The sign problem in Lattice QCD

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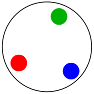
Scope of lattice QCD simulations: Physics of color singlets

* "One-body" physics: confinement
  hadron masses
  form factors, etc..



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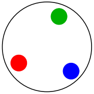

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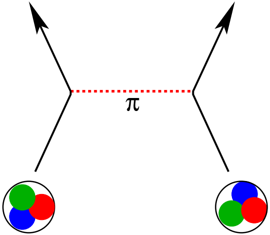
pioneers Hatsuda et al, Savage et al



hard-core
+
pion exchange?

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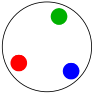

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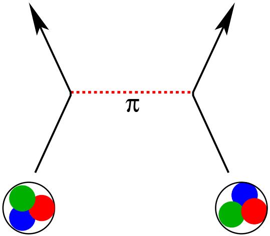

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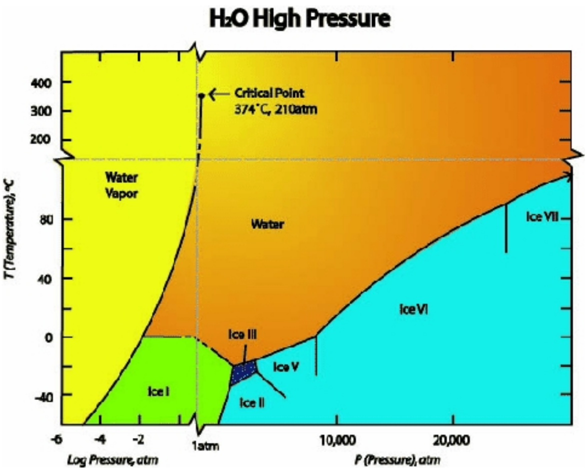
*** Many-[composite]-body physics: nuclear matter
phase diagram vs (temperature T, density $\leftrightarrow \mu_B$)

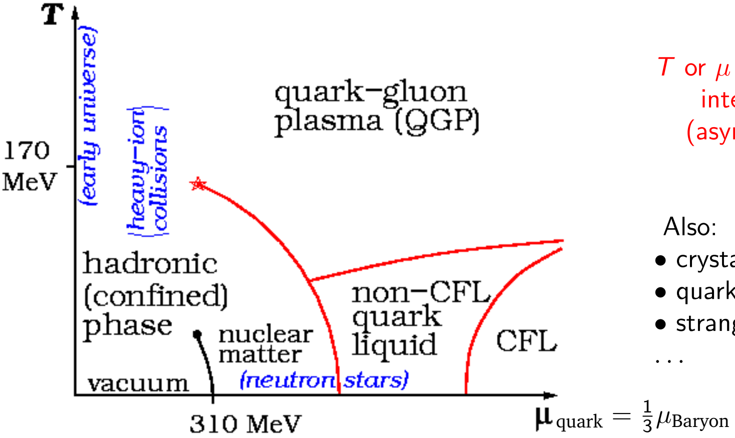
Water changes its state when heated or compressed



What happens to quarks and gluons when heated or compressed?

The phase diagram of QCD according to Wikipedia

Current conjecture



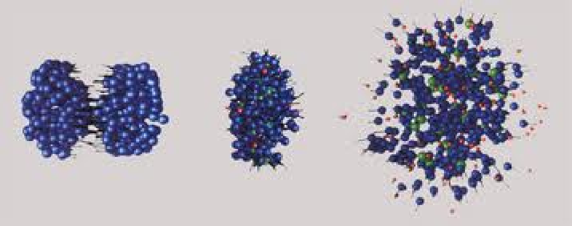
T or $\mu \to \infty$:
interaction weak
(asymptotic freedom)

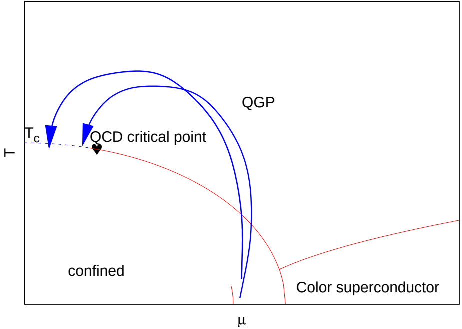
Also:

- crystal phase(s)
- quarkyonic phase
- strangelets
- ...

A vast world to explore and map out!
A "race" between experiment (heavy-ion collisions) & theory (lattice QCD)

Heavy-ion collisions





Knobs to turn:
- atomic number of ions
- collision energy $\sqrt{s}$

So far, no sign of QCD critical point (esp. RHIC beam energy scan)

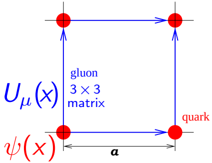
Lattice QCD

Degrees of freedom at each Euclidean time:

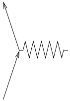
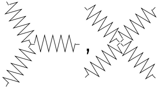
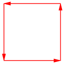
$$\text{positions of } N \text{ particles } (x_1, x_2, .., x_N) \quad \longrightarrow \quad \textit{field } \phi(x)$$

Lattice QCD

space + imag. time $\rightarrow$ $4d$ hypercubic grid:



$$Z = \int \mathcal{D}U\mathcal{D}\bar\psi\mathcal{D}\psi e^{-S_E[\{U,\bar\psi,\psi\}]}$$

