Interaction *vs* correlation effects in many-body systems

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Outline

- ⋆ Defining correlations: quite an elusive issue
  - correlations in the absence of interaction
  - interaction without correlations
- ⋆ Theoretical description of correlations
  - particle and hole propagation in interacting many-body systems
- ⋆ Empirical evidence of nucleon-nucleon correlations
  - nucleon knockout processes
  - Final State Interactions (FSI) in inclusive processes
  - the EMC effect
- ⋆ Summary & Outlook

Defining correlations

- ⋆ Consider a system of N interacting particle described by the wave function $\Psi(x_1, \ldots, x_N)$, with $x_i \equiv (\mathbf{r}_i, \sigma_i)$
- ⋆ Probability of finding particles $1, \ldots, n$ at positions $\mathbf{r}_1, \ldots, \mathbf{r}_n$

$$\rho^{(n)}(\mathbf{r}_1, \ldots, \mathbf{r}_n) = \frac{N!}{(N-n)!} \sum_{\sigma_1,\ldots,\sigma_N} \int d\mathbf{r}_{n+1} \ldots d\mathbf{r}_N \, |\Psi(x_1, \ldots, x_N)|^2$$

- ⋆ Particles 1 and 2 are correlated if

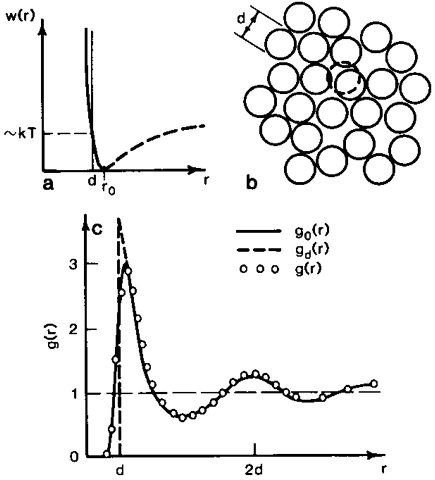
$$\rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2) \neq \rho^{(1)}(\mathbf{r}_1)\rho^{(1)}(\mathbf{r}_2)$$

- ⋆ The quantity

$$g(\mathbf{r}_1, \mathbf{r}_2) = \frac{\rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2)}{\rho^{(1)}(\mathbf{r}_1)\rho^{(1)}(\mathbf{r}_2)}$$

provides a measure of correlations in coordinate space

The archetype corelated system: the Van der Waals liquid



- Equation of state at particle density ρ and temperature T

$$P = \frac{\rho T}{1 - \rho b} - a\rho^2 \ ,$$

  - $b \propto d^3$ is the "excluded volume"
  - $a \sim$ integral of the attractive part of the interaction

- The *full* Van der Waals potential provides a good description of atomic systems. Hovever, its use in perturbation theory involves non trivial problems.

Correlations in the non interacting Fermi gas

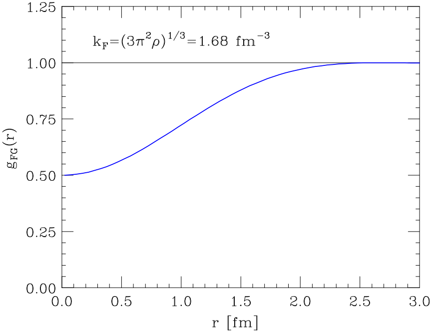
★ Enter Pauli's principle. Consider the ground state of a translationally invariant fermion system at density $\rho = N/V = k_F^3/3\pi^2$

$$\Psi_0(x_1, \ldots, x_N) = \frac{1}{N!} \ \det[\phi_{\alpha_i}(x_i)] \ , \ \phi_{\alpha_i}(x_i) = \frac{1}{V^{1/2}} \ e^{i\mathbf{k}\cdot\mathbf{r}_i} \chi_{\sigma_i} \ , \ |\mathbf{k}_i| < k_F$$

