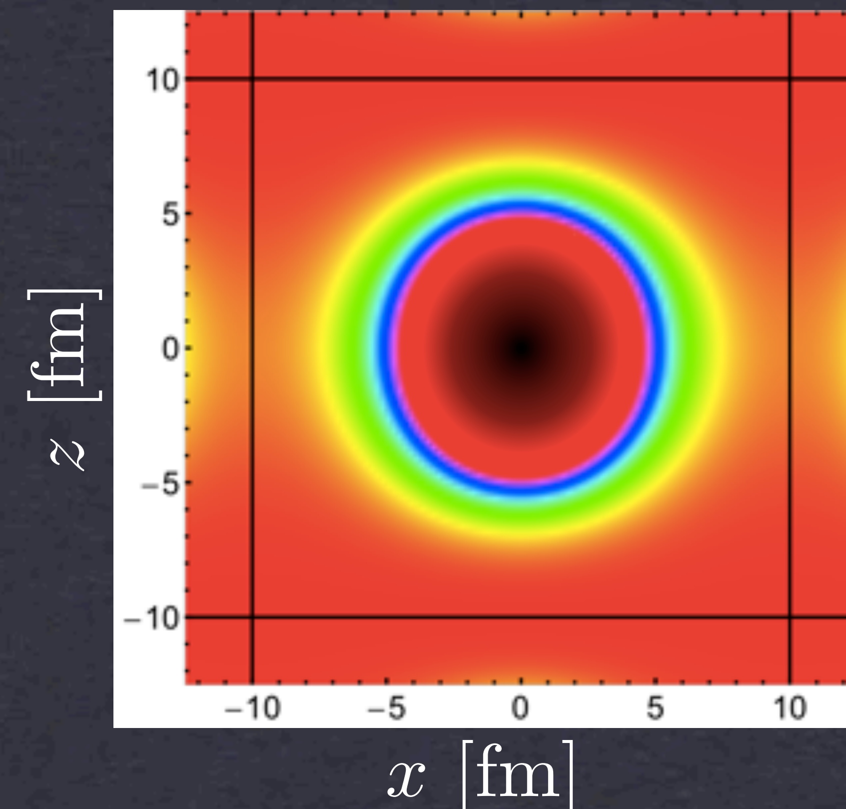
HOW DOES DEUTERON EVOLVE AS A FUNCTION OF VOLUME?

$\mathbf{P} = \frac{2\pi}{L}(0,0,1)$

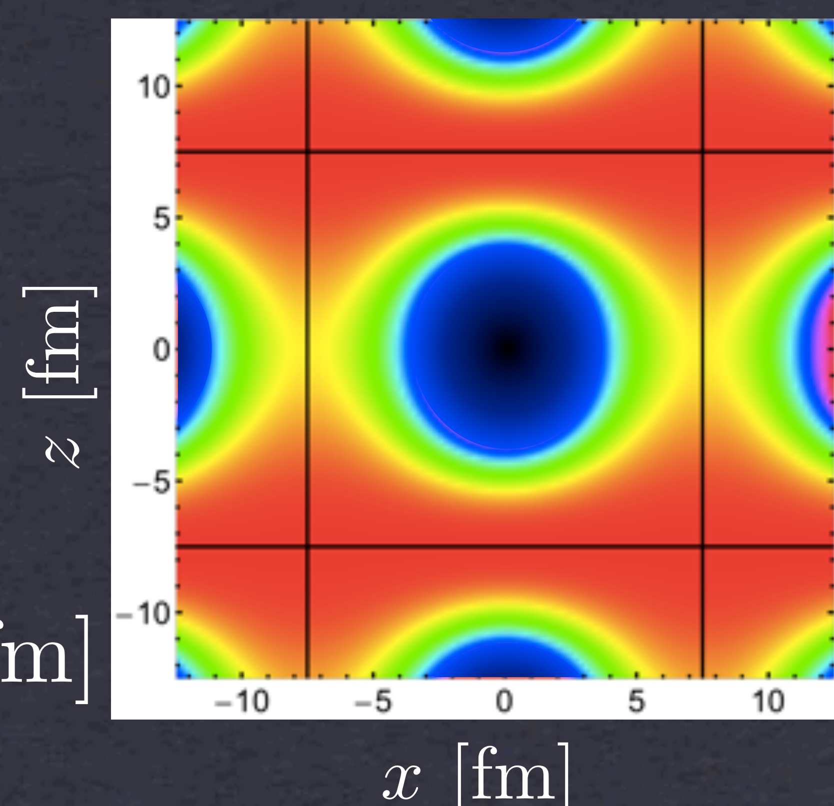
$\mathbb{E}$ irrep



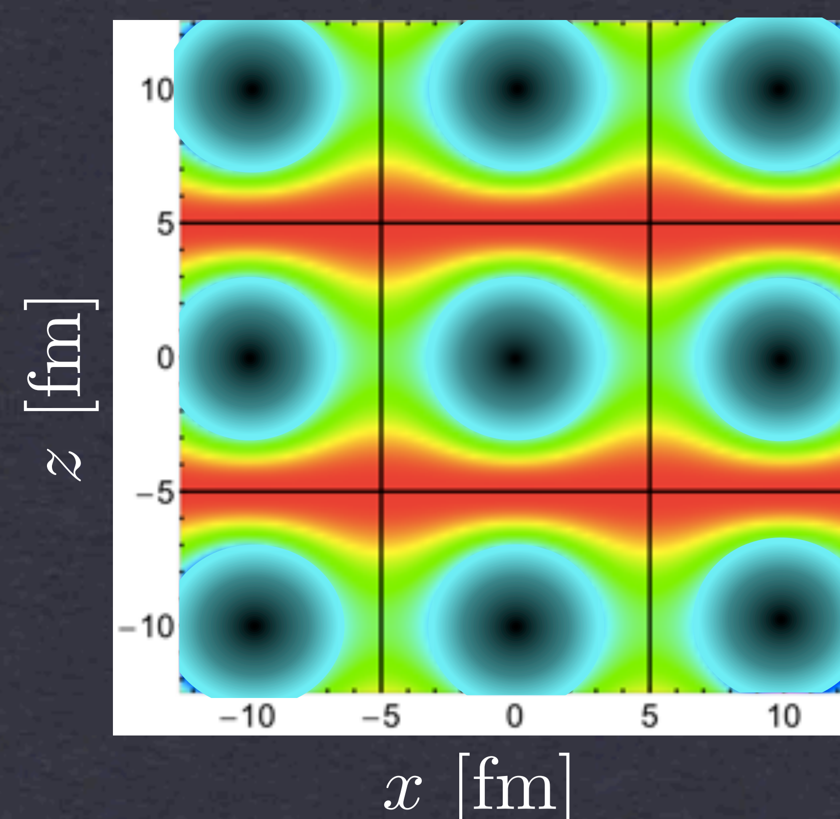
L = 30 [fm]



L = 20 [fm]



L = 15 [fm]

