Nonrelativistic Renormalization

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July 2013

A History (of Sorts)

An example:

$$\frac{g_e}{2} = 1 + \frac{\alpha_0}{2\pi} + \infty\alpha_0^2 + \infty^2\alpha_0^3 + \cdots$$

but

$$\alpha_0 = \frac{\alpha}{1 - \infty\alpha - \infty^2\alpha^2 - \cdots}$$

$$= \alpha\,(1 + \infty\alpha + \cdots)$$

Taylor expansion in powers of $\infty\alpha$.

implies

$$\frac{g_e}{2} = 1.000579826087\ldots$$

(experiment $\Rightarrow$ 1.000579826087...)

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$$\frac{g_e}{2} = 1 + \frac{\alpha_0}{2\pi} + \infty\alpha_0^2 + \infty^2\alpha_0^3 + \cdots$$

but

$$\begin{aligned}\alpha_0 &= \frac{\alpha}{1 - \infty\alpha - \infty^2\alpha^2 - \cdots} \\ &= \alpha\,(1 + \infty\alpha + \cdots)\end{aligned}$$

Taylor expansion in powers of $\infty\alpha$.

implies

$$\frac{g_e}{2} = 1.000579826087\ldots$$

(experiment $\Rightarrow$ 1.000579826087...)

$$\times \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} \left[-4(d-2)A_1 + \cdots - \frac{4(d-2)}{4\sum^2} C_2 \right] \tag{8-124}$$

It is then a pure matter of patience to compute

$$\begin{aligned}(4\pi y)^{d/2} \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} A_1 = {} & k_\rho k_\sigma \frac{\alpha_{23}\alpha_{14} + 2(\alpha_1 - \alpha_3)(\alpha_4 - \alpha_2)}{2y\sum^2}(d-2) \\ & + \delta_{\rho\sigma} \Bigg[(k^2)^2 z(1-z)(\alpha+z)(1-\alpha-z) \\ & + \frac{k^2}{2y} \left\{ 2\hat{\alpha}_{23}(1-\hat{\alpha}_{23})[\alpha + 2z - 2z(\alpha+z)] + \frac{2(\alpha_1-\alpha_3)(\alpha_4-\alpha_2)}{\sum^2} \right\} \\ & - \frac{k^2 d}{2y} [(\alpha+z)(1-\alpha-z)\hat{\alpha}_{23}^2 + (1-\hat{\alpha}_{23})^2 z(1-z)] \\ & + \frac{d(d+2)}{4y^2} \hat{\alpha}_{23}^2 (1-\hat{\alpha}_{23})^2 \Bigg]\end{aligned}$$

$$\begin{aligned}(4\pi y)^{d/2} \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} B_1 = {} & (2-d)(2\alpha^2 - 1)k_\rho k_\sigma \\ & - (d-2)\delta_{\rho\sigma} \left\{ \frac{1}{y} + k^2[(\alpha+z)(1-\alpha-z) + z(1-z)] - \frac{d}{2y}[\hat{\alpha}_{23}^2 + (1-\hat{\alpha}_{23})^2] \right\} \\ & + 2\delta_{\rho\sigma} \left[k^2(1-2z)(1-2z-2\alpha) - 2\hat{\alpha}_{23}(1-\hat{\alpha}_{23})\frac{d}{y} \right]\end{aligned}$$

$$(4\pi y)^{d/2} \int \frac{d^d q'}{(2\pi)^d} e^{-yq'^2} A_2 = \frac{d-2}{2y} \hat{\alpha}_{23}(1-\hat{\alpha}_{23})(k_\rho k_\sigma - k^2 \delta_{\rho\sigma})$$

