Toward a complete decomposition of nucleon spin

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1. Introduction

current status and homework of nucleon spin problem

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma^Q + \Delta g + \text{Orbital Angular Momenta ?}$$

(1) $\Delta\Sigma^Q$: fairly precisely determined ! $(\sim 1/3)$

(2) Δg : likely to be small , but large uncertainties

⇓

What carries the remaining 2 / 3 of nucleon spin ?

quark OAM ? gluon spin ? gluon OAM ?

⇓

To answer this question unambiguously, we cannot avoid to clarify

- What is a **precise (QCD) definition** of each term of the decomposition ?
- How can we extract individual term by means of **direct measurements** ?

especially controversy are **orbital angular momenta !**

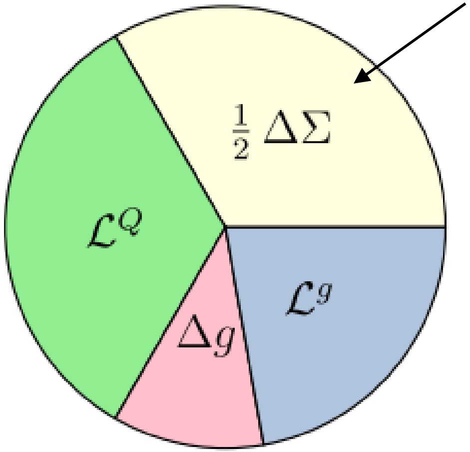
2. Nucleon spin decomposition problem

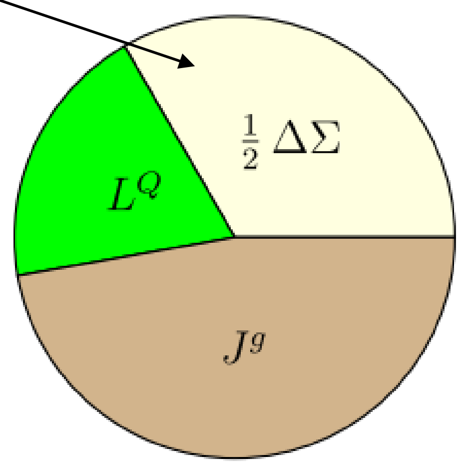
Two popular decompositions of the nucleon spin

Jaffe-Manohar

Ji

common





$$
\begin{aligned}
\boldsymbol{J}_{QCD} &= \int \psi^\dagger \frac{1}{2}\boldsymbol{\Sigma}\,\psi\, d^3x \\
&+ \int \psi^\dagger\, \boldsymbol{x} \times \frac{1}{i}\nabla\,\psi\, d^3x \\
&+ \int \boldsymbol{E}^a \times \boldsymbol{A}^a\, d^3x \\
&+ \int E^{ai}\, \boldsymbol{x} \times \nabla\, A^{ai}\, d^3x
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{J}_{QCD} &= \int \psi^\dagger \frac{1}{2}\boldsymbol{\Sigma}\,\psi\, d^3x \\
&+ \int \psi^\dagger\, \boldsymbol{x} \times \frac{1}{i}\boldsymbol{D}\,\psi\, d^3x \\
&+ \int \boldsymbol{x} \times (\boldsymbol{E}^a \times \boldsymbol{B}^a)\, d^3x
\end{aligned}
$$

Each term is not separately gauge-invariant !

No further decomposition of J^g !

First, pay attention to the difference of quark OAM parts

$$\boldsymbol{L}_Q(\mathrm{JM}) \sim \psi^\dagger \boldsymbol{x} \times \boldsymbol{p}\, \psi \qquad \boldsymbol{L}_Q(\mathrm{Ji}) \sim \psi^\dagger \boldsymbol{x} \times (\boldsymbol{p} - g\, \boldsymbol{A})\, \psi$$

canonical OAM — **dynamical OAM**

($\boldsymbol{p}$: **canonical momentum**) — ($\boldsymbol{p} - g\,\boldsymbol{A}$: **dynamical momentum**)

not gauge invariant ! — gauge invariant !

gauge principle

observables must be gauge-invariant !

- Observability of canonical OAM has been questioned.
- On the other hand, it has been long known that the dynamical quark OAM corresponds to observables through GPDs. (X. Ji, 1997)

Chen-Wang-Goldman proposed a new gauge-invariant complete decomposition

X.-S. Chen et al., Phys. Rev. Lett. 103, 062001 (2009) ; 100, 232002 (2008).

basic idea

$$A^{\mu} = A^{\mu}_{phys} + A^{\mu}_{pure}$$

which is a kind of generalization of the decomposition of photon field into the transverse and longitudinal components in QED :

$$\boldsymbol{A}_{phys} \Leftrightarrow \boldsymbol{A}_{\perp}, \qquad \boldsymbol{A}_{pure} \Leftrightarrow \boldsymbol{A}_{||}$$

Chen et al.'s decomposition (in "generalized Coulomb gauge")

$$\begin{aligned} \boldsymbol{J}_{QCD} &= \int \psi^{\dagger} \frac{1}{2} \boldsymbol{\Sigma} \psi \, d^3x + \int \psi^{\dagger} \boldsymbol{x} \times (\boldsymbol{p} - g \, \boldsymbol{A}_{pure}) \, \psi \, d^3x \\ &+ \int \boldsymbol{E}^a \times \boldsymbol{A}^a_{phys} \, d^3x + \int E^{aj} \, (\boldsymbol{x} \times \nabla) \, \boldsymbol{A}^{aj}_{phys} \, d^3x \\ &= \boldsymbol{S}'^q + \boldsymbol{L}'^q + \boldsymbol{S}'^g + \boldsymbol{L}'^g \end{aligned}$$

- Each term is separately gauge-invariant ! (under residual gauge transformation)
- It reduces to gauge-variant **Jaffe-Manohar decomposition** in a particular gauge !

$$\boldsymbol{A}_{pure} = 0, \qquad \boldsymbol{A} = \boldsymbol{A}_{phys}$$

Chen et al.'s papers arose **quite a controversy** on the **feasibility** of complete decomposition of nucleon spin.

- X. Ji, Phys. Rev. Lett. 104 (2010) 039101 : 106 (2011) 259101.
- S. C. Tiwari, arXiv:0807.0699.
- X. S. Chen et al., arXiv:0807.3083 ; arXiv:0812.4336 ; arXiv:0911.0248.
- Y. M. Cho et al., arXiv:1010.4336 ; arXiv:1102.1130.
- X. S. Chen et al., Phys.Rev. D83 (2011) 071901.
- E. Leader, Phys. Rev. D83 (2011) 096012.
- Y. Hatta, Phys. Rev. D84 (2011) 041701R.
- P .M. Zhang and D. G. Pak, arXiv:1110.6516
- H.-W. Lin and K.-F. Liu, arXiv:1111.0678
- Y. Hatta, arXiv : 1111.3547 [hep-ph]

  ……………………………

We believe that we have arrived at **one** (**satisfactory)** **solution** to the problem, step by step, through the following three papers :

(i) M. W., Phys. Rev. D81 (2010) 114010.
(ii) M. W., Phys. Rev. D83 (2011) 014012.
(iii) M. W., Phys. Rev. D84 (2011) 037501.

In the paper (i), we have shown that the way of gauge-invariant decomposition of nucleon spin is not necessarily unique, and proposed another G.I. decomposition :

$$\boldsymbol{J}_{QCD} = \boldsymbol{S}^q + \boldsymbol{L}^q + \boldsymbol{S}^g + \boldsymbol{L}^g$$

where

$$\boldsymbol{S}^q = \int \psi^\dagger \frac{1}{2} \boldsymbol{\Sigma} \psi \, d^3x$$

$$\boldsymbol{L}^q = \int \psi \, \boldsymbol{x} \times (\boldsymbol{p} - g \, \boldsymbol{A}) \, \psi \, d^3x$$

$$\boldsymbol{S}^g = \int \boldsymbol{E}^a \times \boldsymbol{A}^a_{phys} \, d^3x$$

$$\boldsymbol{L}^g = \int E^{aj} \, (\boldsymbol{x} \times \nabla) \, A^{aj}_{phys} \, d^3x + \int \rho^a \, (\boldsymbol{x} \times \boldsymbol{A}^a_{phys}) \, d^3x$$

"potential angular momentum"

- The quark part of our decomposition is common with the **Ji decomposition**.
- The quark and gluon intrinsic spin parts are common with the **Chen decomp**.
- A crucial difference with the Chen decomp. appears in the orbital parts

$$\boldsymbol{L}^q + \boldsymbol{L}^g = \boldsymbol{L}'^q + \boldsymbol{L}'^g$$

$$\boldsymbol{L}^g - \boldsymbol{L}'^g = - (\boldsymbol{L}^q - \boldsymbol{L}'^q) = \int \rho^a \, (\boldsymbol{x} \times \boldsymbol{A}^a_{phys}) \, d^3x$$

The QED correspondent of this term is the orbital angular momentum carried by electromagnetic potential, appearing in the famous Feynman paradox.

An **arbitrariness** of the spin decomposition arises, because this potential angular momentum term is solely gauge-invariant !

$$\int \rho^a \, \boldsymbol{x} \times \boldsymbol{A}^a_{phys} \, d^3x \;=\; g \int \psi^\dagger(x) \, \boldsymbol{x} \times \boldsymbol{A}_{phys}(x) \, \psi(x) \, d^3x$$

$$\rightarrow \; \text{gauge invariant}$$

since

$$\boldsymbol{A}_{phys}(x) \;\rightarrow\; U^\dagger(x) \, \boldsymbol{A}_{phys}(x) \, U(x)$$

$$\psi^\dagger(x) \;\rightarrow\; \psi^\dagger(x) \, U^\dagger(x), \qquad \psi(x) \;\rightarrow\; U(x) \, \psi(x)$$

This means that one has a freedom to shift this potential OAM term to the quark OAM part in **our decomposition**, which leads to the **Chen decomposition**.

$$\boldsymbol{L}^q \, (\text{Ours}) \; + \; \text{potential angular momentum}$$

$$= \int \psi^\dagger \, \boldsymbol{x} \times (\boldsymbol{p} - g \, \boldsymbol{A}) \, \psi \, d^3x \; + \; g \int \psi^\dagger \, \boldsymbol{x} \times \boldsymbol{A}_{phys} \, \psi \, d^3x$$

$$= \int \psi^\dagger \, \boldsymbol{x} \times (\boldsymbol{p} - g \, \boldsymbol{A}_{pure}) \, \psi \, d^3x \quad = \quad \boldsymbol{L}'^q \, (\text{Chen})$$

Next, **in the paper (ii),** we found that we can make a covariant extension of the gauge-invariant decomposition of nucleon spin.