- Discretized action S_E:
  -  $\longrightarrow$ $\bar\psi(x)U_\mu(x)\psi(x+\hat\mu) + h.c.,$ — Dirac operator $\bar\psi \not{D}\, \psi$
  -  $\longrightarrow$ $\beta\, \mathrm{ReTr} U_P,\ U_P$ plaquette matrix  Yang-Mills action $\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$
- Monte Carlo: with Grassmann variables $\psi(x)\psi(y) = -\psi(y)\psi(x)$??
  Integrate out analytically (Gaussian) $\rightarrow$ determinant *non-local*

$$\mathrm{Prob}(\mathrm{config}\{U\}) \propto \det{}^2 \not{D}(\{U\})\ e^{+\beta\sum_P \mathrm{ReTr} U_P}$$ real non-negative when $\mu = 0$

Lattice QCD Monte Carlo: sources of errors

- Systematic errors:

$$\begin{cases} L \to \infty, & \text{thermodynamic limit} \\ a \to 0, & \text{continuum limit} \\ m_q \searrow m_{\text{phys}} & \end{cases}$$

Extrapolations guided by analytic ansätze (asymptotic freedom, χPT)

- Statistical (Monte Carlo) errors: $\propto 1/\sqrt{\#\text{configs.}}$

30 years of steady progress since Mike Creutz, 1980:

Both errors have been shrinking thanks to hardware + algorithmic progress

→ Universal tool for *static, equilibrium* properties of QFT

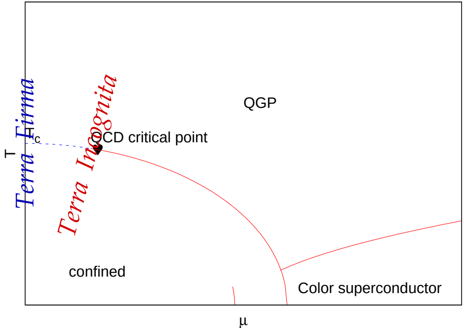
Non-zero chemical potential $\mu \Longrightarrow$ complex determinant

- $\mu > 0$ favors quarks over anti-quarks, ie. breaks charge-conjugation symmetry
- Charge conjugation $\sim$ complex conjugation $\longrightarrow$ det $\neq$ det* when $\mu > 0$
- Formally: γ_5-hermiticity $\rightarrow$ $\det \not{D}(\mu) = \det^* \not{D}(-\mu^*)$

  determinant real only if $\mu = 0$ (or $i\mu_i$), otherwise complex

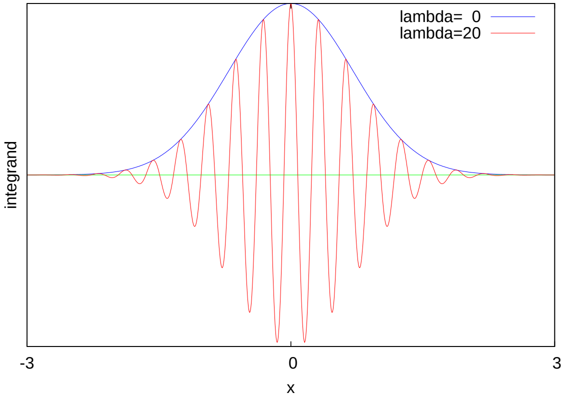
$$\Downarrow$$

$\mu \neq 0$: no probabilistic interpretation of $\det \not{D} \in C$, ie. "sign problem"



Sampling oscillatory integrands

- Example: $Z(\lambda) = \int dx \exp(-x^2 + \mathbf{i}\lambda\mathbf{x}) = \int dx \exp(-x^2) \cos(\lambda x)$



- $Z(\lambda)/Z(0) = \exp(-\lambda^2/4)$: exponential cancellations

$\rightarrow$ truncating deep in the tail at $x \sim \lambda$ gives $\mathcal{O}(100\%)$ error

"Every x is important" $\leftrightarrow$ How to sample?

Reweighting and optimal sampling of oscillatory integrand

- To study: $Z_f \equiv \int dx\ f(x)$, $\quad f(x) \in \mathbf{R}$, with $f(x)$ sometimes negative

  Sample w.r.t. auxiliary partition function $Z_g \equiv \int dx\ g(x)$, $\quad g(x) \geq 0\ \forall x$

$$\langle W \rangle_f \equiv \frac{\int dx\ W(x) f(x)}{\int dx\ f(x)} = \frac{\frac{1}{Z_g} \int dx\ W(x) \frac{f(x)}{g(x)}\ g(x)}{\frac{1}{Z_g} \int dx\ \frac{f(x)}{g(x)}\ g(x)} = \boxed{\frac{\langle W \frac{f}{g} \rangle_g}{\langle \frac{f}{g} \rangle_g}}$$

"reweighting"

$\frac{f}{g}$ is "reweighting factor"

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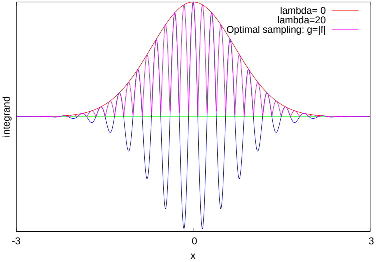
- Optimal g? Minimize relative fluctuations of denom. $\rightarrow$ $\boxed{g(x) = |f(x)|}$, $\frac{f}{g} = \mathrm{sign}(f)$

$$\langle W \rangle_f = \frac{\langle W \mathrm{sign}(f) \rangle_{|f|}}{\langle \mathrm{sign}(f) \rangle_{|f|}}$$