Consequently,

$$\begin{aligned}\Gamma_{\rho\sigma}^{(A_2+B_2)} = {} & -\frac{e^4}{(4\pi)^d} 4(6-d)(d-2)(k^2\delta_{\rho\sigma} - k_\rho k_\sigma) \int_0^1 \frac{dx'}{x'^2} \int_0^\infty \frac{d\alpha_1 \cdots d\alpha_4 \alpha_{23}\alpha_{14}(1-x')}{2\sum^{d/2+2} y^{d/2}} \\ & \times \exp\left[-\frac{\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{41} - x'(\alpha_1\alpha_3 - \alpha_2\alpha_4)^2}{\sum \alpha_{23}\alpha_{14}} k^2 \right] \\ = {} & -\frac{e^4}{(4\pi)^d} 2(6-d)(d-2)(k^2\delta_{\rho\sigma} - k_\rho k_\sigma)(k^2)^{d-4}\Gamma(4-d) \\ & \times \int_0^1 dx' \, x'^{d/2-2}(1-x') \int_0^1 \frac{d\alpha_1 \cdots d\alpha_4 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}{(\alpha_{14}\alpha_{23})^{3d/2-5}} \\ & \times [\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{41} - x'(\alpha_1\alpha_3 - \alpha_2\alpha_4)^2]^{d/2-4}\end{aligned}$$

After the change of variables,

$$\alpha_1 = \beta u \qquad \alpha_2 = (1-\beta)v \qquad \alpha_3 = (1-\beta)(1-v) \qquad \alpha_4 = \beta(1-u)$$

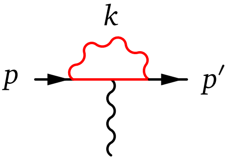
$$\alpha_{14} = \beta \qquad \alpha_{23} = 1 - \beta$$

$$\begin{aligned}[\alpha_{12}\alpha_{23}\alpha_{34}\alpha_{41} - x'(\alpha_1\alpha_3 - \alpha_2\alpha_4)^2] = {} & \beta(1-\beta)\{[\beta(1-u) + (1-\beta)(1-v)] \\ & \times [\beta u + (1-\beta)v] - x'\beta(1-\beta)(u-v)^2\}\end{aligned}$$

$$\int_0^1 d\alpha_1 \cdots d\alpha_4 \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)F(\alpha_i) = \int_0^1 d\beta\, \beta(1-\beta) \int_0^1 du \int_0^1 dv\, F(\alpha_i)$$

The Problem

Eg)



$= \int d^4k \ldots$

$\Rightarrow$ Integral diverges from $k \to \infty$ states.

$\Rightarrow$ $k \to \infty$ states infinitely important?

$\Rightarrow$ $k \to \infty$ not just QED $\Rightarrow$ need to understand string theory (or. . . ?) in order to calculate anything?? Disaster???

The Solution: UV Cutoff

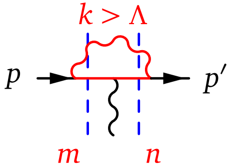
- Introduce UV cutoff: omit all states with $k > \Lambda$ from theory.
- Choose $\Lambda =$ boundary between known and unknown physics $\Rightarrow \Lambda \not\to \infty$!!
- Use for $p \ll \Lambda$ where

$$p = \text{typical momentum in process of interest,}$$

- Fixes infinities, but ...

What is left out?

Eg)



$k > \Lambda \gg p, p' \Rightarrow$ states m, n far off shell ($\Delta E \approx \Lambda$).

$\Rightarrow$ m, n very shortlived (uncertainty principle):

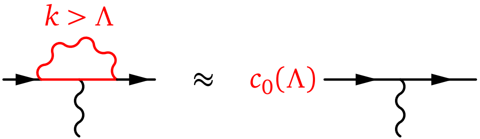
$$\Delta t \approx \frac{1}{\Delta E} \approx \frac{1}{\Lambda}.$$

$\Rightarrow$ Interaction occurs over very small region:

$$\Delta x \approx \frac{1}{\Lambda} \ll \frac{1}{p}.$$

$\Rightarrow$ Interactions effectively local compared to $\lambda \approx 1/p$.

⇒ Can mimic piece of theory excluded by cutoff with new local interaction:



⇒ Add $k > \Lambda$ physics back in by adding

$$\delta\mathscr{L} \equiv c_0(\Lambda)\,\overline{\psi}\not{A}\psi$$

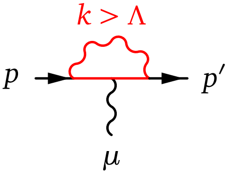
to the cutoff Lagrangian (much simpler)!