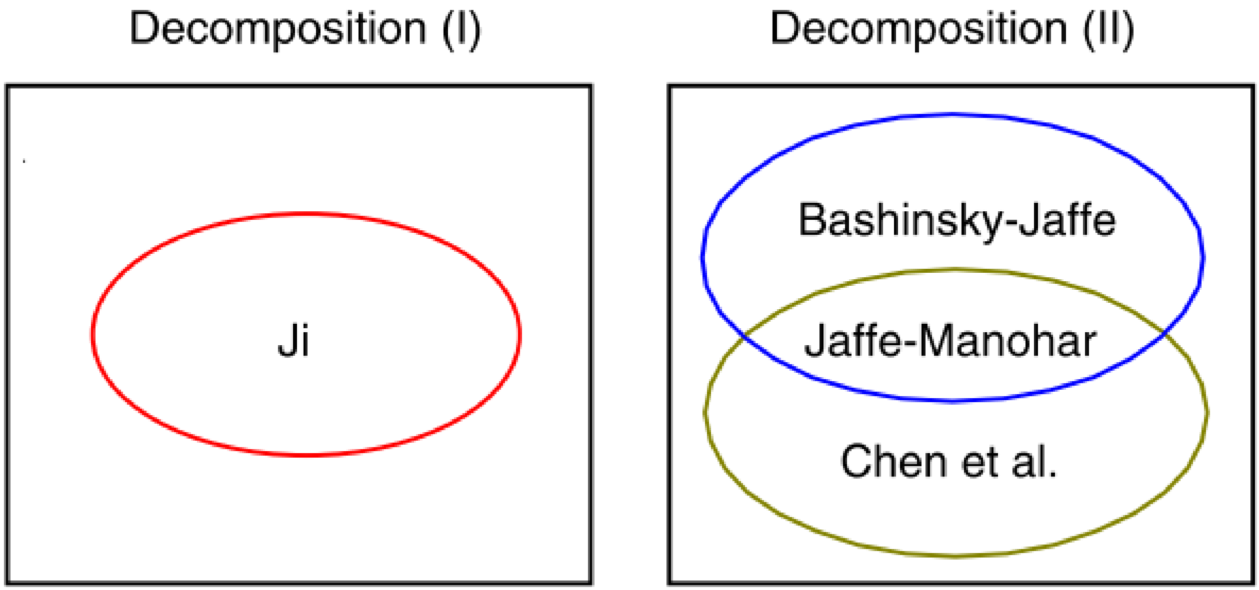
covariant generalization of the decomposition has several advantages.

(1) It is useful to find relations to high-energy DIS observables.

(2) It is essential to prove Lorentz frame-independence of the decomposition.

(3) It generalizes and unifies the nucleon spin decompositions in the market.

Basically, we find two physically nonequivalent decompositions (I) and (II) .



Our starting point is the decomposition of gluon field, similar to Chen et al.

$$A^\mu = A^\mu_{phys} + A^\mu_{pure}$$

Different from their treatment, we impose the following general conditions alone :

$$F^{\mu\nu}_{pure} \equiv \partial^\mu A^\nu_{pure} - \partial^\nu A^\mu_{pure} - i\, g\, [A^\mu_{pure}, A^\nu_{pure}] = 0$$

and

$$A^\mu_{phys}(x) \rightarrow U(x)\, A^\mu_{phys}(x)\, U^{-1}(x)$$

$$A^\mu_{pure}(x) \rightarrow U(x) \left(A^\mu_{pure}(x) - \frac{i}{g}\, \partial^\mu \right) U^{-1}(x)$$

- Actually, these conditions are not enough to fix gauge uniquely !
- However, the **point of our analysis** is that we can postpone a concrete gauge-fixing until later stage, while accomplishing a gauge-invariant decomposition of $M^{\mu\nu\lambda}$ based on the **above general conditions** alone.

Again, we find the way of gauge-invariant decomposition is not unique.

decomposition (I) & decomposition (II)

Gauge-invariant decomposition (II) : covariant generalization of Chen et al's

$$M^{\mu\nu\lambda}_{QCD} = M'^{\mu\nu\lambda}_{q-spin} + M'^{\mu\nu\lambda}_{q-OAM} + M'^{\mu\nu\lambda}_{g-spin} + M'^{\mu\nu\lambda}_{g-OAM}$$
$$+ \text{ boost } + \text{ total divergence}$$

with

$$M'^{\mu\nu\lambda}_{q-spin} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\,\bar{\psi}\,\gamma_\sigma\,\gamma_5\,\psi$$
$$M'^{\mu\nu\lambda}_{q-OAM} = \bar{\psi}\,\gamma^\mu\,(\,x^\nu\,i\,D^\lambda_{pure} - x^\lambda\,i\,D^\nu_{pure}\,)\,\psi$$

$$M'^{\mu\nu\lambda}_{g-spin} = 2\,\mathrm{Tr}\,\{\,F^{\mu\lambda}\,A^\nu_{phys} - F^{\mu\nu}\,A^\lambda_{phys}\,\}$$
$$M'^{\mu\nu\lambda}_{q-OAM} = 2\,\mathrm{Tr}\,\{\,F^{\mu\alpha}\,(\,x^\nu\,D^\lambda_{pure} - x^\lambda\,D^\nu_{pure}\,)\,A^{phys}_\alpha\,\}$$

This decomposition reduces to any ones of Bashinsky-Jaffe, of Chen et al., and of Jaffe-Manohar, after an appropriate gauge-fixing in a suitable Lorentz frame, which reveals that **these 3 decompositions are all gauge-equivalent** !

These 3 are **physically equivalent** decompositions !

Gauge-invariant decomposition (I) :

The difference with the decomposition (II) resides in **OAM parts** !

$$M^{\mu\nu\lambda} = M^{\mu\nu\lambda}_{q-spin} + M^{\mu\nu\lambda}_{q-OAM} + M^{\mu\nu\lambda}_{g-spin} + M^{\mu\nu\lambda}_{g-OAM}$$
$$+ \ \text{boost} \ + \ \text{total divergence}$$

with

full covariant derivative

$$M^{\mu\nu\lambda}_{q-spin} = M'^{\mu\nu\lambda}$$
$$\boxed{M^{\mu\nu\lambda}_{q-OAM}} = \bar{\psi}\,\gamma^{\mu}\,(\,x^{\nu}\,i\,D^{\lambda} - x^{\lambda}\,i\,D^{\nu}\,)\,\psi \neq \boxed{M'^{\mu\nu\lambda}_{q-OAM}}$$

$$M^{\mu\nu\lambda}_{g-spin} = M'^{\mu\nu\lambda}_{g-spin}$$
$$\boxed{M^{\mu\nu\lambda}_{g-OAM}} = \boxed{M'^{\mu\nu\lambda}_{g-OAM}} + 2\,\mathrm{Tr}\,[\,(\,D_{\alpha}\,F^{\alpha\mu}\,)\,(\,x^{\nu}\,A^{\lambda}_{phys} - x^{\lambda}\,A^{\nu}_{phys}\,)\,]$$

⇧

covariant generalization of potential OAM !

It was sometimes criticized that there are so many decompositions of nucleon spin.

$$\begin{aligned}\frac{1}{2} &= \frac{1}{2}\Sigma + L_Q + \Delta g + L_g \\ &= \frac{1}{2}\Sigma' + L'_Q + \Delta g' + L'_g \\ &= \frac{1}{2}\Sigma'' + L''_Q + \Delta g'' + L''_g \\ &\vdots\end{aligned}$$

However, this is not true any more. One should recognize now that there are only two physically nonequivalent decompositions !

Decomposition (I)	Decomposition (II)
extension of **Ji's decomp.** including gluon part	nontrivial gauge-invariant extension of **Jaffe-Manohar's decomp.**
dynamical OAMs	**"canonical" OAMs**

Since both decompositions are gauge-invariant, there is a possibility that they both correspond to observables !

A clear relation with observables was first obtained for the **decomposition (I)**.

The **keys** are the following identities, which hold in our decomposition (I) :

quark :
$$x^\nu T_q^{\mu\lambda} - x^\lambda T_q^{\mu\nu} = M_{q-spin}^{\mu\nu\lambda} + M_{q-OAM}^{\mu\nu\lambda} + \text{total divergence}$$

and

gluon :
$$x^\nu T_g^{\mu\lambda} - x^\lambda T_g^{\mu\nu} - \text{boost} = M_{g-spin}^{\mu\nu\lambda} + M_{g-OAM}^{\mu\nu\lambda} + \text{total divergence}$$

with

$$T_{QCD}^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu} \quad : \quad \text{Belinfante symmetric form}$$

Evaluating the nucleon forward M.E. of the $(\mu\nu\lambda) = (012)$ component (in rest frame) or $(\mu\nu\lambda) = (+12)$ component (in IMF) of the above equalities, we can prove the following crucial relations :

For the **quark part**

$$
\begin{aligned}
L_q &= \frac{1}{2}\int_{-1}^{1} x\,[\,H^q(x,0,0) + E^q(x,0,0)\,]\,dx - \frac{1}{2}\int_{-1}^{1} \Delta q(x)\,dx \\
&= J_q - \frac{1}{2}\Delta q \\
&= \langle p\uparrow |\, M^{012}_{q-OAM} \,| p\uparrow\rangle
\end{aligned}
$$

with

$$
M^{012}_{q-OAM} = \bar{\psi}\left(\boldsymbol{x}\times\frac{1}{i}\boldsymbol{D}\right)^3\psi \neq \begin{cases} \bar{\psi}\left(\boldsymbol{x}\times\frac{1}{i}\nabla\right)^3\psi \\ \bar{\psi}\left(\boldsymbol{x}\times\frac{1}{i}\boldsymbol{D}_{pure}\right)^3\psi \end{cases}
$$

$$\Downarrow$$

In other words

the quark OAM extracted from the combined analysis of GPD and polarized PDF is "dynamical OAM" (or "mechanical OAM") not "canonical OAM" !

This conclusion is nothing different from Ji's claim !

For the **gluon part** (this is totally new)

$$
\begin{aligned}
L_g &= \frac{1}{2}\int_{-1}^{1} x\,[\,H^g(x,0,0) \;+\; E^g(x,0,0)\,]\,dx \;-\; \int_{-1}^{1} \Delta g(x)\,dx \\
&= J_g \;-\; \Delta g \\
&= \langle p\uparrow \,|\, M^{012}_{g-OAM} \,|\, p\uparrow\rangle
\end{aligned}
$$

with

$$
\begin{aligned}
M^{012}_{g-OAM} \;&=\; 2\,\mathrm{Tr}\,[\,E^j\,(\boldsymbol{x}\times \boldsymbol{D}_{pure})^3\,A^{phys}_j\,] \qquad : \quad \text{canonical OAM} \\
&+\; 2\,\mathrm{Tr}\,[\,\rho\,(\boldsymbol{x}\times \boldsymbol{A}_{phys})^3\,] \qquad : \quad \text{potential OAM term}
\end{aligned}
$$

The gluon OAM extracted from the combined analysis of GPD and polarized PDF contains "potential OAM" term, in addition to "canonical OAM" !

It is natural to call the whole part the gluon "**dynamical OAM**" .

We want to make several **important remarks** on our decomposition.

♣ Our decomposition is Lorentz-frame independent !

This should be clear from the fact that the (G)PDFs appearing in the r.h.s. of our sum rules are manifestly Lorentz-invariant quantities !

Goldman argued that the nucleon spin decomposition is frame-dependent !

- T. Goldman, arXiv:1110.2533.

This is generally true, but our interest here is the longitudinal spin sum rule.

♣ The longitudinal spin decomposition is certainly frame-independent !

Leader recently proposed a sum rule for transverse angular momentum.

- E. Leader, arXiv:1109.1230.

$$\langle J_T(quark) \rangle \;=\; \frac{1}{2\,M} \left[P_0 \int_{-1}^{1} x\, E^q(x,0,0)\, dx \;+\; M \int_{-1}^{1} x\, H^q(x,0,0)\, dx \right]$$

It is clear that this sum rule **does not** have a frame-independent meaning !

Underlying reason why the longitudinal spin sum rule is Lorentz-frame independent seems clear.

