Towards a relativistic, model-independent relation between the finite-volume spectrum and three-particle scattering amplitudes

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based on unpublished work with
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Outline

- Introduction
- Review of two-particle case
- Three-particle case
- Threshold expansion

Introduction

Maiani Testa no-go theorem says that one cannot get S-matrix elements (away from threshold) from infinite-volume Euclidean-time correlators.[1]

In finite volume the no-go theorem does not apply.

Indeed, Lüscher derived a relation between

finite-volume spectrum of QCD Hamiltonian (below four pion masses)

and

phase shift for elastic two-pion scattering.[2][3][4]

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[1] Maiani, L. & Testa, M. *Phys.Lett.* **B245,** 585–590 (1990).

[2] Luescher, M. *Commun. Math. Phys.* **104,** 177 (1986).

[3] Luescher, M. *Commun. Math. Phys.* **105,** 153–188 (1986).

[4] Luescher, M. *Nucl. Phys.* **B354,** 531–578 (1991).

Introduction

As is emphasized on the workshop webpage, there has so far been no lattice calculation of S-matrix elements above inelastic threshold.

Here one should distinguish between
a) systems with multiple, strongly-coupled, **two-particle** channels
b) systems with one or more, strongly-coupled, $(N > 2)$-**particle** channels

In the first case, the formalism for determining the S-matrix from the finite-volume spectrum is well understood.[5][6][7]

[5] Bernard, V. *et al.* *JHEP* **1101,** 019 (2011).
[6] Briceno, R. A. & Davoudi, Z. arXiv:1204.1110 [hep-lat] (2012).
[7] Hansen, M. T. & Sharpe, S. R. *Phys.Rev.* **D86,** 016007 (2012).

Introduction

Important progress has also been made for the simplest ($N > 2$)-particle cases:

two-to-three and **three-to-three scattering**.[8][9]

However, a **relativistic**, **model-independent** relation between the finite-volume spectrum and S-matrix elements for three-particle states is still unavailable.

This is the subject of this talk.

[8] Polejaeva, K. & Rusetsky, A. *Eur.Phys.J.* **A48,** 67 (2012).

[9] Briceno, R. A. & Davoudi, Z. arXiv:1212.3398 [hep-lat] (2012).

Outline

- Introduction
- **Review of two-particle case**
- Three-particle case
- Threshold expansion

Finite-volume set-up

Here finite volume means...

- finite, cubic spatial volume (extent L)
- periodic boundary conditions
- time direction infinite.

Assume L large enough to ignore exponentially suppressed (e^{-mL}) corrections.

Assume continuum field theory throughout.

Allow non-zero total momentum in finite-volume frame...

- total energy E
- total momentum $\vec{P}$ $\quad \left(\vec{P} = \frac{2\pi \vec{n}_P}{L} \qquad \vec{n}_P \in \mathbb{Z}^3 \right)$
- CM frame energy E^* $\quad \left(E^* = \sqrt{E^2 - \vec{P}^2} \right)$

Finite-volume set-up

For this talk, the spectrum is the relevant observable of the finite-volume theory.

Thinking of $\{L, \vec{n}_P\}$ as fixed, we denote the CM frame spectrum by

$$E_k^* \qquad \text{with} \qquad k = 1, 2, 3, \cdots .$$

Particle content set-up

Restrict particle content to a single scalar with mass m. So we work throughout with **identical particles**.

Assume...

- G-parity like symmetry, prevents even/odd coupling
- physics captured by summing, to all orders, a perturbative expansion of some local relativistic field theory

Require $E^* < 4m$.

Particle content set-up

For $E^* < 4m$ the only on-shell, G-parity-even states are two-particle states. So determining the S-matrix means determining the two-to-two scattering amplitude

$$i\mathcal{M}(\hat{k}^{*'}, \hat{k}^*) \equiv 4\pi Y^*_{\ell',m'}(\hat{k}^{*'}) i\mathcal{M}_{\ell',m';\ell,m} Y_{\ell,m}(\hat{k}^*)\,.$$

Note

$$i\mathcal{M}_{\ell',m';\ell,m} = i\mathcal{M}^{\ell,m}\delta_{\ell',\ell}\delta_{m',m}\,,$$

(no sum).

Statement of the problem

Want to relate $\mathcal{M}_{\ell',m';\ell,m}$ to the discrete spectrum of the finite-volume theory

$$E_k^* \qquad \text{for} \qquad k = 1, 2, 3, \cdots$$

at a given $\{L, \vec{n}_P\}$.

Method given here is due to Kim, Sachrajda and Sharpe.[10]

[10] Kim, C. *et al.* *Nucl.Phys.* **B727,** 218–243 (2005).

Derivation

For a given $\{L, \vec{n}_P\}$, the two-particle energies of the finite-volume theory are the values of E which are poles in

$$C_L(E, \vec{P}) \equiv \int_L d^4x \; e^{-i\vec{P}\cdot\vec{x}+iEt} \langle \Omega | T\sigma(x)\sigma^\dagger(0)|\Omega\rangle \,.$$

Here $\sigma(x)$ is an operator which couples to two particle states.