"*put sign in observable*"



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$$\langle W \rangle_f = \frac{\langle W \mathrm{sign}(f) \rangle_{|f|}}{\langle \mathrm{sign}(f) \rangle_{|f|}}$$ "*put sign in observable*"

- $$\langle \mathrm{sign}(f) \rangle_{|f|} = \frac{\int dx\ \mathrm{sign}(f(x)) |f(x)|}{\int dx\ |f(x)|} = \frac{Z_f}{Z_g} = \exp(-\frac{V}{T} \underbrace{\Delta f(\mu^2, T)}_{\text{diff. free energy dens.}})$$, exponentially small

  Each meas. of $\frac{f}{g}$ gives value $\pm 1 \Longrightarrow$ statistical error $\approx \frac{1}{\sqrt{\#\ \text{meas.}}}$

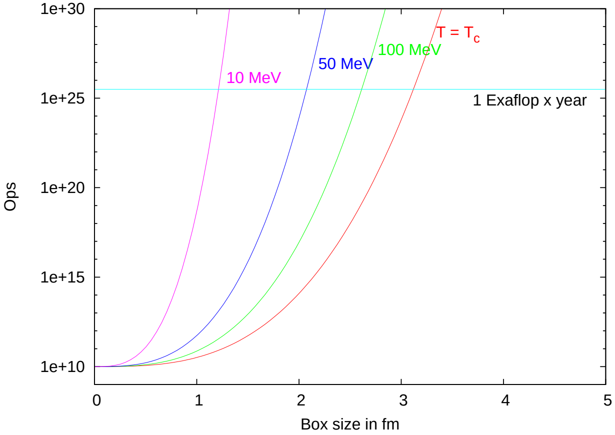
  Constant relative accuracy $\Longrightarrow$ need statistics $\propto \exp(+2\frac{\mathbf{V}}{\mathbf{T}} \Delta f)$

  Large V, low T inaccessible: signal/noise ratio degrades exponentially

$\boxed{\Delta f \text{ measures severity of sign pb.}}$

The CPU effort grows *exponentially* with L^3/T

CPU effort to study matter at nuclear density in a box of given size
Give or take a few powers of 10...



Crudely based on:
- 10 sec on 1GF laptop for 2^4 lattice, $a = 0.1$ fm
- effort $\propto \exp(2\frac{V}{T}\underbrace{\rho_{\rm nucl.}(m_B - 3/2m_\pi)}_{\Delta f})$

Sampling for QCD at finite μ

- QCD: sample with $|\mathrm{Re}(\det(\mu)^{N_f})|$ optimal, but not equiv. to Gaussian integral
  Can choose instead: $|\det(\mu)|^{N_f}$, i.e. "phase quenched"
  $|\det(\mu)|^{N_f} = \det(+\mu)^{\frac{N_f}{2}} \det(-\mu)^{\frac{N_f}{2}}$, ie. isospin chemical potential $\mu_u = -\mu_d$
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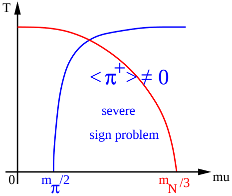

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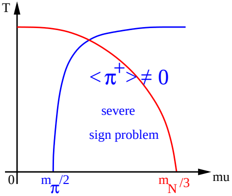

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  $Z_{\mathrm{QCD}} \leftrightarrow Z_{|\mathrm{QCD}|}$ by changing fermion b.c. $\Rightarrow$ ratio UV-finite
  For $T,\ \mu \ll m_\rho$, analytic results via RMT/χPT — Splittorff, Verbaarschot et al.

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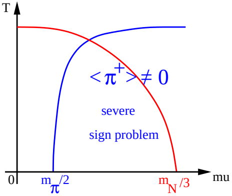

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- Can improve by incorporating baryons via HRG $\rightarrow$ Prediction: 1005.0539

$\langle \mathrm{sign} \rangle \gtrsim 0.1 \Leftrightarrow \mathcal{O}(10)$ baryons max. at $T \lesssim T_c$ (less as $T \searrow$, hardly more as $V \nearrow$)

Reweighting strategies

- Sample phase-quenched $|\det(\mu)|$ + reweight with $e^{i\theta}$ → only 0711.0023, 1111.6363

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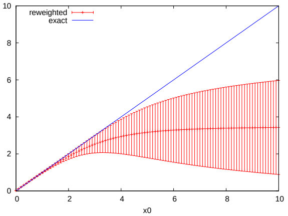
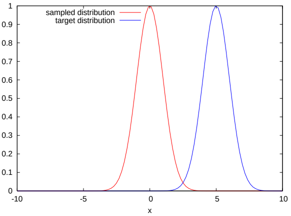
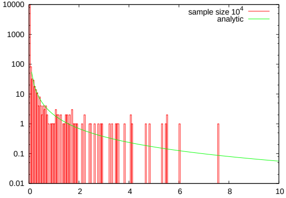
- Sample $\mu = 0$ ensemble? worse, because reweighting factor fluctuates also in magnitude $\rightarrow$ increased statistical errors
- Further danger: "overlap pb." between sampled and reweighted ensembles $\rightarrow$ WRONG estimates in reweighted ensemble for finite statistics
- Example: sample $\exp(-\frac{x^2}{2})$, reweight to $\exp(-\frac{(x-x_0)^2}{2})$ $\rightarrow$ $\langle x \rangle = x_0$?



