One- and Two-Neutron Halos in Effective Field Theory

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1804

Work done in collaboration with
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Philipp Hagen
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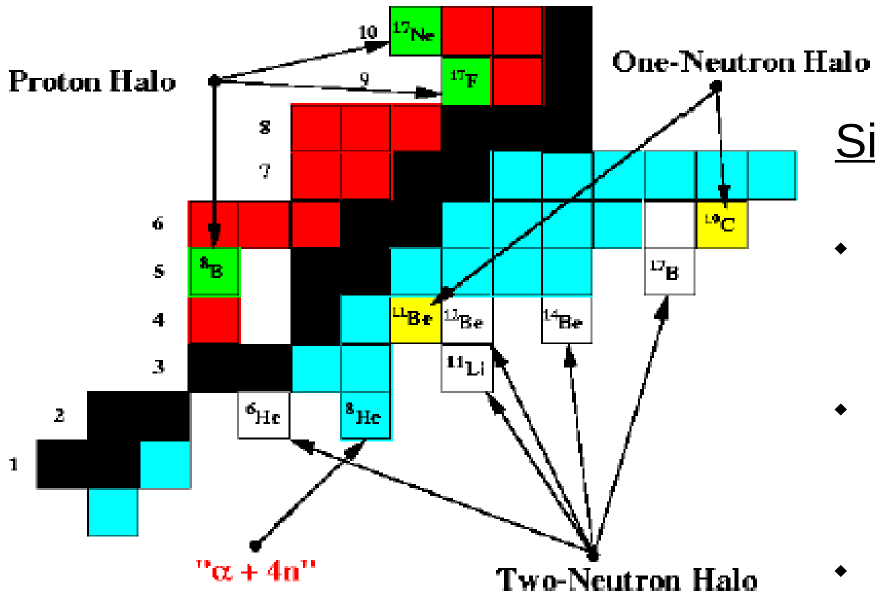
Outline

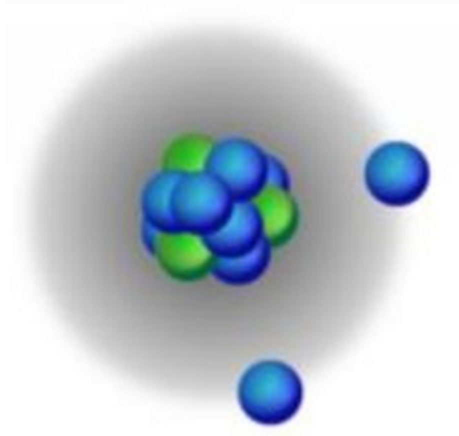
- Neutron Halos
  - Overview, motivation
  - Some experimental results
- EFT For One-Neutron Halos
  - Analysis of experimental data on Carbon-19
- EFT For Two-Neutron Halos
  - Implications of a measurement of the Carbon-22 matter radius
  - Coulomb dissociation of Carbon-22



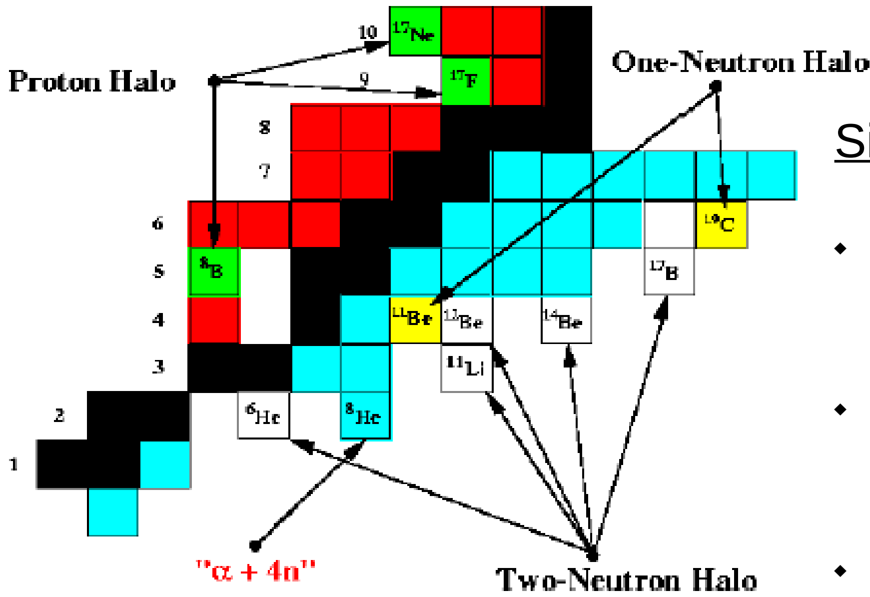
<u>Signature</u>

- Low separation energy for one or more neutrons, core tightly bound
- Large cross section for transfer and break-up reactions, large matter radius
- Enhanced charge radius and dipole moment

http://www.nupecc.org/



Alan Stonebraker for *APS Physics*



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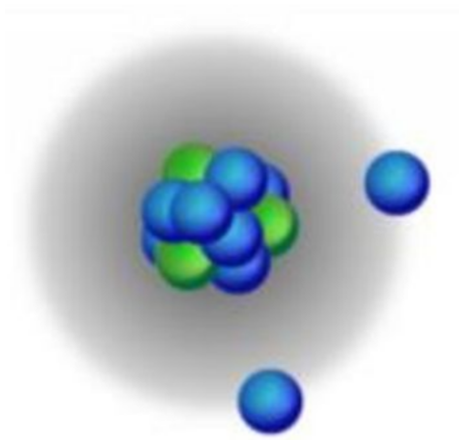
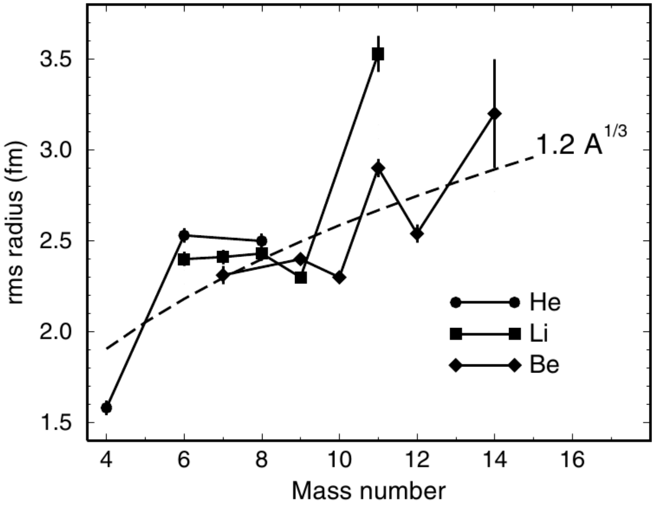


Alan Stonebraker for *APS Physics*

Motivation

- Nuclear reactions of astrophysical significance
- Nuclear structure away from the line of stability
- “Universality” – connection to other systems with large scattering length (nucleons, cold atoms near Fesbach resonance...)

Al-Khalili and Tostevin (1996)
Al-Khalili et al (1996)



Matter radii of nuclei deduced by Glauber model calculations from reaction cross section data.

