AT THE THRESHOLD OF THE EFIMOV EFFECT

AU SEUIL DE L'EFFET EFIMOV

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OUTLINE OF THE TALK

- **Basic facts and physical motivation**
- **The physical model and the integral equation to solve**
- **Useful limiting cases, their solution**
- **The analytical results**
- **A numerical study**

BASIC FACTS AND PHYSICAL MOTIVATION

THE EFIMOV EFFECT

Relevant regime:

- a resonant s-wave binary interaction between particles
- assume infinite scattering length, no two-body bound state

Then the Efimov effect may occur:

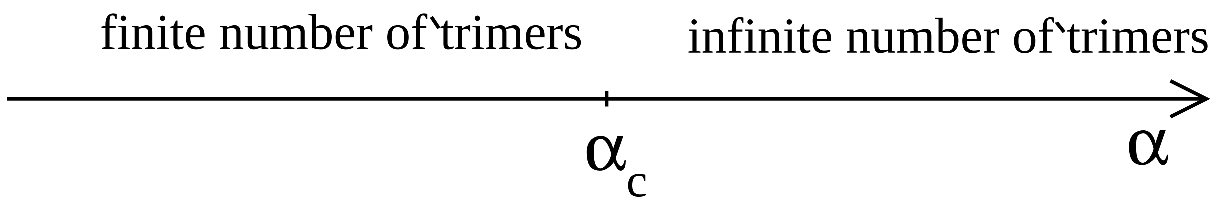
- an infinite number of trimer states
- the spectrum is asymptotic to a geometric sequence, in the limit of a large quantum number n:

$$E_n \underset{n\to+\infty}{\sim} E_{\text{glob}} e^{-2\pi n/|s|}$$

- the exponent $s \in i\mathbb{R}^+$ is given by Efimov zero-range theory, contrarily to E_{glob}
- spectrum becomes geometric, as in zero-range theory, when de Broglie wavelength $\gg$ interaction ranges

A PARTICULARLY INTERESTING CASE

There exists a control parameter α allowing one to continuously switch on/off the Efimov effect



How does the system evolve from a finite number to an infinite number of trimer states ?

Simple facts:

- **The efimovian states cannot emerge from $E = -\infty$ [any physical spectrum is bounded from below], they shall emerge from $E = 0$**
- **close to threshold, the efimovian states are in the zero-range regime so their spectrum shall be entirely geometric**

- behavior of exponent s known, vanishes as $(\alpha - \alpha_c)^{1/2}$:

$$\Lambda(s, \alpha) = 0$$

with Λ even function of s. At threshold, collision in $s = 0$ of two real $(\alpha < \alpha_c)$ or imaginary $(\alpha > \alpha_c)$ roots:

$$\frac{1}{2} s^2 \partial_s^2 \Lambda(0, \alpha_c) + (\alpha - \alpha_c) \partial_\alpha \Lambda(0, \alpha_c) = O(\alpha - \alpha_c)^2$$

- Does E_{glob} also vanish or diverge at the threshold, with some critical exponent ?

Our goal here:

- Answer this question quantitatively on a simple but realistic model: the infinitely narrow Feshbach resonance
- Then analytic techniques exist to calculate E_{glob}, as done for three bosons (Gogolin, Mora, Egger, 2008).
- Also three-body losses suppressed in that limit

THE PHYSICAL MODEL AND THE INTEGRAL EQUATION TO SOLVE

CONFIGURATION & PREDICTIONS OF EFIMOV THEORY

Make Efimov effect avoidable thanks to Pauli exclusion principle:

- polarized fermions do not interact in s-wave
- so take two same-spin-state fermions of mass m resonantly interacting ($1/a = 0$) with an impurity of mass M
- Control parameter is mass ratio $\alpha = m/M$: no Efimov effect if α not too large (Efimov, 1973)