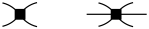
We now calculate the finite-volume corrections to C_L, to all orders in perturbation theory.

Derivation

$C_L(E, \vec{P})$ is equal to a sum of all Feynman diagrams built from...

- endcaps $\sigma(q)$ and $\sigma^\dagger(q')$. These are regular functions of momentum, determined by the specific form of the operators.

- arbitrary even vertices



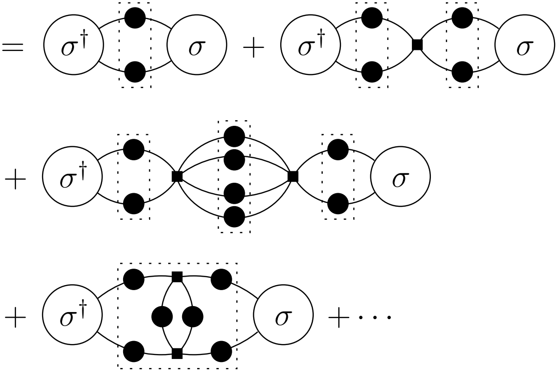
- fully dressed propagators

$$= i\frac{z(q)}{(q^0)^2 - \vec{q}^{\,2} - m^2 + i\epsilon}$$

Derivation

Schematically

$$C_L(E, \vec{P}) =$$



Finite volume in loops

Finite volume is incorporated by summing (instead of integrating) over spatial components of loop momenta

$$\frac{1}{L^3}\sum_{\vec{k}}\int\frac{dk^0}{2\pi} \quad \text{where} \quad \vec{k}=\frac{2\pi\vec{n}}{L}\,, \quad \vec{n}\in\mathbb{Z}^3\,.$$

It turns out that, unless propagators go on-shell, one can replace

$$\frac{1}{L^3}\sum_{\vec{k}}\int\frac{dk^0}{2\pi}\longrightarrow\int\frac{d^4k}{(2\pi)^4}$$

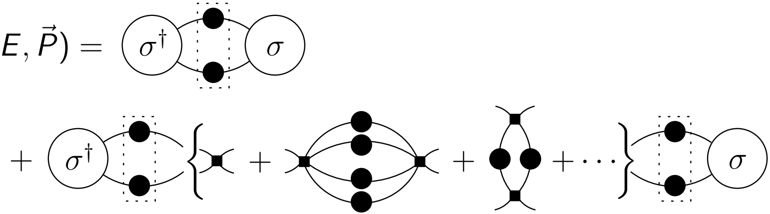
and only incur exponentially suppressed error (take this to be negligible).[11]

[11] Luescher, M. *Commun. Math. Phys.* **104,** 177 (1986).

Derivation

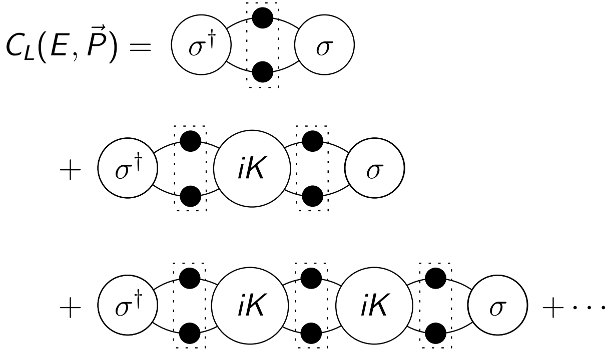
For the values of E^* being considered, only two propagators can go on shell.

$C_L(E, \vec{P}) =$

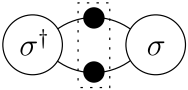


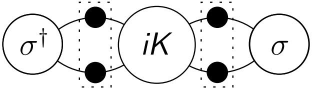
$+\cdots$

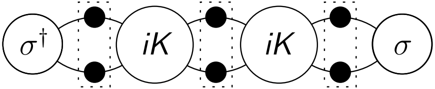
Derivation


$C_L(E, \vec{P}) =$
$\sigma^\dagger$
σ
$+$
$\sigma^\dagger$
iK
σ
$+$
$\sigma^\dagger$
iK
iK
σ
$+ \cdots$

Derivation

$C_L(E, \vec{P}) =$ 

$+$ 

$+$  $+ \cdots$

Let's focus on the first term.

Finite-volume effects in first term

Defining $\omega_q \equiv \sqrt{\vec{q}^2 + m^2}$, the first term is

$$\mathcal{X}_L \equiv \frac{1}{2}\frac{1}{L^3}\sum_{\vec{q}} \int \frac{dq^0}{2\pi} \frac{iz(q) iz(P-q)\sigma(q)\sigma^\dagger(q)}{[(q^0)^2 - (\omega_q - i\epsilon)^2][(E-q^0)^2 - (\omega_{P-q} - i\epsilon)^2]}$$

$$= \frac{1}{2}\frac{1}{L^3}\sum_{\vec{q}} \sigma^*(\hat{q}^*) \frac{i}{2\omega_q 2\omega_{P-q}(E - \omega_q - \omega_{P-q})} \sigma^{\dagger *}(\hat{q}^*)$$

$$+ \frac{1}{L^3}\sum_{\vec{q}} \text{finite function of } \vec{q}\,,$$

where in the second line we have evaluated the q^0 integral.
This can be done via contour integration or alternatively via time-ordered perturbation theory.