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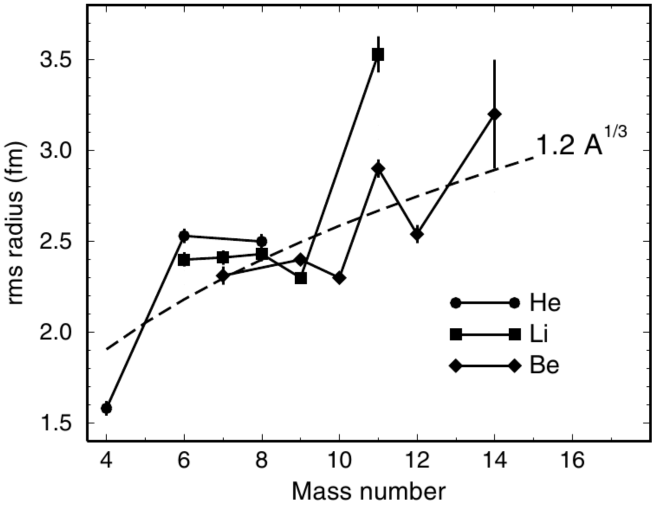
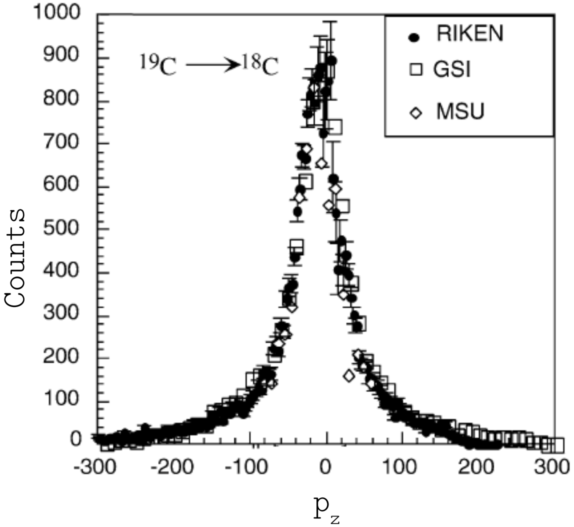


Matter radii of nuclei deduced by Glauber model calculations from reaction cross section data.

Chiba et al (2004)



Momentum distribution of ^{18}C from neutron removal of ^{19}C.

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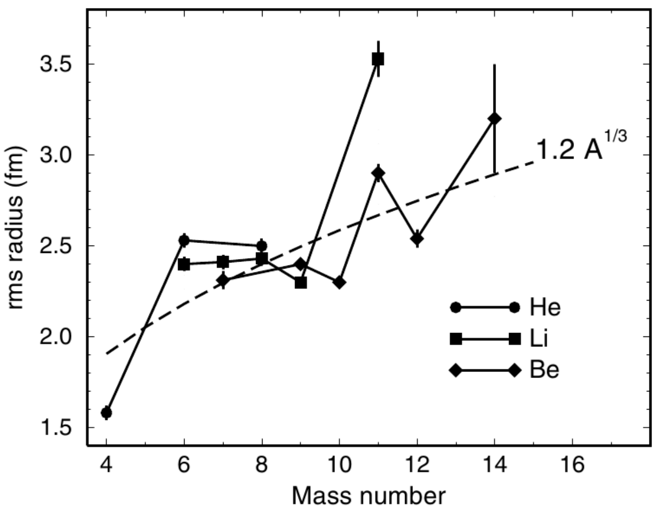

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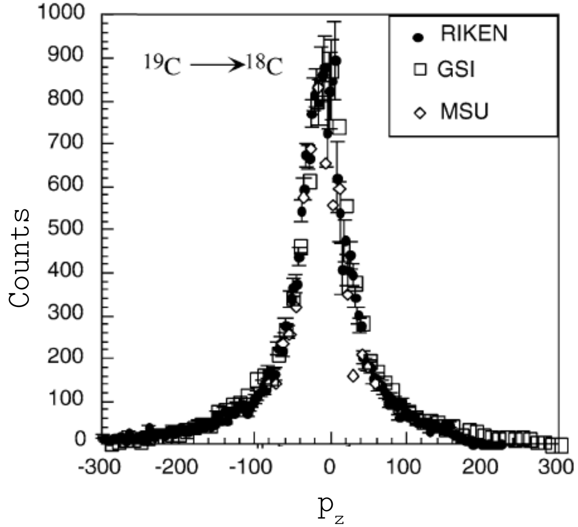
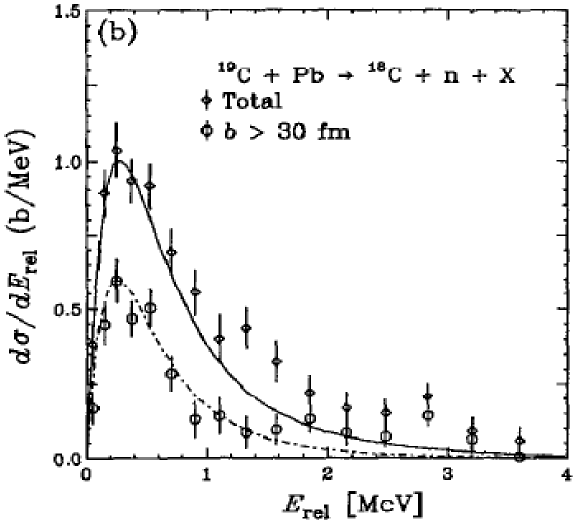


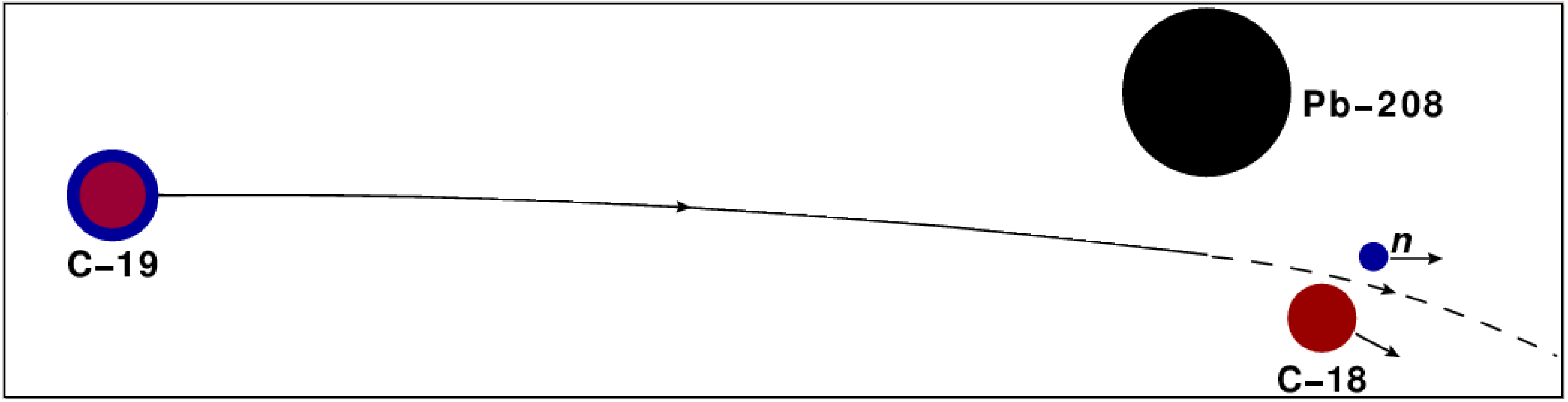
Momentum distribution of ^{18}C from neutron removal of ^{19}C.