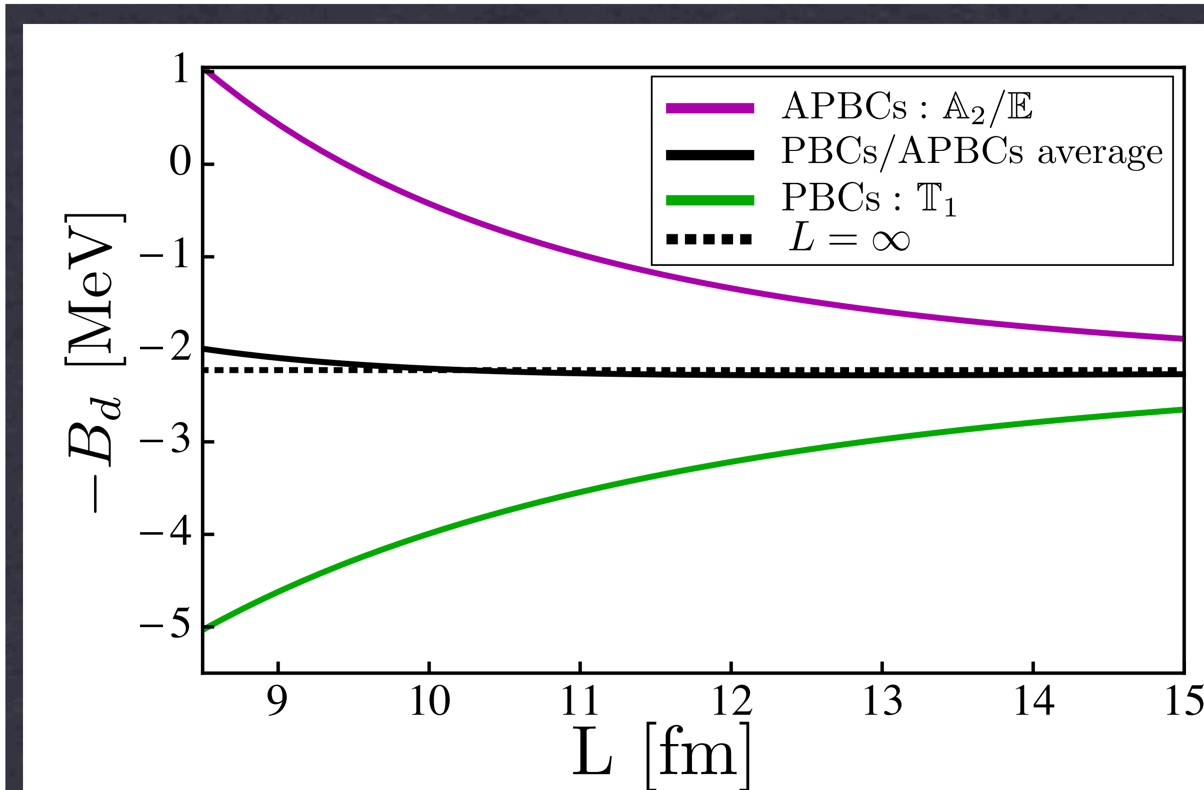


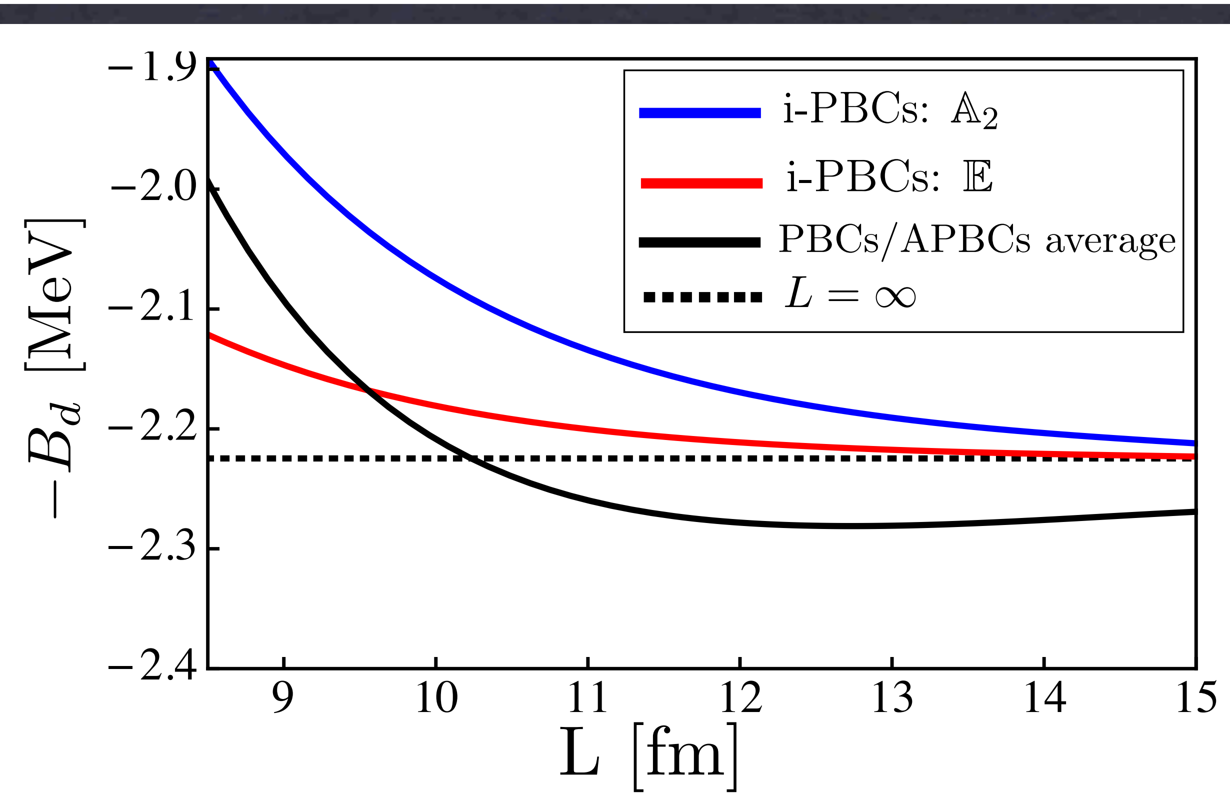
L = 10 [fm]

DEUTERON BINDING ENERGY
TWISTED BOUNDARY CONDITIONS AND VOLUME IMPROVEMENT

$$\psi(\mathbf{x}+\mathbf{n}\mathrm{L}) = e^{i\theta\cdot\mathbf{n}}\psi(\mathbf{x})$$

Bedaque, arXiv:0402051.

