

Universal behaviour of four-boson systems from a functional renormalisation group

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Results from:

Jaramillo Avila and Birse, arXiv:1304.5454

Schmidt and Moroz, arXiv:0910.4586

Krippa, Walet and Birse, arXiv:0911.4608

Birse, Krippa and Walet, arXiv:1011.5852

Background

Ideas of effective field theory and renormalisation group

- well-developed for few-nucleon and few-atom systems
 - rely on separation of scales
 - Wilsonian RG used to derive power counting
- classify terms as perturbations around fixed point (or limit cycle)

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Many unsuccessful attempts to extend to dense matter

- but no separation of scales
- other EFT's for interacting Fermi systems exist
(Landau Fermi liquid, Ginsburg-Landau theory)
- but parameters have no simple connection to underlying forces

EFTs based on contact interactions

- not well suited for standard many-body methods
- switch to lattice simulation or look for some more heuristic approach
- based on field theory
- can be matched onto EFT's for few-body systems
(input from 2- and 3-body systems in vacuum)

Try functional renormalisation group (“exact” RG)

- based on Wilsonian RG approach to field theories
- successfully applied to various systems in areas from condensed-matter physics to quantum gravity
[version due to Wetterich (1993)]

Outline

- Functional RG
- Spin- $\frac{1}{2}$ fermions
 - Dimer-dimer scattering
- Bosons
 - Efimov physics
 - 4-body systems

Functional RG

Version based on the effective action $\Gamma[\phi_c]$

- start from generating function $W[J]$ defined by

$$e^{iW[J]} = \int D\phi e^{i(S[\phi] + J \cdot \phi - \frac{1}{2} \phi \cdot R \cdot \phi)}$$

- $R(q, k)$: regulator function
suppresses modes with momenta $q \lesssim k$ (“cutoff scale”)
- only modes with $q \gtrsim k$ integrated out
- $W[J]$ becomes full generating function as $k \rightarrow 0$

Legendre transform \rightarrow effective action

$$\Gamma[\phi_c] = W[J] - J \cdot \phi_c + \frac{1}{2} \phi_c \cdot R \cdot \phi_c \quad \text{where} \quad \phi_c = \frac{\delta W}{\delta J}$$

(generating function for 1-particle-irreducible diagrams)

Γ evolves with scale k according to

$$\partial_k \Gamma = -\frac{i}{2} \text{Tr} \left[(\partial_k R) \left(\Gamma^{(2)} - R \right)^{-1} \right] \quad \text{where} \quad \Gamma^{(2)} = \frac{\delta^2 \Gamma}{\delta \phi_c \delta \phi_c}$$

$(\Gamma^{(2)} - R)^{-1}$: propagator of boson in background field ϕ_c
(one-loop structure but still exact)

Evolution interpolates between “bare” classical action at large scale K and full 1PI effective action as $k \rightarrow 0$ (thresholds etc ...)

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Functional differential equation

- hard/impossible solve in general
- work with truncated ansatz for Γ
- local action expanded in powers of derivatives
(cf low-energy EFTs, but don't know *a priori* if we have a consistent power counting)

Derivative expansion may be good at starting scale K

- use power counting of EFT to determine relevant terms
(or use this RG to find that power counting in scaling regime)
- but no guarantee that it remains good for $k \rightarrow 0$
(can't be for scattering amplitudes at energies above threshold:
cuts \rightarrow nonanalytic behaviour)

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\rightarrow need consistency checks:

stability against adding extra terms to ansatz
stability against changes in form of regulator

- use this to optimise choice of regulator [Litim, Pawłowski]

Two species of fermion

Fermion field: $\psi(x)$ (spin- $\frac{1}{2}$ atoms or neutrons)

Boson “dimer” field: $\phi(x)$ (strongly interacting pairs)