▹ Statistical correlations are described by the function ($r = |\mathbf{r}_1 - \mathbf{r}_2|$)

$$g_{FG}(r) = \frac{\rho^{(2)}(r)}{\rho^2} = 1 - \frac{1}{2}\ell^2(k_F r)$$

$$\ell(x) = 3\frac{\sin x - x\cos x}{x^3}$$



Coordinate vs momentum space

- ⋆ Bottom line: correlations are best defined in coordinate space.
- ⋆ To see this, consider the non interacting Fermi gas again. The joint probability of finding two particles with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ is

$$n_{FG}(\mathbf{k}_1,\mathbf{k}_2) = \theta(k_F - |\mathbf{k}_1|)\theta(k_F - |\mathbf{k}_2|)\left[1 - \frac{1}{N}\frac{\rho}{2}\,(2\pi)^3\delta(\mathbf{k}_1 - \mathbf{k}_2)\right]$$

- ⋆ *In the absence of long range order*, a similar result holds true in interacting systems

$$n(\mathbf{k}_1,\mathbf{k}_2) = n(\mathbf{k}_1)n(\mathbf{k}_2)\,[1 + O\,(1/N)]$$

- ⋆ In momentum space, non trivial correlations effects on $n(\mathbf{k}_1,\mathbf{k}_2)$ vanish in the $N \to \infty$ limit. However, correlations strongly affect the behaviour of $n(\mathbf{k})$ at $|\mathbf{k}| > k_F$.

Interaction without correlation: the mean field picture

- ⋆ Dynamical correlations are induced by two-body interactions described by the potential v_{ij} appearing in the N-particle hamiltonian

$$H = \sum_{i=1}^{N} -\frac{\nabla_i^2}{2m} + \sum_{j>i=1}^{N} \mathrm{v}_{ij} \ ,$$

- ⋆ The mean field approximation is based on the replacements

$$\sum_{j>i=1}^{N} \mathrm{v}_{ij} \to \sum_{i=1}^{N} U_i \quad , \qquad H \to \sum_{i=1}^{N} h_i = \sum_{i=1}^{N} \left(-\frac{\nabla_i^2}{2m} + U_i \right)$$

implying

$$H|\Psi_0\rangle = E_0|\Psi_0\rangle \to h_i|\phi_i\rangle = \epsilon_i|\phi_i\rangle$$

★ Within the mean field approximation

$$E_0 = \sum_{i \in \{F\}} \epsilon_i \quad , \quad \Psi(x_1, \ldots, x_N) = \det[\phi_i(x_i)]$$

$$\rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = \sum_{i,j \in \{F\}} \phi_i^\dagger(\mathbf{r}_1)\phi_j^\dagger(\mathbf{r}_2)\left[\phi_i(\mathbf{r}_1)\phi_j(\mathbf{r}_2) - \phi_j(\mathbf{r}_1)\phi_i(\mathbf{r}_2)\right]$$

★ The mean field approach provides a remarkably accurate description of a variety of properties of interacting many-body systems. However, one should keep in mind that

- dynamical correlations are not taken into acount
- including their effects as corrections to the mean field approximation may be highly misleading, as the definition of the mean field itself is model dependent

★ Theoretical studies aimed at pinning down the role of correlations should be carried out within *ab initio* many body approaches

Model independent determination of correlations

- Definition of Green's function

$$iG(x - x') = \langle 0 | T[\hat{\psi}(x)\hat{\psi}^\dagger(x')] | 0 \rangle$$

After Fourier transformation ($\eta = 0^+$)

$$G(\mathbf{k}, E) = \sum_n \left\{ \frac{|\langle n_{(N+1)}(\mathbf{k}) | a^\dagger_{\mathbf{k}} | 0_N \rangle|^2}{E - (E_n - E_0) + i\eta} + \frac{|\langle n_{(N-1)}(-\mathbf{k}) | a_{\mathbf{k}} | 0_N \rangle|^2}{E + (E_n - E_0) - i\eta} \right\}$$

$$= G_p(\mathbf{k}, E) + G_h(\mathbf{k}, E) = \int dE' \left[\frac{P_p(\mathbf{k}, E')}{E - E' + i\eta} + \frac{P_h(\mathbf{k}, E')}{E + E' - i\eta} \right]$$

- Spectral functions of hole and particle states

$$P_h(\mathbf{k}, E) = \sum_n |\langle n_{(N-1)}(\mathbf{k}) | a_{\mathbf{k}} | 0_N \rangle|^2 \delta(E - E_n + E_0) = \frac{1}{\pi} \, \mathrm{Im} \, G_h(\mathbf{k}, E)$$

$$P_p(\mathbf{k}, E) = \sum_n |\langle n_{(N+1)}(\mathbf{k}) | a^\dagger_{\mathbf{k}} | 0_N \rangle|^2 \delta(E + E_n - E_0) = \frac{1}{\pi} \, \mathrm{Im} \, G_p(\mathbf{k}, E)$$