Nakamura et al, RIKEN (2003)



^{19}C break-up on Pb. Curves are calculated using Woods-Saxon wavefunction at $S_n = 0.53$ MeV.

Theory of Coulomb Dissociation Experiments



- Direct reaction. Eikonal or semiclassical approximation.

- Perturbation theory to first order. Higer orders small. Typel and Baur (2001,2008)

- Virtual photons → real photons $$\sigma = \sum_{\pi L} \int \frac{\mathrm{d}\omega}{\omega} \, N_{\pi L}(\omega) \; \sigma_{\gamma}^{\pi L}(\omega)$$

- Dipole excitation, e.g. higher multipoles smaller by factor of 10^5 for ^{11}Li. Bertulani (2009)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E} = \frac{16\pi^3}{9} \alpha N_{E1}(B+E) \frac{\mathrm{d}B(E1)}{e^2 \mathrm{d}E}$$

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Halo EFT

Bertulani, Hammer and van Kolck (2002)
Bedaque, Hammer and van Kolck (2003)

- Degrees of freedom: halo neutron and the core.
- Symmetries: invariance under Galilean transformation, translation, rotation...
- Exploit separation of scales: $\sqrt{(mB)} \sim M_{lo} << M_{hi} \sim R^{-1}$.
- Systematic expansion in M_{lo}/M_{hi}.
- Short distance physics (at scale M_{hi} and beyond) of the core unresolved, but its impact on low energy observables taken care of by renormalization.

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^{19}C: $J^{\pi} = 1/2^{+}$, B = 0.58 MeV

^{18}C: R = 2.7 fm†, $J^{\pi} = 0^{+}$, E^{*} = 1.62 MeV

NNDC, BNL
†Simple estimate based 1.2 $A^{1/3}$ law

$M_{lo}/M_{hi} \sim 0.5$

$$\mathcal{L} = N^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m}\right) N + c^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M}\right) c$$
$$+ d^\dagger \left[\eta \left(i\partial_0 + \frac{\nabla^2}{2(M+m)}\right) - \Delta\right] d - g \left[d^\dagger N c + c^\dagger N^\dagger d\right]$$

Kaplan, Savage and Wise (1998);
Gegelia (1998); van Kolck (1998);
Birse, McGovern and Richardson (1998)

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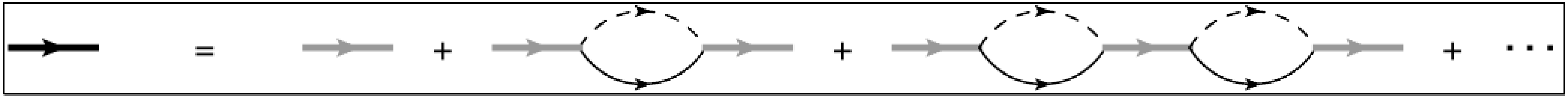

$$T = -\frac{2\pi}{\mu} \frac{1}{-\frac{1}{a} + \frac{1}{2} r_0 k^2 - ik}$$

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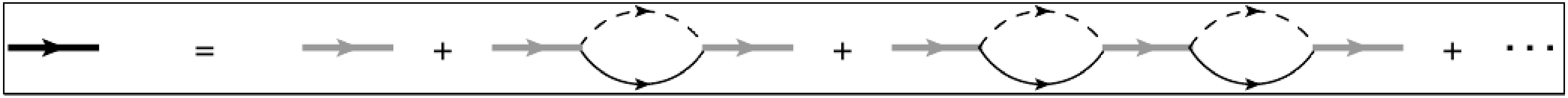

$$T = -\frac{2\pi}{\mu} \frac{1}{-\frac{1}{a} + \frac{1}{2} r_0 k^2 - ik}$$

$$a = \left(\frac{2\pi}{\mu g^2}\Delta + \kappa\right)^{-1} \qquad r_0 = -\eta \frac{2\pi}{\mu^2 g^2}$$

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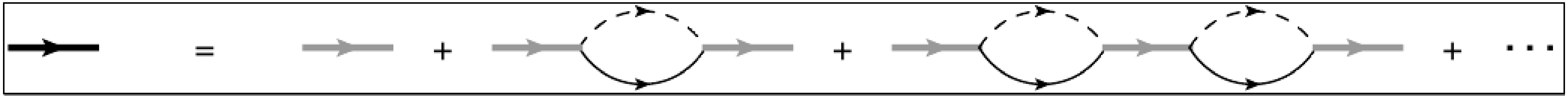

$$T = -\frac{2\pi}{\mu} \frac{1}{-\frac{1}{a} + \frac{1}{2} r_0 k^2 - ik}$$

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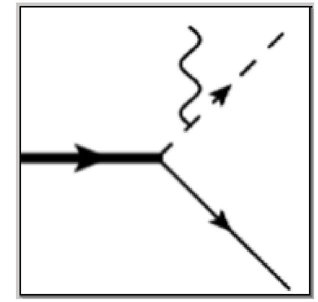
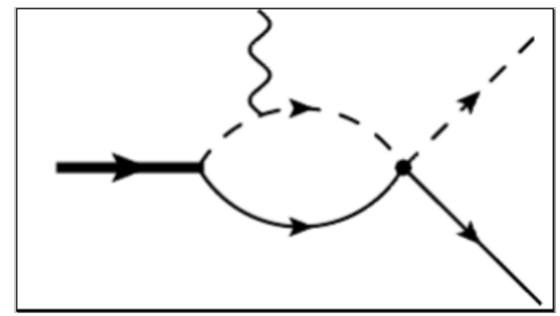
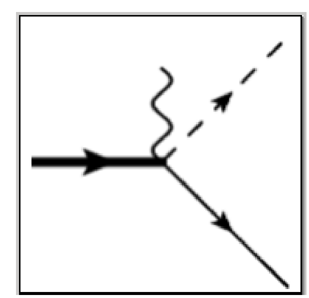
Assume naturalness: shape parameter, *P*, enters at N3LO. Stay at N2LO.