Even more interesting: a sequence of efimovian thresholds

- in the sectors of increasing odd angular momenta:

$$\alpha_c^{(l=1)} = 13.60696\ldots \quad \alpha_c^{(l=3)} = 75.99449\ldots$$
$$\alpha_c^{(l=5)} = 187.9583\ldots \quad \alpha_c^{(l=7)} = 349.6384\ldots$$

- no Efimov effect for even angular momenta

- Simple Born-Oppenheimer explanation:

$$\psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{R}) \approx \phi(\mathbf{r}_1-\mathbf{r}_2)\Phi(\mathbf{R};\mathbf{r}_1,\mathbf{r}_2)$$

where the ground-state wavefunction Φ of the impurity at fixed fermionic positions is a symmetric function of these positions, of eigenenergy $-\hbar^2 C^2/(2Mr_{12}^2)$ [here $C = \exp(-C)$], which leads to the effective potential

$$V_{\text{eff}}(r_{12}) = \frac{\hbar^2 l(l+1)}{mr_{12}^2} - \frac{\hbar^2 C^2}{2Mr_{12}^2} \overset{\text{Efimov}}{\equiv} \frac{\hbar^2(s_l^2-1/4)}{mr_{12}^2}$$

We have results beyond the Born-Oppenheimer approximation:

At bounded distance from threshold:

$$|s_l|^2 \underset{l\to\infty}{=} \frac{1}{2}C^2(\alpha-\alpha_c^{(l)})[1+O(1/l)]$$

$$\frac{1}{2}C^2\alpha_c^{(l)} \underset{l\to\infty}{=} \left(l+\frac{1}{2}\right)^2 + \frac{17-C^2}{12} - \frac{7}{6}\left(C+\frac{1}{C+1}\right) + O(1/l)$$

WHICH IMPURITY-FERMION INTERACTION

- A Feshbach resonance: two-channel model
- in the open channel, van de Waals interaction of length b and non-resonant scattering length $a_{\rm bg} \approx b$
- infinitely narrow: take limit $b \to 0$ for fixed (rather than diverging) interchannel coupling Λ. Then corresponding Feshbach length R_* does not vanish. E. g. for $|a_{\rm bg}| \ll b$:

$$R_* \simeq \frac{\pi \hbar^4}{\Lambda^2 \mu^2}$$

- R_* gives the effective range of the binary interaction:

$$f_k = \frac{-1}{ik + k^2 R_*}$$

Ansatz for the trimer state of energy $E = -\hbar^2 q^2/(2\mu) < 0$:

$$|\psi_{3\,\mathrm{at}}\rangle = \int \frac{\prod_{i=1}^{3} d^3k_i}{[(2\pi)^3]^3}(2\pi)^3\delta(\sum_{i=1}^{3}\mathbf{k}_i)A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)a^\dagger_{\mathbf{k}_1}c^\dagger_{\mathbf{k}_2}c^\dagger_{\mathbf{k}_3}|0\rangle$$

$$|\psi_{1\,\mathrm{at}+1\,\mathrm{mol}}\rangle = \int \frac{d^3k}{(2\pi)^3}B(\mathbf{k})b^\dagger_{-\mathbf{k}}c^\dagger_{\mathbf{k}}|0\rangle$$

- Integral equation from Schrödinger's equation:

$$\left[q_{\mathrm{rel}}(k) + q^2_{\mathrm{rel}}(k)R_*\right]D(\mathbf{k}) = -\int \frac{d^3k'}{2\pi^2}\frac{D(\mathbf{k}')}{q^2+k^2+k'^2+\frac{2\alpha}{1+\alpha}\mathbf{k}\cdot\mathbf{k}'}$$