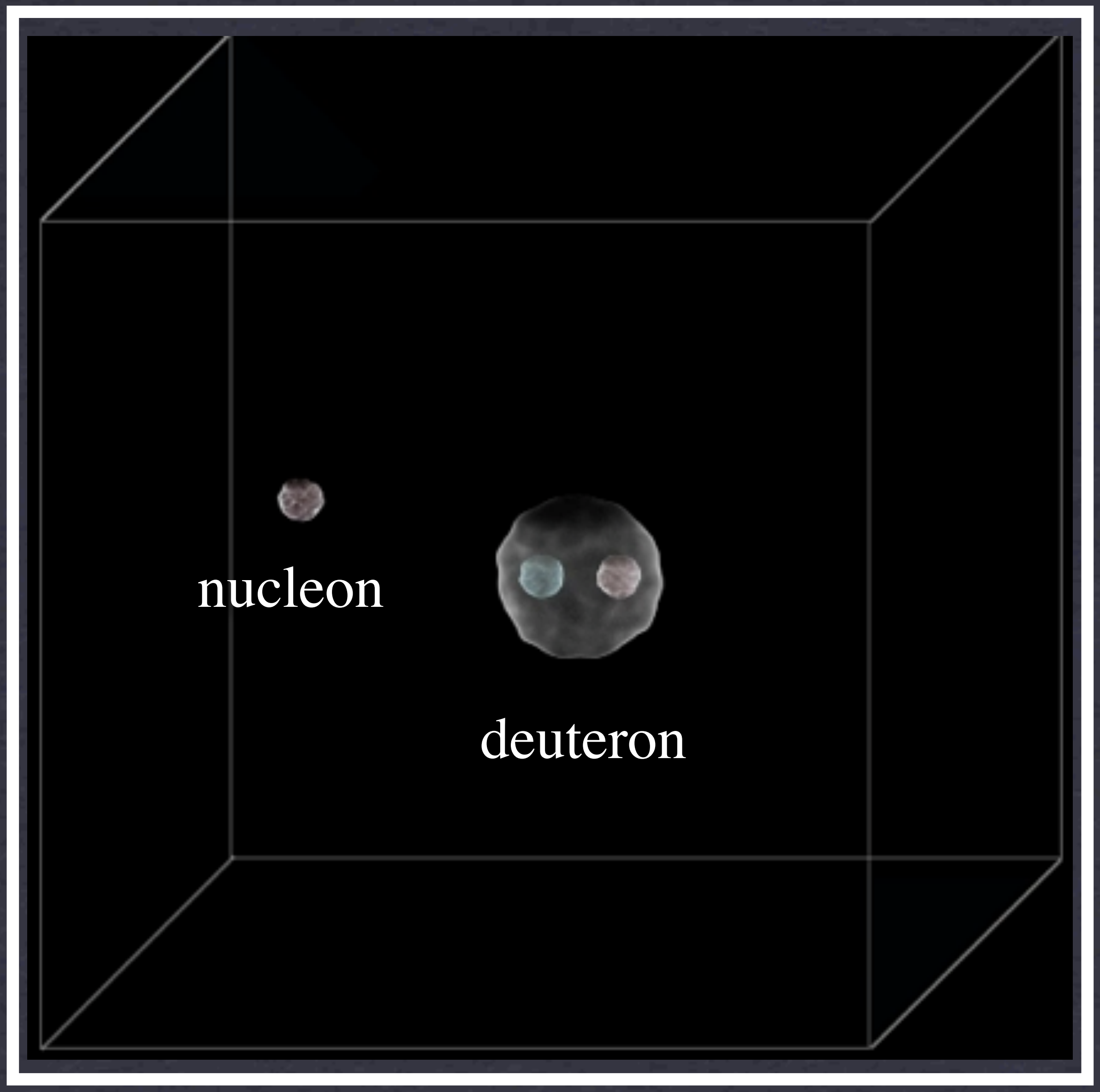




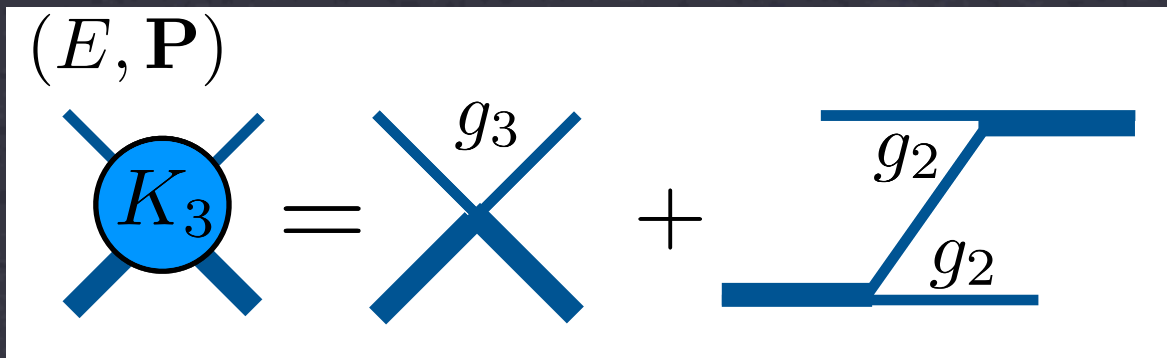
3. R. Briceno, ZD, T. Luu and M. J. Savage, Phys. Rev. D 89, 074509.

R. Briceno, Phys. Rev. D 89, 074507.

THREE-BODY SECTOR

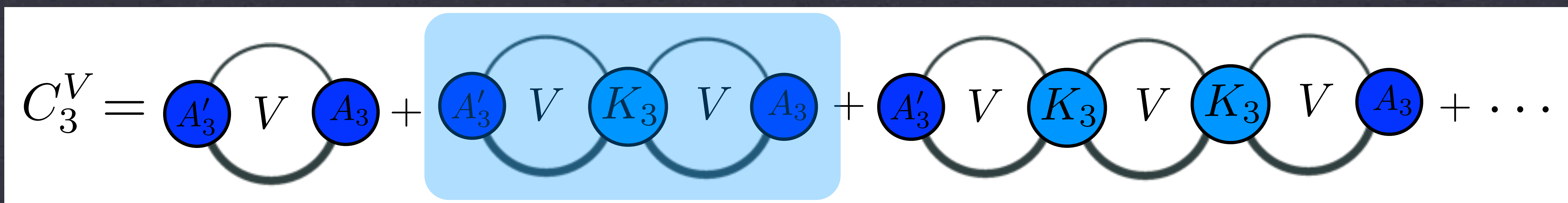


THREE-BODY CORRELATION FUNCTIONS WITH DIMER FIELD



Kinematic region below four-particle threshold

Expand the correlation function in powers of kernel



$$\frac{1}{L^6}\sum_{\mathbf{q}_1,\mathbf{q}_2} A_3(\mathbf{q}_1)\, i\mathcal{D}^V\!\left(E-\frac{q_1^2}{2m}, |\mathbf{P}-\mathbf{q}_1|\right) iK_3(\mathbf{q}_1,\mathbf{q}_2;\mathbf{P},E)\, i\mathcal{D}^V\!\left(E-\frac{q_2^2}{2m}, |\mathbf{P}-\mathbf{q}_2|\right) A_3'(\mathbf{q}_2)$$

$$-\int \frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3} \quad ?$$

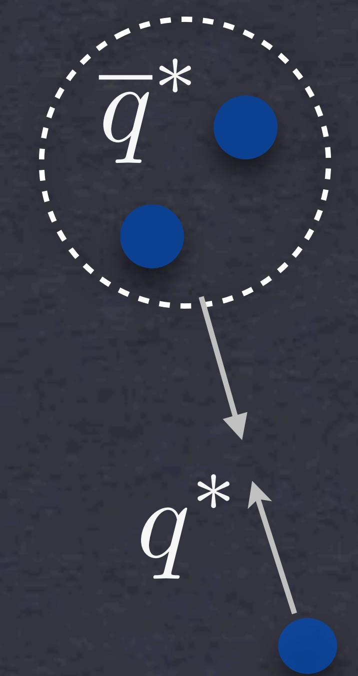
$$iK_3(\mathbf{q}_1,\mathbf{q}_2;\mathbf{P},E) \equiv -ig_3 - \frac{ig_2^2}{E-\frac{\mathbf{q}_1^2}{2m}-\frac{\mathbf{q}_2^2}{2m}-\frac{(\mathbf{P}-\mathbf{q}_1-\mathbf{q}_2)^2}{2m}+i\epsilon}$$

The poles of three-body kernel cancel with zero's of full FV dimer propagator!