Local (nonrelativistic) ansatz for action in vacuum: 2-body sector

$$\begin{aligned} & \Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger; k] \\ &= \int d^4x \left[\psi(x)^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi(x) \right. \\ & \quad \left. + Z_\phi(k) \phi(x)^\dagger \left(i\partial_0 + \frac{\nabla^2}{4M} \right) \phi(x) - u_1(k) \phi(x)^\dagger \phi(x) \right. \\ & \quad \left. - g \left(\frac{i}{2} \phi(x)^\dagger \psi(x)^T \sigma_2 \psi(x) + \text{H.c.} \right) \right] \end{aligned}$$

g : AA→D coupling

$u_1(k)$: dimer self-energy (u_1/g^2 : only physical parameter)

$Z_\phi(k)$: dimer wave-function renormalisation

Evolution equation

$$\partial_k \Gamma = +\frac{i}{2} \text{Tr} \left[(\partial_k \mathbf{R}_F) \left((\Gamma^{(2)} - \mathbf{R})^{-1} \right)_{FF} \right] \\ -\frac{i}{2} \text{Tr} \left[(\partial_k \mathbf{R}_B) \left((\Gamma^{(2)} - \mathbf{R})^{-1} \right)_{BB} \right]$$

$\Gamma^{(2)}$: matrix of second derivatives of the action

(Gorkov-like form: ψ and ψ^\dagger as independent variables \rightarrow factors of $\frac{1}{2}$)

“Skeleton” diagram for driving terms in evolution of 2-body parameters



(need to insert $\partial_k \mathbf{R}_F$ on one internal line)

Expand in powers of energy $\rightarrow \partial_k u_1, \partial_k Z_\phi$

3-body sector: AD contact interaction

$$\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger; k] = \dots - \lambda(k) \int d^4x \psi^\dagger(x) \phi^\dagger(x) \phi(x) \psi(x)$$

Evolution of λ driven by terms corresponding to skeletons



- AD contact interaction
- single-A exchange between dimers
(cf Faddeev and STM equations)

4-body sector: DD→DD, DD→DAA, DAA→DAA terms

[Birse, Krippa and Walet (2010); cf Schmidt and Moroz (2009): bosons]

$$\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger; k] = \dots - \int d^4x \left[\frac{1}{2} u_2(k) (\phi^\dagger \phi)^2 \right. \\ \left. + \frac{1}{4} v(k) (i\phi^{\dagger 2} \phi \psi^T \sigma_2 \psi + \text{H.c.}) \right. \\ \left. + \frac{1}{4} w(k) \phi^\dagger \phi \psi^\dagger \sigma_2 \psi^{\dagger T} \psi^T \sigma_2 \psi \right]$$

- dimer “breakup” terms allow 3-body physics to feed in properly (cf Faddeev-Yakubovski)

→ coupled evolution equations for u_2, v, w (27 distinct skeletons)

Regulators

- fermions: sharp cutoff

$$R_F(\mathbf{q}, k) = \frac{k^2 - q^2}{2M} \theta(k - q)$$

- pushes states with $q > k$ up to energy $k^2/2M$
- nonrelativistic version of “optimised” cutoff [Litim (2001)]
- fastest convergence at this level of truncation
- bosons

$$R_B(\mathbf{q}, k) = Z_\phi(k) \frac{(c_B k)^2 - q^2}{4M} \theta(c_B k - q)$$

- c_B : relative scale of boson cutoff
- optimised choice $c_B = 1$ [cf Pawłowski (2007)]
(no mismatch between fermion and boson cutoffs)

Also examined smooth cutoffs – more convenient in dense matter

Initial conditions

As $k \rightarrow \infty$ boson field purely auxiliary

- $Z_\phi(k) \rightarrow 0$
- $u_1(K)$ chosen so that in physical limit ($k \rightarrow 0$)

$$u_1(0) = -\frac{Mg^2}{4\pi a_0} \quad a_0: \text{AA scattering length}$$

- other couplings λ, u_2, v, w also vanish as $k \rightarrow \infty$
- either set $Z_\phi(K) = 0$ etc at large starting scale K
or match on to K^{-n} behaviour in scaling regime $K \gg 1/a_0$