where $D(\mathbf{k}) \simeq B(\mathbf{k})$ for $|a_{\mathrm{bg}}| \ll b$

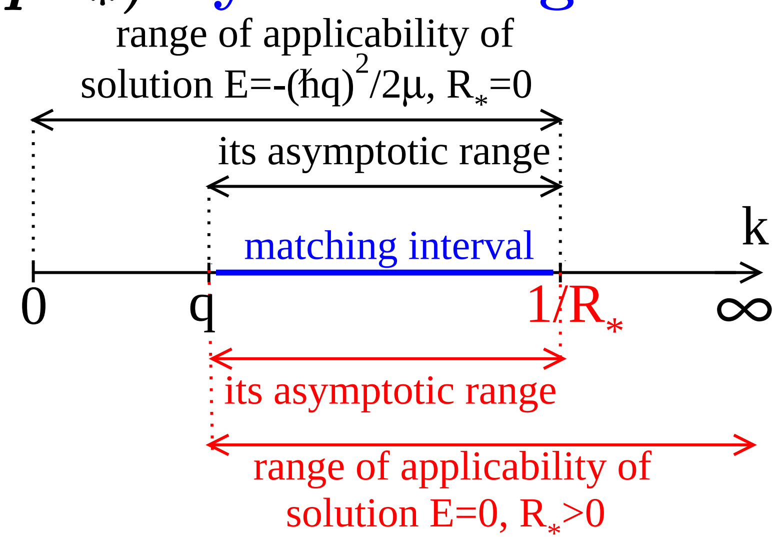
- effective relative wavenumber between impurity and fermion:

$$q_{\mathrm{rel}}(k) = [q^2 + \frac{1+2\alpha}{(1+\alpha)^2}k^2]^{1/2}$$

- At fixed angular momentum: $D(\mathbf{k}) = d(k)Y_l^0(\hat{\mathbf{k}})$

USEFUL LIMITING CASES, THEIR SOLUTION

We shall obtain the trimer energies analytically with a relative error $O(qR_*)$ by matching two solutions:



When $qR_* \ll 1$ there exists a momentum interval where both solutions are applicable and are in their $k \to \infty$ and $k \to 0$ asymptotic regimes. Matchable asymptotic forms:

$$k^2 d(k) \underset{k/q\to\infty}{=} e^{i\theta_<}(k/q)^s + \text{c.c.} + O(k/q)^2$$

$$k^2 d(k) \underset{kR_*\to 0}{=} e^{i\theta_>}(kR_*)^s + \text{c.c.} + O(kR_*)$$

HOW TO SOLVE ?

$E < 0, R_* = 0$:

- Fourier transform the real space Efimov solution

$E = 0, R_* > 0$ (Gogolin, Mora, Egger, 2008):

- integral term is scaling invariant. Change of variable $x = \ln(kR_* \cos\nu)$ [where $\nu = \arcsin\frac{\alpha}{1+\alpha}$ is mass angle] makes it translationally invariant: setting $k^2 d(k) = F(x)$,

$$0 = (1 + e^x)F(x) + (K * F)(x)$$

- Fourier transform with respect to x:

$$0 = \tilde{F}(S+i) + \Lambda_l(iS, \alpha)\tilde{F}(S)$$

- Infinite product representation of $s \mapsto \Lambda_l(s)$ over its roots and poles. Then solution for $\tilde{F}(S)$ is an infinite product of ratios of Γ functions $[\Gamma(z+1) = z\Gamma(z)]$

THE ANALYTICAL RESULTS

Exact value of the global energy scale:

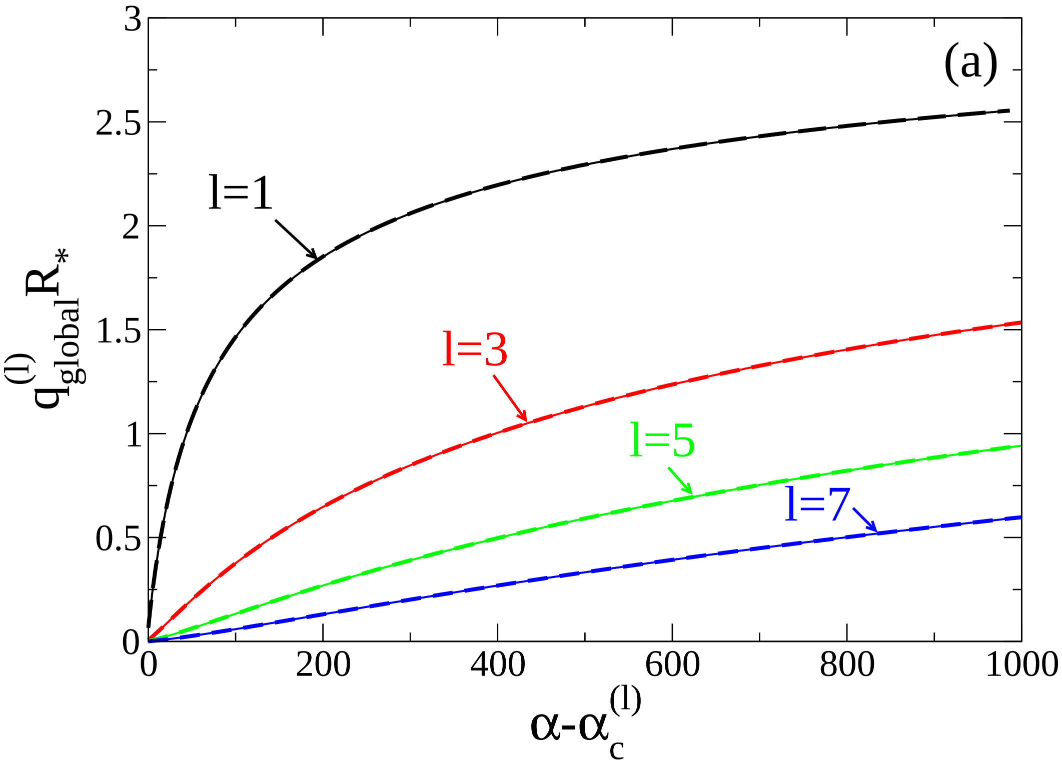
$$E^{(l)}_{\rm glob} = -\frac{2\hbar^2}{\mu R_*^2} e^{2\theta_l/|s_l|} \equiv -\frac{\hbar^2 q^{(l)2}_{\rm glob}}{2\mu}$$

$$\theta_l = \mathrm{Im}[\ln\Gamma(1+s_l)+\ln\Gamma(1+2s_l)+2\ln\Gamma(l+1-s_l)+\ln\Gamma(l+2-s_l)]$$
$$+\int_0^{|s_l|} dS \ln\left[\Lambda_l(iS,\alpha)\,\frac{S^2+(l+1)^2}{S^2-|s_l|^2}\right]$$
$$+\sum_{k\geq 1} \frac{(-1)^k B_{2k}}{(2k)!}\frac{d^{2k-1}}{dS^{2k-1}}\left\{\ln\left[\Lambda_l(iS,\alpha)\,\frac{S^2+(l+1)^2}{S^2-S_l^2}\right]\right\}_{S=|s_l|}$$

Finite limit at threshold:

$\theta_l/|s_l| \rightarrow 3\psi(1)-2\psi(l+1)-\psi(l+2)+\sum_j[\psi(x_j)+\psi(1+x_j)-\psi(l+1+2j)-\psi(l+2+2j)]$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function and the sum is taken over the positive roots of $\Lambda_l(x,\alpha_c^{(l)})$