$(M_{lo}/M_{hi})^{-1}$	LO	
$(M_{lo}/M_{hi})^{2}$	N3LO	
$(M_{lo}/M_{hi})^{4}$	N5LO	

cf. Beane and Savage (2001); Hammer and Phillips (2011); Rupak and Higa (2011); Rupak, Fernando and Vaghani (2012) for similar analysis and calculations with other nuclei

Extracting Effective Range Parameters

$$\frac{\mathrm{d}B(E1)}{e^2\mathrm{d}E} = \frac{12}{\pi^2}\frac{\mu^3}{M^2}Z^2\frac{\gamma_0}{1-r_0\gamma_0}\frac{p^3}{\left(\gamma_0^2+p^2\right)^4},$$

cf. Bertulani and Baur (1988) for LO result

$$\frac{1}{a}+\frac{1}{2}r_0\gamma_0^2-\gamma_0=0; \quad ANC=\sqrt{\frac{2\gamma_0}{1-r_0\gamma_0}}$$

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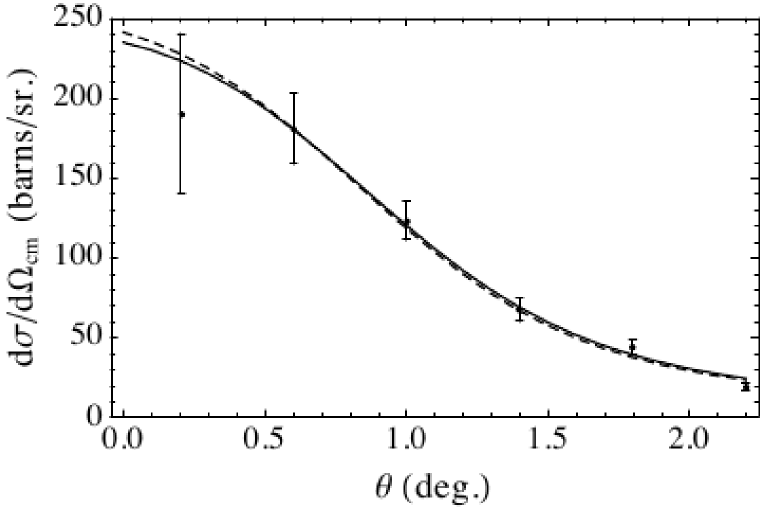
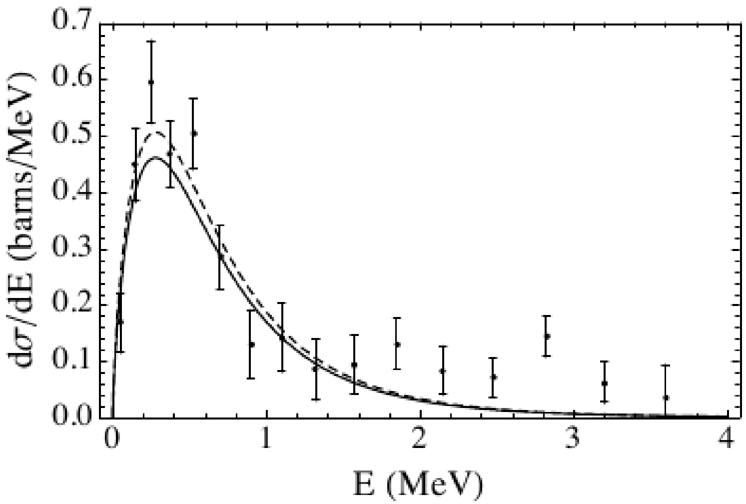


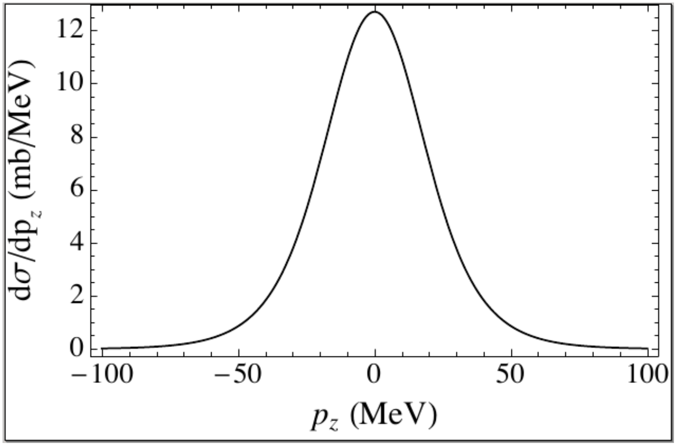


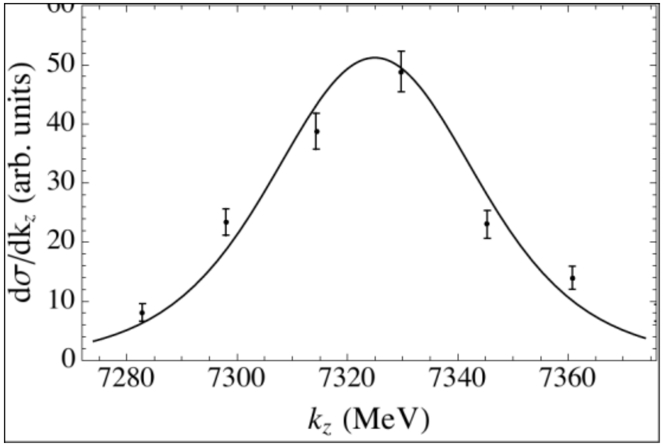
Data: Nakamura et al, RIKEN (1999,2003);
Calculation: Acharya and Phillips (2013)

$$a = (7.75 \pm 0.35(\text{stat.}) \pm 0.3(\text{EFT}))\ \text{fm}; \ r_0 = (2.6^{+0.6}_{-0.9}(\text{stat.}) \pm 0.1(\text{EFT}))\ \text{fm}$$

$$B = (575 \pm 55(\text{stat.}) \pm 20(\text{EFT}))\ \text{keV}$$

Prediction: Momentum Distribution





Data: Bazin et al, NSCL (1998);
Calculation: Acharya and Phillips (2013)

- Width sensitive to *B* ; *ANC* only affects height.
- Data with normalization unavailable for high *Z* target. Nuclear break-up background too strong for low *Z* ones.
- Uncertainty in absolute energy scale → also fit position → width is the only prediction.

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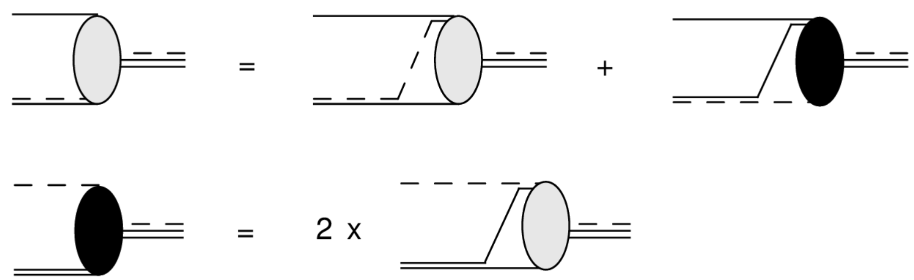
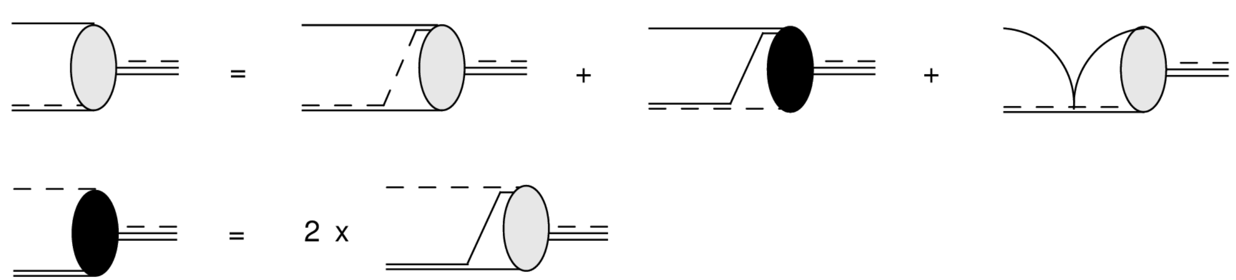
Two-Neutron Halos