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$$= \frac{1}{2}\frac{1}{L^3}\sum_{\vec{q}} \sigma^*(\hat{q}^*)\frac{i}{2\omega_q 2\omega_{P-q}(E-\omega_q-\omega_{P-q})}\sigma^{\dagger *}(\hat{q}^*)$$

$$+ \frac{1}{L^3}\sum_{\vec{q}} \text{finite function of } \vec{q}\,,$$

where in the second line we have evaluated the q^0 integral.
This can be done via contour integration or alternatively via time-ordered perturbation theory.

We have also introduced $\sigma^{\dagger *}(\hat{q}^*)$, which is just $\sigma^\dagger(q)$ **restricted to on-shell momenta**.

Finite-volume effects in first term

Next we subtract the infinite-volume version of the same diagram from both sides to get

$$\mathcal{X}_L - \mathcal{X}_\infty = \sigma_{\ell',m'}[-F_{\ell',m';\ell,m}]\sigma^\dagger_{\ell,m}\,.$$

Here we have defined

$$\sigma^{\dagger *}(\hat{q}^*) \equiv \sigma^\dagger_{\ell,m}\sqrt{4\pi}Y^*_{\ell,m}(\hat{q}^*)$$

$$-F_{\ell',m';\ell,m} \equiv \frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{q}} - \int \frac{d^3q}{(2\pi)^3}\right] \frac{i4\pi Y_{\ell',m'}(\hat{q}^*)Y^*_{\ell,m}(\hat{q}^*)}{2\omega_q 2\omega_{P-q}(E-\omega_q-\omega_{P-q}+i\epsilon)}\,.$$

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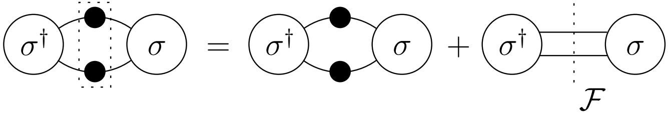
Diagrammatic representation

$$\mathcal{X}_L \quad = \quad \mathcal{X}_\infty \quad + \sigma_{\ell',m'}[-F_{\ell',m';\ell,m}]\sigma^\dagger_{\ell,m}\,,$$



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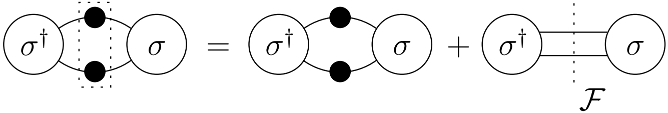
Diagrammatic representation

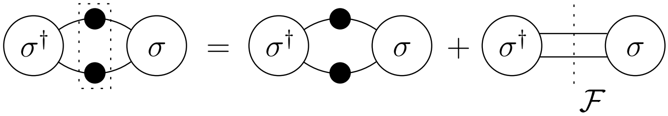
$$\mathcal{X}_L \qquad = \qquad \mathcal{X}_\infty \qquad + \sigma_{\ell',m'}[-F_{\ell',m';\ell,m}]\sigma^\dagger_{\ell,m}\,,$$



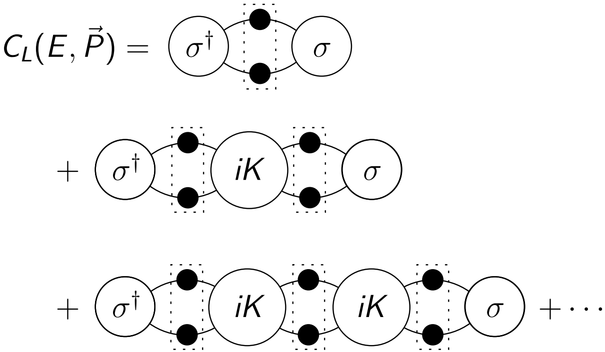
In the term with F only the on-shell values of the σs are needed.