4. R. Briceno and ZD, Phys. Rev. D 87, 094507.

DIMER-PARTICLE CORRELATION FUNCTION
QUANTIZATION CONDITION I

In CM frame



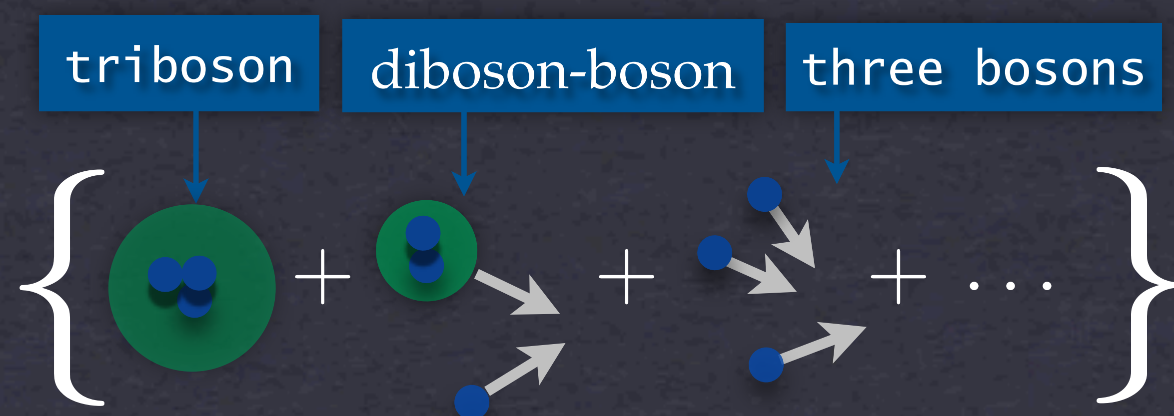
Only Luscher poles matter

(I) $$\overline{q}^*_\kappa \cot \delta_d = 4\pi \; c_{00}^{(\frac{2\mathbf{P}}{3} - q^*_\kappa)}(\overline{q}^2_\kappa)$$

$$\overline{q}^{*2}_\kappa = mE^* - \frac{3}{4} q^{*2}_\kappa$$

COUPLED-CHANNELS

Three-particle states



Power-law corrections

$$\{\overline{q}^*_\kappa, q^*_\kappa\} = \{\left(\overline{q}^*_0, \sqrt{\frac{4}{3}(mE^* - \overline{q}^{*2}_0)}\right), \left(\overline{q}^{*2}_1, \sqrt{\frac{4}{3}(mE^* - \overline{q}^*_1)}\right), \ldots, \left(\overline{q}^*_{N_{E^*}}, \sqrt{\frac{4}{3}(mE^* - \overline{q}^{*2}_{N_{E^*}})}\right)\}$$

Off-shell states $\quad mE^* < \frac{3}{4} q^{*2}_\kappa$

Exponential corrections

DIMER-PARTICLE CORRELATION FUNCTION
QUANTIZATION CONDITION II

(II) $$\mathrm{Det}(1+\tilde{\mathcal{M}}^{\infty}_{\mathrm{V}}\delta\tilde{\mathcal{G}}^{\mathrm{V}}) \equiv \det_{\mathrm{oc}}[\det_{\mathrm{pw}}(1+\tilde{\mathcal{M}}^{\infty}_{\mathrm{V}}\delta\tilde{\mathcal{G}}^{\mathrm{V}})] = 0$$

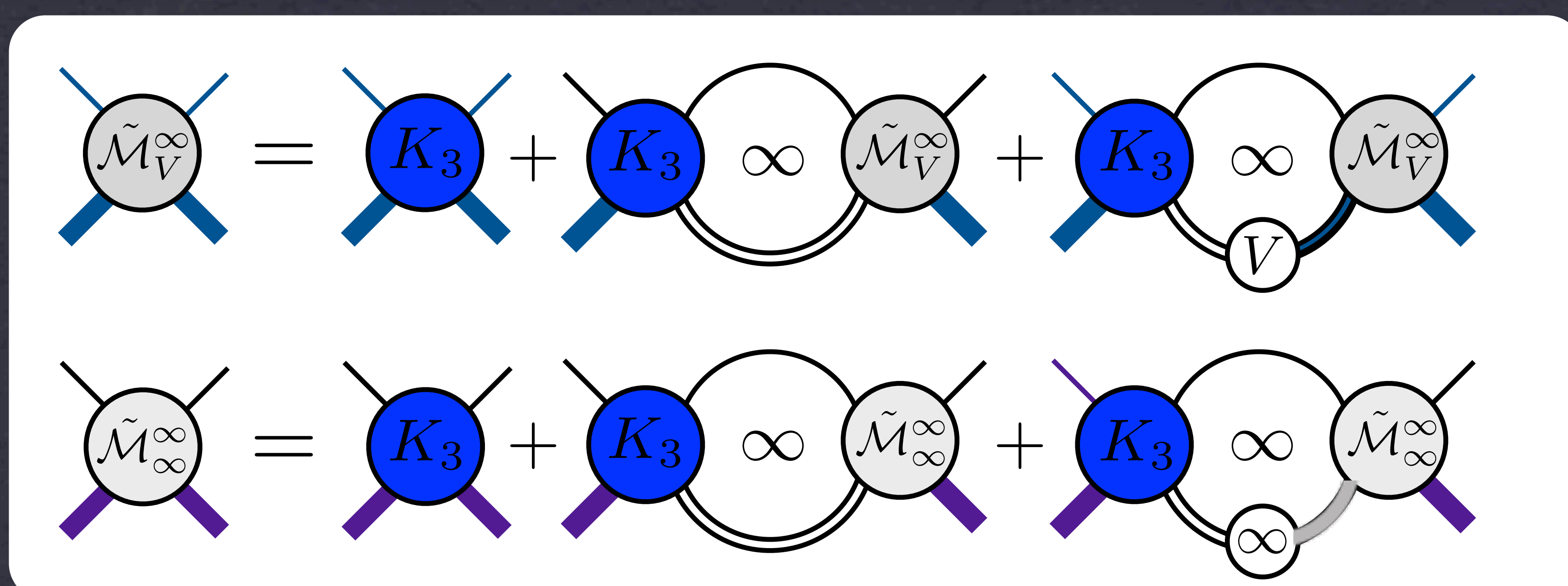
Determinant over open kinematic channels

Determinant over partial-wave channels of boson-dimer state

$$\tilde{\mathcal{M}}^{\infty}_{V}(\mathbf{p},\mathbf{k};\mathbf{P},E) = \tilde{\mathcal{M}}^{\infty}_{\infty}(\mathbf{p},\mathbf{k};\mathbf{P},E) - \int \frac{d^3q}{(2\pi)^3}\tilde{\mathcal{M}}^{\infty}_{\infty}(\mathbf{p},\mathbf{q};\mathbf{P},E)\,\delta\mathcal{D}^{V}\left(E-\frac{q^2}{2m},|\mathbf{P}-\mathbf{q}|\right)\tilde{\mathcal{M}}^{\infty}_{V}(\mathbf{q},\mathbf{k};\mathbf{P},E)$$