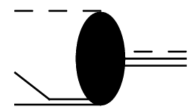
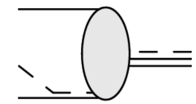
$$|\Psi\rangle = \hat{G}_0 \sum_{i=1}^{3} \hat{V}_i |\Psi\rangle; \; |F_i\rangle \equiv \hat{G}_0 \sum_{j \neq i} \hat{V}_j |\Psi\rangle \Rightarrow |F_i\rangle \equiv \hat{G}_0 \sum_{j \neq i} \hat{t}_j |F_j\rangle$$

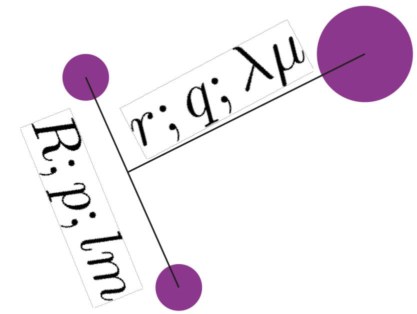


- At LO, dressed two-body propagators are renormalized by using two-body scattering lengths as input.
- Three-body contact interaction enters at LO.

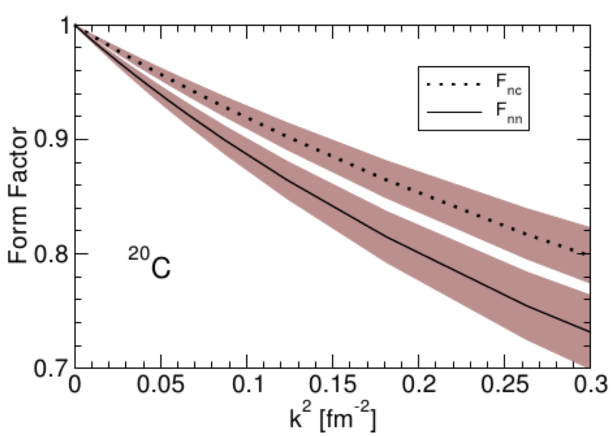
Bedaque, Hammer and van Kolck (1998)



$\Psi(\vec{p},\vec{q}) =$  + 2 x 



$$\mathcal{F}(k^2) = \int \mathrm{d}^3p \int \mathrm{d}^3q \; \Psi\left(\vec{p},\vec{q}\right) \; \Psi\left(\vec{p}-\vec{k},\vec{q}\right) = 1 - \frac{1}{6}k^2\langle r^2\rangle + \ldots$$



Canham and Hammer (2008)

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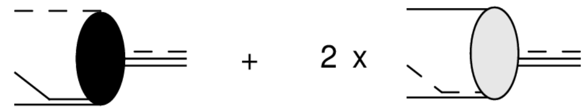

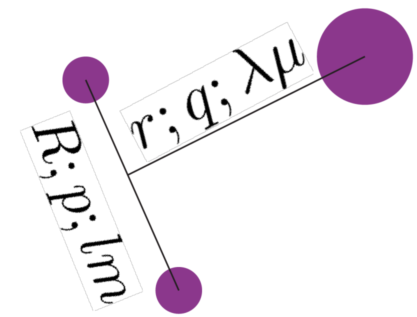

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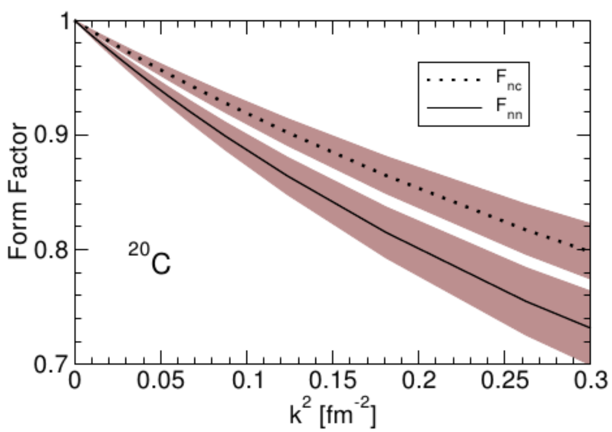

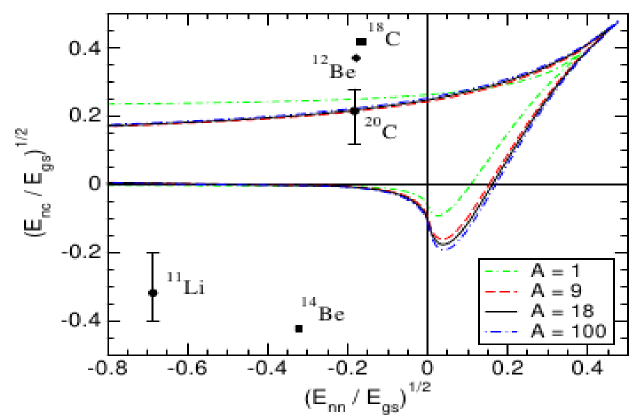

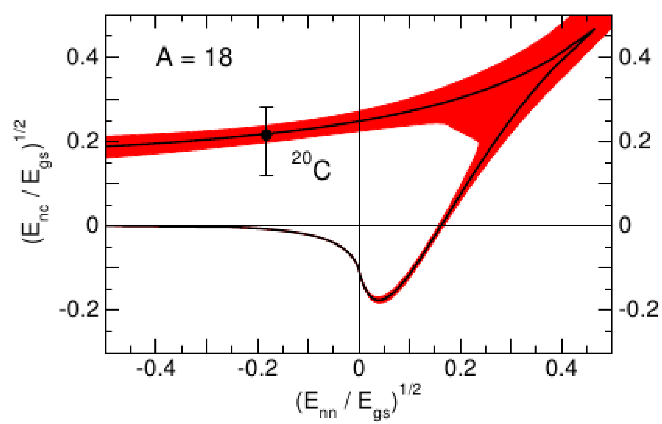

Canham and Hammer (2008)

The Point Core Limit

$$mB\langle r_0^2\rangle \equiv f\left(\frac{E_{\rm nn}}{B}, \frac{E_{\rm nc}}{B}; A\right); \quad B = S_{\rm 2n}$$