Derivation

Substitute

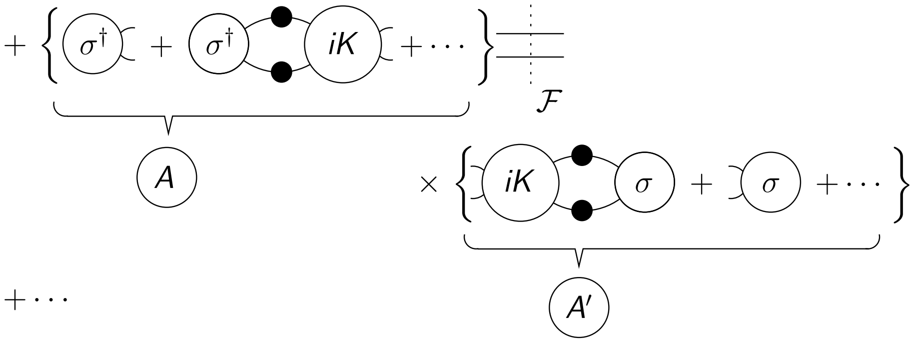


into



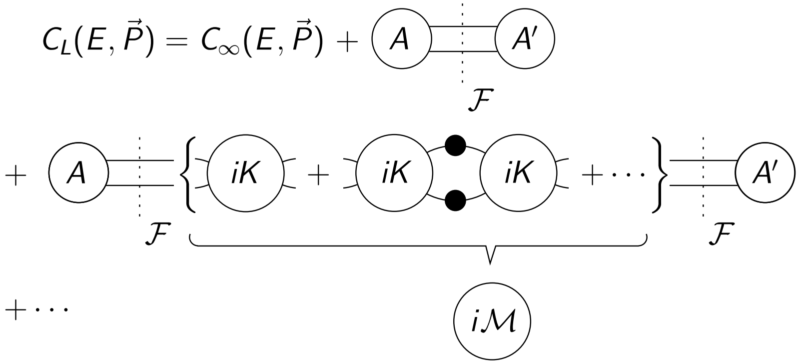
Derivation

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P})$$



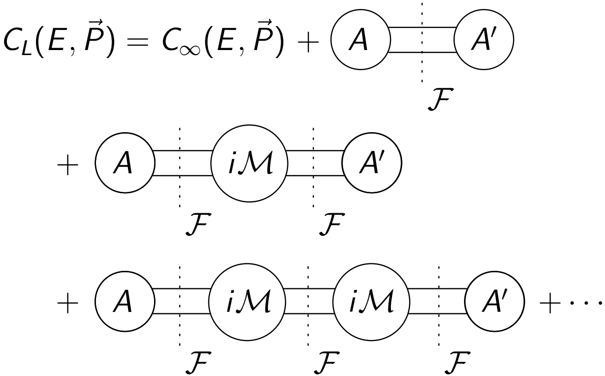
Derivation

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P}) +$$



$+\cdots$

Derivation


$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) +$
A
A′
$\mathcal{F}$
+
A
$i\mathcal{M}$
A′
$\mathcal{F}$
$\mathcal{F}$
+
A
$i\mathcal{M}$
$i\mathcal{M}$
A′
$+\cdots$
$\mathcal{F}$
$\mathcal{F}$
$\mathcal{F}$

Result

We conclude

$$C_L(E,\vec{P}) - C_\infty(E,\vec{P}) = -\sum_{n=0}^{\infty} A' F[-i\mathcal{M}F]^n A = -A' \frac{1}{F^{-1} + i\mathcal{M}} A$$

So at given values of $\{L, \vec{n}_P\}$, the spectrum is just the set

$$E_k^* \qquad \text{with} \qquad k = 1, 2, 3, \cdots$$

for which

$$\det(F^{-1} + i\mathcal{M}) = 0\,.$$

Comments on result

$$\det(F^{-1} + i\mathcal{M}) = 0\,.$$

$i\mathcal{M}_{\ell',m';\ell,m}$ is diagonal (rotational invariance of infinite-volume).

$F_{\ell',m';\ell,m}$ is not diagonal (rotational invariance broken by finite-volume).

Despite $F_{\ell',m';\ell,m}$ not being diagonal, if $i\mathcal{M}_{\ell',m';\ell,m}$ is negligible above some $\ell_{\max}$, then F can also be truncated.

In particular, if the s-wave dominates we get

$$F^{-1}_{00;00} + i\mathcal{M}_{00;00} = 0\,.$$

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- **Three-particle case**
- Threshold expansion

New set-up

Only two changes to the set-up:

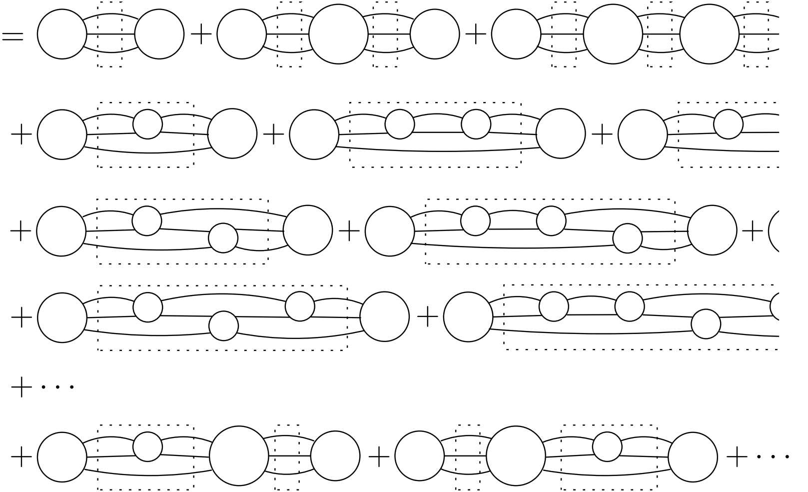
1. Consider new energy range $3m < E^* < 5m$.
2. Choose σ operators in correlator $C_L(E, \vec{P})$ to now couple to odd-particle-number states.