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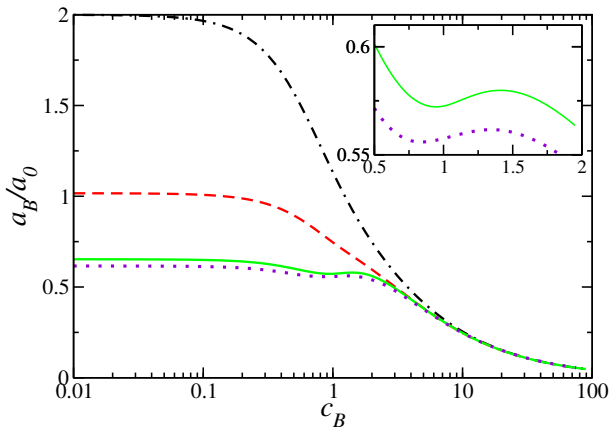
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Expansion point for $a_0 > 0$: dimer binding energy $\mathcal{E}_D = -1/(Ma_0^2)$

- external boson lines carry $P_0 = \mathcal{E}_D$
- external fermion lines carry $P_0 = \mathcal{E}_D/2$
(below all thresholds)

Results: DD scattering length



- black: “minimal” action – only two-body and DD vertex u_2
- red adds three-body coupling λ
- green: full local four-body action, includes v, w
- purple: similar but using smooth cutoff

Comments

- results seem to converge as more terms are included
- converge to value only weakly dependent on cutoff
(very little variation over range $0 \leq c_B \lesssim 2$)
- stationary very close to expected “optimum” $c_B = 1$
- incomplete actions \rightarrow strong dependence on c_B around $c_B = 1$

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Final result

- $a_B/a_0 \simeq 0.58 \pm 0.02$
- agrees well with full few-body result $a_B/a_0 = 0.6$
[Petrov, Salomon and Shlyapnikov (2004)]

Bosons

More interesting: Efimov effect in 3-body sector

Very similar action and evolution equations

- 3-body coupling λ periodic under scaling k by factor e^{π/s_0} where $s_0 = 0.92503$ [Schmidt and Moroz (2009)]
- agrees with Efimov $s_0 = 1.00624$ to $< 10\%$
- no sign of 4-body bound states at this truncation [numerical integration requires some care – poles in λ]

Introduce trimer field $\chi(x)$

- include energy dependence associated with 3-body bound states
- obtain equations with structure like Faddeev-Yakubovsky [coupled DD, AT channels]

Effective action

$$\begin{aligned}
 & \Gamma_k[\psi, \psi^*, \phi, \phi^*, \chi, \chi^*] \\
 &= \int d^4x \left[\psi^* \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi + Z_d \phi^* \left(i\partial_0 + \frac{\nabla^2}{4m} \right) \phi + Z_t \chi^* \left(i\partial_0 + \frac{\nabla^2}{6m} \right) \chi \right. \\
 &\quad - u_d \phi^* \phi - u_t \chi^* \chi - \frac{g}{2} (\phi^* \psi \psi + \psi^* \psi^* \phi) - h (\chi^* \phi \psi + \phi^* \psi^* \chi) \\
 &\quad - \lambda \phi^* \psi^* \phi \psi \\
 &\quad - \frac{U_{dd}}{2} (\phi^* \phi)^2 - \frac{V_d}{4} (\phi^* \phi^* \phi \psi \psi + \phi^* \psi^* \psi^* \phi \phi) - \frac{W}{4} \phi^* \psi^* \psi^* \phi \psi \psi \\
 &\quad - u_{tt} \chi^* \psi^* \chi \psi - \frac{U_{dt}}{2} (\phi^* \phi^* \chi \psi + \chi^* \psi^* \phi \phi) \\
 &\quad \left. - \frac{V_t}{2} (\phi^* \psi^* \psi^* \chi \psi + \chi^* \psi^* \phi \psi \psi) \right]
 \end{aligned}$$