N.B. $\mathscr{L}^{(\Lambda)} + \delta\mathscr{L}$ then has interaction $e(\Lambda)\,\overline{\psi}\not{A}\psi$ where

$$e(\Lambda) \equiv e_0 + c_0(\Lambda) \;=\; \text{"running coupling."}$$

More Accuracy

Taylor expand in p/Λ, p'/Λ:



$$= c_0(\Lambda)\ \bar{u}\gamma_\mu u + \frac{c_1(\Lambda)}{\Lambda}\ \bar{u}\sigma_{\mu\nu}(p-p')^\nu u + \frac{c_2(\Lambda)}{\Lambda^2}(p-p')^2\ \bar{u}\gamma_\mu u + \ldots$$

$\Rightarrow$ Add more corrections to $\mathscr{L}^{(\Lambda)}$:

$$\frac{c_1(\Lambda)}{\Lambda} \quad \overline{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi \qquad \text{for } p/\Lambda$$

$$\frac{c_2(\Lambda)}{2\Lambda^2} \quad \overline{\psi}i\partial_\mu F^{\mu\nu}\psi \qquad \text{for } (p/\Lambda)^2$$

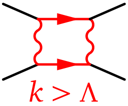
$$\vdots$$

N.B.

- Operators all local $\equiv$ polynomial in ψ, A_μ, and ∂_μ (Taylor expansion!).
- Infinitely many operators but only need first few since

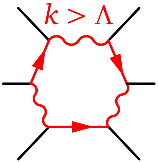
$$\frac{p}{\Lambda} \ll 1.$$

Only other amplitude important in order $1/\Lambda^2$ is



$$\rightarrow \qquad \frac{d(\Lambda)}{\Lambda^2}(\overline{\psi}\gamma\psi)^2 + \cdots$$

Eg)



$$\rightarrow \qquad \frac{f(\Lambda)}{\Lambda^5}(\overline{\psi}\gamma\psi)^3 + \cdots$$

$$\int_\Lambda d^4k \left(\frac{1}{\not{k}}\right)^3 \left(\frac{1}{k^2}\right)^3 \sim \frac{1}{\Lambda^5}$$

Summary: Renormalization Theory

- UV cutoff $\Rightarrow$ omit $k > \Lambda$ states
  $\Rightarrow$ no infinities
  $\Rightarrow$ no string/M theory needed!

- Add local universal correction terms, with theory-specific coupling coefficients, to $\mathscr{L}^{(\Lambda)}$ to mimic effects of $k > \Lambda$ physics (all we need to know about it).

- Only a finite number of correction terms needed for given accuracy, $(p/\Lambda)^n$.

$\Rightarrow$ **Arbitrary precision with finite Λ!**

G.P. Lepage, *What is Renormalization?*, arXiv:hep-ph/0506330

Why is QED renormalizable?

QED = low-energy approximation to complex super-theory (strings? branes? SUSY?) with threshold Λ.

$\Rightarrow$

$$\mathscr{L}_{\text{QED}}^{(\Lambda)} = \mathscr{L}_R + \frac{c_1 \overline{\psi} \sigma \cdot F \psi}{\Lambda} + \frac{c_2 \overline{\psi} \partial \cdot F \cdot \gamma \psi}{\Lambda^2} + \ldots$$

"Renormalizable" theory.

Λ is boundary between old and new physics.

Cutoff restricts theory to region of validity.

Due to new dynamics at $k > \Lambda$.

Terms really there, but only affect results in order $p/\Lambda \ll 1$.

$\Rightarrow$ Theory *appears* to be renormalizable!

Example: QED in Atoms — Ps, H, He . . .

Non-relativistic Expansion (eg, Ps)

- Probability $P(p_e \geq m_e) \sim \alpha^5 \Rightarrow$ very non-relativisitc.
- Expand in powers of $p/m_e = v$:

$$H = \frac{\mathbf{p}^2}{m_e} - \frac{\alpha}{r} + \delta H_{\text{rel}}$$

where

$$\delta H_{\text{rel}} = -\frac{\mathbf{p}^4}{4m_e^3} + \frac{\mathbf{p}^6}{8m_e^5} + \cdots \qquad \leftarrow \text{from } \sqrt{\mathbf{p}^2 + m_e^2}$$

$$+ \frac{\pi\alpha}{m_e^2}\delta^{(3)}(\mathbf{r}) + \frac{\alpha}{4m_e^2}\frac{\mathbf{L}\cdot(\boldsymbol{\sigma}_e + \sigma_{\bar{e}})}{r^3} + \cdots$$

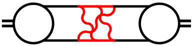
- Good approximation through $O(v^2)$, but . . .