Now the important finite-volume corrections to $C_L(E, \vec{P})$ are from diagrams with **three** on-shell particles.

New skeleton expansion

We jump straight to the new skeleton expansion, which displays all of the important finite-volume corrections to $C_L(E, \vec{P})$.

$C_L(E, \vec{P}) =$

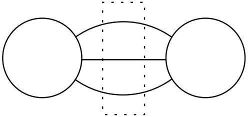


$+ \cdots$

$+ \cdots$

No two-to-two insertions

As a warm-up, consider the subset of diagrams with no two-to-two insertions. The simplest of these is the free particle diagram.

$\mathcal{V}_L \equiv$ 

We find

$$\mathcal{V}_L = \mathcal{V}_\infty + \sigma_{n'}[-U_{n';n}]\sigma_n^\dagger \,,$$

where we have defined

$$-U_{n';n} \equiv \frac{1}{6}\left[\frac{1}{L^6}\sum_{\vec{q}}\sum_{\vec{k}} - \int \frac{d^3q}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\right] \times \frac{iB_{n'}(\Omega)B_n^*(\Omega)}{2\omega_q 2\omega_k 2\omega_{P-q-k}(E-\omega_q-\omega_k-\omega_{P-q-k})}\,.$$

B_n spans the momentum space of three particles with total energy-momentum $E, \vec{P}$.

No two-to-two insertions

We can use the same identity everywhere in the set with no two-to-two insertions. We find

$$C_L^{[\mathrm{No}\ 2\to 2]}(E,\vec{P}) - C_\infty^{[\mathrm{No}\ 2\to 2]}(E,\vec{P}) = \\ - A'^{[\mathrm{No}\ 2\to 2]} \frac{1}{U^{-1} + i\mathcal{M}_{3\to 3}^{[\mathrm{No}\ 2\to 2]}} A^{[\mathrm{No}\ 2\to 2]} \,,$$

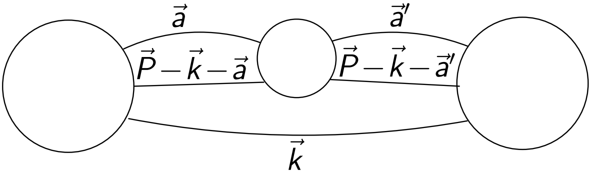
where $\left[i\mathcal{M}_{3\to 3}^{[\mathrm{No}\ 2\to 2]} \right]_{n';n}$ is defined via

$$i\mathcal{M}_{3\to 3}^{[\mathrm{No}\ 2\to 2]}(\Omega',\Omega) \equiv B^*_{n'}(\Omega') \left[i\mathcal{M}_{3\to 3}^{[\mathrm{No}\ 2\to 2]} \right]_{n';n} B_n(\Omega) \,.$$

This is just the sum of all amputated, on-shell six-point diagrams with no two-to-two insertions.

One two-to-two insertion

This is the simplest diagram with one two-to-two insertion.



Here we have a propagator that appears in two different sets. This means that the identity from the free diagram cannot be used.

One two-to-two insertion

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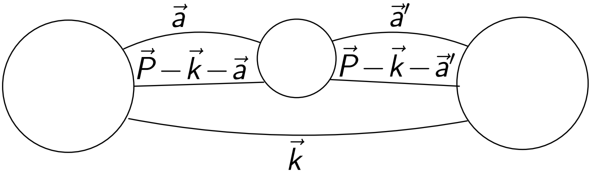

Here we have a propagator that appears in two different sets. This means that the identity from the free diagram cannot be used.

We can however separately work out the singularity structure of this diagram after time-component integration.

Time ordered perturbation theory is a perfect tool for this task.

One two-to-two insertion

Schematically, the answer turns out to be

$$\frac{1}{L^9} \sum_{\vec{k},\vec{a}',\vec{a}} \left\{ \sigma \frac{i}{E - 3\omega'} iK \frac{i}{E - 3\omega} \sigma^\dagger + \sigma \frac{i}{E - 3\omega'} iK [\text{reg}] \sigma^\dagger \right.$$
$$\left. + \sigma [\text{reg}] iK \frac{i}{E - 3\omega} \sigma^\dagger + \sigma [\text{reg}] iK [\text{reg}] \sigma^\dagger \right\},$$

where [reg] stands for known regular functions. Also we have introduced the shorthand

$$3\omega \equiv \omega_a + \omega_{P-k-a} + \omega_k$$
$$3\omega' \equiv \omega_{a'} + \omega_{P-k-a'} + \omega_k \,.$$

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Schematically, the answer turns out to be

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$$\left.+\sigma[\mathrm{reg}]iK\frac{i}{E-3\omega}\sigma^\dagger+\sigma[\mathrm{reg}]iK[\mathrm{reg}]\sigma^\dagger\right\},$$

where [reg] stands for known regular functions. Also we have introduced the shorthand

$$3\omega \equiv \omega_a+\omega_{P-k-a}+\omega_k$$
$$3\omega' \equiv \omega_{a'}+\omega_{P-k-a'}+\omega_k\,.$$