The OAM component along the longitudinal direction comes from the motion in the perpendicular plane to this axis, and such transverse motion is not affected by the Lorentz boost along this axis.

$$\begin{aligned} M^{+12}_{q-OAM} \;&=\; \frac{1}{2}\,\bar{\psi}\,\gamma^{+}\,(\,x^{1}\,i\,\partial^{2} \;+\; x^{2}\,i\,\partial^{1}\,)\,\psi \\ &+\; g\,\bar{\psi}\,\gamma^{+}\,(\,x^{1}\,A^{2}_{\perp} \;-\; x^{2}\,A^{1}_{\perp}\,)\,\psi \end{aligned}$$

$$\begin{cases} x_0 \;\rightarrow\; x_0' \;=\; \gamma\left(x_0 \;-\; \frac{v}{c}\,x_3\right) \\ x_1 \;\rightarrow\; x_1' \;=\; x_1 \\ x_2 \;\rightarrow\; x_2' \;=\; x_2 \\ x_3 \;\rightarrow\; x_3' \;=\; \gamma\left(x_3 \;-\; \frac{v}{c}\,x_0\right) \end{cases} \qquad \begin{cases} A_0 \;\rightarrow\; A_0' \;=\; \gamma\left(A_0 \;-\; \frac{v}{c}\,A_{||}\right) \\ A_1 \;\rightarrow\; A_1' \;=\; A_1 \\ A_2 \;\rightarrow\; A_2' \;=\; A_2 \\ A_{||} \;\rightarrow\; A_{||}' \;=\; \gamma\left(A_{||} \;-\; \frac{v}{c}\,A_0\right) \end{cases}$$

with

$$\gamma \;=\; \frac{1}{\sqrt{1-(v\,/\,c)^{2}}}$$

Although our decomposition seems satisfactory in many respects, one subtle question remained. It is a role of **quantum-loop effects.**

[remaining important question]

Is ΔG gauge-invariant even at quantum level ? ⇨ **delicate question**

In fact, it was often claimed that ΔG has its meaning only in the LC gauge and in the infinite-momentum frame (for instance, by X. Ji and P. Hoodbhoy).

More specifically, in

- P. Hoodbhoy, X. Ji, and W. Lu, Phys. Rev. D59 (1999) 074010.

they claim that ΔG evolves differently in the LC gauge and the Feynman gauge.

However, the gluon spin operator used in their Feynman gauge calculation is

$$M^{+12}_{g-spin} = 2\,\mathrm{Tr}\,[\,F^{+1}\,A^2 - F^{+2}\,A^1\,]$$

which is delicately different from our gauge-invariant gluon spin operator

$$M^{+12}_{g-spin} = 2\,\mathrm{Tr}\,[\,F^{+1}\,A^2_{phys} - F^{+2}\,A^1_{phys}\,]$$

The problem is how to introduce this difference in the Feynman rule of evaluating 1-loop anomalous dimension of the quark and gluon spin operator.

This problem was attacked and solved in our 3rd paper

(iii) M. W., Phys. Rev. D84 (2011) 037501.

♣ We find that the calculation in the Feynman gauge (as well as in any covariant gauge including the Landau gauge) reproduces the answer obtained in the LC gauge, which is also the answer obtained by the Altarelli-Parisi method.

♣ So far, a direct check of the answer of Altarelli-Pasiri method for the evolution equation of ΔG within the operator-product-expansion (OPE) framework was limited to the LC gauge calculation, just because it was believed that there is no gauge-invariant definition of gluon spin in the OPE framework.

♣ This is the reason why the question of gauge-invariance of ΔG has been left in unclear status for a long time !

♣ Now we can definitely say that the gauge-invariant gluon spin operator appearing in our nucleon spin decomposition (although nonlocal) certainly provides us with a satisfactory operator definition of gluon spin operator (with gauge-invariance), which has been searched for nearly 40 years.

[**A natural question**] **Why can we observe "dynamical OAM"** ?

- motion of a charged particle in static electric and magnetic fields

(See the textbook of J.J. Sakurai, for instance.)

$$\boldsymbol{E} \ = \ - \nabla \phi, \quad \boldsymbol{B} \ = \ \nabla \times \boldsymbol{A}$$

Hamiltonian

$$H \ = \ \frac{1}{2\,m} \, (\boldsymbol{p} - e\,\boldsymbol{A})^2 \ + \ e\,\phi$$

Heisenberg equation

$$\frac{d x_i}{dt} \ = \ \frac{1}{i\,\hbar} [\, x_i, H \,] \ = \ \frac{p_i - e\,A_i}{m}$$

One finds

$$\boldsymbol{\Pi} \ \stackrel{def}{\equiv} \ m \, \frac{d \boldsymbol{x}}{dt} \ = \ \boldsymbol{p} - e\,\boldsymbol{A} \ \neq \ \boldsymbol{p}$$

$\boldsymbol{\Pi}$: mechanical (or dynamical) momentum

$\boldsymbol{p}$: canoninal momentum

Equation of motion

$$m \frac{d^2 \boldsymbol{x}}{dt^2} = \frac{d\boldsymbol{\Pi}}{dt} = e \left[\boldsymbol{E} + \frac{1}{2} \left(\frac{d\boldsymbol{x}}{dt} \times \boldsymbol{B} - \boldsymbol{B} \times \frac{d\boldsymbol{x}}{dt} \right) \right]$$

- ♣ What appears in Newton-Lorentz equation is dynamical momentum $\boldsymbol{\Pi}$ not canonical one $\boldsymbol{p}$.

- ♣ "Equivalence principle" of Einstein dictates that the "flow of inertia mass" can in principle be detected by using gravitational force as a probe.

- ♣ Naturally, the gravitational force is too weak to be used as a probe of mass flow in microscopic system.

- ♣ However, remember the fact that the 2nd moments of unpolarized GPDs are also called the gravito-electric and gravito-magnetic form factors.

- ♣ The fact that the dynamical OAM as well as dynamical linear momentum can be extracted from GPD analyses is therefore not a mere accident !

After establishing satisfactory natures of the decomposition (I), now we come to discussing another decomposition (II).

According to Chen-Wang-Goldman, the greatest advantage of the decomposition (II) is that their OAM operator $\boldsymbol{L}' \equiv -i\,\boldsymbol{x} \times (\nabla - i\,g\,\boldsymbol{A}_{pure})$ satisfies

$$\boldsymbol{L}' \times \boldsymbol{L}' = i\,\boldsymbol{L}' \quad \text{due to} \quad \nabla \times \boldsymbol{A}_{pure} = 0$$

It was claimed that this is crucial for its physical interpretation as an OAM.

However, this is not necessarily true, as discussed in

"**Commutation rules and eigenvalues of spin and orbital angular momentum of radiation fields**", S.J. Van Enk, G. Nienhuis, J. of Modern Optics, 41 (1994)963.

(I will discuss it later, if time allows.)

Then, the claimed superiority of decomposition (II) over (I) is not actually present.

Nevertheless, since the decomposition (II) is also gauge-invariant, there still remains a possibility that it can be related to observables.

Recently, Hatta made important step toward this direction.

- Y. Hatta, arXiv : 1111.3547.

based on his formal decomposition formula

- Y. Hatta, P. R. D84, 041701 (R) (2011).

$$A^\mu(x) \;=\; A^\mu_{phys}(x) \;+\; A^\mu_{pure}(x)$$

$$A^\mu_{phys}(x) \;=\; -\int dy^- \, \mathcal{K}(y-x)\, \mathcal{W}^-_{xy}\, F^{+\mu}(y^-, \boldsymbol{x})\, \mathcal{W}^-_{yx}$$

$$A^\mu_{pure}(x) \;=\; -\frac{i}{g}\, \mathcal{W}^-_{x,\pm\infty}\, \mathcal{W}_{\pm\infty} \left(\mathcal{W}^-_{x,\pm\infty}\, \mathcal{W}_{\pm\infty} \right)^\dagger$$

where

$$\mathcal{W}^-_{xy} \;\equiv\; \mathcal{P} \exp \left(-i\, g \int_{y^-}^{x^-} A^+(y'^-, \boldsymbol{x})\, dy'^- \right)$$

$\mathcal{W}_{\pm\infty}$: Wilson line in the spatial ($\boldsymbol{x}$) direction at $x^- = \pm\infty$)

$\mathcal{K}(y^-)$ is either of the followings depending on the choice of LC gauge

$$\mathcal{K}(y^-) \;=\; \frac{1}{2}\epsilon(y^-), \quad \text{or} \quad \theta(y^-), \quad \text{or} \quad -\,\theta(-\,y^-)$$

Starting from a gauge-invariant expression of the Wigner distribution (or generalized transverse-momentum-dependent PDF) as follows :

$$\boxed{f_L(x, q_T, \Delta)} \equiv \int \frac{dz^- \, d^2 z_T}{(2\pi)^3} \, e^{i\,x\,\bar{P}^+ z^- - i\,q_T\, z_T}$$
$$\times \langle P' S' \,|\, \bar{\psi}\left(-\frac{z^-}{2}, -\frac{z_T}{2}\right) \gamma^+ \mathcal{W}^-_{-\frac{z}{2}, \pm\infty} \mathcal{W}^T_{-\frac{z_T}{2}, \frac{z_T}{2}} \mathcal{W}^-_{\pm\infty, \frac{z}{2}} \psi\left(\frac{z^-}{2}, \frac{z_T}{2}\right) |\, PS \rangle$$

$\mathcal{W}^T$: Wilson line in the transverse direction at $x^- = -\pm\infty$

he showed the relation

$$\epsilon^{ij} L_{\text{“}can\text{”}} = \frac{1}{2P^+} \lim_{\Delta \to 0} \frac{\partial}{\partial\, i\, \Delta^i} \langle P' S' \,|\, \bar{\psi}(0)\, \gamma^+ \left(i\, \overrightarrow{D}^j_{pure} - i\, \overleftarrow{D}^j_{pure} \right) \psi(0) \,|\, PS \rangle$$
$$= \lim_{\Delta \to 0} \frac{\partial}{\partial\, i\, \Delta^i} \int dx\, d^2 q_T \; q^i_T \; f_L(x, q_T, \Delta)$$
$$= \epsilon^{ij} \frac{S^+}{P^+} \frac{1}{2} \int dx\, d^2 q_T \; q_T^2 \; f_{1L}(x, q_T, 0)$$

What is it ?

where

$$f_L(x, q_T, \Delta) \sim \frac{i}{\bar{P}^+} \epsilon^{+-ij} \, \bar{S}^+ \, q_{Ti} \, \Delta_j \, f_{1L}(x, q_T^2, \xi) + \cdots$$

“canonical” OAM $\Longleftrightarrow$ matrix element of a manifestly gauge invariant operator

Wigner function (S. Meissner, A. Metz, and M. Schlegel, JHEP08(2009)056)

$$W^{[\gamma^+]}(x, \xi, \boldsymbol{q}_T^2, \boldsymbol{q}_T \cdot \boldsymbol{\Delta}_T, \boldsymbol{\Delta}_T^2; \eta)$$
$$= \frac{1}{2} \int \frac{dz^- \, d^2 z_T}{(2\pi)^2} e^{k \cdot z} \langle p', \lambda' | \bar{\psi}\left(-\frac{z}{2}\right) \gamma^+ \mathcal{W}\left(-\frac{z}{2}, \frac{z}{2} | n\right) \psi\left(\frac{z}{2}\right) | p, \lambda \rangle_{z^+ = 0}$$
$$= \frac{1}{2M} \bar{u}(p', \lambda') \left[F_{1,1} + \frac{i \sigma^{i+} q_T^i}{P^+} F_{1,2} + \frac{i \sigma^{i+} \Delta_T^i}{P^+} F_{1,3} + \frac{i \sigma^{ij} q_T^i \Delta_T^j}{M^2} \boxed{F_{1,4}} \right] u(p, \lambda)$$

forward limit ($\Delta \rightarrow 0$)

$$f_1(x, \boldsymbol{q}_T^2) = F_{1,1}^e(x, 0, \boldsymbol{q}_T^2, 0, 0), \qquad f_{1T}^{\perp}(x, \boldsymbol{q}_T^2, ; \eta) = - F_{1,2}^o(x, 0, \boldsymbol{q}_T^2, 0, 0; \eta)$$

$F_{1,3}, F_{1,4}$ term : vanish !