Diagonal in angular momentum

Mixes the three particle states



BOUND-STATE PARTICLE SCATTERING

RECOVERING LUESCHER

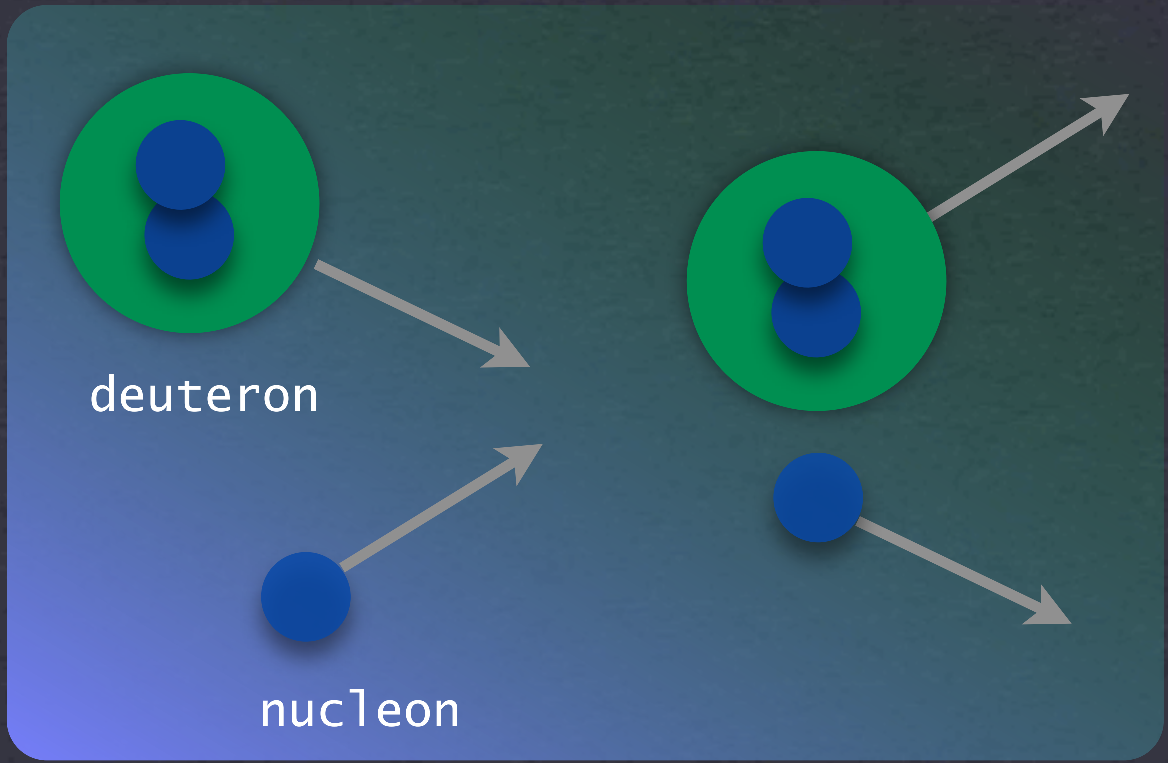
S-wave boson-diboson elastic scattering amplitude

$$\mathcal{M}_{Bd} = \frac{3\pi}{m}\frac{1}{q_0^* \cot\delta_{Bd} - iq_0^*}$$

$$\tilde{\mathcal{M}}_V^\infty \text{ vs. } \tilde{\mathcal{M}}_\infty^\infty \equiv \mathcal{M}_{Bd} \ ?$$

Key: Diboson is a compact object in sufficiently large volumes

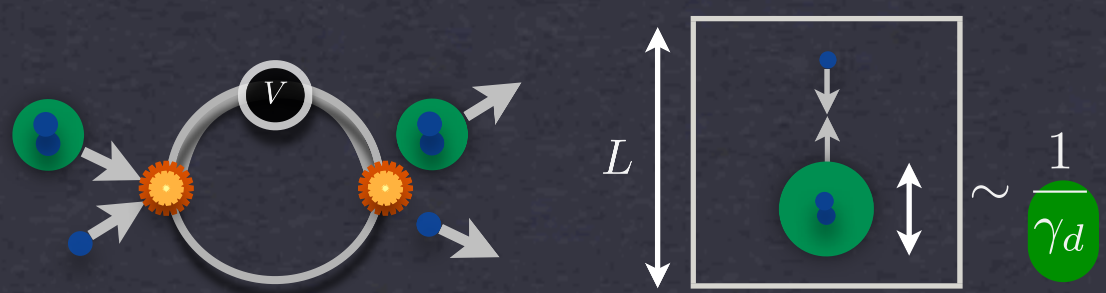
$$\bar{q}_0^* = i\gamma_d + \mathcal{O}(e^{-\gamma_d L}/L)$$



$$q_0^* \cot\delta_{Bd} = 4\pi\ c_{00}^P(q_0^*) + \eta\frac{e^{-\gamma_d L}}{L}$$

A coefficient that needs to be fit to data

$$q_0^* = \sqrt{\frac{4}{3}}(mE^* - \bar{q}_0^{*2})$$



Diboson infinite volume binding momentum

Rokash, et al, arXiv:1308.3386 (2013).

BOUND-STATE PARTICLE SCATTERING

RECOVERING LUESCHER

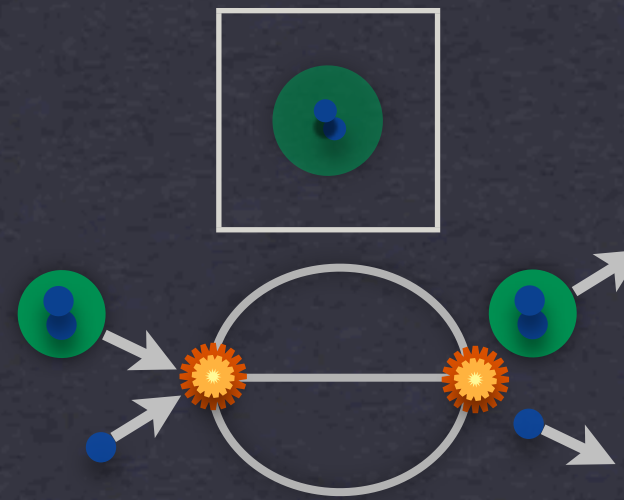
Other sources of systematics to the Luescher approximation

NLO correction due to size of diboson

$$\mathcal{O}\left(e^{-\sqrt{2}\gamma_d L}/L\right)$$

First off-shell state ignored

$$\mathcal{O}\left(\frac{e^{-\sqrt{\frac{4}{3}(q_1^{*2}-mE^*)}L}}{L}\right)$$



Partial-wave mixing, S-wave dimer?

$$(J_d, J_{Bd}) = \{(0,0),(2,0),(4,0),(0,4),(2,4),(2,6),\ldots\}$$

$$(J_d, J_{Bd}) = \{(0,0),(0,1),(2,0),(2,1),(0,2),(2,2),\ldots\}$$

Triton binding energy?

$$\gamma_{Bd} + q^*_{Bd}\cot\delta_{Bd}\big|_{q^{*2}_{Bd}=-\gamma^2_{Bd}} = \mathcal{O}(e^{-\gamma_{Bd}L}/L)$$