- Too singular: $\delta^3(\mathbf{r})$, $1/r^3$ cause UV divergences in 2^{nd}-order; $\mathbf{p}^6$, $\nabla^2\delta^3(\mathbf{r})$ diverge in 1^{st} order.

  $\Rightarrow$ Treat relativity exactly? Eg, use Bethe-Salpeter equation:

$$(i\partial_e \cdot \gamma - m_e)(i\partial_{\bar{e}} \cdot \gamma - m_e)\psi(x_e, x_{\bar{e}}) = \int d^4y_e d^4y_{\bar{e}}\, K(x_e, x_{\bar{e}}; y_e, y_{\bar{e}})\, \psi(y_e, y_{\bar{e}})$$

  - ◇ Nonperturbative, but QED renormalization implemented order-by-order in Feynman perturbation theory.
  - ◇ Bound states offshell $\Rightarrow$ K highly gauge-dependent: *e.g.*,



$$\sim \begin{cases} \alpha^3 m_e & \text{in Feynman gauge,} \\ \alpha^5 m_e & \text{in Coulomb gauge.} \end{cases}$$

  $\Rightarrow$ Bound states directly in QED is bad idea!

- Nonrelativistic bound states involve multiple, widely separated scales: $K \sim mv^2$, $P \sim mv$, m where $v \approx \alpha$.

- α expansion of E's, Γ's not so convergent.

Eg)

$$\Gamma_{\text{O-Ps}} = \Gamma \left(1 - 3\alpha + 4\alpha^2 - 20\alpha^3 + \cdots\right)$$

$\alpha^4 \ln^3 \alpha$

$\frac{3}{2\pi} \ln^2 \alpha + 0.7 \ln \alpha$

$\rightarrow$ Due to multi-scales: $\ln^2(K/m) \ldots$

Nonrelativistic Effective Theory

UV cutoff $\Lambda \sim m_e$ for $p_e \ll m_e$ problems (eg, atoms).

- Cutoff prevents infinities.
- Nonrelativisitic electrons: $p_e \sim m_e v \ll m_e$
  $\Rightarrow$ no pair creation $\Rightarrow$ don't need QFT for e.
- Hard photons: $p_\gamma \sim p_e \Rightarrow E_\gamma \gg E_e$ (by factor $1/v$)
  $\Rightarrow \gamma$ highly virtual, short-lived
  $\Rightarrow$ replace by instantaneous potentials $V(\mathbf{r})$, $\boldsymbol{\sigma} \cdot \mathbf{p} V'(\mathbf{r})$. . . .
- Soft photons: $p_\gamma \sim v p_e \Rightarrow E_\gamma \sim E_e$
  Probablility $P(e\gamma_{\text{soft}}) \sim \alpha^3$ is very small.
  Replace by E-dependent potential $\delta V_{\text{soft}}(E)$.

  N.B., no photons left $\Rightarrow$ gauge invariant.

QED → nonrelativistic Schrödinger theory

$$H = \sum_i \frac{p_i^2}{2m_i} + V(\mathbf{r}_j, \mathbf{p}_j \ldots) + \delta V_{\text{soft}}(E)$$

UV Cutoff

$$\frac{1}{r} \xrightarrow{\text{F.T.}} \frac{4\pi}{q^2}$$

$$\xrightarrow{\text{cutoff}} \frac{4\pi}{q^2}\,\mathrm{e}^{-q^2/2\Lambda^2} \qquad (\text{cutoff} \Rightarrow q < \Lambda)$$

$$\xrightarrow{\text{F.T.}} \frac{\mathrm{erf}(r\Lambda/\sqrt{2})}{r} \qquad \left(\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}dt\right)$$

$\Rightarrow$ Analytic at $r = 0$, and $1/r$ at large r.

Note: One of infinitely many choices.