Within the framework of light-cone quark model (non-gauge theory)

- C. Lorce and B. Pasquini, P.R. D84, 014015 (2011).

$$L_{can} = - \int dx \, d^2 q_T \, \frac{\boldsymbol{q}_T^2}{M^2} \boxed{F_{1,4}^q(x, 0, q_T^2, 0, 0)}$$

This is just the sum rule, to which Hatta gave gauge-invariant meaning.

really observable ?

3. Short summary

♣ We have established the existence of two physically nonequivalent decompositions of the nucleon spin, the **decompositions (I) and (II)**, with particular emphasis upon the existence of two kinds of OAM, i.e.

"canonical" OAM & dynamical OAM

and also

"canonical" momentum & dynamical momentum

♣ It was shown that the dynamical OAMs of quarks and gluons appearing in the decomposition (I) can in principle be extracted model-independently from combined analysis of GPD and polarized DIS measurements.

♣ It is important to recognize that this longitudinal spin decomposition, which we have derived, has Lorentz-frame independent meaning !

♣ Besides, the sum rule persists even at quantum level !

♣ This means that we now have at least one satisfactory solution to the nucleon spin decomposition problem.

♣ On the other hand, Hatta's recent work opened up a possibility that the OAM appearing in the decomposition (II) may also be related to observables.

♣ Since the relation between the OAM appearing in the decomposition (I) and the observables is already known, this means that we may be able to isolate the correspondent of "potential angular momentum" term appearing in Feynman's paradox of electodynamics.

$$L_{pot} \;=\; L_{dyn} \;-\; L_{\text{“}can\text{”}}$$

♣ However, one must be careful about the presence of very delicate problem on the sum rules containing generalized (and ordinary) TMDs.

♣ Once quantum loop effects is included, the very existence of TMDs satisfying gauge-invariance and factorization (universality or process independence) at the same time is being questioned !

$L_{\text{“}can\text{”}}$ ⇨ Is process-independent extraction possible ?

Still a challenging open question !

4. Some important lessons from QED

4.1. What is "potential angular momentum" ?

We have shown that the **key quantity**, which distinguishes the two nucleon spin decompositions, is what-we-call the "**potential angular momentum**" term.

To understand its **physical meaning**, it is instructive to study easier QED case.

Let's start with the Hamiltonian of a system of charged particles and photons.

$$H = \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \frac{1}{2} \int d^3 r \, [\, \boldsymbol{E}^2 + \boldsymbol{B}^2 \,]$$

longitudinal-transverse decomposition [(ex.) $\boldsymbol{E}_{||} = - \nabla A^0$ in Coulomb gauge]

$$\boldsymbol{E} = \boldsymbol{E}_{||} + \boldsymbol{E}_{\perp}, \qquad \boldsymbol{B} = \boldsymbol{B}_{\perp}$$

$$\begin{aligned} H &= \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \frac{1}{2} \int d^3 r \, \boldsymbol{E}_{||}^2 + \frac{1}{2} \int d^3 r \, [\, \boldsymbol{E}_{\perp}^2 + \boldsymbol{B}^2 \,] \\ &= \sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + V_{coul} + H_{trans} \end{aligned}$$

total momentum

$$\begin{aligned}
\boldsymbol{P} &= \sum_i m_i \dot{\boldsymbol{r}}_i + \int d^3r\, \boldsymbol{E} \times \boldsymbol{B} \\
&= \sum_i m_i \dot{\boldsymbol{r}}_i + \int d^3r\, \boldsymbol{E}_{\parallel} \times \boldsymbol{B}_{\perp} + \int d^3r\, \boldsymbol{E}_{\perp} \times \boldsymbol{B}_{\perp} \\
&= \sum_i m_i \dot{\boldsymbol{r}}_i + \boldsymbol{P}_{long} + \boldsymbol{P}_{trans} \\
&= \sum_i m_i \dot{\boldsymbol{r}}_i + \sum_i q_i \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) + \boldsymbol{P}_{trans}
\end{aligned}$$

potential momentum : *a la* Konopinski

momentum associates with the longitudinal field of the particle i

Which of particle or photon should it be attributed to ?

Combining it with the **mechanical momentum** gives **canonical momentum** $\boldsymbol{p}_i$.

$$\boldsymbol{p}_i \equiv m_i \dot{\boldsymbol{r}}_i + q_i \boldsymbol{A}_{\perp}(\boldsymbol{r}_i)$$

Using the latter

$$\boldsymbol{P} = \sum_i \boldsymbol{p}_i + \boldsymbol{P}_{trans}$$

total angular momentum

$$\begin{aligned} \boldsymbol{J} &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \int d^3r \, \boldsymbol{r} \times [\boldsymbol{E} \times \boldsymbol{B}] \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \int d^3r \, \boldsymbol{r} \times [\boldsymbol{E}_{||} \times \boldsymbol{B}_\perp] + \int d^3r \, \boldsymbol{r} \times [\boldsymbol{E}_\perp \times \boldsymbol{B}_\perp] \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \boldsymbol{J}_{long} + \boldsymbol{J}_{trans} \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \sum_i \boldsymbol{r}_i \times q_i \boldsymbol{A}_\perp(\boldsymbol{r}_i) + \boldsymbol{J}_{trans} \end{aligned}$$

↘

what-we-call the "**potential angular momentum**"

angular momentum associates with the longitudinal field of the particle i

Again, combining this term with the **mechanical angular momentum,** by using

$$\boldsymbol{p}_i \equiv m_i \dot{\boldsymbol{r}}_i + q_i \boldsymbol{A}_\perp(\boldsymbol{r}_i)$$

we find that

$$\boldsymbol{J} = \boxed{\sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i} + \boldsymbol{J}_{trans}$$

canonical OAM

We therefore find the following simpler-looking relations.

$$\boldsymbol{P} = \sum_i \boldsymbol{p}_i + \boldsymbol{P}_{trans}$$

$$\boldsymbol{J} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i + \boldsymbol{P}_{trans}$$

compare !

At first sight, it appears to indicate physical superiority of canonical momentum and canonical angular momentum over the mechanical ones.

However, it is not true, as is clear from the following expression for H .

total Hamiltonian

$$H = \boxed{\sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2} + V_{coul} + H_{trans} \neq \boxed{\sum_i \frac{\boldsymbol{p}_i^2}{2\,m}} + H_{trans}$$

$\sum_i \frac{1}{2} m_i \dot{\boldsymbol{r}}_i^2$: mechanical kinetic energy

$\sum_i \frac{\boldsymbol{p}_i^2}{2\,m}$: what physical meaning ?

Hydrogen atom (in Coulomb gauge)

$$H = \frac{1}{2} m \dot{\boldsymbol{r}}^2 + V_{Coul} + H_{trans} = H_0 + H_{trans} + H_{int}$$

$$H_0 = \frac{\boldsymbol{p}^2}{2m} + V_{Coul}(\boldsymbol{r})$$

$$H_{trans} = \sum_{\boldsymbol{k}} \sum_{\lambda=1,2} \hbar \omega_{\boldsymbol{k}} \, a^{\dagger}_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda}$$

interaction term !

$$H_{int} = \frac{e}{2m} [\boldsymbol{p} \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r}) + \boldsymbol{A}_{\perp}(\boldsymbol{r}) \cdot \boldsymbol{p}] + \frac{e^2}{2m} \boldsymbol{A}_{\perp}(\boldsymbol{r}) \cdot \boldsymbol{A}_{\perp}(\boldsymbol{r})$$

general form of eigen-states : $|\psi_n\rangle \otimes |\{n_{\boldsymbol{k},\lambda}\}\rangle$

$$H_0 |\psi_n\rangle = E_n |\psi_n\rangle$$

$$|\{n_{\boldsymbol{k},\lambda}\}\rangle = \prod_{\alpha} |n_{\boldsymbol{k}_\alpha,\lambda_\alpha}\rangle$$

In the usual description of hydrogen atom, we do not include

Fock components of transverse photons !

$$|\{n_{\boldsymbol{k},\lambda}\}\rangle \Rightarrow |0\rangle_{photon}$$

eigen equation of hydrogen atom (relativistic version)

$$H\,\psi_n = \left(\frac{\boldsymbol{\alpha}\cdot\nabla}{i} + \beta\, m - \frac{\alpha}{r}\right)\psi_n = E_n\,\psi_n$$

eigen wave function

$$\psi^l_{jm} = \begin{pmatrix} i\,\dfrac{G_{lj}(r)}{r}\,\varphi^l_{jm} \\ \dfrac{F_{lj}(r)}{r}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\,\varphi^l_{jm} \end{pmatrix}$$

where

$$\varphi^l_{jm} = \begin{cases} \varphi^{(+)}_{jm} & \text{if } j = l + 1/2 \\ \varphi^{(-)}_{jm} & \text{if } j = l - 1/2 \end{cases} \qquad \text{with} \quad \boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\,\varphi^{(+)}_{jm} = \varphi^{(-)}_{jm}$$

spin and orbital angular momentum

$$\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\Sigma = \begin{pmatrix} \boldsymbol{L} & 0 \\ 0 & \boldsymbol{L} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

We know that

$$[\boldsymbol{J}, H] = 0 \quad \text{but} \quad [\boldsymbol{L}, H] \neq 0, \; [\Sigma, H] \neq 0$$

Expectation value

$$\langle O \rangle \equiv \langle \psi^l_{jm} \,|\, O \,|\, \psi^l_{jm} \rangle$$

It holds that

$$\langle L_3 \rangle = m \left\{ \frac{2j-1}{2j} \int_0^\infty G_{lj}^2 \, dr + \frac{2j+3}{2(j+1)} \int_0^\infty F_{lj}^2 \, dr \right\}$$

$$\left\langle \frac{1}{2} \Sigma_3 \right\rangle = m \left\{ \frac{1}{2j} \int_0^\infty G_{lj}^2 \, dr - \frac{1}{2(j+1)} \int_0^\infty F_{lj}^2 \, dr \right\}$$

$$\langle J_3 \rangle = m \int_0^\infty \left[G_{lj}^2 + F_{lj}^2 \right] dr = m$$

Electron alone saturates the spins of hydrogen atom (and any atoms) !

No transverse photon Fock components !

$$\boldsymbol{L}_{can} = \boldsymbol{r} \times \boldsymbol{p} \quad \Longleftrightarrow \quad \boldsymbol{L}_{dyn} = \boldsymbol{r} \times (\boldsymbol{p} - e\,\boldsymbol{A}_\perp)$$

No difference !

Totally different from the nucleon spin problem of QCD.

Strongly-coupled gauge system !

The meaning of what-we-call the "potential angular momentum" seems clear now !

$$\begin{aligned} \boldsymbol{J} &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \int d^3r \, \boldsymbol{r} \times [\boldsymbol{E} \times \boldsymbol{B}] \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \int d^3r \, \boldsymbol{r} \times [\boldsymbol{E}_{\parallel} \times \boldsymbol{B}_{\perp}] + \int d^3r \, \boldsymbol{r} \times [\boldsymbol{E}_{\perp} \times \boldsymbol{B}_{\perp}] \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \boldsymbol{J}_{long} + \boldsymbol{J}_{trans} \\ &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i + \sum_i \boldsymbol{r}_i \times q_i \, \boldsymbol{A}_{\perp}(\boldsymbol{r}_i) + \boldsymbol{J}_{trans} \end{aligned}$$

↘

"**potential angular momentum**"

- It represents angular momentum associates with the longitudinal part of electric field generated by color-charged quarks !

- We attribute it to the nature of gluons, while Chen et al. to that of quarks.

- Since the choice is in a sense a matter of taste, any further claim on a superiority of one choice must be done in reference to relations with observables.