- Estimated $\langle x \rangle$ saturates at largest sampled x-value
- Error estimate too small



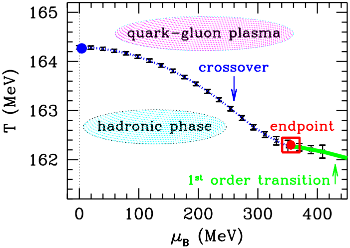
Insufficient overlap ($x_0 = 5$)

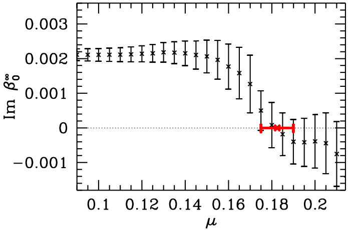


Very non-Gaussian distribution of reweighting factor Log-normal Kaplan et al.

Reweighting from $\mu = 0$: multi-parameter

- Fodor & Katz: sample $(\mu\!=\!0, \beta\!=\!\beta_c)$ and reweight with $\left(\frac{\det(\mu)}{\det(\mu=0)} \times e^{-\Delta\beta S_{YM}}\right)$ along pseudo-critical line $T_c(\mu)$
  - fluctuations in reweighting factor compensate between det and S_{YM}
  - improved (ensured?) overlap: both phases sampled





hep-lat/0402006 (physical quark masses, $N_t = 4$) $\rightarrow$ $(\mu_E^q, T_E) = (120(13), 162(2))$MeV

- Abrupt qualitative change near μ_E:

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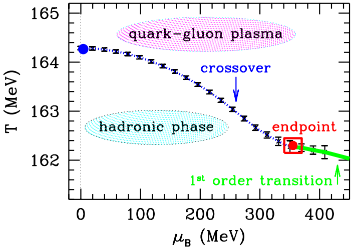

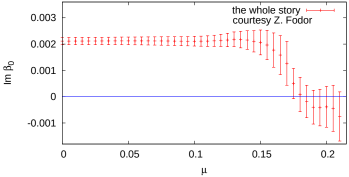

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abrupt change of physics or breakdown of reweighting ? (see later)

- Revival (fast det Wilson fermions): Ukawa et al., Nakamura et al.

Change of strategy

Reweighting gives exact answer in small volumes (work $\sim \exp(V)$) in principle

In practice: may fail without letting you know!

Try instead: approximate answer in large volume ?

And – perhaps – full confidence in results

Consider expansion parameter $\frac{\mu}{T} \lesssim 1$:

- Truncated Taylor expansion about $\mu = 0$
- Imaginary μ + polynomial fit + analytic continuation

Taylor expansion of pressure

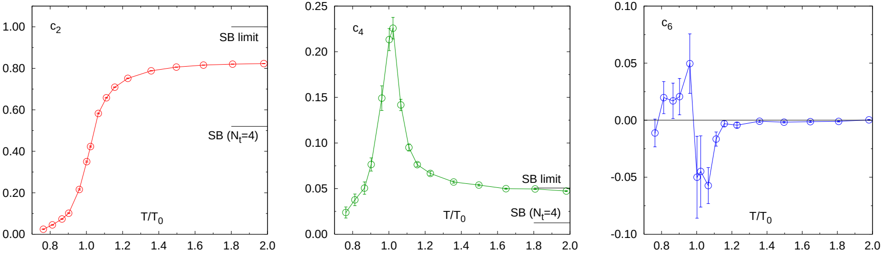
F. Karsch et al.

$$P(T,\mu) = \underbrace{P(T,\mu=0)}_{\text{indep. calc.}} + \Delta P(T,\mu),$$

$$\frac{\Delta P(T,\mu)}{T^4} = \sum_{k=1} c_{2k}(T) \left(\frac{\mu}{T}\right)^{2k}$$

$$c_{2k} = \langle \mathrm{Tr}(\text{degree } 2k \text{ polynomial in } \not{D}^{-1}, \frac{\partial \not{D}}{\partial \mu}) \rangle_{\mu=\mathbf{0}} \rightarrow \text{vanilla HMC}$$

- From $\{c_{2k}\}$, obtain all thermodynamic info: EOS *and* $T_c(\mu)$ *and* crit. pt. *and* ...
- As $\frac{\mu}{T}$ increases, need higher-order c_{2k}'s to control truncation error



C. Schmidt, hep-lat/0610116

Taylor expansion: nitty-gritty

$$
\begin{aligned}
c_6 \to \frac{\partial^6 \ln\det M}{\partial\mu^6} \quad = \quad & \mathrm{tr}\left(M^{-1}\frac{\partial^6 M}{\partial\mu^6}\right) - 6\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^5 M}{\partial\mu^5}\right) \\
& -15\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) - 10\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
& +30\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) + 60\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
& +60\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) + 30\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& -120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
& -180\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& -90\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& +360\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& -120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right).
\end{aligned}
$$

Now estimate all Traces by sandwiching between noise vectors:

$\mathrm{Tr}\mathcal{O} = \langle \eta^\dagger \mathcal{O} \eta \rangle_\eta$, where $\langle \eta_x^\dagger \eta_y \rangle = \delta_{xy} \longrightarrow$ GPU farm

Complexity of Taylor expansion approach?

