Lattice QCD Calculations of Generalized Form Factors with Dynamical Fermions

Sergey N. Syritsyn

Lawrence Berkeley National Laboratory
Nuclear Science Division

INT Workshop *"Orbital angular momentum in QCD"*
Seattle, Feb 7, 2012

QCD on a Lattice: Numerical Feynman Integration

Monte Carlo sampling with $\mathrm{Prob}\big[U,\psi,\bar\psi\big] \sim e^{-S[U,\psi,\bar\psi]}$

$$\langle\mathcal{O}\rangle = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar\psi\ \mathcal{O}\big[U,\psi,\bar\psi\big]\ e^{-S[U,\psi,\bar\psi]}$$

$$= \int \mathcal{D}U\ \tilde{\mathcal{O}}\ \Pi_f\big[\not{D} + m_f\big]\ e^{-S_g[U]} \to \frac{1}{N}\sum^{N} \tilde{\mathcal{O}}\big[U\big]$$

QCD on a Lattice: Numerical Feynman Integration

Monte Carlo sampling with Prob$\big[U,\psi,\bar\psi\big] \sim e^{-S[U,\psi,\bar\psi]}$

$$\langle\mathcal{O}\rangle = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar\psi\ \mathcal{O}\big[U,\psi,\bar\psi\big]\ e^{-S[U,\psi,\bar\psi]}$$

$$= \int \mathcal{D}U\ \tilde{\mathcal{O}}\ \Pi_f\big[\not{D}+m_f\big]\ e^{-S_g[U]} \to \frac{1}{N}\sum^{N}\tilde{\mathcal{O}}\big[U\big]$$

- Euclidean QFT: $\begin{cases} x^0\equiv t & \to -ix_4\equiv -i\tau \\ p^0\equiv E & \to ip_4 \\ \langle N(t)\bar N(0)\rangle & \to e^{-E\tau} \end{cases}$

QCD on a Lattice: Numerical Feynman Integration

Monte Carlo sampling with Prob$\left[U,\psi,\bar\psi\right] \sim e^{-S[U,\psi,\bar\psi]}$

$$\langle\mathcal{O}\rangle = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar\psi\ \mathcal{O}\left[U,\psi,\bar\psi\right]\ e^{-S[U,\psi,\bar\psi]}$$

$$= \int \mathcal{D}U\ \tilde{\mathcal{O}}\ \Pi_f\left[\not{D}+m_f\right]\ e^{-S_g[U]} \to \frac{1}{N}\sum^{N}\tilde{\mathcal{O}}[U]$$

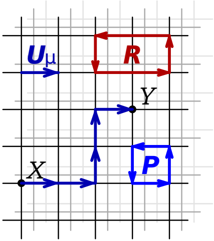
- Euclidean QFT: $\begin{cases} x^0 \equiv t & \to -ix_4 \equiv -i\tau \\ p^0 \equiv E & \to ip_4 \\ \langle N(t)\bar N(0)\rangle & \to e^{-E\tau} \end{cases}$
- Fields on a discrete space-time grid:

$$A^a_\mu(x) \to U_{x,\mu} = \mathcal{P}e^{-i\int_x^{x+\hat\mu} dx\cdot(A^a\frac{\lambda^a}{2})}$$

$$\left(D_\mu\varphi\right)_x \to \frac{1}{a}\left(U_{x,\mu}\varphi_{x+\hat\mu} - \varphi_x\right)$$

$$S_g\left[A_\mu\right] \sim (F^a_{\mu\nu})^2 \to A\,\mathrm{Tr}(P) + B\,\mathrm{Tr}(R)$$

- Fermions on a lattice: pick two from
  no "doublers"; chiral symmetry; economy.