4.2. Delicacies of spin and OAM decomposition of photons

- S.J. Van Enk and G. Nienhuis, J. of Optics 41 (1994) 963.

total angular momentum of photon

$$\boldsymbol{J} \;=\; \int d^3 r \; \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B})$$

decomposition of $\boldsymbol{E}$ (gauge-invariant)

$$\boldsymbol{E} \;=\; \boldsymbol{E}_{||} \;+\; \boldsymbol{E}_{\perp} \qquad \left(ex. \; \boldsymbol{E}_{||} \;=\; - \nabla A^0 \quad \text{in Coulomb gauge} \right)$$

corresponding decomposition of $\boldsymbol{J}$

$$\begin{aligned} \boldsymbol{J} \;&=\; \int d^3 r \, \boldsymbol{r} \times (\boldsymbol{E}_{||} \times \boldsymbol{B}) \;+\; \int d^3 r \, \boldsymbol{r} \times (\boldsymbol{E}_{\perp} \times \boldsymbol{B}) \\ &\equiv\; \boldsymbol{J}_{long} \;+\; \boldsymbol{J}_{trans} \end{aligned}$$

where

$$\boldsymbol{J}_{long} \;=\; \int d^3 r \; \rho \, (\boldsymbol{r} \times \boldsymbol{A}_{\perp}) \quad : \quad \text{potential angular momentum}$$

$$\begin{aligned} \boldsymbol{J}_{trans} \;&=\; \int d^3 r \, E^{\perp}_l \, (\boldsymbol{r} \times \nabla) \, A^{\perp}_l \;+\; \int d^3 r \, \boldsymbol{E}_{\perp} \times \boldsymbol{A}_{\perp} \\ &=\; \boldsymbol{L} \;+\; \boldsymbol{S} \end{aligned}$$

Introduce transverse mode functions $\boldsymbol{F}_\lambda$ with polarization λ .

$$\nabla^2 \boldsymbol{F}_\lambda = -k^2 \boldsymbol{F}_\lambda, \quad \nabla \cdot \boldsymbol{F}_\lambda = 0$$
$$\langle \boldsymbol{F}_\lambda | \boldsymbol{F}_{\lambda'} \rangle \equiv \int d^3r \, \boldsymbol{F}_\lambda \cdot \boldsymbol{F}_{\lambda'} = \delta_{\lambda\lambda'}$$

mode expansion of transverse electromagnetic field

$$\boldsymbol{A}_\perp = \sum_\lambda \sqrt{\frac{\hbar}{2\,\omega_\lambda}} [\, a_\lambda \boldsymbol{F}_\lambda + a_\lambda^\dagger \boldsymbol{F}_\lambda^* \,]$$
$$\boldsymbol{E}_\perp = \sum_\lambda i \sqrt{\frac{\hbar\,\omega_\lambda}{2}} [\, a_\lambda \boldsymbol{F}_\lambda - a_\lambda^\dagger \boldsymbol{F}_\lambda^* \,] \qquad [a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}$$
$$\boldsymbol{B}_\perp = \sum_\lambda i \sqrt{\frac{\hbar}{2\,\omega_\lambda}} [\, a_\lambda \nabla \times \boldsymbol{F}_\lambda + a_\lambda^\dagger \nabla \times \boldsymbol{F}_\lambda^* \,]$$

It follows that

$$\boldsymbol{S} = \int d^3r \, \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp = \frac{1}{2} \sum_{\lambda,\lambda'} [a_\lambda^\dagger a_{\lambda'} + a_{\lambda'} a_\lambda^\dagger] \, \langle \boldsymbol{F}_\lambda | \hat{\boldsymbol{S}} | \boldsymbol{F}_{\lambda'} \rangle$$
$$\boldsymbol{L} = \int d^3r \, E_l^\perp (\boldsymbol{r} \times \nabla) A_l^\perp = \frac{1}{2} \sum_{\lambda,\lambda'} [a_\lambda^\dagger a_{\lambda'} + a_{\lambda'} a_\lambda^\dagger] \, \langle \boldsymbol{F}_\lambda | \hat{\boldsymbol{L}} | \boldsymbol{F}_{\lambda'} \rangle$$

with

$$\hat{\boldsymbol{L}} = -i\,\hbar\,(\boldsymbol{r} \times \nabla), \quad (\hat{\boldsymbol{S}})_{ij} = -i\,\hbar\,\varepsilon_{ijk}$$

$\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{L}}$ satisfy the familiar C.R.'s

$$[\,\hat{S}_i, \hat{S}_j\,] \;=\; i\,\hbar\,\hat{S}_k, \quad [\,\hat{L}_i, \hat{L}_j\,] \;=\; i\,\hbar\,\hat{L}_k$$

However, what correspond to **observables** are not $\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{L}}$ but $\boldsymbol{S}$ and $\boldsymbol{L}$, since the latter are operators acting on **physical Fock space**.

What are the C.R.'s of $\boldsymbol{S}$ and $\boldsymbol{L}$ like, then ?

choose circularly polarized plane waves as field modes

$$\boldsymbol{F}_\lambda \;=\; \frac{1}{\sqrt{V}}\,\boldsymbol{\varepsilon}_{\boldsymbol{k},s}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}} \qquad (s \;=\; \pm 1)$$

in this case

$$\boldsymbol{S} \;=\; \sum_{\boldsymbol{k}} \frac{\hbar\,\boldsymbol{k}}{k}\,(\,a^\dagger_{\boldsymbol{k},1}\,a_{\boldsymbol{k},1} \;-\; a^\dagger_{\boldsymbol{k},-1}\,a_{\boldsymbol{k},-1}\,)$$

so that

$$[S_i, S_j] \;=\; 0$$

This means that $\boldsymbol{S}$ does not generate general rotations of photon polarization.

Only the components of the operator $\boldsymbol{S}$ along $\boldsymbol{k}$ is a true spin angular momentum operator, because only this component generate spin rotation.

What about C.R. of $\boldsymbol{L}$?

- First, total $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ must obey the standard C.R.

$$[J_i, J_j] \;=\; i\,\hbar\,\varepsilon_{ijk}\,J_k$$

- Second, $\boldsymbol{S}$ and $\boldsymbol{L}$ must transform as **vector** under rotation, so that

$$[J_i, S_j] \;=\; i\,\hbar\,\varepsilon_{ijk}\,S_k$$
$$[J_i, L_j] \;=\; i\,\hbar\,\varepsilon_{ijk}\,L_k$$

- Combining these relations with $[S_i, S_j] \;=\; 0$, it follows that

$$[L_i, L_j] \;=\; i\,\hbar\,\varepsilon_{ijk}\,(L_k \;-\; S_k)$$
$$[L_i, S_j] \;=\; i\,\hbar\,\varepsilon_{ijk}\,S_k$$

This means that $\boldsymbol{L}$ also does not have a meaning of normal OAM, even though it can be measured !

(ex.) orbital angular momentum of paraxial laser beam, etc.

All these **delicacies of photon spin decomposition** comes from the fact that there is **no rest frame** for **massless photon** !

[Backup Slides]

[Backup Slide 1] Chen et al.'s decomposition of linear momentum

$$\boldsymbol{P}_{QCD} = \int \psi^\dagger \frac{1}{i} \boldsymbol{D}_{pure} \psi \, d^3x + \int E^i \, \mathcal{D}_{pure} \, A^i_{phys} \, d^3x$$

where

$$\boldsymbol{D}_{pure} = \nabla - i\, g\, \boldsymbol{A}_{pure}, \quad \mathcal{D}_{pure} = \nabla - i\, g\, [\, \boldsymbol{A}_{pure}, \cdot\,]$$

This decomposition is different from the standardly-accepted decomposition

$$\boldsymbol{P}_{QCD} = \int \psi^\dagger \frac{1}{i} \boldsymbol{D} \, \psi \, d^3x + \int \boldsymbol{E} \times \boldsymbol{B} \, d^3x$$

and they claim that it leads to the following nonstandard prediction for the asymptotic values of quark and gluon momentum fractions :

$$\lim_{Q^2 \to \infty} \langle x \rangle^Q = \frac{3\, n_f}{\frac{1}{2} n_g + 3\, n_f} \overset{n_f = 6}{\simeq} 0.82$$

$$\lim_{Q^2 \to \infty} \langle x \rangle^g = \frac{\frac{1}{2} n_g}{\frac{1}{2} n_g + 3\, n_f} \overset{n_f = 6}{\simeq} 0.18$$

However, this claim is probably **wrong**, as we shall discuss below !

existing decomposition of OCD energy momentum tensor

$T^{\mu\nu}_{QCD}$	$T^{\mu\nu}_q$	$T^{\mu\nu}_g$
(1) standard	$\frac{1}{2}\bar{\psi}\,(\gamma^\mu\, i\, D^\nu + \gamma^\nu\, i\, D^\mu)\,\psi$	$2\,\mathrm{Tr}\,[F^{\mu\alpha}\,F^\nu_\alpha]$ $+\frac{1}{2}\,g^{\mu\nu}\,\mathrm{Tr}\,F^2$
(2) Jaffe-Manohar	$\frac{1}{2}\bar{\psi}\,(\gamma^\mu\, i\,\partial^\nu + \gamma^\nu\, i\,\partial^\mu)\,\psi$	$-\,\mathrm{Tr}\,[\,F^{\mu\alpha}\,\partial^\nu\,A_\alpha + F^{\nu\alpha}\partial^\mu\,A_\alpha\,]$ $+\frac{1}{2}\,g^{\mu\nu}\,\mathrm{Tr}\,F^2$
(3) Chen et al.	$\frac{1}{2}\bar{\psi}\,(\gamma^\mu\, i\, D^\nu_{pure} + \gamma^\nu\, i\, D^\mu_{pure})\,\psi$	$-\,\mathrm{Tr}\,[F^{\mu\alpha}\,D^\nu_{pure}\,A_{\alpha,phys} + F^{\nu\alpha}\,D^\mu_{pure}\,A_{\alpha,phys}\,]$ $+\frac{1}{2}\,g^{\mu\nu}\,\mathrm{Tr}\,F^2$
(4) Ours	$\frac{1}{2}\bar{\psi}\,(\gamma^\mu\, i\, D^\nu + \gamma^\nu\, i\, D^\mu)\,\psi$	$-\,\mathrm{Tr}\,[\,F^{\mu\alpha}\,D^\nu_{pure}\,A_{\alpha,phys} + F^{\nu\alpha}\,D^\mu_{pure}\,A_{\alpha,phys}\,]$ $-\,\mathrm{Tr}\,[\,D_\alpha\,F^{\mu\alpha}\,A^\nu_{phys} + D_\alpha\,F^{\nu\alpha}\,A^\mu_{phys}\,]$ $+\frac{1}{2}\,g^{\mu\nu}\,\mathrm{Tr}\,F^2$

generalized potential momentum term !

What do these decompositions mean for the **momentum sum rule** of QCD ?

Take light-cone (LC) gauge $(A^+ = 0)$

$$A^+_{phys} \rightarrow 0, \quad A^+_{pure} \rightarrow 0$$

$$D^+ \equiv \partial^+ - i\,g\,A^+ \rightarrow \partial^+, \quad D^+_{pure} \equiv \partial^+ - i\,g\,A^+_{pure} \rightarrow \partial^+$$

$$F^{+\alpha} = \partial^+ A^\alpha - \partial^\alpha A^+ + g\,[A^+, A^\alpha] \rightarrow \partial^+ A^\alpha$$

T^{++} component in **any of the 4 decompositions** then reduce to

$$T^{++} = i\,\psi^\dagger_+ \partial^+ \psi_+ + \mathrm{Tr}\,(\partial^+ \boldsymbol{A}_\perp)^2$$

Interaction-dependent part drops in the LC gauge and infinite-momentum frame !