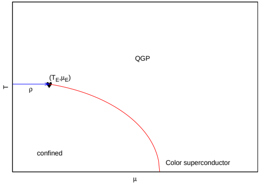
Effects of increasing Taylor order k:

- $c_{2k} = \langle \mathrm{Tr}(\text{ degree } 2k \text{ polynomial in } \not{D}^{-1}, \frac{\partial \not{D}}{\partial \mu})\rangle_{\mu=\mathbf{0}}$ $\rightarrow$ nb. terms $\sim 6^{2k}$
- Cancellations: c_{2k} finite as $V \rightarrow \infty$, but sum of terms, each possibly $\sim V^{2k}$

ie. the sign problem fights back!

- c_{2k} obtained as average over less and less Gaussian dist. $\rightarrow$ stat. error?
- $c_{2k} \sim 2k$-point function $\rightarrow$ need larger volumes

Current best: $N_t = 6$, 8th order (c_2, c_4, c_6, c_8) Gavai & Gupta, 0806.2233



Need *much* higher order

to estimate convergence radius $\rightarrow$ critical point

Karsch, Schaefer et al, 1009.5211

Imaginary μ: similar but simpler – probably cheaper

- Simulate at several values of $\mu = i\mu_I$: no sign pb.
- Fit $\langle \mathcal{O} \rangle(\mu_I) = \sum_k d_k \left(\frac{\mu_I}{T}\right)^k$
  - For pressure, take eg. $\mathcal{O} = n_B = \frac{\partial P}{\partial \mu_B}$ and integrate fitted polynomial
  - Analytic continuation trivial: $i\mu_I \to \mu$
  - Stat. error analysis simple: data at different μ_I's uncorrelated
  - Systematic error: order of truncation, fitting range

No free lunch: fit insensitive to d_k because $\left(\frac{\mu_I}{T}\right)^k \ll 1$

Advantage over Taylor expansion: milder V-dependence?

- $\left|\frac{\mu_I}{T}\right| < \frac{\pi}{3}$, Roberge-Weiss singularity
  Conformal mapping to unit disk — Morita et al., 1008.4549

Other approaches

- Canonical ensemble: $Z_Q = \int d(\frac{\mu_I}{T}) e^{i\frac{\mu_I}{T}Q} Z(\mu = i\mu_I)$

  now with fast and accurate Fourier transform

  Alford, Wilczek et al., PdF & Kratochvila, K.F. Liu et al., Nakamura, Wenger,..

- Density of states a.k.a. histogram method:

$$Z(\mu) = \int dx \underbrace{\int \mathcal{D}U \; \delta(W(U) - x) \det(\mu) e^{-S_{YM}}}_{\rho(W;x)}$$

$2d$ variant $\rho(P, \left|\frac{\det(\mu)}{\det(0)}\right|)$ Ejiri et WHOT-QCD

Considerable technical progress but no breakthrough so far

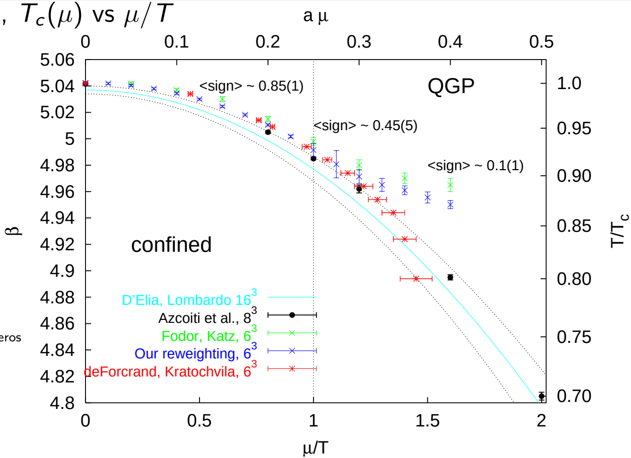
- Complex Langevin? progress: now *converges* to the right or wrong answer

  Aarts, Seiler, Stamatescu

Valuable crosschecks

All methods agree for $\mu/T \lesssim \mathcal{O}(1)$ on small lattices

Here, $T_c(\mu)$ vs μ/T



imaginary μ
2 param. imag. μ
dble reweighting, LY zeros
Same, susceptibilities
canonical

$N_f = 4$ staggered,
$am_q = 0.05$, $N_t = 4$
PdF & Kratochvila
LAT05

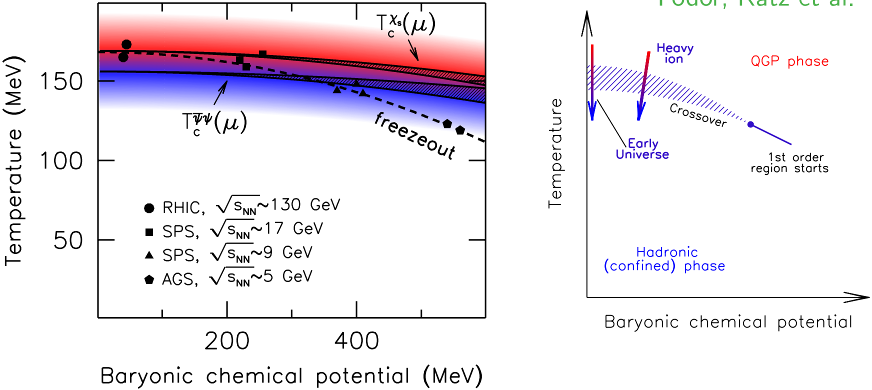
More recent crosschecks (Wilson fermions):

- Reweighting $\leftrightarrow$ Taylor expansion — Nagata & Nakamura
- Reweighting $\leftrightarrow$ canonical — Takeda, Kuramashi & Ukawa

State of the art

- Curvature of crossover $T_c(\mu)$ in continuum limit (4 deriv. of P)

Fodor, Katz et al.