Cutoff $\Rightarrow$ errors of $\mathcal{O}((p/\Lambda)^n)$. Remove errors order-by-order using local correction terms (mimic excluded $k > \Lambda$ physics):

$$
\begin{aligned}
V_{\rm eff}(\mathbf{r}) \quad = \quad & -\frac{\alpha}{r}\,\mathrm{erf}(r\Lambda/\sqrt{2}) \\
& + c\,\delta^3_\Lambda(\mathbf{r})/\Lambda^2 \qquad \leftarrow \text{removes } \mathcal{O}(p/\Lambda)^2 \text{ errors} \\
& + d_1\,\nabla^2\delta^3_\Lambda(\mathbf{r})/\Lambda^4 + d_2\,\nabla\cdot\delta^3_\Lambda(\mathbf{r})\nabla/\Lambda^4 \quad \leftarrow \text{removes } \mathcal{O}(p/\Lambda)^4 \\
& + \cdots \\
& + g\,\nabla^n\delta^3_\Lambda(\mathbf{r})/\Lambda^{n+2} \qquad \leftarrow \ \delta^3_\Lambda(\mathbf{r}) \equiv \frac{\mathrm{e}^{-(r\Lambda)^2/2}\,\Lambda^3}{(2\pi)^{3/2}} \\
& + \cdots
\end{aligned}
$$

G.P. Lepage, *How to renormalize the Schrödinger Equation*, arXiv:nucl-th/9706029.

Eg) Nonperturbative Positronium

Calculate O-Ps decay rate in two steps.

1) Define Hamiltonian ($\equiv$ QED) with finite cutoff Λ built in.

$$H^{(\Lambda)} = \frac{\mathbf{p}^2}{m} - \frac{\mathbf{p}^4}{4m^3} + V + iW$$

Decay → iW

Define V,W by matching $T = V + iW + V(E - H_0)^{-1}V + \cdots$ to QED Feynman diagrams for $e\bar{e}$ scattering, order-by-order in α and p/Λ.

R. Hill and G.P. Lepage, Phys Rev D62, 111301,2000; hep-ph/000327.

$$\langle \mathbf{l}|V|\mathbf{k}\rangle = -\frac{4\pi\alpha}{|\mathbf{k}-\mathbf{l}|^2}\,\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2} \qquad \leftarrow \mathcal{O}(\alpha^2 m)$$

$$\mathcal{O}(\alpha^4 m) \rightarrow \quad + \left[\frac{\pi\alpha}{m^2}\frac{(l^2-k^2)^2}{|\mathbf{k}-\mathbf{l}|^4} - \frac{4\pi\alpha}{2\Lambda^2} + \cdots\right]\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}$$

$$\mathcal{O}(\alpha^5 m) \rightarrow \quad + \frac{8\pi\alpha}{3\pi m^2}\,\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}\langle \mathbf{p}\cdot(H-E)\ln(\Lambda/(H-E))\cdot\mathbf{p}\rangle + \cdots$$

(Lamb Shift)

$$+ \frac{\alpha^2}{m^2}\left[\frac{14}{3}\ln(|\mathbf{l}-\mathbf{k}|/m) - \frac{74}{15} - \frac{16}{3}\ln 2 + D\right]\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}$$

Match $e\bar{e} \rightarrow e\bar{e}$

$$D = -\sqrt{\pi}\left[\frac{-121}{36}\frac{\Lambda}{m} - 9\frac{m}{\Lambda} + \frac{5}{3}\left(\frac{m}{\Lambda}\right)^2\right] - \frac{16}{3}\ln\frac{\Lambda}{m}$$

Decay piece:

$$\langle \mathbf{l}|W|\mathbf{k}\rangle = \left[A + B\,\frac{|\mathbf{k}-\mathbf{l}|^2}{m^2}\right]\,\mathrm{e}^{-|\mathbf{k}-\mathbf{l}|^2/2\Lambda^2}$$

$A^{(0)}(1+\alpha A^{(1)}+\cdots)$ from matching Born series for T with QED.