AD interaction λ regenerated by evolution even if zero initially
[unlike AA scattering]

→ introduce running trimer field [cf Gies and Wetterich (2002)]

$$\partial_k \chi = \zeta_1 \phi \psi + \zeta_2 \psi^\dagger \chi \psi + \zeta_3 \psi^\dagger \phi \phi + \zeta_4 \psi^\dagger \phi \psi \psi$$

where $\zeta_1 = -\partial_k \lambda / 2h$ to cancel running of λ

- other terms do same for four-atom couplings v_d , w and v_t
- additional piece in evolution equation

$$\partial_k \Gamma = -\frac{i}{2} \text{Tr} \left[(\partial_k \mathbf{R}) \left((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1} \right) \right] + \frac{\delta \Gamma}{\delta \chi} \cdot \partial_k \chi$$

- can keep λ , v_d , w and $v_t = 0$ for all k

3-body sector

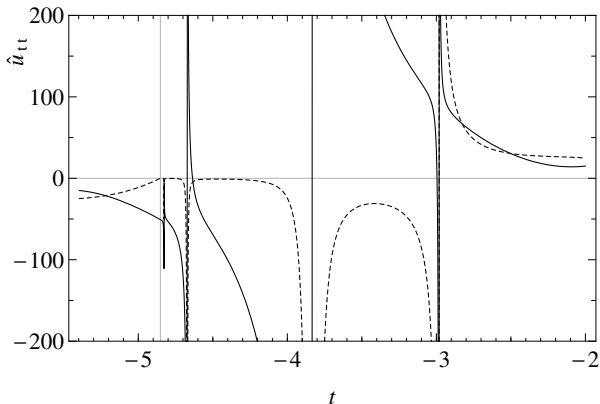
Coupled equations for $u_t(k)$, $Z_t(k)$ and $h^2(k)$

Scaling limit $k \gg 1/|a_0|$

- couplings oscillate sinusoidally with $\ln k$
 - poles in AD scattering amplitude h^2/u_t
(values of k where 3-body bound states appear at zero energy)
- tower of Efimov states with $s_0 = 0.92503$
momentum scale factor $e^{\pi/s_0} = 29.2$ (exact: 22.7)
- tower cuts off when $k \sim 1/a_0$

4-body sector

3-body couplings \rightarrow cyclic behaviour in $u_{dd}(k)$, $u_{dt}(k)$, and $v_t(k)$
One Efimov cycle of rescaled $\hat{u}_{tt}(k)$ as a function of $t = \ln(k/K)$



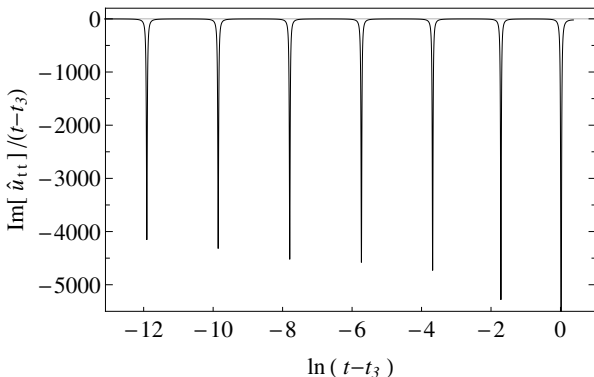
solid: real, dashed: imaginary

vertical grey line: AT threshold passes through zero energy

Comments

- imaginary part appears at AT threshold $t = t_3 \simeq -4.85$
- 4-body bound states below AT threshold $t \simeq -3.83, -4.67, \dots$
(decay to deeper trimer + free atom \rightarrow finite widths)
- unphysical singularity from zero of $h^2(k)$ at $t \simeq -3.0$
(end of region within cycle where $h^2(k)$, $Z_t(k)$ have opposite signs \rightarrow trimer “ghost”)