QCD on a Lattice: Numerical Feynman Integration

Monte Carlo sampling with Prob$[U,\psi,\bar\psi] \sim e^{-S[U,\psi,\bar\psi]}$

$$\langle\mathcal{O}\rangle = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar\psi\ \mathcal{O}[U,\psi,\bar\psi]\ e^{-S[U,\psi,\bar\psi]}$$

$$= \int \mathcal{D}U\ \tilde{\mathcal{O}}\ \Pi_f[\not{D}+m_f]\ e^{-S_g[U]} \rightarrow \frac{1}{N}\sum^{N}\tilde{\mathcal{O}}[U]$$

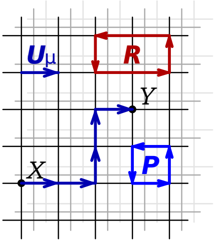
- Euclidean QFT: $\begin{cases} x^0 \equiv t & \rightarrow -ix_4 \equiv -i\tau \\ p^0 \equiv E & \rightarrow ip_4 \\ \langle N(t)\bar N(0)\rangle & \rightarrow e^{-E\tau} \end{cases}$

- Fields on a discrete space-time grid:

$$A^a_\mu(x) \rightarrow U_{x,\mu} = \mathcal{P}e^{-i\int_x^{x+\hat\mu} dx\cdot(A^a\frac{\lambda^a}{2})}$$

$$(D_\mu\varphi)_x \rightarrow \frac{1}{a}(U_{x,\mu}\varphi_{x+\hat\mu}-\varphi_x)$$

$$S_g[A_\mu] \sim (F^a_{\mu\nu})^2 \rightarrow A\,\mathrm{Tr}(P) + B\,\mathrm{Tr}(R)$$



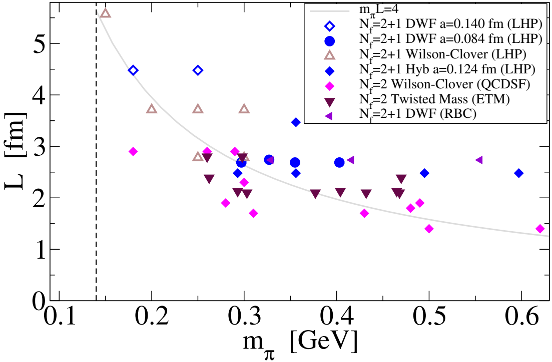
- Fermions on a lattice: pick two from
  no "doublers"; chiral symmetry; economy.

- Tune $(\alpha_S^{\text{lat}}, am_{ud}, am_s)$ to reproduce *e.g.* (m_π, m_K, m_Ω).

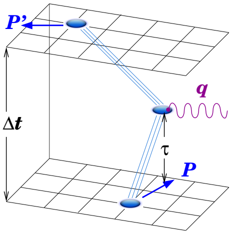
Lattice QCD is a Hard Problem

Solving QCD numerically is hard because

- light quarks are *expensive*: cost $\sim \frac{1}{m_\pi}$
- need large physical size of the box $L \gtrsim \frac{4}{m_\pi}$
- have to take continuum limit $a \to 0$, $L_{\text{lat}} = \frac{L}{a} \to \infty$
- chiral symmetry is expensive to preserve in lattice regularization



Hadron Matrix Elements



Extract $\langle P'|\mathcal{O}|P\rangle$ from 3-pt correlators

$$\sum_{\vec{x},\vec{y}} e^{-i\vec{P}'\vec{x}+i\vec{q}\vec{y}} \langle N(\Delta t,\vec{x})\,\mathcal{O}(\tau,\vec{y})\,\bar{N}(0)\rangle$$

where for the proton

$$N_\alpha = \epsilon^{abc} u^a_\alpha \left[(u^b)^T C\gamma_5 d^c\right]$$

All QCD states are present:

$$\langle N(\Delta t)\,\mathcal{O}(\tau)\,\bar{N}(0)\rangle \sim \sum_{m,n} \sqrt{Z_m}\cdot e^{-E_m(\Delta t-\tau)}\cdot \mathcal{O}_{mn}\cdot e^{-E_n\tau}\cdot \sqrt{Z_n}^\dagger$$