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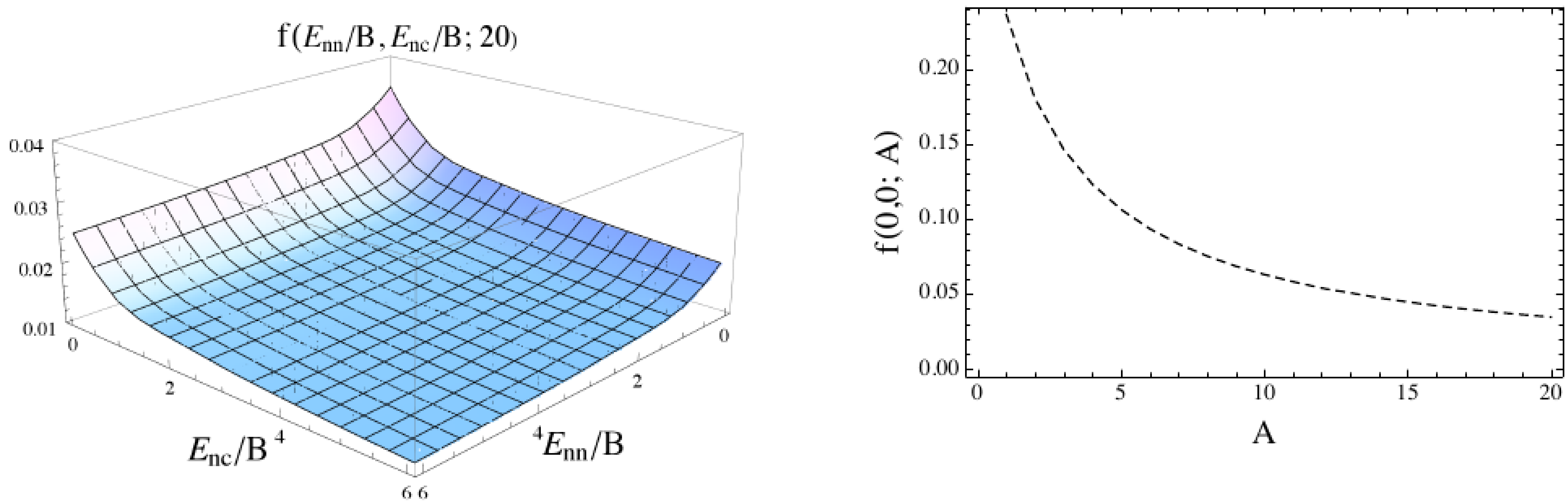

cf. Yamashita et al (2004) for an earlier attempt

Selected for a Viewpoint in *Physics*

PRL **104,** 062701 (2010) PHYSICAL REVIEW LETTERS week ending 12 FEBRUARY 2010

Observation of a Large Reaction Cross Section in the Drip-Line Nucleus ^{22}C

K. Tanaka,[1] T. Yamaguchi,[2] T. Suzuki,[2] T. Ohtsubo,[3] M. Fukuda,[4] D. Nishimura,[4] M. Takechi,[4,1] K. Ogata,[5] A. Ozawa,[6]

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	^{20}C	^{21}C	^{22}C
	bound	unbound	bound
Spin and Parity	0^+	$1/2^+$	0^+
Binding/Virtual Energy	S_n = 2.9 MeV NNDC, BNL (2013)	E_{nc} = ?	S_{2n} = 0.42(94) MeV Horiuchi and Suzuki (2006) S_{2n} = -0.14(46) MeV Gaudefroy et al (2012)
RMS matter radius	2.97(5) fm Ozawa et al (2001)	–	5.4(9) fm Tanaka et al, RIKEN (2010)

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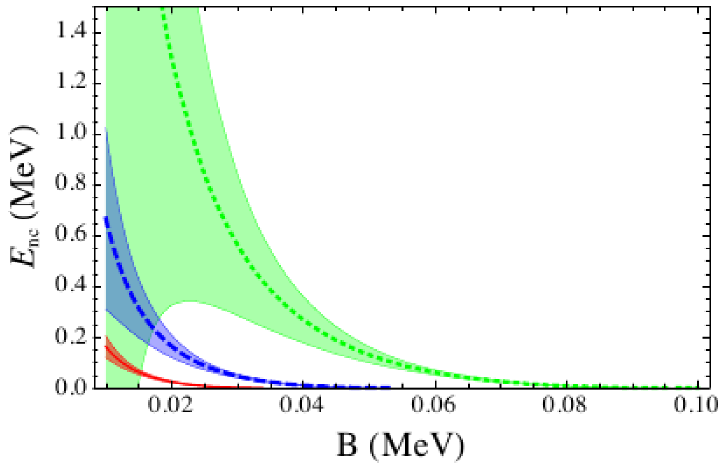
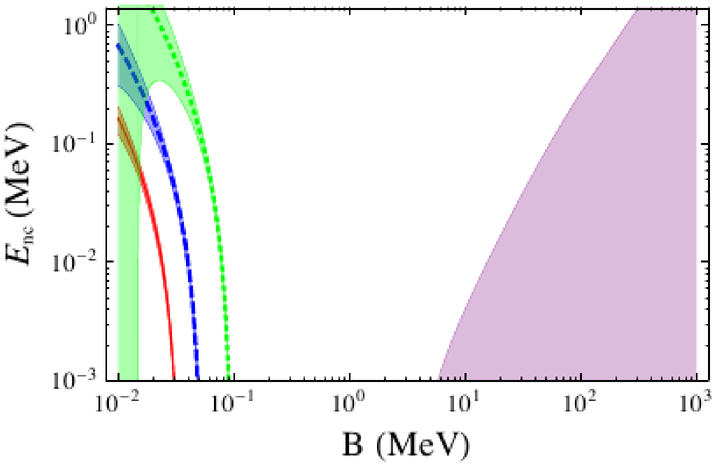
- $\surd(mS_{2n}[^{22}\mathrm{C}]) \sim M_{lo}$,$\surd(mS_n[^{20}\mathrm{C}])$, $(\surd\langle r^2[^{20}\mathrm{C}]\rangle)^{-1} \sim M_{hi}$
- E_{nc} unknown → treat as free parameter; $\surd(mE_{nc})$ as M_{lo} .
- $B = S_{2n}[^{22}\mathrm{C}]$ not well constrained by experiments → Treat as free parameter.

Acharya, Ji and Phillips (2013)





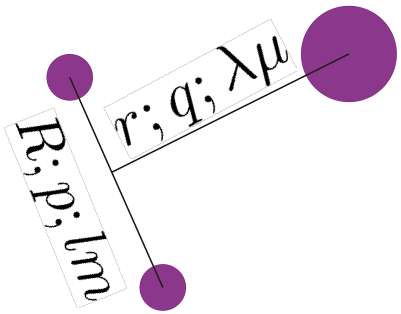
- 1-σ experimental error bar → B < 100 keV
- Excited Efimov states not possible unless E_{nc} < 1 keV.
- $|a_{nc}|$ < 2.8 fm. Mosby et al (2013)

cf. Hagen, Hagen, Platter and Hammer for study of Efimov states in Ca isotopic chain using Halo EFT, coupled cluster theory and interactions from Chiral EFT.