Thus, from

- **Jaffe** -

$$\langle P_\infty \,|\, T^{++} \,|\, P_\infty \rangle \,/\, 2\,(P^+_\infty)^2 = 1$$

we obtain the standard momentum sum rule of QCD : $\langle x \rangle^q + \langle x \rangle^g = 1$

Even Chen decomposition gives the standard sum rule, contrary to their claim !

The point is that the difference between

$$T_q'^{++} = \frac{1}{2}\bar{\psi}(\gamma^+ i\partial^+ + \gamma^+ i\partial^+)\psi \quad : \quad \text{canonical momentum}$$

$$T_q^{++} = \frac{1}{2}\bar{\psi}(\gamma^+ i D^+ + \gamma^+ i D^+)\psi \quad : \quad \text{dynamical momentum}$$

does not appear in the **longitudinal momentum sum rule**, since $A^+ = 0$!

However, this is not the case for the **angular momentum sum rule**.

In fact, the difference between

$$M_{q-OAM}'^{\mu\nu\lambda} = \frac{1}{2}\bar{\psi}\gamma^\mu(x^\nu i\partial^\lambda + x^\lambda i\partial^\nu)\psi \quad : \quad \text{canonical OAM}$$

$$M_{q-OAM}^{\mu\nu\lambda} = \frac{1}{2}\bar{\psi}\gamma^\mu(x^\nu i D^\lambda + x^\lambda i D^\nu)\psi \quad : \quad \text{dynamical OAM}$$

does not vanish even in LC gauge and IMF, since

$$M_{q-OAM}^{+12} - M_{q-OAM}'^{+12} = g\,\bar{\psi}\gamma^+(x^1 A_\perp^2 - x^2 A_\perp^1)\psi$$

physical components, which cannot be transformed away by any gauge transformation !

[Backup Slide 2] gauge-invariance of the evolution of gluon spin

quark and gluon spin operator in our GI decomposition

$$M^{+12}_{q-spin} = \bar{\psi}\,\gamma^+\,\gamma_5\,\psi,$$
$$M^{+12}_{g-spin} = 2\,\mathrm{Tr}\left[F^{+1}\,A^2_{phys} - F^{+2}\,A^1_{phys} \right]$$

a little more explicit form

$$M^{+12}_{g-spin} = V_A + V_B + V_C$$

with

$$V_A = (\partial^+ A^1_a)\,A^2_{a,phys} - (\partial^+ A^2_a)\,A^1_{a,phys}$$
$$V_B = -\left[(\partial^1 A^+_a)\,A^2_{a,phys} - (\partial^2 A^+_a)\,A^1_{a,phys} \right]$$
$$V_C = g\,f_{abc}\,A^+_b \left[A^1_c\,A^2_{a,phys} - A^2_c\,A^1_{a,phys} \right]$$

- In the LC gauge $(A^+ = 0)$, only the V_A term survives !
- The question is how to introduce this unique feature of our gluon spin operator into Feynman rules for evaluating relevant anomalous dimensions !

The gluon propagator in general covariant gauge

$$D_{ab}^{\mu\nu}(k) = \frac{i\,\delta_{ab}}{k^2 + i\epsilon} \sum_{\lambda=1}^{4} \varepsilon^{\mu}(k,\lambda)\,\varepsilon^{\nu}(k,\lambda)$$
$$= \frac{i\,\delta_{ab}}{k^2 + i\varepsilon} \left(-g^{\mu\nu} + (1-\xi)\frac{k^{\mu}k^{\nu}}{k^2 + i\varepsilon} \right)$$

arbitrary gauge parameter

$\xi = 1 \Leftrightarrow$ Feynman gauge

Since one of the gluon field appearing in our gluon spin operator is its physical part, we must replace the gluon propagator by

$$\frac{i\,\delta_{ab}}{k^2 + i\epsilon} \sum_{\lambda=1}^{2} \varepsilon^{\mu}(k,\lambda)\,\varepsilon^{\nu}(k,\lambda)$$

when one of the endpoint of gluon propagator is obtained by the contraction with the physical part of gluon in our gluon spin operator.

Here, we need a sum of the product of gluon polarization vectors over two physical polarization states (not including the scalar and longitudinal polarization states).

The answer is given in the textbook of **Bjorken and Drell** :

$$T^{\mu\nu} \equiv \sum_{\lambda=1}^{2} \varepsilon^{\mu}(k,\lambda)\,\varepsilon^{\nu}(k,\lambda) = -g^{\mu\nu} + \frac{k^{\mu}n^{\nu} + n^{\mu}k^{\nu}}{n\cdot k} - n^2 \frac{k^{\mu}k^{\nu}}{(n\cdot k)^2}$$

where n being an arbitrary 4-vector subject to the conditions :

$$n\cdot\varepsilon = 0, \qquad n\cdot k \neq 0$$

For practical calculation, it is convenient to take n to be a light-like 4-vector satisfying $n^2 = 0$.

In this case, the modified gluon propagator reduces to

$$\tilde{D}^{\mu\nu}_{ab}(k) \equiv \frac{i\,\delta_{ab}}{k^2 + i\epsilon} \sum_{\lambda=1}^{2} \varepsilon^{\mu}(k,\lambda)\,\varepsilon^{\nu}(k,\lambda) = \frac{i\,\delta_{ab}}{k^2 + i\epsilon}\left(-g^{\mu\nu} + \frac{k^{\mu}n^{\nu} + n^{\mu}k^{\nu}}{n\cdot k} \right)$$

which precisely coincides with the gluon propagator in the LC gauge.

This does not mean we are working in the LC gauge from the very beginning.

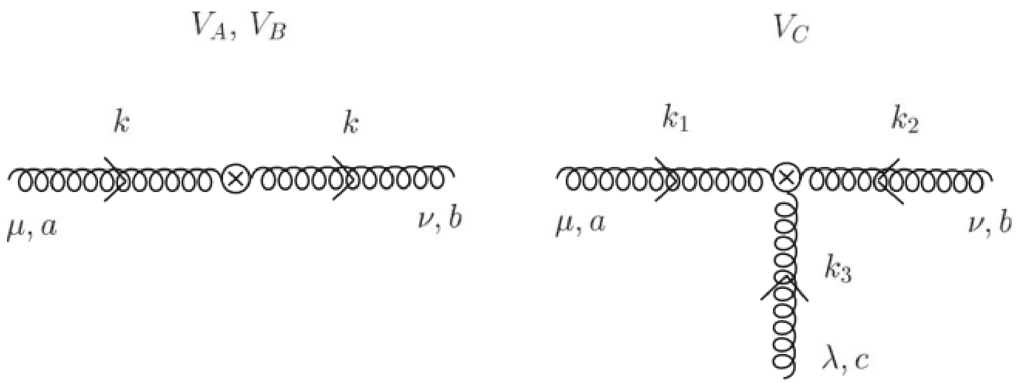
In fact, if we did so, there would be no contributions to anomalous dimensions from the operators V_B and V_C .

It is crucial to use the above modified propagator only when one of the endpoint of the gluon propagator is obtained by the contraction with the physical part of A_μ in our gluon spin operator.

In other places, one should use the standard gluon propagator, which, for instance in the Feynman gauge, is given by

$$D^{\mu\nu}_{ab}(k) \;=\; \frac{i\,\delta_{ab}}{k^2 + i\epsilon}\,(\,-\,g^{\mu\nu}\,)$$

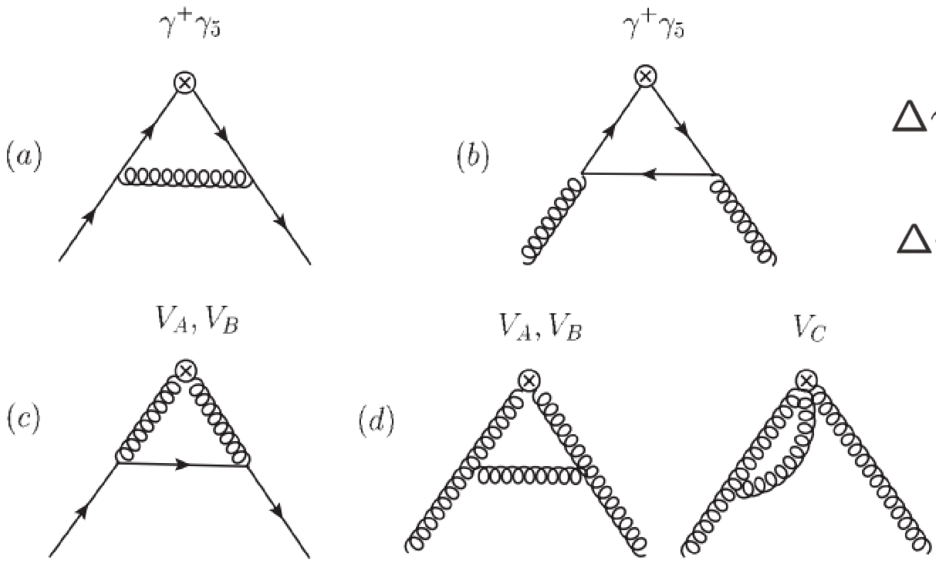
The momentum space vertex operators for the gluon spin



$$
\begin{aligned}
V_A &= i\, k^+ \,(\, g^{\mu 1}\, g^{\nu 2} - g^{\mu 2}\, g^{\nu 1}\,)\, P_T^{\nu}\, \delta_{ab} \\
&- \quad (\mu \leftrightarrow \nu), \\
V_B &= -\, i\, g^{\mu +}\,(\, k^1\, g^{\nu 2} - k^2\, g^{\nu 1}\,)\, P_T^{\nu}\, \delta_{ab} \\
&- \quad (\mu \leftrightarrow \nu), \\
V_C &= g\, f_{abc}\, g^{\lambda +}\,(\, g^{\mu 1}\, g^{\nu 2} - g^{\mu 2}\, g^{\nu 1}\,)\,(\, P_T^{\mu} + P_T^{\nu}\,) \\
&+ \; g\, f_{abc}\, g^{\mu +}\,(\, g^{\nu 1}\, g^{\lambda 2} - g^{\nu 2}\, g^{\lambda 1}\,)\,(\, P_T^{\nu} + P_T^{\lambda}\,) \\
&+ \; g\, f_{abc}\, g^{\nu +}\,(\, g^{\lambda 1}\, g^{\mu 2} - g^{\lambda 2}\, g^{\mu 1}\,)\,(\, P_T^{\lambda} + P_T^{\mu}\,)
\end{aligned}
$$

Here, P_T^{ν} is a sort of projection operator, which reminds us of the fact that we must use the modified gluon propagator , whenever it contains the Lorentz index ν .