Analytic structure of the Green's function

- In interacting systems, the Green's function (e.g. for hole states) can be written in terms of the particle self energy $\Sigma(\mathbf{k}, E)$

$$G_h(\mathbf{k}, E) = \frac{1}{E - |\mathbf{k}|^2/2m - \Sigma(\mathbf{k}, E)}$$

- Landau's quasiparticle picture: isolate contributions of $1h$ (*bound*) intermediate states, exhibiting poles at energies ϵ_k, given by $\epsilon_k = |\mathbf{k}|^2/2m + \text{Re}\,\Sigma(\mathbf{k}, \epsilon_k)$, as $\text{Im}\,\Sigma(\mathbf{k}, E) \to 0$ (Fermi surface)
- The resulting expression is

$$G_h(\mathbf{k}, E) = \frac{Z_k}{E - \epsilon_k - iZ_k\,\text{Im}\,\Sigma(\mathbf{k}, e_k)} + G_h^B(\mathbf{k}, E)$$

where $Z_k = |\langle -\mathbf{k}|a_{\mathbf{k}}|0\rangle|^2$, and $G_h^B(\mathbf{k}, E)$ is a smooth contribution, arising from $2h - 1p, 3h - 2p, \ldots$ (*continuum*) intermediate states

Correlated Basis Functions (CBF) approach

- ⋆ Correlated states obtained from Fermi gas states through the transformation

$$|n\rangle = \frac{F}{\langle n_{FG}|F^{\dagger}F|n_{FG}\rangle}|n_{FG}\rangle \quad , \quad F = \mathcal{S}\prod_{j>i} f_{ij}$$

- ⋆ The two-nucleon correlation operator reflects the complexity of the nucleon-nucleon (NN) force [spin-isospin (ST) dependent, non central]

$$f_{ij} = \sum_{TS}\left[f_{TS}(r_{ij}) + \delta_{S1}f_{Tt}(r_{ij})S_{ij}\right]P_{TS}$$

$$P_{TS} \ : \ \text{spin} - \text{isospin projectors} \ , \ S_{ij} = \sigma_i^{\alpha}\sigma_j^{\beta}\left(3r_{ij}^{\alpha}r_{ij}^{\beta} - \delta^{\alpha\beta}\right)$$

- ⋆ Shapes of f_{TS}, f_{tT} determined from minimization of ground state energy

CBF perturbation theory

- Split the hamiltonian according to

$$H = H_0 + H_I$$

$$\langle m|H_0|n\rangle = \delta_{mn}\langle m|H|n\rangle \quad , \quad \langle m|H_I|n\rangle = (1-\delta_{mn})\langle m|H|n\rangle$$

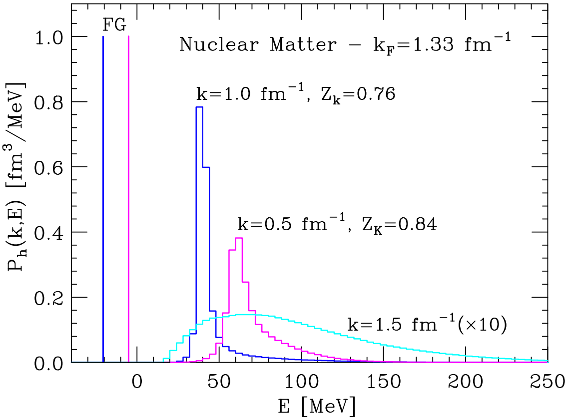
- If correlated states have large overlaps with the eigenstates of the hamiltonian, the matrix elements of H_I are small and perturbation theory can be used to obtain, e.g., the ground state from

$$|\widetilde{0}\rangle = \sum_m (-)^m \left(\frac{H_I - \Delta E_0}{H_0 - E_0^V} \right)^m |0\rangle$$