Excited states can lead to systematic bias in m.e. :

$$\bar{N}_{\text{lat}}|\Omega\rangle = |N\rangle + C|X\rangle, \quad \Delta M = M_X - M_N,$$

$$\langle N|\mathcal{O}|N\rangle_{\text{lat}} \cong \langle N|\mathcal{O}|N\rangle + |C|^2\langle X|\mathcal{O}|X\rangle e^{-\Delta M\cdot\Delta t} + \text{“tails”}$$

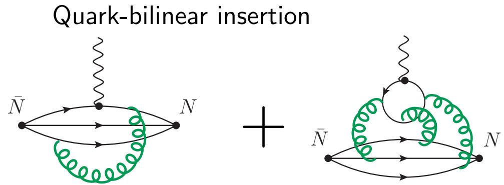
Signal / noise $\sim e^{-(M_N-\frac{3}{2}m_\pi)\cdot\Delta t}$

Hadron Matrix Elements are Challenging

- **2008:** The first calculation of hadron spectrum by the Budapest-Marseille-Wuppertal collaboration [Dürr et al, Science, 322:1224 (2008)]

- Hadron Structure $\gg$ Hadron Spectrum

$$\langle P'|\bar{q}\Gamma q|P\rangle =$$



- Disconnected contractions are noisy
- Gluon operators are noisy (especially with dynamical fermions)

Only quenched calculations (no dynamical fermions) have been performed for gluon and disconnected quark EM tensors.

Quark GPDs

- Generalized Parton Distributions

$$\langle P'|\mathcal{O}^{[\gamma^5]}(x)|P\rangle \to \{\mathcal{H},\mathcal{E},\tilde{\mathcal{H}},\tilde{\mathcal{E}}\}(x,\xi,q^2),$$

$$\mathcal{O}^{[\gamma^5]}(x) = \int \frac{d\lambda}{2\pi}\, e^{2i\lambda x}\, \bar{q}_{(-\lambda n)}\Big[\not{n}\,[\gamma^5]\,\mathcal{W}(-\lambda n,\lambda n)\Big] q_{(\lambda n)}$$

- Moments $\mathcal{O}_n = \int dx\, x^n \mathcal{O}(x) \to \bar{q}\gamma^+(i\overleftrightarrow{D}^+)^n q$
  may be computed on a lattice using *local* operators

$$\mathcal{O}_n^{[\gamma^5]} = \bar{q}\Big[\gamma_{\{\mu_1}\,[\gamma^5]\,i\overleftrightarrow{D}_{\mu_2}\cdots i\overleftrightarrow{D}_{\mu_n\}}\Big] q$$

  and reduced to Generalized Form Factors

  $\langle P'|\mathcal{O}_n|P\rangle \longrightarrow \{A_{ni}, B_{ni}, C_n, \tilde{A}_{ni}, \tilde{B}_{ni}\}(Q^2)$

Twist-2 Operators on a Hypercubic Lattice

Mellin moments of GPDs $\Longleftrightarrow$ symmetric, trace $= 0$ quark operators:

- In continuum: Lorentz symmetry preserves operators from mixing
- On a lattice: Hybercubic group has 20 irreducible representations

$$
\begin{array}{lrl}
n=1 & \bar{q}\gamma_\mu q & \to \mathbf{4}_1^- \\
n=2 & \bar{q}\big[\gamma_{\{\mu} i\overleftrightarrow{D}_{\nu\}} - \langle\mathrm{Tr}\rangle\big]q & \to \mathbf{3}_1^+ \oplus \mathbf{6}_3^+ \\
n=3 & \bar{q}\big[\gamma_{\{\mu} i\overleftrightarrow{D}_{\nu} i\overleftrightarrow{D}_{\rho\}} - \langle\mathrm{Tr}\rangle\big]q & \to \mathbf{8}_1^- \oplus \color{red}{\mathbf{4}_1^-} \oplus \mathbf{4}_2^- \\
n=4 & \bar{q}\big[\gamma_{\{\mu} i\overleftrightarrow{D}_{\nu} i\overleftrightarrow{D}_{\rho} i\overleftrightarrow{D}_{\sigma\}} - \langle\mathrm{Tr}\rangle\big]q & \to \color{red}{\mathbf{1}_1^+} \oplus \color{red}{\mathbf{3}_1^+} \oplus \color{red}{\mathbf{6}_3^+} \oplus \mathbf{2}_1^+ \oplus \mathbf{1}_2^+ \oplus \mathbf{6}_1^+ \oplus \mathbf{6}_2^+ \\
\cdots & &
\end{array}
$$