Infinite tower of 4-body bound states below AT threshold



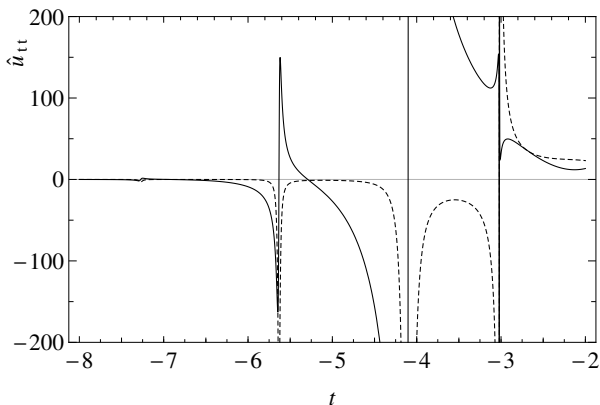
Double exponential pattern \sim super-Efimov effect

[Nishida, Moroz and Son (2013)]

- but may not survive in physical limit $k \rightarrow 0$
- 4-body states may move relative to AT threshold, become virtual

Final cycle of $\hat{u}_{tt}(k)$ for finite $a_0 < 0$

tuned so that last three-body state appears at $k = 0$ ($t = -\infty$)



Three 4-body states, at $t = -4.1, -5.6$ and -7.1

(consistent with theorem of Amado and Greenwood)

AA scattering length corresponding to zero-energy 3-body state: a_3
(results just shown)

Further decrease in a : 4-body states cross zero energy at

$$a_4^{(0)}/a_3 \simeq 0.438, \quad a_4^{(1)}/a_3 \simeq 0.877, \quad a_4^{(2)}/a_3 \simeq 0.9967$$

Two lowest states: ratios within 5% of exact results

[von Stecher et al (2009); Deltuva (2010)]

Third state extremely weakly bound

- if real: challenge to observe numerically and experimentally
- could be artefact of truncation
(Efimov cycles too long: scale factor ~ 30 instead of 23)

Scaling regime

Scales at which 3- and 4-body states appear

- double exponential form

$$k_4^{(n)} = k_3 \exp \left[\alpha e^{-\beta n} \right]$$

- Ratios of scales given by universal relation

$$k_4^{(n+1)} / k_4^{(n)} = \left(k_3 / k_4^{(n)} \right)^{1 - \exp(-\beta)}$$

Similar to universal scaling function found by Hadizadeh *et al* but

- different functional form
- no new 4-body scale parameter
(α fixed, independent of the initial conditions)

Summary

Applications of functional RG to 3- and 4-body systems

- local effective action, “optimised” cutoff
- keeping all local terms in 4-body sector

Fermions

- results for dimer-dimer scattering length stable against variation of cutoff, agree with direct calculations

Bosons

- dynamical trimer field to match structure of Faddeev-Yakubovsky equations
- imaginary parts of 4-body couplings from each AT threshold
- infinite tower of 4-body bound states below each AT threshold
- double exponential (super-Efimov) pattern
- finite 2-body scattering length: three states in last cycle

Super-Efimov effect

Relies on being close to fixed point with complex scaling

Example for fewer-body coupling g^2 at nontrivial fixed point

$$\frac{dv}{dt} = ag^4 + bg^2v + cv^2$$

with $b^2 - 4ac < 0 \rightarrow$ imaginary scaling dimension

Now consider g^2 marginal: $g^2 = g_0^2/t$ with $t = \ln(k/k_0)$

and define $\hat{v} = tv$

$$t \frac{d\hat{v}}{dt} = ag_0^4 + (1 + bg_0^2)\hat{v} + c\hat{v}^2$$

\rightarrow cyclic behaviour in $\ln t = \ln(\ln(k/k_0))$ if

$$\left(\frac{1}{g_0^2} + b\right)^2 - 4ac < 0$$

4 bosons – close to AAD Efimov cycle [Deltuva (2012)]