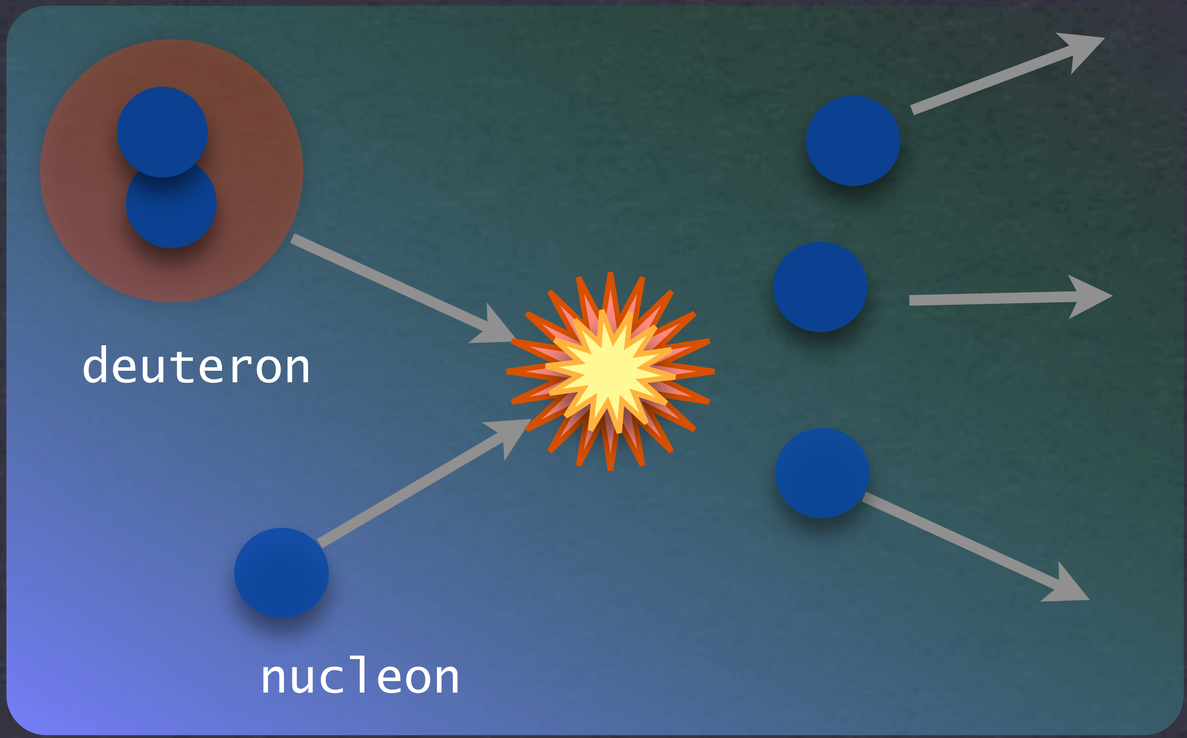
RECOMBINATION AND BREAK UPS?

NO ALGEBRAIC EQUATION EXISTS

Just above the threshold

1 **2**

$$\left\{ \left(\bar{q}_0^*, \sqrt{\frac{4}{3}(mE^* - \bar{q}_0^{*2})} \right), \left(\bar{q}_1^{*2}, \sqrt{\frac{4}{3}(mE^* - \bar{q}_1^*)} \right) \right\}$$



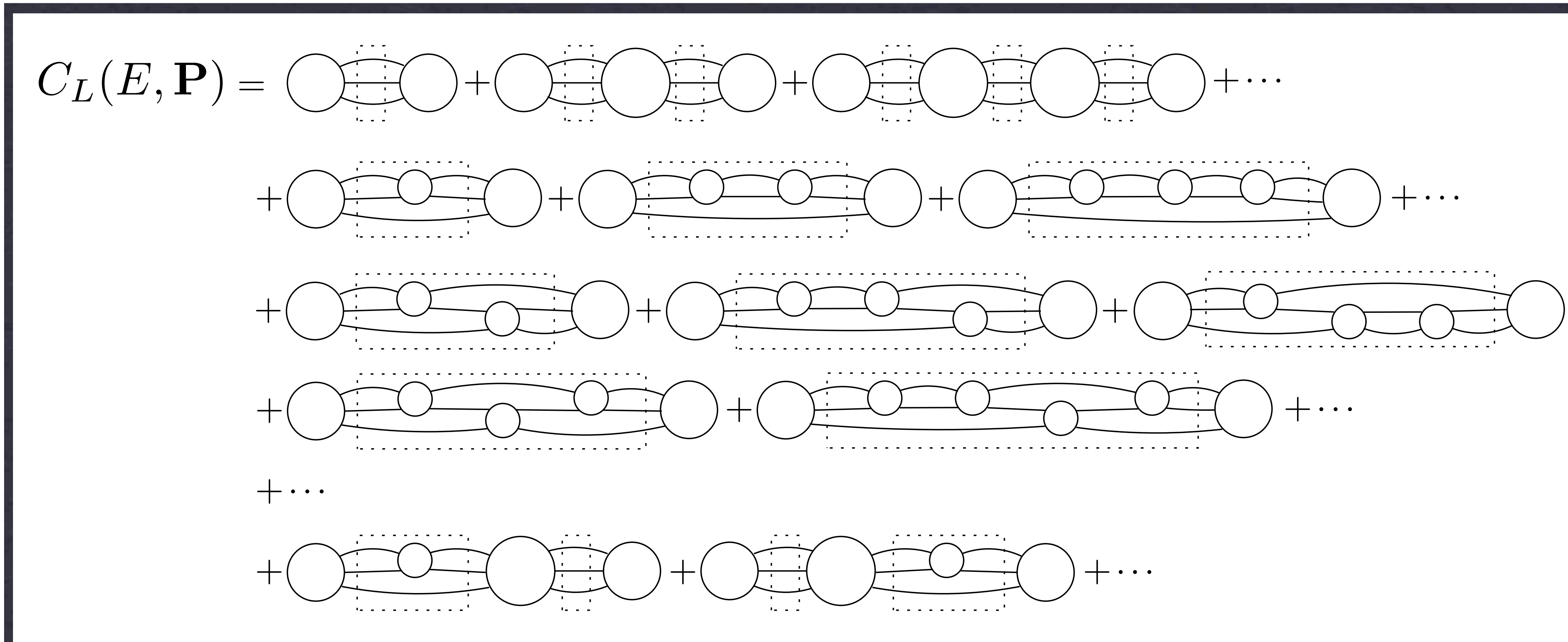
A coupled-channels problem

$$(1 + \tilde{\mathcal{M}}^{\infty}_{V,Bd-Bd}\, \delta\tilde{\mathcal{G}}^{V}_{Bd})(1 + \tilde{\mathcal{M}}^{\infty}_{V,BBB-BBB}\, \delta\tilde{\mathcal{G}}^{V}_{BBB}) = |\tilde{\mathcal{M}}^{\infty}_{V,Bd-BBB}|^2\, \delta\tilde{\mathcal{G}}^{V}_{Bd}\, \delta\tilde{\mathcal{G}}^{V}_{BBB}$$

Relates to physical scattering amplitudes through an integral equation

THREE-PARTICLE QUANTIZATION CONDITION
ALTERNATIVE APPROACHES

Hansen, Sharpe, arXiv:1311.4848..

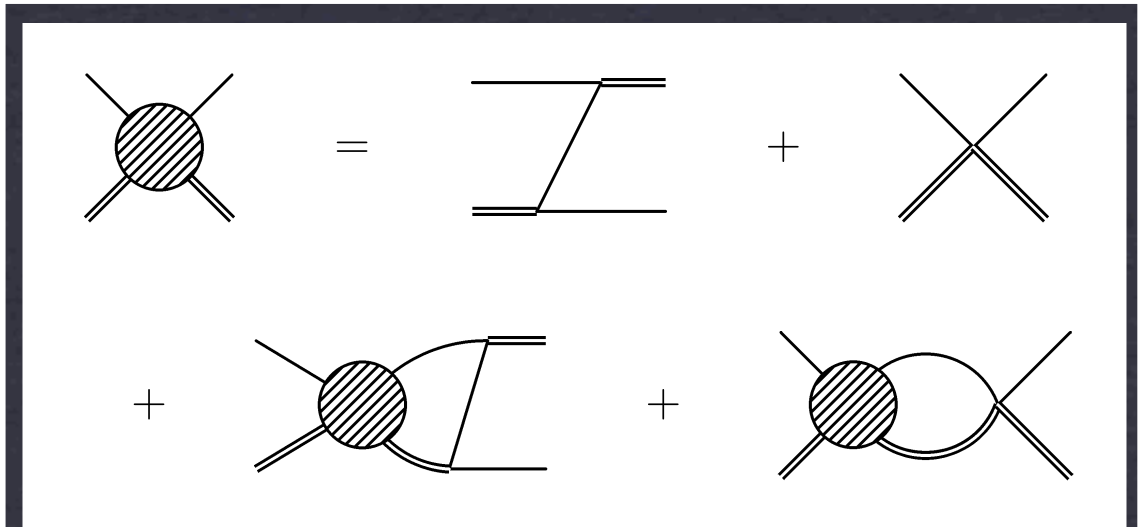


Relativistic model-independent formalism

Non-algebraic in nature

Reproduces perturbative results of Beane, Detmold and Savage (2007) and Tan (2008) up to $\mathcal{O}(1/L^6)$