Coulomb dissociaton of Carbon-22

cf. Ershov et al (2012) for a non-EFT calculation (hyperspherical harmonic model)
Hagen, Platter and Hammer (2013) for charge form factor calculation

$$\hat{d} = 2eZr\mathrm{Y}_1^0(\hat{r})/(A+2)$$
$$\mathcal{M}_{\mathrm{PWIA}} = \langle pq; lm\lambda\mu|\ \hat{d}\ |\Psi_{in}\rangle$$



Coulomb dissociaton of Carbon-22

cf. Ershov et al (2012) for a non-EFT calculation (hyperspherical harmonic model)
Hagen, Platter and Hammer (2013) for charge form factor calculation

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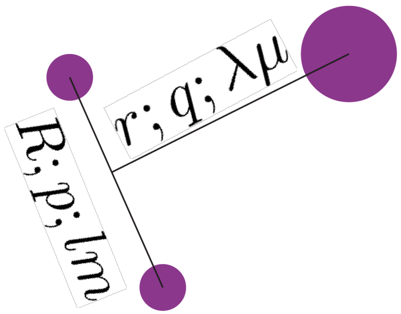
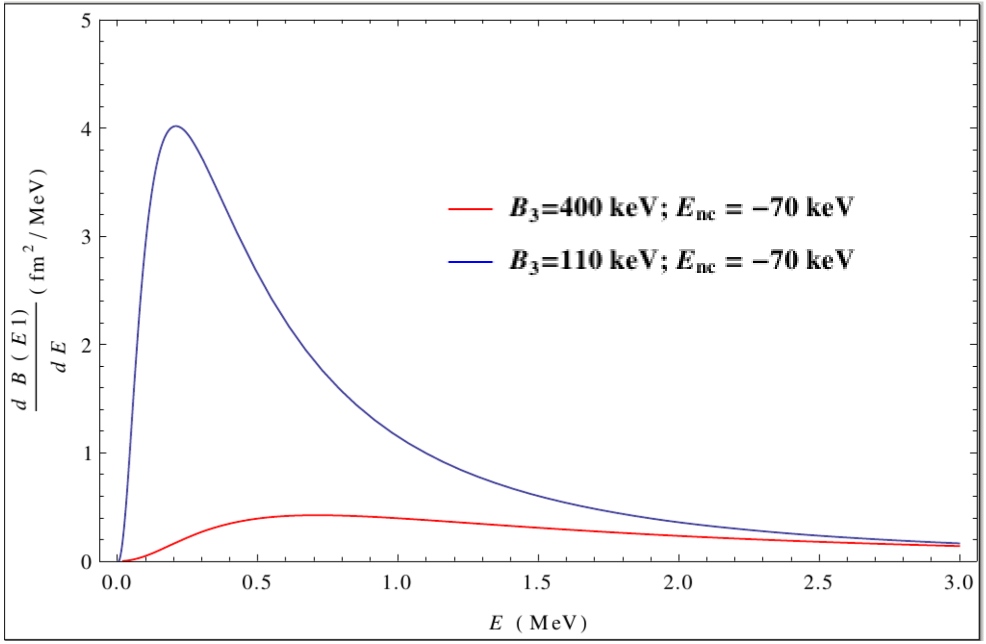
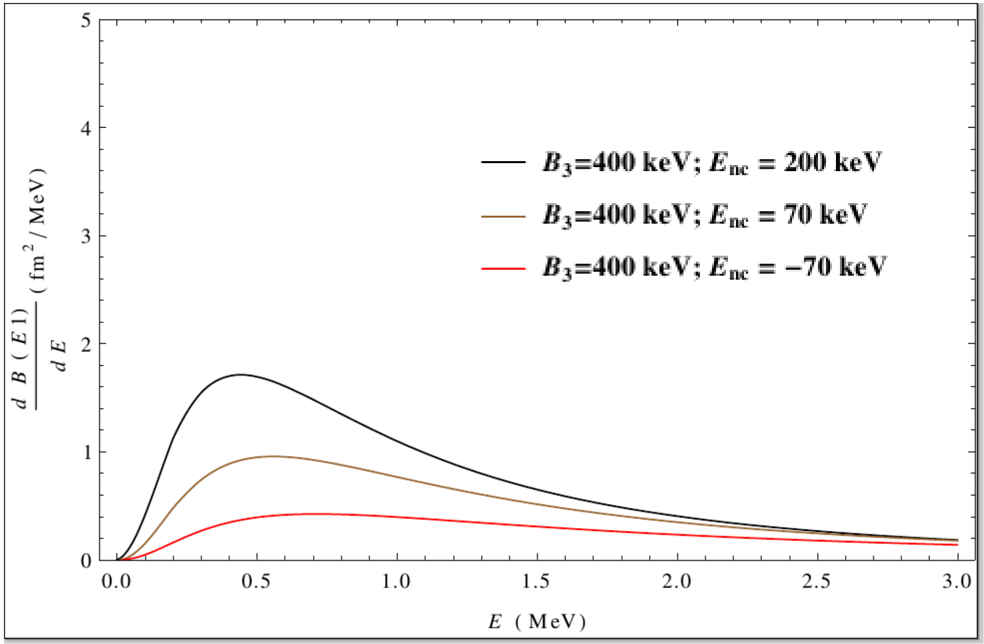
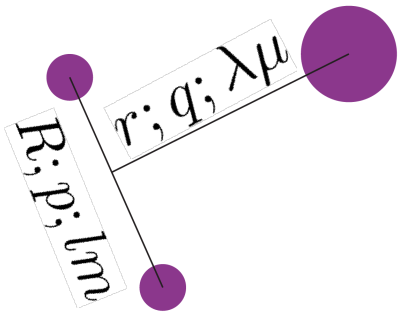






Final State Interactions

- $|\,l - \lambda\,| \leq 1 \leq l + \lambda$ in the final state.



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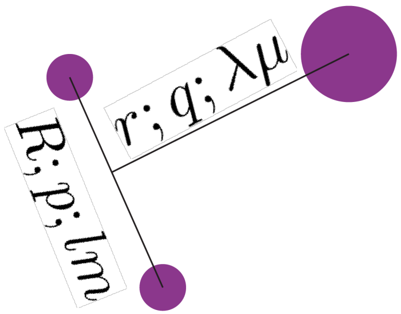

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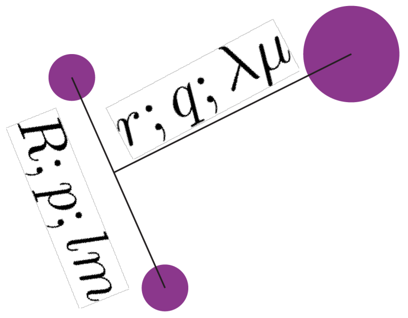

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Work In Progress
Check Back Soon

Conclusions and Outlook

- We applied Halo EFT to study Coulomb dissociation of ^{19}C and determined the S_n and the *ANC* of the ^{18}C – *n* system with high accuracy. S_n agrees with momentum distribution data; *ANC* remains to be tested.

- 1-σ experimental error on the matter radius of ^{22}C puts an upper bound of about 100 keV on its S_{2n} .

- Absence of low lying virtual states in ^{21}C rules out Efimov states in ^{22}C.