- $T_c(\mu)$ flatter than experimental heavy-ion freeze-out curve (different things)
- Different definitions of $T_c(\mu)$ do not meet: no signal of critical point for $\frac{\mu_q}{T} \lesssim \mathcal{O}(1)$

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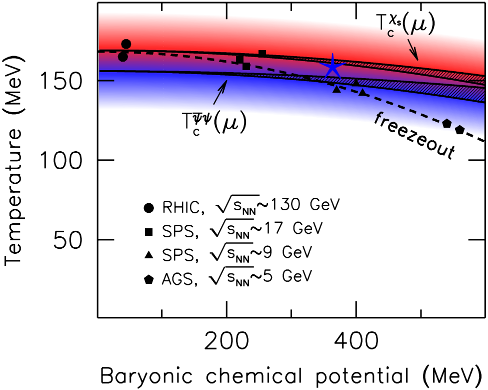

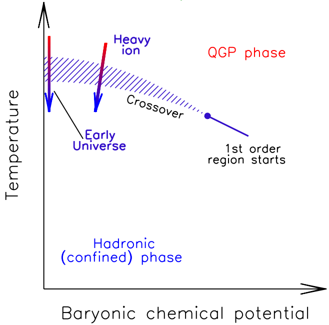

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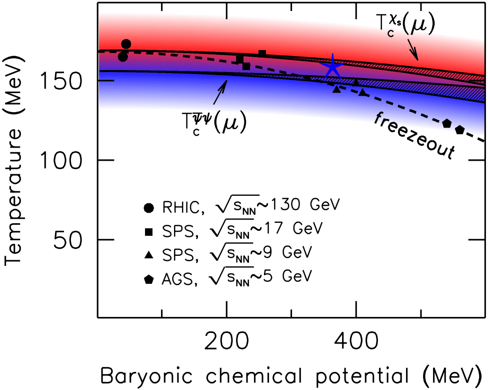

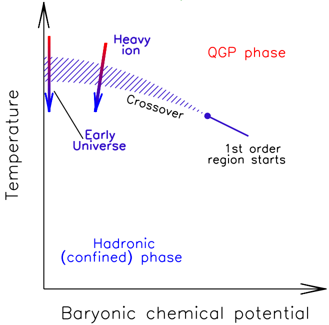

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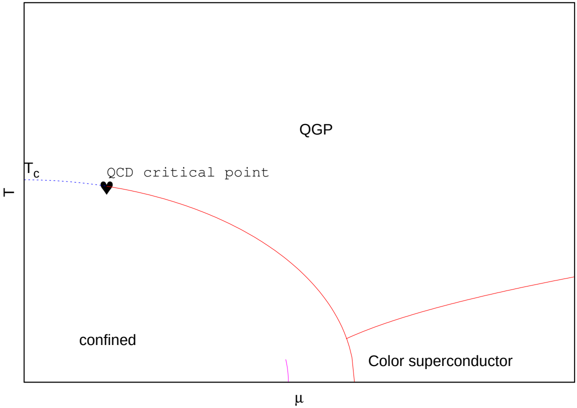
- Next order (8 deriv. of P) on coarse lattice: weakening of transition PdF & Philipsen

Possible phase diagrams of QCD

Where is the QCD critical point?



Exciting!

Possible phase diagrams of QCD

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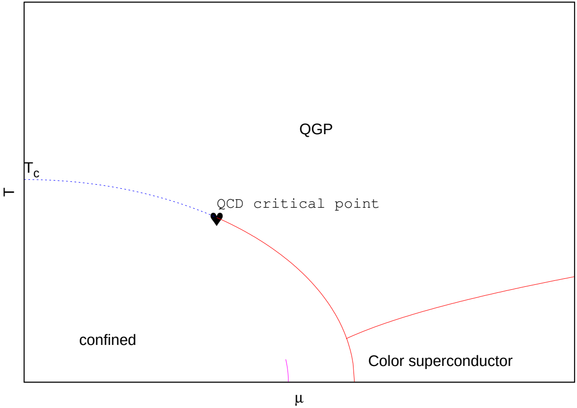

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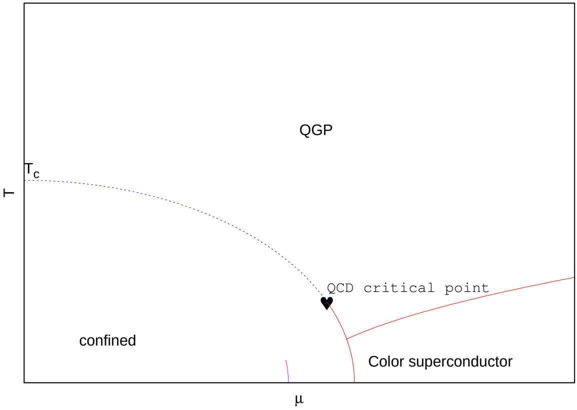