Not shown here:

a) Factors of $1/(2\omega)$

b) Momentum dependence of σ and iK

(coordinates shared with singularity are on-shell)

b) Functional forms of various [reg]s

One two-to-two insertion

Schematically, the answer turns out to be

$$\frac{1}{L^9}\sum_{\vec{k},\vec{a}',\vec{a}}\left\{\sigma\frac{i}{E-3\omega'}iK\frac{i}{E-3\omega}\sigma^\dagger+\sigma\frac{i}{E-3\omega'}iK[\mathrm{reg}]\sigma^\dagger\right.$$
$$\left.+\sigma[\mathrm{reg}]iK\frac{i}{E-3\omega}\sigma^\dagger+\sigma[\mathrm{reg}]iK[\mathrm{reg}]\sigma^\dagger\right\}.$$

How can we generalize our summation of three-to-three insertions

$$-A'^{[\mathrm{No}\ 2]}\frac{1}{U^{-1}+i\mathcal{M}_{3\to3}^{[\mathrm{No}\ 2]}}A^{[\mathrm{No}\ 2]},$$

to include diagrams with two-to-two insertions?

One two-to-two insertion

Schematically, the answer turns out to be

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$$\left.+\,\sigma[\mathrm{reg}]iK\frac{i}{E-3\omega}\sigma^\dagger+\sigma[\mathrm{reg}]iK[\mathrm{reg}]\sigma^\dagger\right\}.$$

How can we generalize our summation of three-to-three insertions

$$-A'^{[\mathrm{No\ 2}]}\frac{1}{U^{-1}+i\mathcal{M}_{3\to3}^{[\mathrm{No\ 2}]}}A^{[\mathrm{No\ 2}]}\,,$$

to include diagrams with two-to-two insertions?

Consider just the second term

$$\frac{1}{L^6}\sum_{\vec{k},\vec{a}'}\sigma\frac{i}{E-3\omega'}\left[\int_{\vec{a}}iK[\mathrm{reg}]\sigma^\dagger\right].$$

One two-to-two insertion

Subtracting out the infinite-volume (integrated) version of the terms leaves

$$\left[\frac{1}{L^6} \sum_{\vec{k},\vec{a}'} - \int_{\vec{k},\vec{a}'} \right] \sigma \frac{i}{E - 3\omega'} \left[\int_{\vec{a}} iK[\mathrm{reg}] \sigma^\dagger \right] .$$

One two-to-two insertion

Subtracting out the infinite-volume (integrated) version of the terms leaves

$$\left[\frac{1}{L^6}\sum_{\vec{k},\vec{a}'} - \int_{\vec{k},\vec{a}'}\right]\sigma\frac{i}{E-3\omega'}\left[\int_{\vec{a}} iK[\mathrm{reg}]\sigma^\dagger\right] .$$

This term has exactly the right form to be summed into

$$C_L(E,\vec{P}) - C_\infty(E,\vec{P}) = -A'UA + \cdots .$$

(Note that we have dropped the "[No 2]" on A.)

One two-to-two insertion

Subtracting out the infinite-volume (integrated) version of the terms leaves

$$\left[\frac{1}{L^6}\sum_{\vec{k},\vec{a}'} - \int_{\vec{k},\vec{a}'}\right]\sigma \frac{i}{E-3\omega'}\left[\int_{\vec{a}} iK[\mathrm{reg}]\sigma^\dagger\right] .$$

This term has exactly the right form to be summed into

$$C_L(E,\vec{P}) - C_\infty(E,\vec{P}) = -A' U A + \cdots .$$

(Note that we have dropped the "[No 2]" on A.)

But to go from $A^{[\mathrm{No}\ 2]}$ to A we also need terms like

$$\left[\frac{1}{L^6}\sum_{\vec{k},\vec{a}'} - \int_{\vec{k},\vec{a}'}\right]\sigma \frac{i}{E-3\omega'}\left[\int_{\vec{a}} iK \frac{i}{E-3\omega}\sigma^\dagger\right] .$$

Subtracting out this desired term from what we have gives a remainder

$$\left[\frac{1}{L^6}\sum_{\vec{k},\vec{a}'} - \int_{\vec{k},\vec{a}'}\right]\sigma \frac{i}{E-3\omega'}\left[\left[\frac{1}{L^3}\sum_{\vec{a}} - \int_{\vec{a}}\right] iK \frac{i}{E-3\omega}\sigma^\dagger\right] .$$

A big mess

These remainders cannot be summed via natural extensions of the two-particle case.

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The fundamental issue is that we are dealing, for the first time, with a **product of poles that have common coordinates** (in this case $\vec{k}$).

As a result, we have terms that **cannot be factored**. Factoring was key to producing a geometric series that could be summed into a useful result.

This will persist at all orders. In general we have chains of N poles multiplied together, with sums and integrals over common coordinates.

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This will persist at all orders. In general we have chains of N poles multiplied together, with sums and integrals over common coordinates.

Note that this particular diagram should be easy to understand. After all, it is just the two-to-two case in disguise.

New strategy

Do not try to fit into the three-to-three summation. Instead only worry about two-to-two diagrams. Begin by reanalyzing the two particle case, but now with a spectator.