[Göckeler et al, Phys.Rev.D54,5705(1996)]

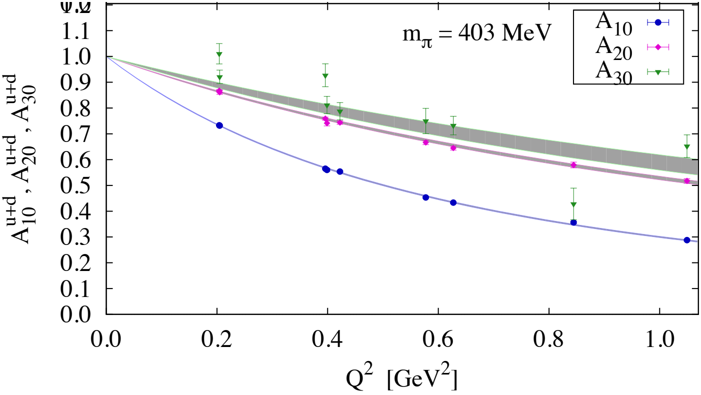
$$\text{Mixing coefficients} \sim \Lambda_{\mathrm{UV}}^{d_1-d_2} = \left(\frac{1}{a}\right)^{d_1-d_2}$$

E.g. for $n=2$ $\mathcal{O}^{\mathrm{lat}} = \mathcal{O}^{\mathrm{phys}} + O(a^2)$

For higher n:

- subtraction with non-perturbative mixing coefficients
- QCD on a more symmetric (Celmaster) 4D lattice

Generalized form factors $A_{n0}(Q^2)$

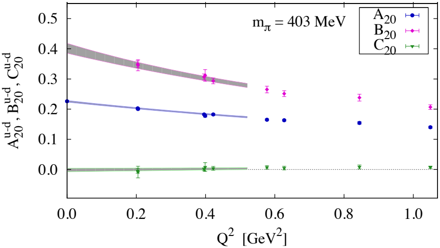


(normalized $A_{n,0}(Q^2)/A_{n,0}(0)$)

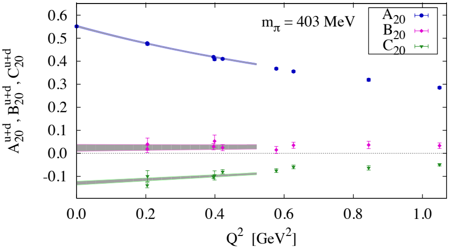
- Noise grows with n
- Generalized radii $\langle r^2_{n=1}\rangle > \langle r^2_{n=2}\rangle > \langle r^2_{n=3}\rangle$

$n = 2$ Gen. Form Factors A_{20}, B_{20}, C_2

isovector



isoscalar



$$|A_{20}^{u+d}| \gg |A_{20}^{u-d}|\,, \qquad |B_{20}^{u-d}| \gg |B_{20}^{u+d}| \approx 0\,, \qquad |C_{2}^{u+d}| \gg |C_{2}^{u-d}| \approx 0\,.$$