The Feynman diagrams contributing to relevant anomalous dimensions



from (a) from (FS)

$$\Delta\gamma_{qq}^{(0)} = \frac{\alpha_S}{2\pi}\cdot\frac{1}{2}C_F + \frac{\alpha_S}{2\pi}\cdot\left(-\frac{1}{2}C_F\right) = 0$$

$$\Delta\gamma_{qG}^{(0)} = 0$$

from V_A from V_B

$$\Delta\gamma_{Gq}^{(0)} = \frac{\alpha_S}{2\pi}\cdot C_F + \frac{\alpha_S}{2\pi}\cdot\frac{1}{2}C_F = \frac{\alpha_S}{2\pi}\cdot\frac{3}{2}C_F$$

\+ field strength (FS) renormalization graphs

from V_A from V_B

$$\Delta\gamma_{GG}^{(0)} = \frac{\alpha_S}{2\pi}\cdot\frac{11}{24}C_A + \frac{\alpha_S}{2\pi}\cdot\left(-\frac{23}{24}C_A\right)$$
$$+ \frac{\alpha_S}{2\pi}\cdot\frac{3}{2}C_A + \frac{\alpha_S}{2\pi}\cdot\left(\frac{5}{6}C_A - \frac{1}{3}n_f\right) = \frac{\alpha_S}{2\pi}\cdot\left(\frac{11}{6}C_A - \frac{1}{3}n_f\right)$$

from V_C from (FS)

[Backup slide 3A]

Eigen-states of spin and orbital angular momentum of photons

mode functions

$$\boldsymbol{F}(k_t, k_z, m, s) \qquad (\, k_t^2 = k^2 - k_z \,)$$

which are simultaneous eigen-functions of the operators

$$\widehat{\boldsymbol{P}}^2, \;\; \widehat{P}_z, \;\; \widehat{J}_z, \;\; \text{and} \;\; \widehat{T}\,\widehat{S}_z\,\widehat{T}$$

where $\widehat{T}\,\widehat{S}_z\,\widehat{T}$ denotes the projection on the space of transverse functions :

$$\widehat{T}\,\widehat{S}_z\,\widehat{T} \quad : \quad \text{reduced spin operator}$$

such that

$$\begin{aligned} \widehat{\boldsymbol{P}}^2\,\boldsymbol{F}_\pm &= \hbar^2\,k^2\,\boldsymbol{F}_\pm \\ \widehat{P}_z\,\boldsymbol{F}_\pm &= \hbar\,k_z\,\boldsymbol{F}_\pm \\ \widehat{J}_z\,\boldsymbol{F}_\pm &= \hbar\,m \\ \widehat{T}\,\widehat{S}_z\,\widehat{T}\,\boldsymbol{F}_\pm &= \pm\,\frac{\hbar\,k_z}{k} \end{aligned}$$

$$\boldsymbol{F}_\pm = \frac{1}{\sqrt{2}}(\,\boldsymbol{F}_x \;\pm\; i\,\boldsymbol{F}_y\,)$$

Angular momentum operators

$$J_z = \int dk_t \int dk_z \sum_{m,s} m\,\hbar\,\hat{N}_\lambda$$

$$S_z = \int dk_t \int dk_z \sum_{m,s} \frac{s\,\hbar\,k_z}{k}\,\hat{N}_\lambda$$

$$L_z = \int dk_t \int dk_z \sum_{m,s} \left(m\,\hbar - \frac{s\,\hbar\,k_z}{k} \right) \hat{N}_\lambda$$

with

$$\hat{N}_\lambda = \hat{a}_\lambda^\dagger\,\hat{a}_\lambda, \qquad \lambda \equiv (k_t, k_z, m, s)$$

Notice that

$m\,\hbar - \dfrac{s\,\hbar\,k_z}{k}$ need not take discrete value !

[Backup slide 3B] On the conservation of angular momentum

total angular momentum of a system of charged-particles and photons

$$\begin{aligned}\boldsymbol{J} &= \sum_i m_i \, \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i + \int d^3r \; \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) \\ &= \boldsymbol{J}_{mech} + (\boldsymbol{J}_{long} + \boldsymbol{J}_{trans})\end{aligned}$$

with

$$\begin{aligned}\boldsymbol{J}_{long} &= \int d^3r \; \rho \, (\boldsymbol{r} \times \boldsymbol{A}_\perp) \\ \boldsymbol{J}_{trans} &= \int d^3r \; E^l_\perp \, (\boldsymbol{r} \times \nabla) \, A^l_\perp + \int d^3r \; \boldsymbol{E}_\perp \times \boldsymbol{A}_\perp\end{aligned}$$

using Newton-Lorentz equation

$$\frac{d}{dt} \boldsymbol{J}_{mech} = \int d^3r \; (\rho \, \boldsymbol{E}_\perp + \boldsymbol{j} \times \boldsymbol{B})$$

using Maxwell equation

$$\begin{aligned}\frac{d}{dt} \boldsymbol{J}_{long} &= - \int d^3r \; \boldsymbol{r} \times (\rho \, \boldsymbol{E}_\perp + \boldsymbol{j}_{\|} \times \boldsymbol{B}) \\ \frac{d}{dt} \boldsymbol{J}_{trans} &= - \int d^3r \; \boldsymbol{r} \times (\boldsymbol{j}_\perp \times \times \boldsymbol{B})\end{aligned}$$

Note that $(d/dt)\,\boldsymbol{J} = 0$, but, with the decompositions

$$\boldsymbol{J} = (\boldsymbol{J}_{mech} + \boldsymbol{J}_{long}) + \boldsymbol{J}_{trans} = \boldsymbol{J}_{can} + \boldsymbol{J}_{trans}$$

$$= \boldsymbol{J}_{mech} + (\boldsymbol{J}_{long} + \boldsymbol{J}_{trans}) = \boldsymbol{J}_{mech} + \boldsymbol{J}_{\gamma}$$

We find that

$$\frac{d}{dt}\boldsymbol{J}_{can} = \int d^3r\, \boldsymbol{r} \times (\boldsymbol{j}_{\perp} \times \boldsymbol{B}) \neq 0$$

$$\frac{d}{dt}\boldsymbol{J}_{trans} = -\int d^3r\, \boldsymbol{r} \times (\boldsymbol{j}_{\perp} \times \boldsymbol{B}) \neq 0$$

not separately conserved !

$$\frac{d}{dt}\boldsymbol{J}_{mech} = \int d^3r\, \boldsymbol{r} \times (\rho\,\boldsymbol{E}_{\perp} + \boldsymbol{j} \times \boldsymbol{B}) \neq 0$$

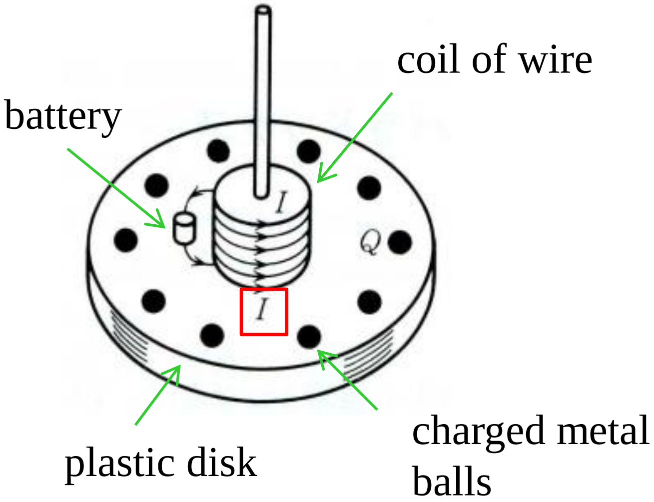
$$\frac{d}{dt}\boldsymbol{J}_{\gamma} = -\int d^3r\, \boldsymbol{r} \times (\rho\,\boldsymbol{E}_{\perp} + \boldsymbol{j}_{\perp} \times \boldsymbol{B}) \neq 0$$

not separately conserved !

[Backup slide 4A] A short review of the **Feynman paradox**

1. Initially, the disk is at rest.

2. Shut off the electric current at some moment.



Question

Does the disk begin to rotate, or does it continue to be at rest ?

Answer (A)

♣ Since an electric current is flowing through the coil, there is a magnetic flux along the axis.

♣ When the current is stopped, due to the electromagnetic induction, an electric field along the circumference of a circle is induced.

♣ Since the charged metal ball receives forces by this electric field, the disk begins to rotate !

Answer (B)

- ♣ Since the disk is initially at rest, its angular momentum is zero.

- ♣ Because of the conservation of angular momentum, the disk continue to be at rest !

⇩

2 totally conflicting answers !

Feynman's paradox

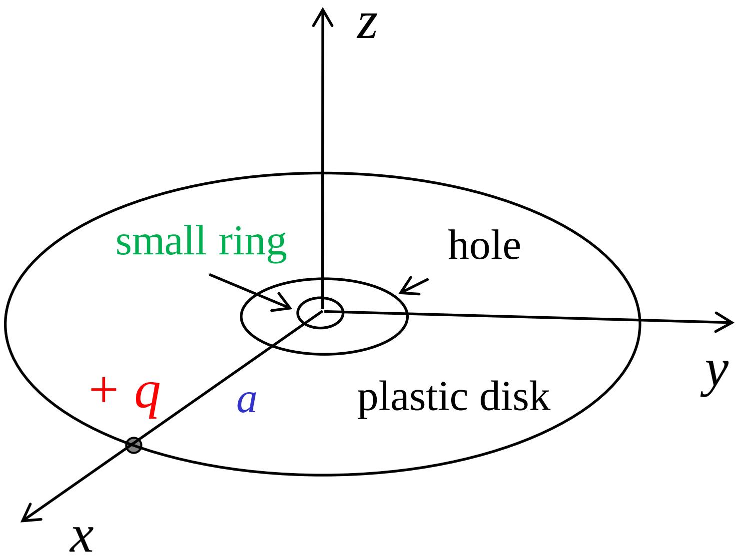
The paradox is resolved, if one takes account of the angular momentum carried by the electromagnetic field or potential generated by an electric current !

$$\boldsymbol{L}_{e.m.} = \int \boldsymbol{r} \times \rho \, \boldsymbol{A} \, d^3 r$$

The answer (A) is correct !

[Backup Slides 4B] A simplified model of the Feynman paradox

- J.M. Aguirregabiria and A. Hernandez, Eur. J. Phys. 2 (1981) 168.



- A current I is flowing in a small (nearly point-like) ring so that it has a magnetic moment

$$\boldsymbol{m} = m\,\boldsymbol{e}_z$$

- A charge $+q$ is located at

$$\boldsymbol{r} = (a, 0, 0)$$

- This disk is initially at rest.

The vector potential $\boldsymbol{A}$ at a point $\boldsymbol{r}$ created by the small ring is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}$$

- Now, the magnetic moment is slowly decreased.
The induced electric fields $\boldsymbol{E} = -\partial\boldsymbol{A}/\partial t$ has a tangential component.

$$E_\phi = -\frac{\mu_0}{4\pi}\frac{\dot{m}}{a^2} \quad \text{at} \quad \boldsymbol{r} = (a, 0, 0)$$

Torque

$$N_z = a \times q\,E_\phi = -\frac{\mu_0}{4\pi}\frac{q\,\dot{m}}{a}$$

When *m* becomes 0, the angular momentum of the disk is

$$L_z = \int N_z \, dt = -\frac{\mu_0 \, q}{4 \, \pi \, a} \int_m^0 \dot{m} \, dt = \frac{\mu_0 \, q \, m}{4 \, \pi \, a}$$

However, since the angular momentum of the disk is initially zero and if it must be conserved, the disk must be at rest.

basically the **Feynman paradox**

We must consider the angular momentum carried by the e.m. field (or potential)

$$\boldsymbol{L}_{e.m.} = \frac{1}{c^2} \int \boldsymbol{r} \times \boldsymbol{S} \, dV = \frac{1}{\mu_0 \, c^2} \int \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) \, dV$$

Using the identity

$$\begin{aligned} \boldsymbol{C} \times (\nabla \times \boldsymbol{D}) + \boldsymbol{D} \times (\nabla \times \boldsymbol{C}) &= (\nabla \cdot \boldsymbol{C}) \boldsymbol{D} + (\nabla \cdot \boldsymbol{D}) \boldsymbol{C} + \nabla \cdot \boldsymbol{T} \\ \boldsymbol{r} \times \nabla \cdot \boldsymbol{T} &= \nabla \cdot \boldsymbol{R} \end{aligned}$$

with

$$\begin{aligned} T_{ij} &= (\boldsymbol{C} \cdot \boldsymbol{D}) \, \delta_{ij} - (C_i \, D_j + C_j \, D_i) \\ R_{ij} &= \varepsilon_j^{kl} \, x_k \, T_{il} \end{aligned}$$

we can write as

$$\boldsymbol{L}_{e.m.} = \varepsilon_0 \int \boldsymbol{r} \times [\boldsymbol{E} \times (\nabla \times \boldsymbol{A})] \, dV$$
$$= \int (\boldsymbol{r} \times \rho \, \boldsymbol{A}) \, dV$$
$$+ \varepsilon_0 \int [(\nabla \cdot \boldsymbol{A}) \, \boldsymbol{r} \times \boldsymbol{E}] \, dV + \varepsilon \int \nabla \cdot \boldsymbol{Q} \, dV$$

with

$$Q_{ij} = \varepsilon_j^{kl} \, x_k \, [(\boldsymbol{E} \cdot \boldsymbol{A}) \, \delta_{li} - (E_l \, A_i + E_i \, A_l)]$$

The 2nd term vanishes, since $\boldsymbol{A}$ satisfies $\nabla \cdot \boldsymbol{A} = 0$.