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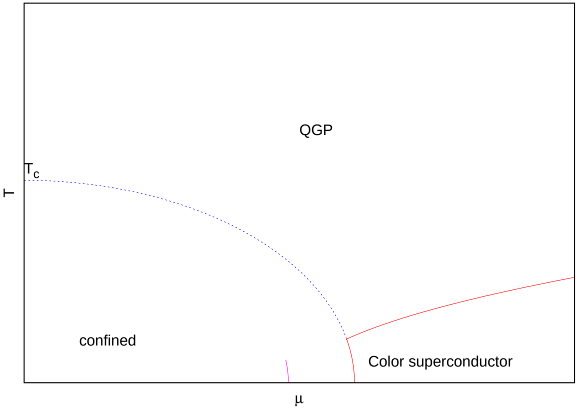

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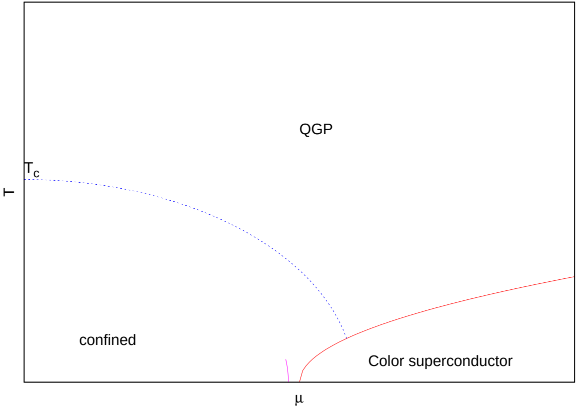

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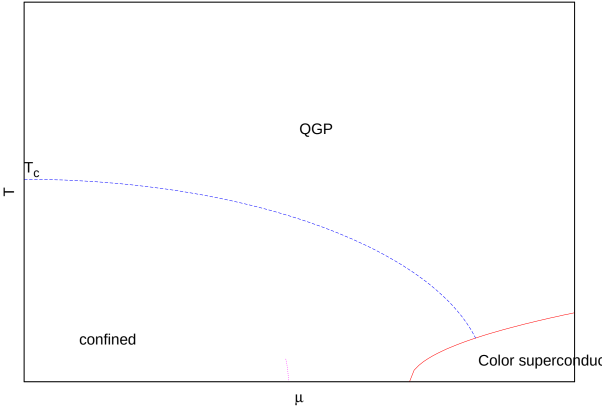

Possible phase diagrams of QCD

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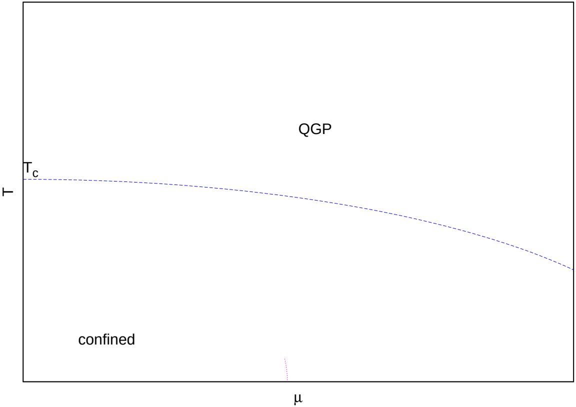


Boring but plausible!

Determining the QCD phase diagram remains just as important

cf. Georges Charpak

How to make the sign problem milder?

- Severity of sign pb. is representation dependent:

$$Z = \mathrm{Tr} e^{-\beta H} = \mathrm{Tr}\left[e^{-\frac{\beta}{N} H} \left(\sum |\psi\rangle\langle\psi|\right) e^{-\frac{\beta}{N} H} \left(\sum |\psi\rangle\langle\psi|\right) \cdots \right]$$

Any complete set $\{|\psi\rangle\}$ will do

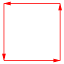
If $\{|\psi\rangle\}$ form an eigenbasis of H, then $\langle\psi_k|e^{-\frac{\beta}{N}H}|\psi_l\rangle = e^{-\frac{\beta}{N}E_k}\delta_{kl} \geq 0 \rightarrow$ no sign pb

- Strategy: choose $\{|\psi\rangle\}$ "close" to physical eigenstates of H

QCD physical states are color singlets $\rightarrow$ Monte Carlo on colored gluon links is bad idea

Usual:
- integrate over quarks analytically $\rightarrow \det(\{U\})$
- Monte Carlo over gluon fields $\{U\}$

Reverse order:
- integrate over gluons $\{U\}$ analytically
- Monte Carlo over quark color singlets (hadrons)

- Caveat: so far, turn off 4-link coupling  in $\beta \sum_P \mathrm{ReTr} U_P$ by setting $\beta = 0$

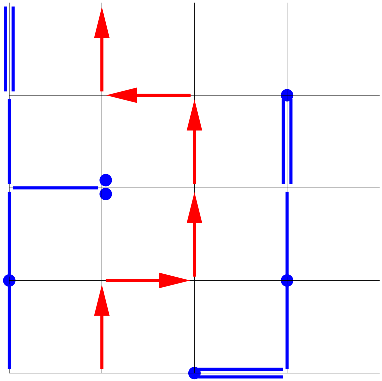
$\beta = 0$: strong-coupling limit $\longleftrightarrow$ continuum limit $(\beta \rightarrow \infty)$

Strong coupling limit at finite density

Chandrasekharan, Wenger, PdF, Ohnishi, ...