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No problem. We find

$$C_L^{2+\mathrm{spec}}(E,\vec{P}) - C_\infty^{2+\mathrm{spec}}(E,\vec{P}) = - \frac{1}{L^3}\sum_{\vec{k}} \frac{1}{2\omega_k} A'(\vec{k}) \frac{1}{[F(\vec{k})]^{-1} + i\mathcal{M}(\vec{k})} A(\vec{k})\,,$$

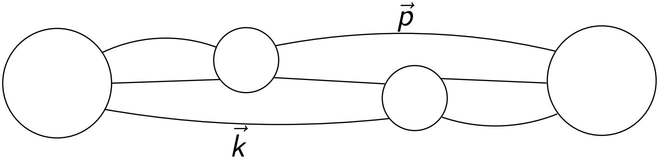
where

$$-F(\vec{k}) = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{a}} - \int_{\vec{a}}\right] \frac{i4\pi Y(\hat{a}^*)Y^*(\hat{a}^*)}{2\omega_a 2\omega_{P-k-a}(E-\omega_k-\omega_a-\omega_{P-k-a})}\,,$$

and $i\mathcal{M}(\vec{k})$ is the two-to-two amplitude with total energy $E-\omega_k$ and total momentum $\vec{P}-\vec{k}$. This manifestly predicts the right spectrum.

Scatter two different pairs

We now consider diagrams with exactly two scattered pairs.



Again we use time-ordered perturbation theory to identify singularities after time component integration.

It turns out that all singularities to the left and right of the switch can be summed just as in the two-particle case. We find

$$\frac{1}{L^6}\sum_{\vec{k},\vec{p}} \frac{1}{2\omega_p 2\omega_k}\Bigg\{ A'[-F(\vec{k})]\frac{1}{1+i\mathcal{M}(\vec{k})F(\vec{k})}$$

$$\times\left[i\mathcal{M}(\vec{k})\frac{i}{2\omega(E-3\omega)}i\mathcal{M}(\vec{p}) + [\mathrm{reg}]\right]\frac{1}{1+F(\vec{p})i\mathcal{M}(\vec{p})}[-F(\vec{p})]A\Bigg\}.$$

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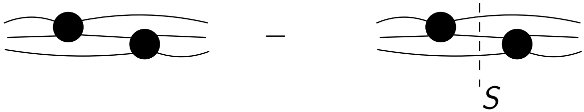
To reach the final expression for these diagrams we substitute

$$[\mathrm{reg}] = i\mathcal{M}_{\mathrm{off}}\frac{iz(P-k-p)}{(P-k-p)^2-m^2}i\mathcal{M}_{\mathrm{off}} - i\mathcal{M}_{\mathrm{on}}\frac{i}{2\omega(E-3\omega)}i\mathcal{M}_{\mathrm{on}}\,,$$

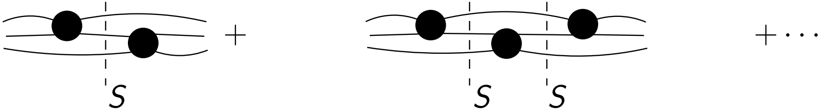


where we have indicated whether the two-to-two amplitudes are on-shell.

Divergence free three-to-three amplitude

It turns out that these regular pieces, which appear for any number of switches, may all be grouped into what we call the **divergence free three-to-three amplitude**

$$i\mathcal{M}_{df,3\to3} \equiv i\mathcal{M}_{3\to3} - \left[i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \int i\mathcal{M}\frac{i}{2\omega(E-3\omega)}\frac{1}{2\omega}i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \cdots \right].$$



Here $i\mathcal{M}_{3\to3}$ is short for $i\mathcal{M}_{3\to3,\ell',m';\ell,m}(\vec{k},\vec{p})$.

So, the in- and the out-states are each parametrized by
one momentum coordinate $\vec{k}$ and
one set of angular-momentum indices ℓ, m.

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$$i\mathcal{M}_{df,3\to3} \equiv i\mathcal{M}_{3\to3}$$
$$-\left[i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \int i\mathcal{M}\frac{i}{2\omega(E-3\omega)}\frac{1}{2\omega}i\mathcal{M}\frac{i}{2\omega(E-3\omega)}i\mathcal{M} + \cdots \right].$$

We emphasize that the original amplitude, $\mathcal{M}_{3\to3}(\vec{k},\vec{p})$, is divergent at $E = \omega_k + \omega_p + \omega_{P-k-p}$.

The physical interpretation of the divergence is that two particles can rescatter and travel arbitrarily far before another two rescatter.

As a result $i\mathcal{M}_{df,3\to3}$ really is a more natural observable to get from lattice simulation.

Setting up final result

Having indicated some of the key complications and a hint of the resolution, we now jump to stating the final answer.