Not real QED behavior but equivalent for low-energy e's, provided A, B correct.

$$\Rightarrow \qquad W(\mathbf{r}) \propto \left[A - B\,\frac{\nabla^2}{m^2}\right]\,\delta^3_\Lambda(\mathbf{r})$$

$$
\begin{aligned}
A^{(1)} &= a_0 + \frac{1}{\sqrt{\pi}}\left[\frac{4}{3}\left(\frac{\Lambda}{m}\right) + 3\left(\frac{\Lambda}{m}\right)^{-1}\right] + \mathscr{O}\left(\frac{\lambda}{m}\right), \\
A^{(2)} &= b_0 - 2a_1 + \frac{1}{\sqrt{\pi}}\left[\frac{4}{3}\left(\frac{\Lambda}{m}\right) + 3\left(\frac{\Lambda}{m}\right)^{-1}\right] a_0 + \frac{1}{3}\ln\frac{\Lambda}{m} \\
&\quad + \frac{1}{\pi\sqrt{\pi}}\left\{\left[-\frac{44\sqrt{6}}{81}\left(\gamma - \ln\frac{2\Lambda^2}{3m^2} - 2\right)\right]\left(\frac{\Lambda}{m}\right)^3\right. \\
&\quad \left. + \left[\frac{7}{3}\ln\frac{\Lambda}{m} + \frac{56\sqrt{6}}{27}\left(\gamma - \ln\frac{2\Lambda^2}{3m^2} - \frac{2}{7}\right) - \frac{37}{15} + \frac{1}{3}\ln 2 - \frac{7}{6}\gamma\right]\left(\frac{\Lambda}{m}\right)\right\} \\
&\quad + \left(\frac{83}{24\pi} - \frac{11\sqrt{3}}{12\pi} + \frac{11}{48}\right)\left(\frac{\Lambda}{m}\right)^2 \\
&\quad + \left(\frac{25}{2\pi} - \frac{4\sqrt{3}}{3\pi} + \frac{17}{18} - \frac{5}{6}\ln 2 - \frac{1}{3}\gamma + \frac{2}{\sqrt{\pi}}\kappa\right) \\
&\quad + \left(\frac{49}{6\pi} - \frac{3\sqrt{3}}{2\pi} - \frac{1}{4}\right)\left(\frac{\Lambda}{m}\right)^{-2} + \mathscr{O}\left(\frac{\lambda}{m}\right).
\end{aligned}
$$

- Given V, W can solve theory even if completely ignorant of QED, renormalization, Effective Field Theory.
    - $\Lambda = m$ (or $m/2$ or $2m$)
      $\Rightarrow$No divergences (V analytic at $r = 0$).
    - Renormalization built in, automatic.
    - High-order QED/relativity built in, automatic.

- Can solve nonperturbatively in V (eg, numerically).
    - $\Rightarrow$ Don't need Rayleigh-Schrödinger perturbation theory.
    - $\Rightarrow$ Move trivially to many-e analysis (He ...).

- No $\ln\alpha$s in V $\Rightarrow$ perturbation theory for V is more convergent than for E_n.
    - $\Rightarrow$ Compute V in (QED) perturbation theory; solve nonperturbatively in V.
    - $\Rightarrow$ Schrödinger equation generates $\ln\alpha$s and resums them automatically.

2) Solve theory:

a) Diagonalize $H^{((\Lambda))}$ (eg, on finite basis set of Gaussians).
$\Rightarrow E_n$s and $|\psi_n\rangle$s.

b) Compute

$$\begin{aligned}\Gamma_n =& -2\langle\psi_n|W|\psi_n\rangle \\ =& \overline{A}\,\langle\psi_n|\mathrm{e}^{-r^2\Lambda^2/2}|\psi_n\rangle - \overline{B}\,\langle\psi_n|\nabla^2\mathrm{e}^{-r^2\Lambda^2/2}|\psi_n\rangle\end{aligned}$$

c) Publish numerical values obtained for Γ_ns.
$\Rightarrow (\Gamma_{1S} = 7.039967(10)\,\mu s^{-1}.)$

Eg) Numerical Analysis Bonus

- $-\alpha/r \to \infty$ as $r \to 0$.
  $\Rightarrow$ Cusp in $\psi(r)$ at $r = 0$.
  $\Rightarrow$ Expansions (eg, on basis set) converge more slowly.

- Gaussian cutoff in $V_{\text{eff}}(r)$ $\Rightarrow$ analytic at $r = 0$.
  $\Rightarrow$ $\psi(r = 0)$ analytic.
  $\Rightarrow$ Expansions converge exponentially faster.

Lamb Shift in H, He

- $N = \#$ basis functions $< \infty$
  $\Rightarrow$ effective cutoff Λ_N.