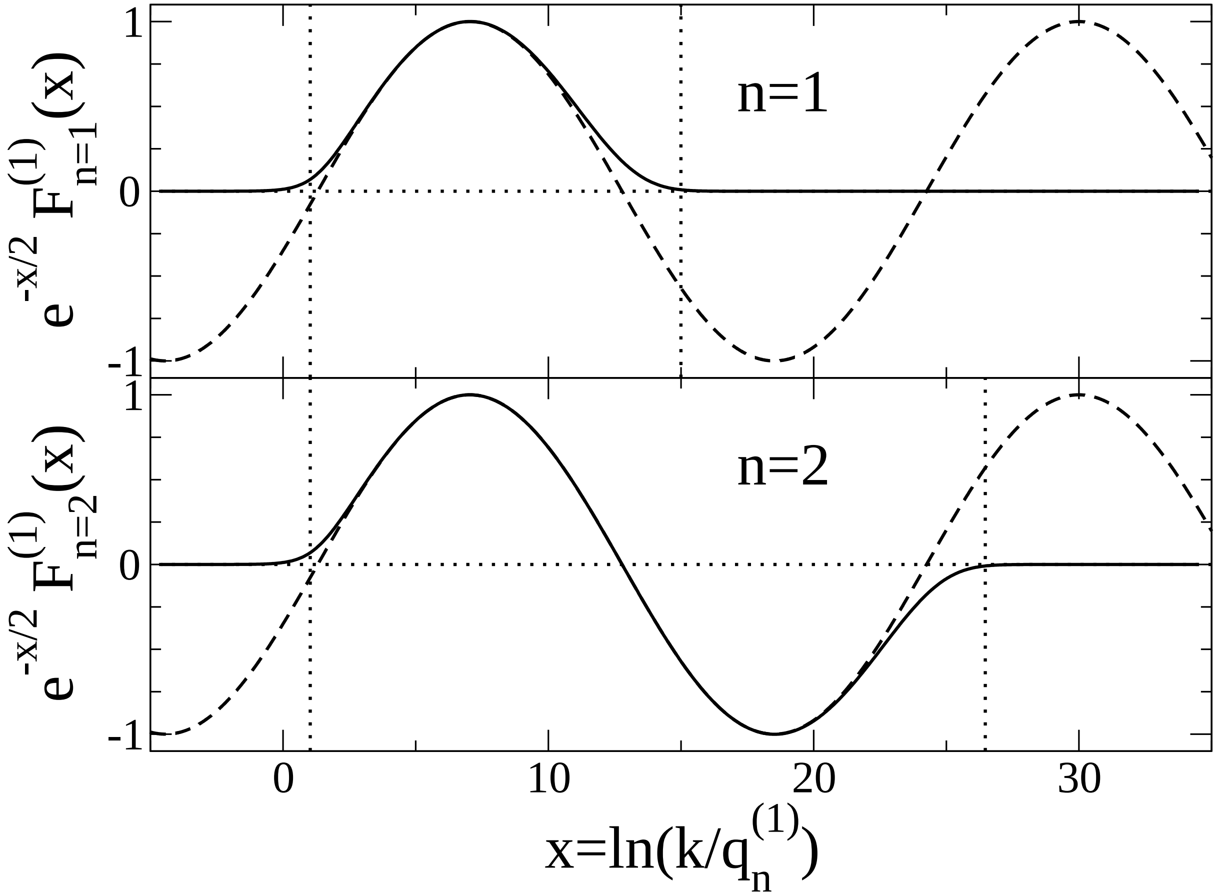
It is an excellent approximation to neglect the sum over $\boldsymbol{k}$: dashed line vs exact solid line. Other results:

$$q^{(l)}_{\rm glob} R_* |_{\rm threshold} \underset{l\to\infty}{\sim} \frac{1+C}{l^3} e^{-3\gamma}$$

$$q^{(l)}_{\rm glob} R_* \underset{\alpha\to\infty}{\longrightarrow} 2(1+C) e^{\int_0^C dx \left(\frac{1}{C}\frac{1+x}{1-xe^x} - \frac{1}{C-x}\right)}$$