- Forthcoming data on Coulomb dissociation of ^{22}C is expected to provide better estimates of the ^{22}C two-neuton separation energy and the ^{21}C virtual state energy.

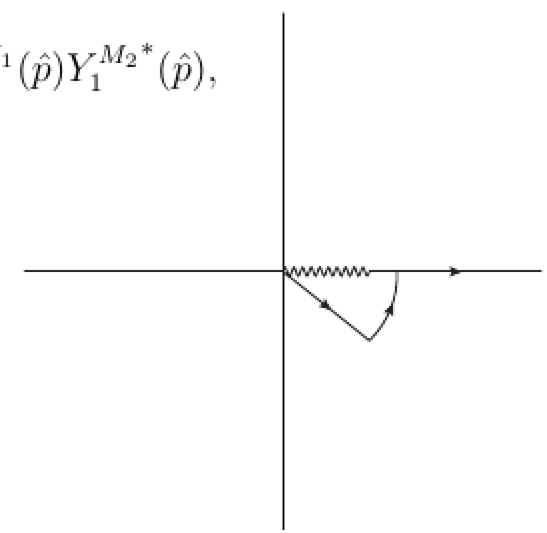
Backups

$$H' = \int d^3 r_1 d^3 r_2 \frac{\rho_1^{ch}(\mathbf{r}_1 - \mathbf{R}_1)\rho_2^{ch}(\mathbf{r}_2 - \mathbf{R}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{Z_1 Z_2 e^2}{R(t)}$$

$$\frac{d\sigma_C}{dE_\gamma}\ (E_\gamma) = \frac{1}{E_\gamma}\ \sum_{\pi L} N_{\pi L}\,(E_\gamma)\ \sigma_\gamma^{\pi L}\ (E_\gamma)$$

$$N_{E1}(\omega, R) = 2\frac{Z_t^2\alpha}{\pi\beta^2}\left(\xi K_0(\xi)K_1(\xi) - \frac{\beta^2}{2}\xi^2\left((K_1(\xi))^2 - (K_0(\xi))^2\right)\right)$$

$$\frac{\mathrm{d}\sigma}{Q\mathrm{d}Q\ \mathrm{d}^3p/(2\pi)^3} = 24\pi^2\frac{Z_t^2\alpha^2}{\gamma^2\beta^2}\omega^2 Z_{eff}^{(1)\,2}\langle r\rangle_{01}^2 \sum_{M_1 M_2} i^{M_1-M_2}\chi_{M_1}(Q)\chi_{M_2}^*(Q)G_{E1M_1}(1/\beta)G_{E1M_2}^*(1/\beta)Y_1^{M_1}(\hat{p})Y_1^{M_2\,*}(\hat{p}),$$



$$B(E1) = \frac{3}{4\pi}\left(\frac{Ze}{A}\right)^2\langle r_1^2 + r_2^2 + 2\boldsymbol{r_1}\cdot\boldsymbol{r_2}\rangle = \frac{3}{\pi}\left(\frac{Ze}{A}\right)^2\langle r_{c,2n}^2\rangle,$$

The kernel of the Faddeev equations involves integrals of the form,

$$\frac{1}{2}\int_{-1}^{1}\mathrm{d}x\; P_n(x)\;\frac{1}{E-\frac{q^2}{a}-\frac{q'^2}{b}-\frac{qq'x}{c}+i\epsilon} = \frac{c}{qq'}\;Q_n\left(\frac{c}{qq'}\left[E-\frac{q^2}{a}-\frac{q'^2}{b}\right]+i\epsilon\right)$$

$$= (-1)^{n+1}\frac{c}{qq'}\;Q_n\left(\frac{c}{qq'}\left[-E+\frac{q^2}{a}+\frac{q'^2}{b}\right]-i\epsilon\right),$$

where q is the external variable and $q\prime$ is the integration variable.

For $n = 0$,

$$Q_0(x \pm i\epsilon) = \begin{cases} \frac{1}{2}\log\frac{|x+1|}{|x-1|}, & |x| > 1 \\ \frac{1}{2}\log\frac{|x+1|}{|x-1|} \mp i\frac{\pi}{2}, & |x| < 1, \end{cases}$$

and for $n = 1$,

$$Q_1(z) = \frac{1}{2}\int_{-1}^{1}\mathrm{d}x\;\frac{x}{z-x} = -1+\frac{z}{2}\int_{-1}^{1}\mathrm{d}x\;\frac{1}{z-x} = -1+zQ_0(z)$$

$$\Rightarrow Q_1(x \pm i\epsilon) = \begin{cases} -1+\frac{x}{2}\log\frac{|x+1|}{|x-1|}, & |x| > 1 \\ -1+\frac{x}{2}\log\frac{|x+1|}{|x-1|} \mp i\frac{\pi}{2}x, & |x| < 1. \end{cases}$$

$$
\begin{aligned}
F_n(q;0010) = \ & \sqrt{\pi}\int_{-1}^{1} \mathrm{d}\,(\hat{q}.\hat{q}')\, P_1(\hat{q}.\hat{q}')G_0^n\left(\pi_1(\mathbf{q},K_n\hat{q}'),q;E\right) t_n(E;K_n)Y_1^0\left(\hat{K}_n\right) \\
& + \int_0^\infty \frac{\mathrm{d}q'\ {q'}^2}{2\pi^2}\,\frac{1}{2}\int_{-1}^{1} \mathrm{d}\,(\hat{q}.\hat{q}')\, P_1(\hat{q}.\hat{q}')G_0^n\left(\pi_1(\mathbf{q},\mathbf{q}'),q;E\right) t_n(E;q')F_n(q';0010) \\
& + \sqrt{\pi}\int_{-1}^{1} \mathrm{d}\,(\hat{q}.\hat{q}')\, P_1(\hat{q}.\hat{q}')G_0^n\left(\pi_0(\mathbf{q},K\hat{q}'),q;E\right) t_c(E;K)Y_1^0\left(\hat{K}\right) \\
& + \int_0^\infty \frac{\mathrm{d}q'\ {q'}^2}{2\pi^2}\,\frac{1}{2}\int_{-1}^{1} \mathrm{d}\,(\hat{q}.\hat{q}')\, P_1(\hat{q}.\hat{q}')G_0^n\left(\pi_0(\mathbf{q},\mathbf{q}'),q;E\right) t_c(E;q')F_c(q';0010),
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
F_c(q;0010) = \ & 2\sqrt{\pi}\int_{-1}^{1} \mathrm{d}(\hat{q}.\hat{q}')P_1(\hat{q}.\hat{q}')G_0^c\left(\pi_2(\mathbf{q},K_n\hat{q}'),q;E\right) t_n(E,K_n)Y_1^0\left(\hat{K}_n\right) \\
& + \int_0^\infty \frac{\mathrm{d}q'\ {q'}^2}{2\pi^2}\int_{-1}^{1} \mathrm{d}\,(\hat{q}.\hat{q}')\, P_1(\hat{q}.\hat{q}')G_0^c\left(\pi_2(\mathbf{q},\mathbf{q}'),q;E\right) t_n(E;q')F_n(q';0010).
\end{aligned}
\tag{2}
$$