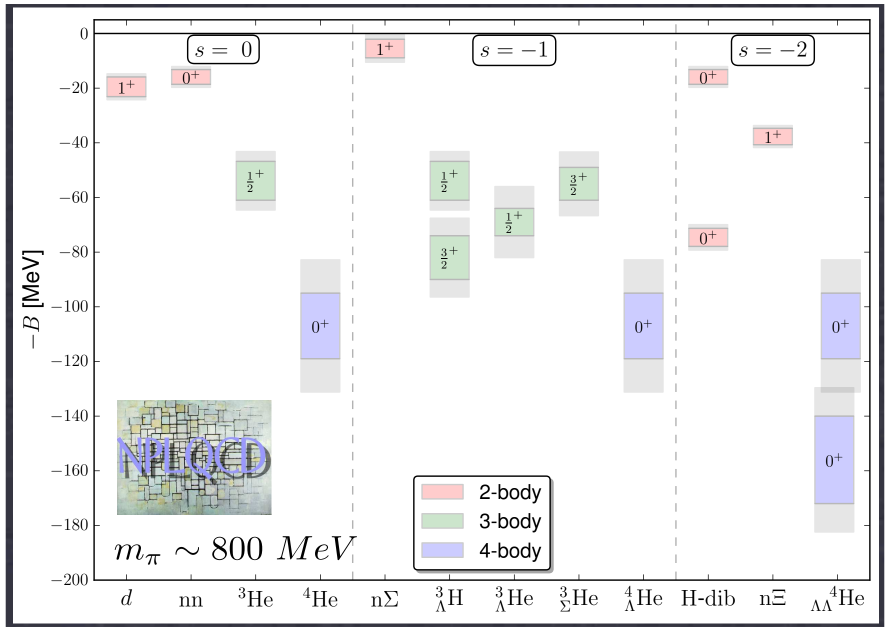
Poeljaeva, Rusetsky, EPJA i 12067 (2012).
Guo, arXiv:1303.3349 (2013).



Kreuzer, Hammer, Phys. Lett. B694: 424 (2011).

NUCLEAR PHYSICS FROM QCD

THE ROADMAP

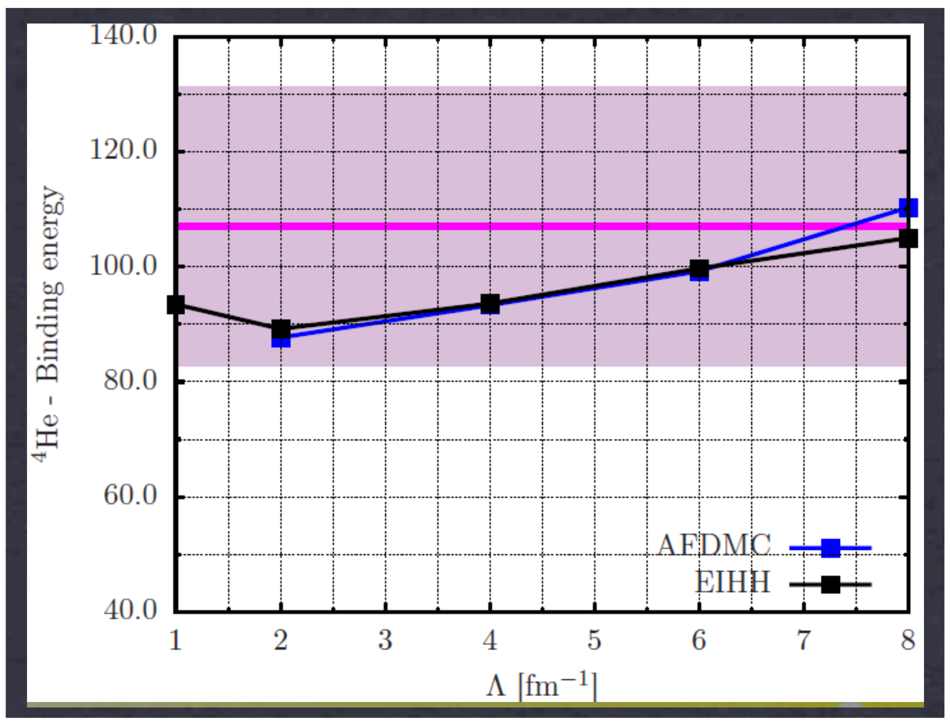


Beane, et al, Phys.Rev. D87 (2013) 034506.

m_π	140	510	805	805
Nucleus	[Nature]	[5]	[6]	[This work]
n	939.6	1320.0	1634.0	1634.0 *
p	938.3	1320.0	1634.0	1634.0
nn	-	7.4 ± 1.4	15.9 ± 3.8	15.9 ± 3.8 *
D	2.224	11.5 ± 1.3	19.5 ± 4.8	19.5 ± 4.8 *
^{3}n	-			-
^{3}H	8.482	20.3 ± 4.5	53.9 ± 10.7	53.9 ± 10.7 *
^{3}He	7.718	20.3 ± 4.5	53.9 ± 10.7	53.9 ± 10.7
^{4}He	28.30	43.0 ± 14.4	107.0 ± 24.2	89 ± 36
^{5}He	27.50			98 ± 39
^{5}Li	26.61			98 ± 39
^{6}Li	32.00			122 ± 50

[5] Yamazaki *et al.* '12
[6] Beane *et al.* '12
[This work] Barnea *et al.* '13

Barnea, et al, arXiv:1311.4966.