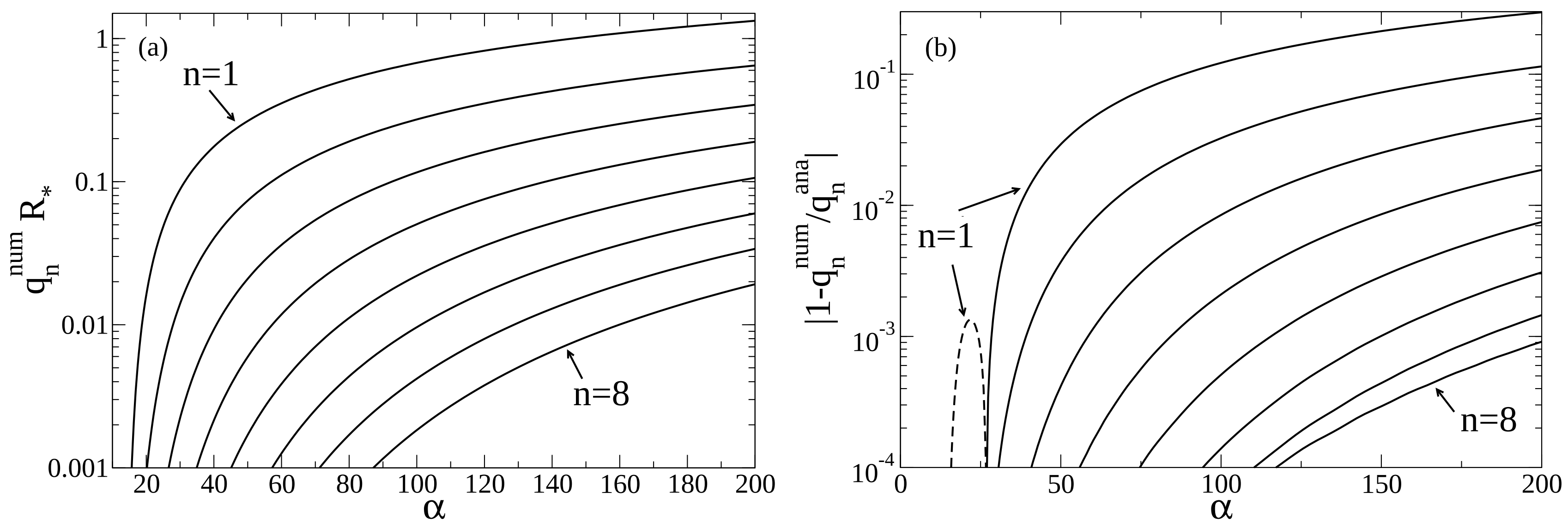
A NUMERICAL STUDY:
BEYOND THE GEOMETRIC SPECTRUM

ANALYTICAL VS NUMERICAL FUNCTIONS ($\alpha = 14$)



Solid line: numerical. Dashed line: asymptotic formula common to $(E < 0, R_* = 0)$ and $(E = 0, R_* > 0)$ analytical solutions. Vertical dotted lines: borders of the matching interval. N.B. $n = 1$ is indeed the ground trimer state.

ANALYTICAL VS NUMERICAL SPECTRA



Relative deviations from geometric spectrum

- at fixed α, $\rightarrow 0$ if $n \rightarrow \infty$
- at fixed n, $\rightarrow \infty$ if $\alpha \rightarrow \infty$

Experimentally accessible range (due to finite a):

- $qR_* > 0.1$ not irrealistic
- On ^{6}Li-^{40}K Feshbach resonance, of width $\Delta B = 1$G, requires 0.3mG B field stabilization

CONCLUSION

- $2+1$ fermionic problem, mass ratio α, narrow Feshbach resonance
- at each Efimov threshold (of odd angular momentum l), the corresponding trimer spectrum is entirely geometric:

$$E_n^{(l)} \underset{\alpha\to\alpha_c^{(l)+}}{\sim} E_{\text{glob}}^{(l)} e^{-2\pi n/|s_l|} \quad \forall n \geq 1$$

  where the ground state trimer is $n=1$
- the exact expression of $E_{\text{glob}}^{(l)}$ shows that it has a finite and non-zero limit at threshold
- opposite limit $\alpha \to +\infty$: spectrum becomes hydrogenoid

$$E_n^{(l)} \underset{\alpha\to\infty}{\sim} -\frac{\hbar^2\alpha}{16\mu R_*^2}\frac{1}{(n+l)^2}$$

  as predicted by the Born-Oppenheimer approximation