$$\Delta E_0 = E_0^V - E_0 = \langle 0|H|0\rangle - E_0$$

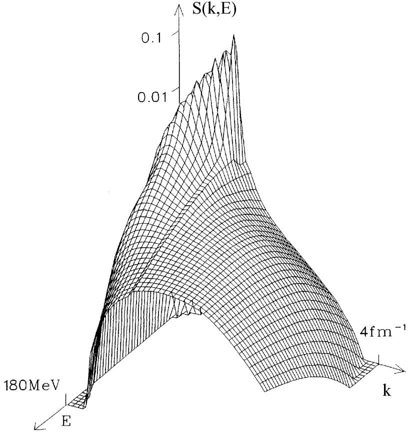
Hole spectral function of nuclear matter from CBF

$$P_h(\mathbf{k}, E) = \frac{1}{\pi} \frac{Z_k^2 \,\text{Im}\, \Sigma(\mathbf{k}, \epsilon_k)}{[E - \mathbf{k}^2/2m - \text{Re}\, \Sigma(\mathbf{k}, E)]^2 + [Z_k \text{Im}\, \Sigma(\mathbf{k}, \epsilon_k)]^2} + P_h^B(\mathbf{k}, E)$$



Spectral function of infinite nuclear matter

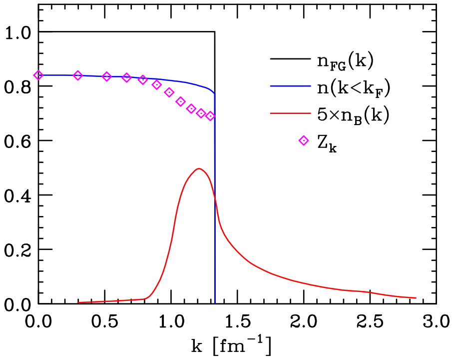
- ⋆ Results obtained using CBF perturbation theory and the U14+TNI hamiltonian



- ⋆ The correlation contribution can be identified by its distinctive energy dependence

- ⋆ In analogy with the spectral function, the momentum distribution can be split into quasi particle (pole) and and correlation (continuum) contributions

$$n(\mathbf{k}) = \int dE\, P(\mathbf{k},E) = Z_k \theta(k_F - |\mathbf{k}|) + \int dE\, P_B(\mathbf{k},E) = Z_k \theta(k_F - |\mathbf{k}|) + n_B(\mathbf{k})$$



Exploiting the (near) universality of correlations

- ⋆ Local density approximation

$$P(\mathbf{k},E) = P_{MF}(\mathbf{k},E) + P_{corr}(\mathbf{k},E)$$

  - ▹ $P_{MF}(\mathbf{k},E) \rightarrow$ from $(e,e'p)$ data

$$P_{MF}(\mathbf{k},E) = \sum_n Z_n |\phi_n(\mathbf{k})|^2 \, F_n(E-E_n)$$

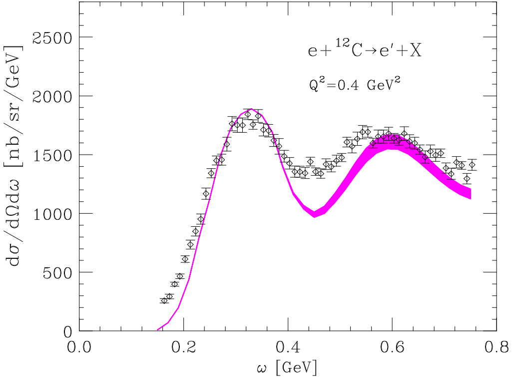
  - ▹ $P_{corr}(\mathbf{p},E) \rightarrow$ from uniform nuclear matter calculations at different densities:

$$P_{corr}(\mathbf{k},E) = \int d^3r \, \rho_A(r) \, P^{NM}_{corr}(\mathbf{k},E;\rho=\rho_A(r))$$

- ⋆ Widely and successfully employed to analize (e,e') data at beam energies $\sim 1GeV$
- ⋆ Warnings: model dependence, chance of double counting

Theory vs data ($E_e = 1.3$ GeV, $\theta_e = 37.5°$)

- ⋆ Note: calculations involve no adjustable parameters



- ⋆ The measured x-section can be described, except in the *dip* region, between the quasi elastic and Δ-production peaks, and the low enrgy loss tail, where FSI (not included) play a role

Correlation effects on the nuclear response

- Consider scattering of a scalar probe, for simplicity

$$\frac{d\sigma}{d\Omega d\omega} \propto S(\mathbf{q},\omega) = \sum_n \langle 0|\rho^\dagger_{\mathbf{q}}|n\rangle\langle n|\rho_{\mathbf{q}}|0\rangle\delta(E_0+\omega-E_n)$$

$$\rho_{\mathbf{q}} = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} \quad , \quad H|0\rangle = E_0|0\rangle \quad , \quad H|n\rangle = E_n|n\rangle$$