- Integrate over U's, then over quarks: *exact* rewriting of $Z(\beta = 0)$

New, discrete degrees of freedom: meson & baryon worldlines



- Constraint at every site (Grassmann):
  3 meson symbols (• $\bar{\psi}\psi$, meson hop)
  or a baryon loop

Point-like, hard-core baryons in pion bath

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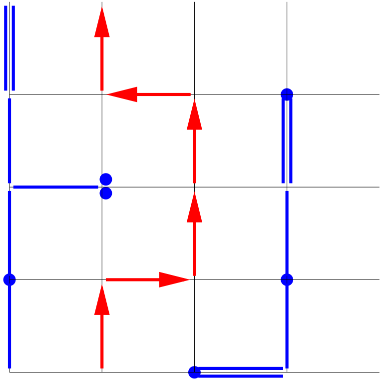

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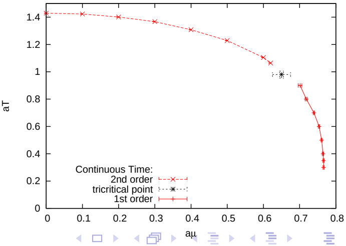
- Sign pb.: Δf reduced by $\mathcal{O}(10^4)$

  $\rightarrow$ full phase diagram



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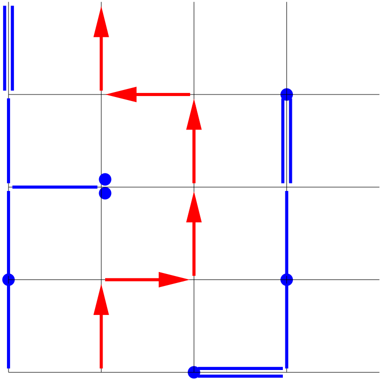

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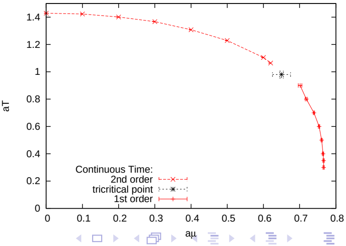


- Baryon: (point-like core + pion cloud)
- Nuclear potential: (hard-core + Yukawa)

Conclusions

- Finite density LQCD suffers from sign problem: $S/N \sim \exp(-\#\text{d.o.f.})$

  $\rightarrow$ only small V, small μ/T

- Simulations still at an early, "experimental" stage
- No reliable indication of QCD critical point [yet]
- Progress: - analytic understanding of severity of sign problem
  - new direction: reverse order of integration (quarks $\leftrightarrow$ gluons)

Finite density QCD is important enough to keep trying

Hardware improvement alone will not suffice

$\downarrow$

A true challenge (= opportunity) for computational physicists!

Backup: complex Langevin 80's revival Aarts, Seiler, Stamatescu, Berges,..

- Real action S: Langevin evolution in Monte-Carlo time τ Parisi-Wu

$\frac{\partial\phi}{\partial\tau} = -\frac{\delta S[\phi]}{\delta\phi} + \eta$, ie. drift force + noise

Can prove: $\langle W[\phi]\rangle_\tau = \frac{1}{Z}\int \mathcal{D}\phi \exp(-S[\phi]) W[\phi]$

- Complex action S ? Parisi, Klauder, Karsch, Ambjorn,..

Drift force complex → complexify field $(\phi^R + i\phi^I)$ and simulate as before

With luck: $\langle W\left[\phi^R + i\phi^I\right]\rangle_\tau = \frac{1}{Z}\int \mathcal{D}\phi \exp(-S[\phi]) W[\phi]$

- Only change since 1980's: adaptive stepsize → runaway sols disappear

- Gaussian example:

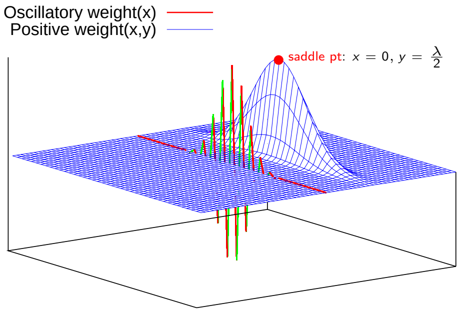
$Z(\lambda) = \int dx \exp(-x^2 + \mathbf{i}\lambda\mathbf{x})$

Complexify:

$\frac{d}{d\tau}(x+iy) = -2(x+iy) + i\lambda + \eta$

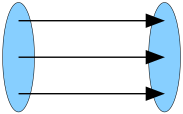
For any observable W,

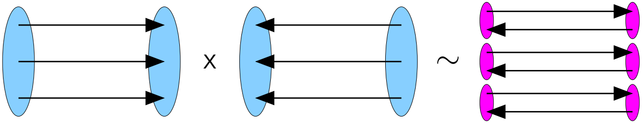
$\langle W(x+iy)\rangle_\tau = \langle W(x)\rangle_Z$



A precursor of the sign problem   Lepage 1989

Signal-to-noise ratio of N-baryon correlator $\propto \exp(-N(m_B - \frac{3}{2}m_\pi)t)$

$C_B(t) =$  $\sim \mathbf{e^{-m_B t}}$

$|C_B(t)|^2 =$  $\sim \mathbf{e^{-3m_\pi t}}$

- Mitigated with variational baryon ops. $\rightarrow m_{\text{eff}}$ plateau for 3 or 4 baryons ?
  Savage et al., 1004.2935
- At least 2 baryons $\rightarrow$ nuclear potential   Aoki, Hatsuda et al., eg. 1007.3559
- Binding energy of ^{3}He, He   Kuramashi, Ukawa et al., 0912.1383

Here, we want a finite baryon *density* $\rightarrow N \propto V$, ie. chem. pot. $\mu \neq 0$