We find

$$C_L(E,\vec{P}) - C_\infty(E,\vec{P}) = -A \frac{1}{F_{\text{three}}^{-1} + i\mathcal{M}_{df,3\to 3}} A\,,$$

where the right-hand-side is a (row) times (matrix) times (column) in the space

$$\text{(momentum } \vec{p} = 2\pi \vec{n}_p/L \text{ of one of the particles)} \times \\ \text{(angular momentum of other two)}\,.$$

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$$(\text{angular momentum of other two})\,.$$

Note that there are a finite number of discrete momenta for which the other two particles are above threshold. As a result truncating the $\vec{p}$ space only incurs exponentially suppressed errors.

It only remains to define $F_{\rm three}$.

The **relativistic**, **model-independent** relation between finite-volume spectrum and scattering amplitudes is

$$\det[F_{\rm three}^{-1} + i\mathcal{M}_{df,3\to3}] = 0\,,$$

where

$$F_{\rm three} \equiv \frac{1}{2\omega L^3}\left[(2/3)iF - \frac{1}{[iF]^{-1} - [1 - i\mathcal{M}iG]^{-1}\, i\mathcal{M}}\right]$$

$$iG_{k,p} = \frac{1}{2\omega_p L^3}\frac{i\sqrt{4\pi}Y(\hat{p}^*)Y^*(\hat{k}^*)}{2\omega_{P-p-k}(E-\omega_p-\omega_k-\omega_{P-p-k})}\,,$$

$$iF_{k,k'} = \delta_{k,k'}\frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{a}} - \int_{\vec{a}}\right]\frac{i\sqrt{4\pi}Y(\hat{a}^*)Y^*(\hat{a}^*)}{2\omega_a 2\omega_{P-k-a}(E-\omega_k-\omega_a-\omega_{P-k-a}+i\epsilon)}\,,$$

$$i\mathcal{M}_{k,k'} = \delta_{k,k'}i\mathcal{M}(\vec{k})\,.$$

Here harmonic indices have been left implicit.

This is the main result of the talk.

Outline

- Introduction
- Review of two-particle case
- Three-particle case
- **Threshold expansion**

Threshold expansion

Expanding our result in $1/L$ about the three particle threshold

$$(E = 3m + \Delta E, \vec{P} = 0)\,,$$

reproduces the result of Beane, Detmold and Savage[12],[13]

$$\Delta E = \frac{12\pi a}{mL^3}\left[1 - \frac{a}{\pi L}\mathcal{I} + \frac{a^2}{\pi^2 L^2}\left[\mathcal{I}^2 + \mathcal{J}\right] + \frac{a^2}{\pi^3 L^3}\left[-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K}\right]\right] + \frac{24\pi^2 a^3 r}{mL^6} + \frac{\overline{\eta}_L}{L^6} + \mathcal{O}(L^{-7})\,,$$

which they found using non-relativistic quantum mechanics. Here

- a is the scattering length
- r is the effective range
- $\overline{\eta}_L$ is an L-dependent three-body interaction coefficient

$$\mathcal{I} = \sum_{\vec{n}\neq 0}^{\Lambda} \frac{1}{\vec{n}^2} - 4\pi\Lambda\,, \qquad \mathcal{J} = \sum_{\vec{n}\neq 0} \frac{1}{\vec{n}^4}\,, \qquad \mathcal{K} = \sum_{\vec{n}\neq 0} \frac{1}{\vec{n}^6}\,.$$

[12] Beane, S. R. *et al.* *Phys.Rev.* **D76,** 074507 (2007).

[13] Detmold, W. & Savage, M. J. *Phys.Rev.* **D77,** 057502 (2008).

Threshold expansion

The expansion is performed by substituting

$$\Delta E = \lambda \Delta E_1 + \lambda^2 \Delta E_2 + \cdots$$

$$\mathcal{M} = \frac{16\pi E^*}{[\tan\delta/p^*]^{-1} - ip^*} = \frac{16\pi E^*}{[-a - a^2 r p^{*2}/2 + \cdots]^{-1} - ip^*}$$

$$\mathcal{M}_{df,3\to3} = -48 m^3 \eta_3 + \cdots .$$

and then solving order by order, taking $\mathcal{O}(\lambda) = \mathcal{O}(a) = \mathcal{O}(\eta_3)$.

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Three comments:

1. In this limit only Y_{00} entries survive in iG and iF. (The limit gives no reduction in $\vec{k}$ space.)

2. η_3 was originally defined as the coefficient of a three-body delta-function potential. The relation to $\mathcal{M}_{df,3\to3}$ is from the Born approximation.

3. There are two additional sums (appearing in both the quantum formalism and ours) that are combined with η_3 to give $\overline{\eta}_L$.

Conclusion

We have given a relativistic, model-independent relation between three-particle S-matrix elements and the finite-volume spectrum.

We have shown that this relation reproduces the three-particle threshold expansion, which has been determined elsewhere using non-relativistic quantum mechanics.

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We have shown that this relation reproduces the three-particle threshold expansion, which has been determined elsewhere using non-relativistic quantum mechanics.

The next step is to map out the spectrum in the full range $3m < E^* < 5m$ for realistic scattering amplitude inputs.

Also interesting would be an attempt to weakly perturb our relation, in order to get a generalization of the Lellouch-Lüscher relation between finite- and infinite-volume weak decay matrix elements.