- Rewrite the response in the form

$$\begin{aligned} S(\mathbf{q},\omega) &= \sum_n |\sum_k \langle n|a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}}|0\rangle|^2 \ \delta(\omega+E_0-E_n) \\ &= \int \frac{dt}{2\pi} \ \mathrm{e}^{i(\omega+E_0)t} \sum_{\mathbf{p},\mathbf{k}} \langle 0|a_{\mathbf{p}+\mathbf{q}} a^\dagger_{\mathbf{p}} \ \mathrm{e}^{-iHt} \ a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}}|0\rangle \end{aligned}$$

- $S(\mathbf{q},\omega)$ can be expressed in terms of interactions and Green functions describing nucleons in particle and hole states

Effects of interactions on the nuclear response

- ⋆ In the absence of correlations, the only possible final states are one particle-one hole states
- ⋆ For example, according to the Fermi gas model

$$M_n = \langle n| \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}+\mathbf{q}} a_{\mathbf{k}} |0\rangle \to M_k = 1 \times \theta(k_F - |\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}| - k_F)$$

$$S(\mathbf{q},\omega) = \sum_{\mathbf{k}} |M_k|^2 \delta(\omega + e_0(\mathbf{k}) - e_0(\mathbf{k}+\mathbf{q})) \quad , \quad e_0(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$$

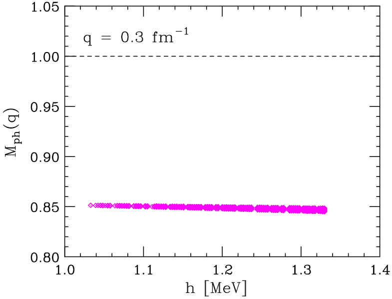
- ⋆ Inclusion of interactions, through the replacement of Fermi gas states with CBF states, leads to a quenching of the transition matrix elements M_k and to a modification of the single particle spectrum $e_0(\mathbf{k})$

Correlations & interaction effects

- ⋆ Isospin symmetric nuclear natter at equilibrium density

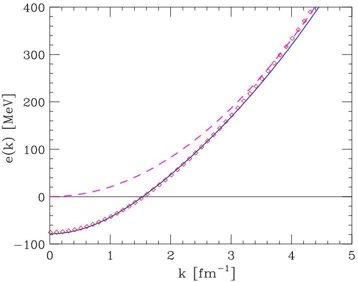
▹ Correlations

$$M_{ph} < 1$$



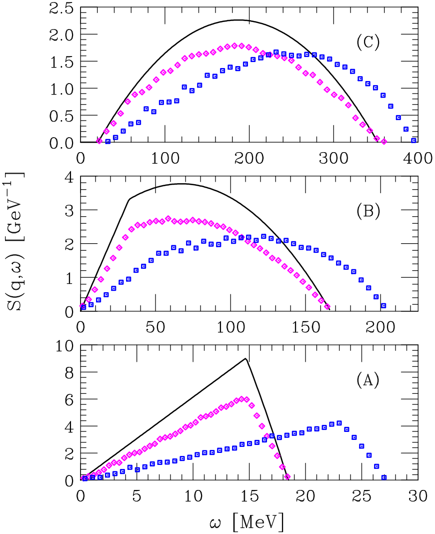
▹ Mean field

$$m \to m^{\star} = \left(\frac{1}{k} \frac{de}{dk} \right)^{-1}$$



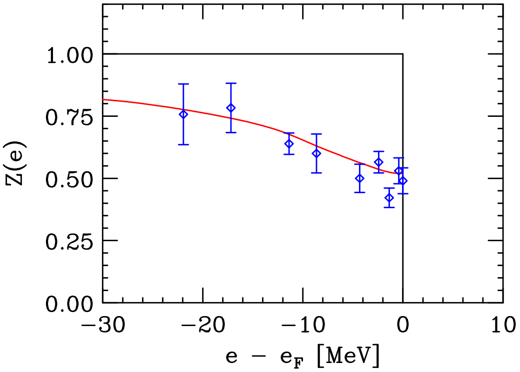
- ⋆ Note that $m^{\star}(k) \neq m$ *does not* measure correlation effects