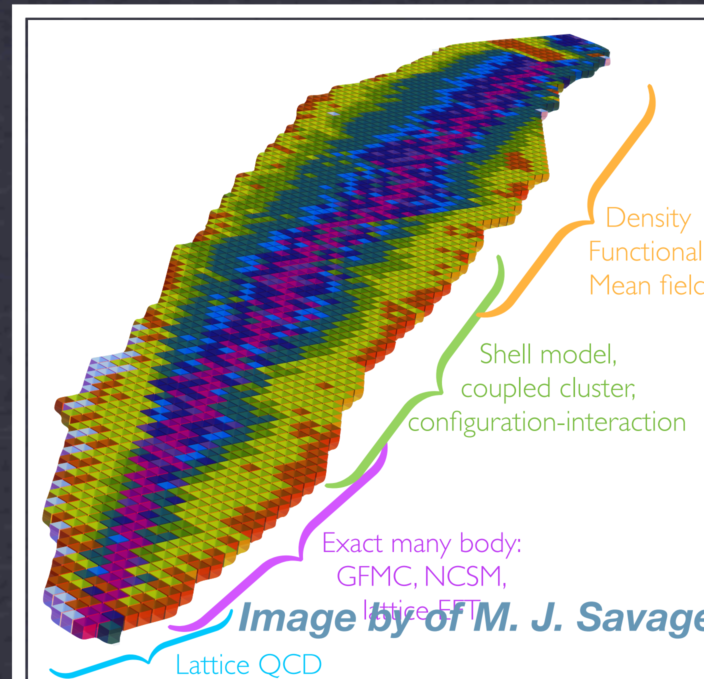
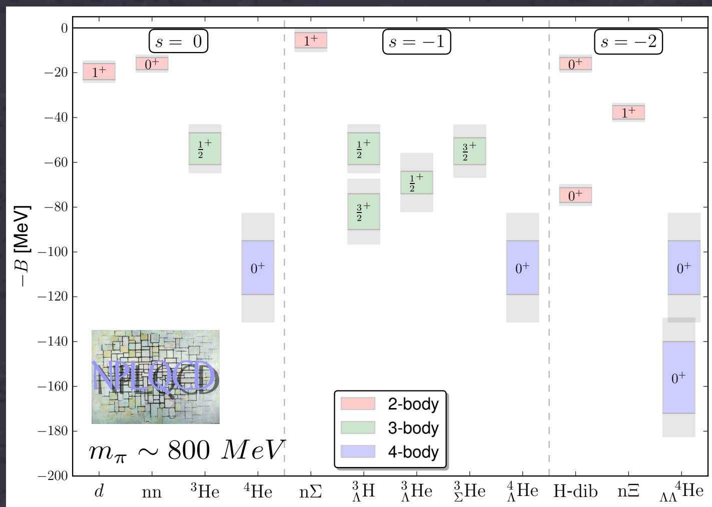
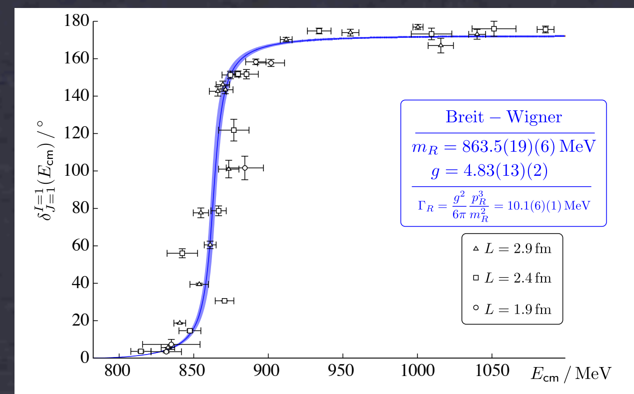


Image by of M. J. Savage

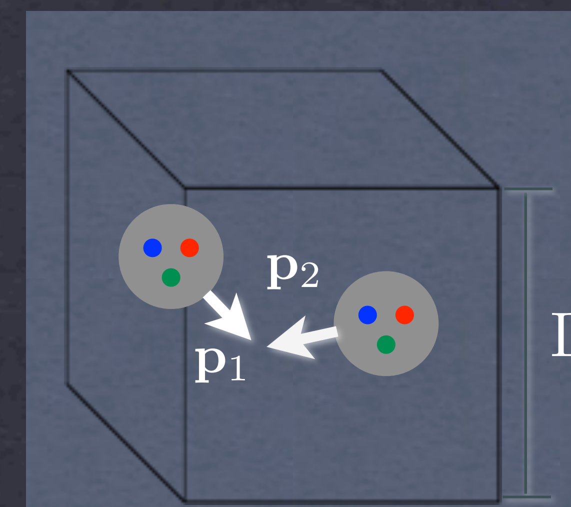
R. Briceno, ZD, T. Luu, Review on the "nuclear reactions from LQCD" workshop, to be released.

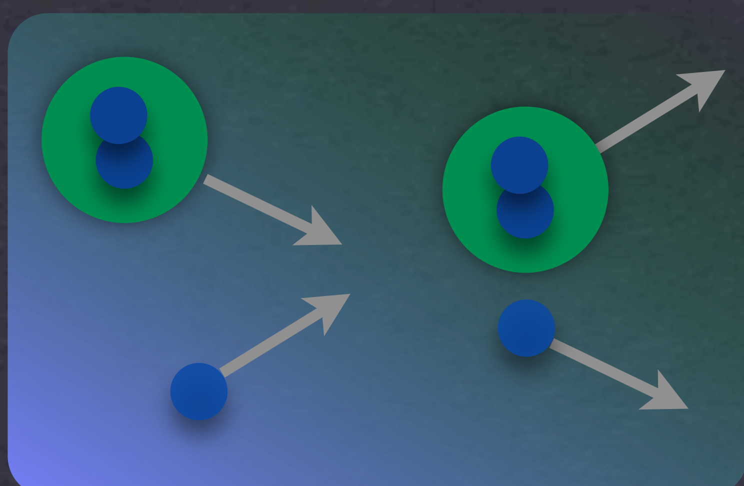
SUMMARY AND CONCLUSION



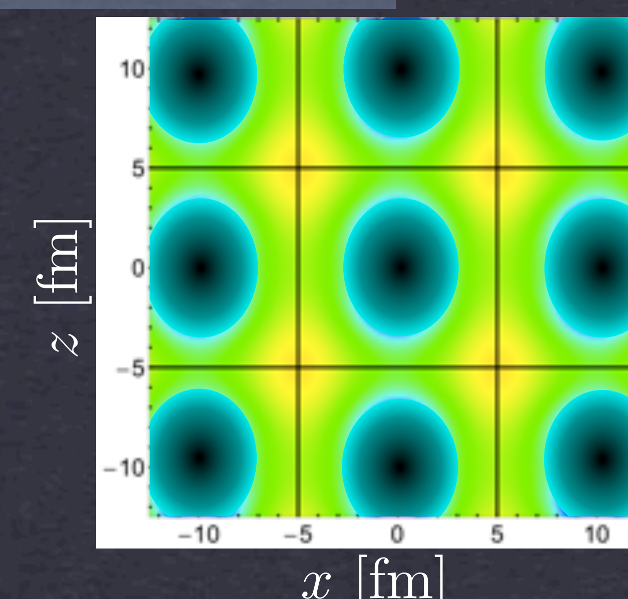


Progress in nuclear multi-particle calculations has been significant.

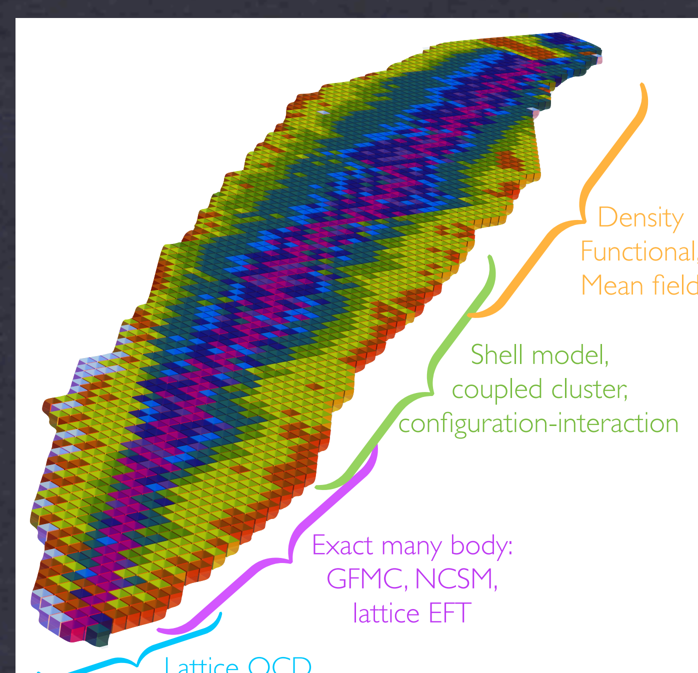




Finite-volume formalism to study three-nucleon systems is under development.



Lattice QCD with the help of effective field theories can be matched into nuclear many-body calculations.



Finite-volume formalism to study two-nucleon systems has been developed.

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MY COLLABORATORS

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THANKS FOR YOUR ATTENTION!