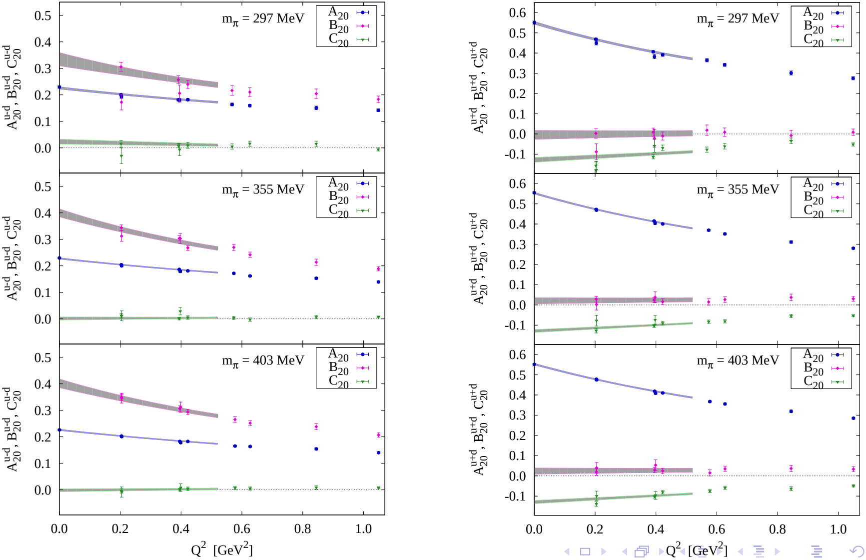
agree with large-N_c scaling [Goeke et at, Prog. Part. Nucl. Phys. 47:401(2001)]

- fit with either *dipole* or *linear* form in $0 \geq Q^2 \geq 0.5$ GeV^2
- extrapolate B_{20} and C_2 to $Q^2 \to 0$
- extract forward values ($Q^2 = 0$) and slopes $d\,\{A_{20}, B_{20}, C_2\}\,/\,d\,Q^2$, $Q^2 \to 0$

$n = 2$ Gen. Form Factors A_{20}, B_{20}, C_2 (cont.)

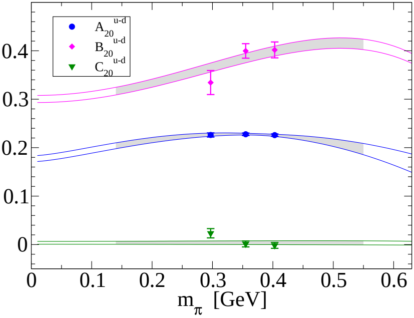
isovector

isoscalar

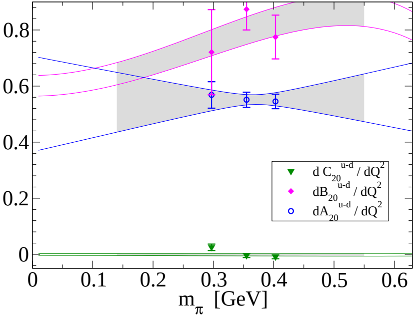


Chiral Extrapolations (isovector part)

forward values $Q^2 \to 0$



$d\{A_{20}, B_{20}, C_2\} / d\,Q^2, \quad Q^2 \to 0$



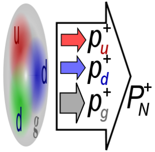
- simultaneously fit *forward values* and *slopes* at $Q^2 \to 0$
- Cov. Baryon χPT [Dorati et al Nucl. Phys. A798:96 (2008)]
- e.g., for $(u-d)$: 12 data points, 8 fit parameters, $\chi^2/dof \approx 1.5$

Quark Momentum and Angular momentum

Quark energy-momentum tensor $T_q^{\mu\nu}$

$$T_q^{\mu\nu} = \bar{q}\Big[\gamma^{\{\mu}\, i\overleftrightarrow{D}^{\nu\}} - \langle trace\rangle\Big] q$$

$$\langle N(P')|T_q^{\mu\nu}|N(P)\rangle \longrightarrow \{A_{20}, B_{20}, C_2\}(Q^2)$$