- (A), (B), (C) → $|\mathbf{q}| = 0.3, 1.8, 3.0\ \mathrm{fm}^{-1}$



Empirical evidence of correlation effects

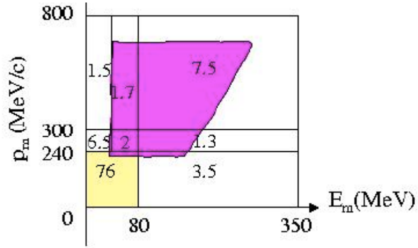
- ⋆ Energy dependence of the spectroscopic strengths of shell model states of ^{208}Pb, measured in high resolution $(e, e'p)$ experiments at NIKHEF-K

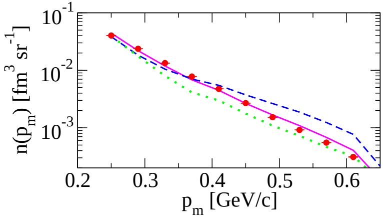


- ⋆ Theory: CBF nuclear matter results corrected for surface effects

Measured correlation strength

- ⋆ The correlation strength in the 2p2h sector has been measured by the JLAB E97-006 Collaboration using a carbon target
- ⋆ Strong energy-momentum correlation: $E \sim E_{thr} + \frac{A-2}{A-1} \frac{\mathbf{k}^2}{2m}$





- ⋆ Measured correlation strength 0.61 ± 0.06, to be compared with the theoretical predictions 0.64 (CBF) and 0.56 (G-Matrix)

FSI in the impulse approximation regime

- ⋆ At momentum transfer $|\mathbf{q}|^{-1} >> 2\pi/d$, d being the average interparticle separation distance

$$S(\mathbf{q},\omega) = \int d^3k dE \; P_h(\mathbf{k},E) P_p(\mathbf{k}+\mathbf{q},\omega-E)$$

  - ▹ P_h → many-body theory
  - ▹ P_p → many-body theory + eikonal approximation (OB, arXiv:1301.3357)

- ⋆ The struck particle travels along a straight trajectory with constant speed v. Its propagation is described by the Green's function ($p = |\mathbf{k}+\mathbf{q}|$)

$$G(\mathbf{r}_\perp, z) = -\frac{i}{\mathrm{v}} \delta(\mathbf{r}_\perp)\theta(z) \exp\left[ipz - \frac{i}{\mathrm{v}} \int_0^z d\zeta \; V(\zeta) \right]$$

with

$$V(\zeta) = \langle 0 | \sum_{j=2}^{N} \Gamma_{\mathbf{p}}(\mathbf{r}_{1j} + \hat{\mathbf{z}}\zeta) | 0 \rangle$$

Correlation effects in FSI

- ⋆ The interaction is described by the Fourier transform of the scattering amplitude

$$\Gamma_{\mathbf{p}}(\mathbf{r}) = -\frac{2\pi}{m} \int \frac{d^3k}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot\mathbf{r}} f_{\mathbf{p}}(\mathbf{k}) \,.$$

with

$$f_p(\mathbf{k}) = \frac{p}{4\pi}\, \sigma_p(\alpha_p + i)\, e^{-\beta_p \mathbf{k}^2}$$

- ⋆ FSI are driven by the quantity

$$V(\zeta) = \int d^3r\, g(r)\, \Gamma_p(\mathbf{r} + \hat{\mathbf{z}}\zeta)$$

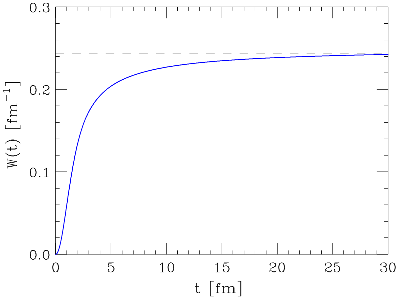
- ⋆ Under the assumptions underlying the eikonal approximation, correlations in coordinate space strongly affect the energy dependence of the spectral function.