- Lamb shift

$\psi^\dagger$  $\psi \quad \sim \quad \psi^\dagger \delta^3(r)\psi$

$$\sim \left| \int^{\Lambda_N} d^3k \psi(k) \right|^2 \qquad \text{where } \psi(k) \to \frac{1}{k^4} \text{ for } k \text{ large}$$

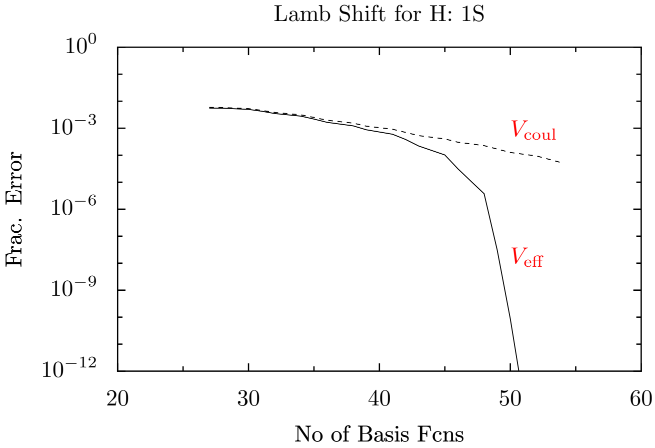
$$\Rightarrow \boxed{\text{error} \propto 1/\Lambda_N^2}$$

- $\psi_{\text{eff}} \sim \mathrm{e}^{-k^2/2\Lambda^2}$ for k large

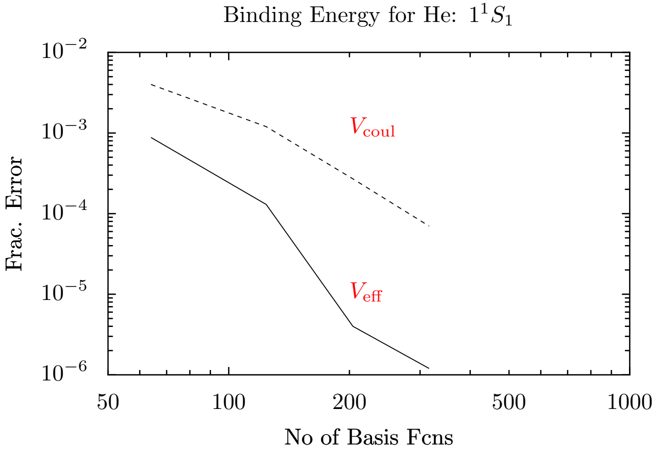
$$\Rightarrow \boxed{\text{error} \propto \mathrm{e}^{-\Lambda_N^2/\Lambda^2}}$$

$\Rightarrow$ Errors exponentially suppressed.

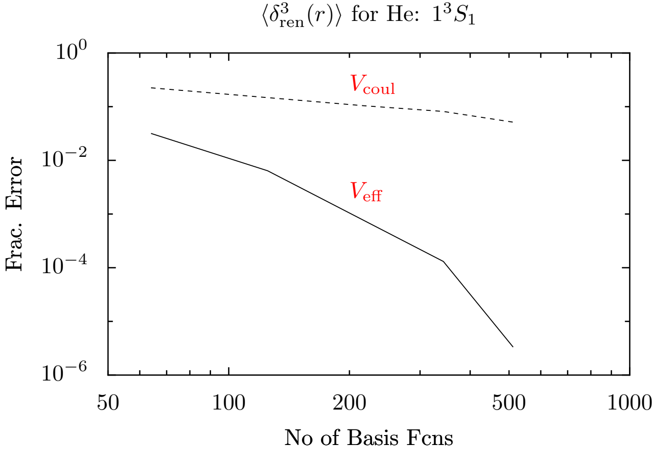
H(1S): Lamb Shift



He(1^1S): Binding Energy



He(1^1S): QED/Relativistic Corrections



Example: μ Decay in Materials

Muon Decay

- Measure μ lifetime $\Gamma_\mu \Rightarrow$ μs stopped in matter.
- Significant number bind to form muonium (μe).
- Effect of binding? Effect of interactions with matter?

A. Czarnecki, G.P. Lepage, and W.J. Marciano, Phys.Rev. D61 (2000) 073001

Simple estimates

- Binding energy reduces phase space
  $\Rightarrow$ correction of $\mathcal{O}(\alpha^2 m_e/m_\mu)$.
  $\Rightarrow$ Important!