Using the Gauss law, the 3rd term also vanishes, since $\boldsymbol{Q} \to 1/r^3$.

Then, noting that $\rho = q \, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{a})$, we get

$$\boldsymbol{L}_{e.m.} = \int (\boldsymbol{r} \times \rho \, \boldsymbol{A}) \, dV = q \, \boldsymbol{r} \times \boldsymbol{A}(\boldsymbol{a})$$

That is

$$\boldsymbol{L}_{e.m.} = \frac{\mu_0 \, q}{4 \, \pi \, a^3} \, \boldsymbol{a} \times (\boldsymbol{m} \times \boldsymbol{a}) = \frac{\mu_0 \, q \, m}{4 \, \pi \, a} \, \boldsymbol{e}_z$$

This exactly coincides with the previously-derived angular momentum of the plastic disk in the final state ! - **OAM carried by e.m. field or potential** -

[Backup Slide 5] Nuclear spin decomposition problem

It is not a well-defined problem, because of the ambiguities of nuclear force.

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2\,m} + \sum_{i<j} V_{NN}(\boldsymbol{r}_i - \boldsymbol{r}_j)$$

To explain it, let us consider the deuteron, the simplest nucleus.

$$H\,\psi_d(\boldsymbol{r}) = E\,\psi_d(\boldsymbol{r})$$

$$\psi_d(\boldsymbol{r}) = \left[u(r) + \frac{S_{12}(\hat{\boldsymbol{r}})}{\sqrt{8}}\, w(r) \right] \frac{\chi}{\sqrt{4\,\pi}}$$

deuteron w.f. and S- and D-state probabilities

$$P_S = \int_0^\infty u^2(r)\, r^2\, dr, \quad P_D = \int_0^\infty w^2(r)\, r^2\, dr$$

angular momentum decomposition of deuteron spin

$$1 = \langle J_3 \rangle = \langle L_3 \rangle + \langle S_3 \rangle = \frac{3}{2} P_D + \left(P_S - \frac{1}{2} P_D \right)$$

We however know the fact that the D-state probability is not a direct observable !

- R.D. Amado, Phys. Rev. C20 (1979) 1473.
- J.L. Friar, Phys. Rev. C20 (1979) 325.

♣ The "interior" of a bound state w.f. cannot be determined empirically.

♣ 2-body unitary transformation arising in the theory of meson-exchange currents can change the D-state probability, while keeping the deuteron observables intact.

♣ The ultimate origin is the non-uniqueness of short range NN potential.

infinitely many phase-equivalent potential !

♣ The D-state probability, for instance, depends on the cutoff Λ of short range physics in an effective theory of 2-nucleon system.

- S.K. Bogner et al., Nucl. Phys. A784 (2007) 79.

⇩

See the figure in the next page !

Deuteron **D-state probability** in an effective theory (Bogner et al., 2007)

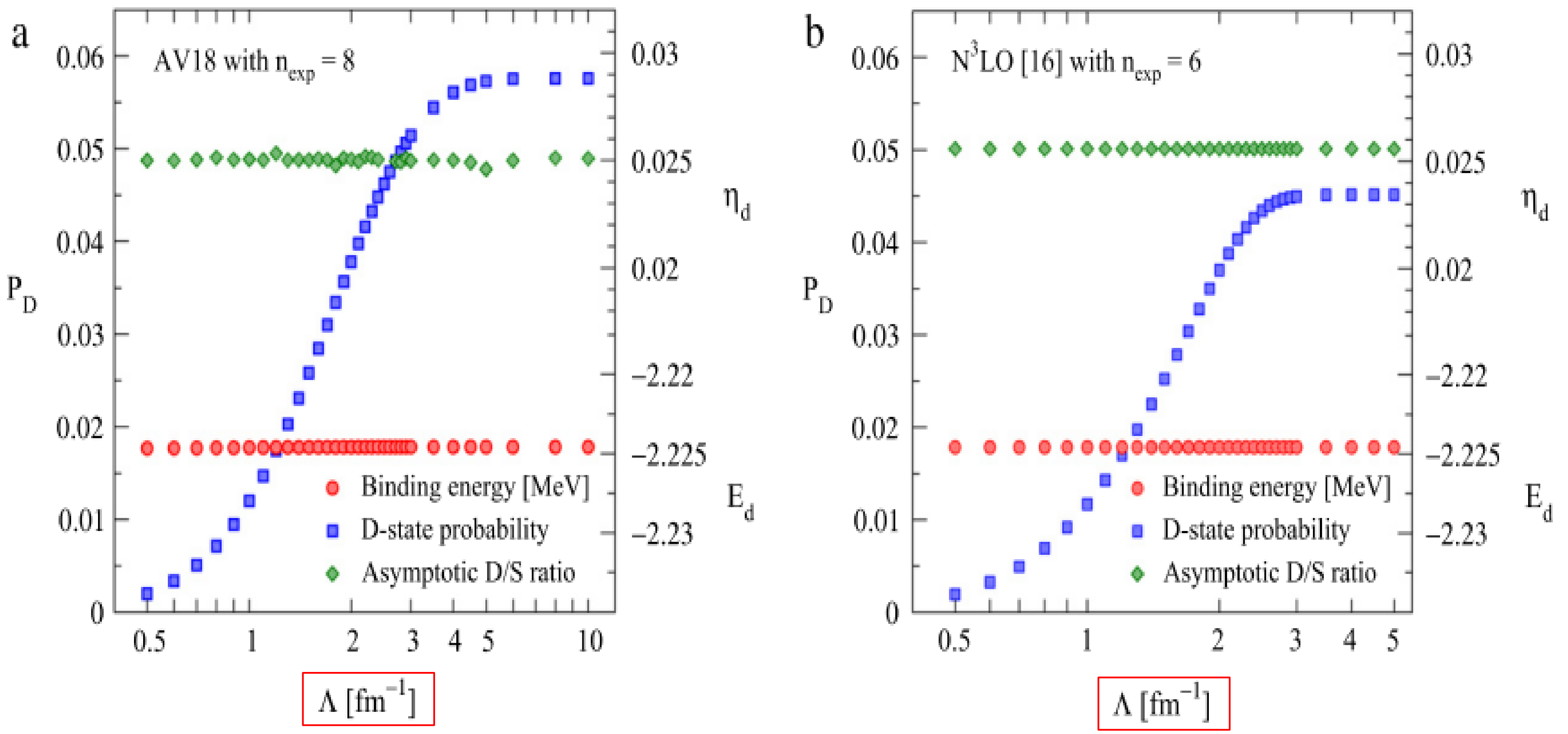


Fig. 57. D-state probability P_D (left axis), binding energy E_d (lower right axis), and asymptotic D/S-state ratio η_d (upper right axis) of the deuteron as a function of the cutoff [6], starting from (a) the Argonne v_{18} [18] and (b) the N^{3}LO NN potential of Ref. [20] using different smooth $V_{\text{low}\,k}$ regulators. Similar results are found with SRG evolution.

Note that the asymptotic D/S ratio corresponds to observables, although the D-state probability not !

[Backup-Slide 6] Check of gauge-invariance of our general decomposition

$$M^{\mu\nu\lambda} = M^{\mu\nu\lambda}_{q-spin} + M^{\mu\nu\lambda}_{q-OAM} + M^{\mu\nu\lambda}_{g-spin} + M^{\mu\nu\lambda}_{g-OAM} + M^{\mu\nu\lambda}_{boost} + \text{total divergence}$$

where

$$M^{\mu\nu\lambda}_{q-spin} = \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\bar{\psi}\gamma_\sigma\gamma_5\psi,$$

$$M^{\mu\nu\lambda}_{q-OAM} = \bar{\psi}\gamma^\mu(x^\nu iD^\lambda - x^\lambda iD^\nu)\psi$$

$$M^{\mu\nu\lambda}_{g-spin} = 2\,\mathrm{Tr}[F^{\mu\lambda}A^\nu_{phys} - F^{\mu\nu}A^\lambda_{phys}],$$

$$M^{\mu\nu\lambda}_{g-OAM} = -2\,\mathrm{Tr}[F^{\mu\alpha}(x^\nu D^\lambda_{pure} - x^\lambda D^\nu_{pure})A^{phys}_\alpha],$$
$$+2\,\mathrm{Tr}[(D_\alpha F^{\alpha\mu})(x^\nu A^\lambda_{phys} - x^\lambda A^\nu_{phys})],$$

$$M^{\mu\nu\lambda}_{boost} = -\frac{1}{2}\mathrm{Tr}\,F^2(x^\nu g^{\mu\lambda} - x^\lambda g^{\mu\nu})$$

under gauge transformation

$$A^\mu_{phys}(x) \rightarrow U(x)\,A^\mu_{phys}(x)\,U^{-1}(x)$$

$$A^\mu_{pure}(x) \rightarrow U(x)\left(A^\mu_{pure}(x) - \frac{i}{g}\partial^\mu\right)U^{-1}(x)$$

gauge-invariant ?

$$M^{\mu\nu\lambda}_{q-spin}, \ M^{\mu\nu\lambda}_{q-OAM} \quad : \quad \text{trivial}$$

$$M^{\mu\nu\lambda}_{g-spin} \quad : \quad \text{almost trivial}$$

$$M^{\mu\nu\lambda}_{g-OAM} \quad : \quad \text{less trivial}$$

However, since

$$\begin{aligned} D^{\lambda}_{pure} A^{phys}_{\alpha} &\equiv \partial^{\lambda} A^{phys}_{\alpha} \ - \ i\,g\,[\,A^{\lambda}_{pure}, A^{phys}_{\alpha}\,] \\ &\rightarrow \partial\,(\,U\,A^{phys}_{\alpha}\,U^{-1}\,) \ - \ i\,g\left[U\left(A^{\lambda}_{pure} - \frac{i}{g}\,\partial^{\lambda}\right)U^{-1}, U\,A^{phys}_{\alpha}\,U^{-1}\right] \\ &= U\,(\,\partial^{\lambda} A^{phys}_{\alpha} \ - \ i\,g\,[\,A^{\lambda}_{pure}, A^{phys}_{\alpha}\,]\,)\,U^{-1} \\ &= U\,D^{\lambda}_{pure}\,A^{phys}_{\alpha}\,U^{-1} \qquad : \quad \text{covariantly transform} \end{aligned}$$

one finds

$$\begin{aligned} M^{\mu\nu\lambda}_{g-OAM} \ = \ &-2\,\mathrm{Tr}\,[\,F^{\mu\alpha}\,(\,x^{\nu}\,D^{\lambda}_{pure} \ - \ x^{\lambda}\,D^{\nu}_{pure}\,)\,A^{phys}_{\alpha}\,], \\ &+2\,\mathrm{Tr}\,[\,(\,D_{\alpha}\,F^{\alpha\mu}\,)\,(\,x^{\nu}\,A^{\lambda}_{phys} \ - \ x^{\lambda}\,A^{\nu}_{phys}\,)\,] \\ \rightarrow & \qquad \text{invariant} \end{aligned}$$