★ Consider the simple case $\alpha_p = \beta_p = 0$, i.e.

$$\mathrm{Im}\,\Gamma_p(\mathbf{r}) = -\frac{1}{2}\rho v \sigma_p \delta(\mathbf{r})$$

The corresponding eikonal phase is

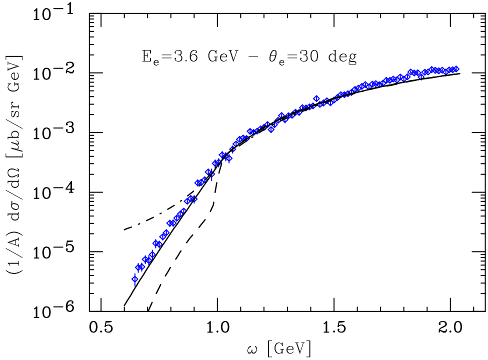
$$W(z) = \int_0^z d\zeta V(\zeta) = \frac{1}{2}\rho\sigma_p \int_0^z d\zeta g(\zeta)$$



★ After Fourier transformation, the z-dependence of W leads to a specific energy dependence of the eikonal spectral function

Preliminary results

★ Isospin symmetric nuclear matter at equilibrium density



★ Main elements of the calculation

- medium modified nucleon-nucleon cross setctions
- nucleon radial distribution function, $g(r)$

Nuclear binding, correlations and the EMC effect

- ⋆ The analysis of the dependence of the slope of the EMC ratio on the average nucleon removal energy, defined as

$$\langle E \rangle = \int d^3k dE \, P(\mathbf{k}, E)$$

requires a level of accuracy not yet achieved for nuclei with $A > 3$

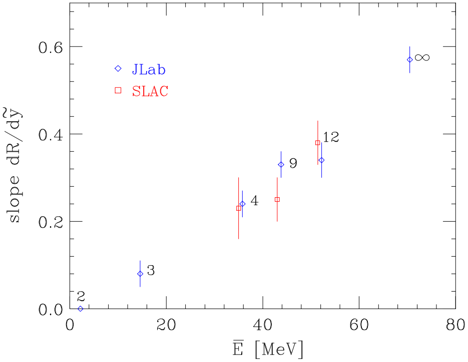
- ⋆ Green's Function Monte Carlo (GFMC) calculations provide the ground state energies, E_0 and the expectation values of the kinetic energy operator, $\langle T \rangle$, of nuclei with $A \leq 12$, obtained from state-of-the-art nuclear hamiltonian
- ⋆ The corresponding average removal energies can be calculated using the GFMC results and the Koltun sum rule, stating that (up to a small correction arising from the three-body potential)

$$\frac{E_0}{A} = \frac{1}{2}\left[\frac{A-2}{A-1}\langle T \rangle - \langle E \rangle\right]$$

- The slope is analyzed in terms of the variable

$$\tilde{y} = \nu - |\mathbf{q}|$$

that can be interpreted as the longitudinal momentum of the struck particle in the target rest frame. Note that $\tilde{y}$ is trivially related to Nachtmann's variable through $\tilde{y} = -\xi/m$.



- OB & I. Sick arXiv:1207.4595

- ⋆ The data shows an excellent correlation with $\langle E \rangle$
- ⋆ The analysis includes the ratio obtained from the extrapolated nuclear matter data. The corresponding removal energy is obtained from the values of E_0 and $\langle T \rangle$ resulting from the CBF calculation of Akmal & Pandharipande
- ⋆ The values of $\langle E \rangle$ employed in the analysis are significantly larger than those used in similar studies. For example, in Carbon the removal energy extracted from $(e, e'p)$ data, corresponding to the shell model states, is $\sim$ 25 MeV, to be compared to the GFMC result $\sim$ 52 MeV
- ⋆ The large values of $\langle E \rangle$ are to be ascribed to strong nucleon-nucleon correlations, leading to the excitation of nucleons to states of high removal energy *and* high momentum

Summary & Outlook

- ⋆ It is long known that correlation effects in nuclei are large. Back in 1952 AD, Blatt & Weiskopf pointed out that:
  - ▹ "The limitation of any independent particle model lies in its inability to encompass the correlation between the positions and spins of the various particles in the system"
- ⋆ While being best defined in coordinate space, correlations manifest themselves in a distinctive energy dependence of the Green's functions.
- ⋆ Pinning down pure correlation effetcs in a model independent fashion requires the calculation of the Green's function within ab initio many-body approaches.
- ⋆ There is ample empirical evidence of important correlation effects from electron-nucleus scattering data. However, the definition of correlation observables remains somewhat elusive.