- Final state interactions
  $\Rightarrow$ $\mathcal{O}(\langle V\rangle/m_\mu) = \mathcal{O}(\alpha^2 m_e/m_\mu)$.
  $\Rightarrow$ Important!

- Except these cancel.
  $\Rightarrow$ Dominant contribution is $\mathcal{O}(\alpha^2 m_e^2/m_\mu^2)$.
  $\Rightarrow$ Unimportant!

NRQED for μ

- Typical $p_\mu \sim \alpha m_e$ for stopped μ in matter.
  $\Rightarrow$ Use NRQED to describe it ($\Rightarrow \Lambda \sim m_\mu$).

- $\mu \to e\nu\overline{\nu} \to \mu \;\Rightarrow$ distances $\sim 1/m_\mu$
  $\Rightarrow$ Include in NRQED as local (non-unitary) terms.

$$
\begin{aligned}
\mathcal{L}_{\text{nrqed}} =& \psi_\mu^\dagger \left\{ iD_t + \frac{\mathbf{D}^2}{2m_\mu} + \cdots \right. && \leftarrow \text{gauge covariant } Ds \\
& + \frac{i\Gamma_\mu}{2} \left(1 + c_1 \frac{\mathbf{D}^2}{2m_\mu^2} + c_2 \frac{3\mathbf{D}^4}{8m_\mu^4} + \cdots \right. && \leftarrow \mu \to e\nu\overline{\nu} \\
& \quad + d_1 \frac{\psi_e^\dagger \psi_e}{m_\mu^3} + d_2 \frac{\sigma \cdot \psi_e^\dagger \sigma \psi_e}{m_\mu^3} + \cdots && \leftarrow \mu e \to \nu\overline{\nu} \\
& \quad \left. \left. + f_1 \frac{e\sigma \cdot \mathbf{B}}{m_\mu^2} + f_2 \frac{e\nabla \cdot \mathbf{E}}{m_\mu^3} + \cdots \right) \right\} \psi_\mu .
\end{aligned}
$$

Note:

- No term $i\Gamma_\mu \psi_\mu^\dagger A^0 \psi_\mu / m_\mu$ (not gauge invariant)
  $\Rightarrow$ No $\alpha^2 m_e / m_\mu$. (Compare $J = 0$ "photon.")

Q. Why no $i\Gamma_\mu \psi_\mu^\dagger i D_t \psi_\mu / m_\mu$?

A. In $\mathscr{L}_{\text{nrqed}}$

$$\psi_\mu^\dagger i D_t \psi_\mu \equiv -\psi_\mu^\dagger \frac{\mathbf{D}^2}{2m} \psi_\mu + \cdots$$

because "equations of motion" are

$$i D_t \psi_\mu = -\frac{\mathbf{D}^2}{2m} \psi_\mu + \cdots .$$

N.B. "Equivalent" not "equal." Prove using field transformation in path integral (change integration variables) $\Rightarrow$ "redundant operators".

- Muon decay from

$$\delta \mathscr{L}_{\text{decay}} \equiv \frac{i\Gamma_\mu}{2}\, \psi_\mu^\dagger \psi_\mu + \mathscr{O}((\alpha m_e/m_\mu)^2\, \Gamma_\mu)$$

Not renormalized; free-μ decay rate in rest frame..

Conserved current; μ number operator.

⇒ Decay rate for *any* state $|\mu\phi\rangle$

$$\langle\mu\phi|\delta\mathscr{L}_{\mathrm{decay}}|\mu\phi\rangle = \frac{i\Gamma_\mu}{2} + \mathscr{O}((\alpha m_e/m_\mu)^2\,\Gamma_\mu)$$

- Here ϕ is e in muonium, conduction band in metal . . . or any other single/multi-electron state in ordinary matter.

⇒ Decay rate of μ unaffected by all ordinary materials at ppb level.

Conclusions

- Uses effective non-relativistic theory:
  - Non-perturbative (eg, numerical) treatment of non-relativistic expansion of relativistic QED.
  - Improved numerical analysis by replacing singular potentials with equivalent UV-regulated potentials.
  - Implications of gauge and other symmetries.

- References:

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