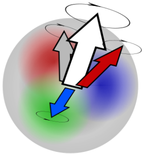


- quark momentum fraction

$$\boxed{\langle x\rangle_q = A_{20}^q(0)}$$

- quark angular momentum [X. Ji '97]:

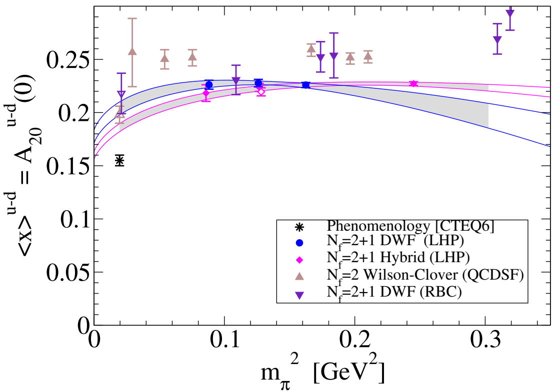
$$\boxed{J_q = \tfrac{1}{2}[A_{20}^q(0) + B_{20}^q(0)]}$$



Separating contributions to nucleon spin:

- quark spin $\boxed{S_q = \tfrac{1}{2}\Sigma_q = \tfrac{1}{2}\langle 1\rangle_{\Delta q}}$
- quark orbital angular momentum $\boxed{L_q = J_q - \tfrac{1}{2}\Sigma_q}$
- gluons : the rest $\boxed{J_{\text{glue}} = \tfrac{1}{2} - \tfrac{1}{2}\Sigma_q - L_q}$

Quark Momentum Fraction $\langle x \rangle_{u-d}$



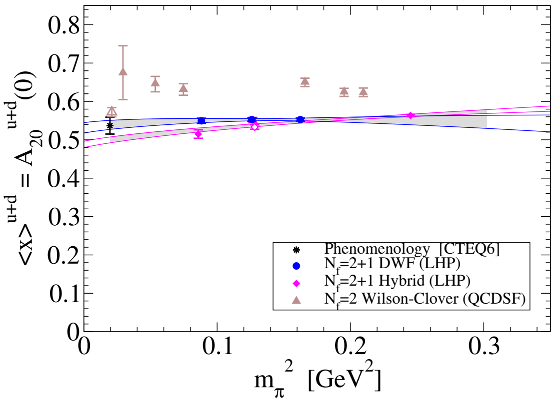
$$\langle x \rangle_q = A_{20}^q(0) = \int dx\, x\, (q(x) + \bar{q}(x))$$

Sources of discrepancy:

- ~~renormalization?~~
- ~~finite volume?~~
- sea quarks?
- fermion action?
- excited states?

- *perturbative* vs. *non-perturbative* renormalization
- agreement between $(2.5\ \text{fm})^3$ and $(3.5\ \text{fm})^3$ at $m_\pi = 350$ MeV
- results are consistently above the phenomenological value by $15 - 25\%$.

Quark Momentum Fraction $\langle x \rangle_{u+d}$



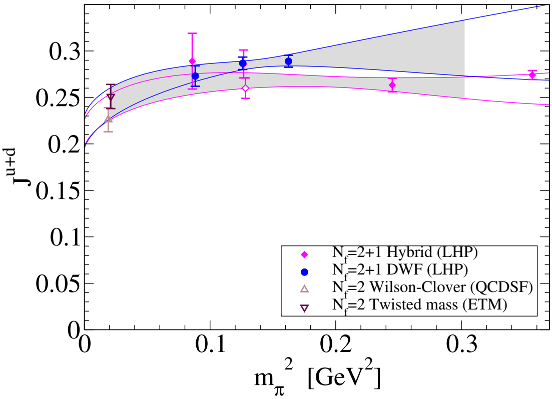
$$\langle x \rangle_q = A_{20}^q(0)$$
$$= \int dx\, x\, \big(q(x) + \bar{q}(x)\big)$$

Sources of discrepancy:

- ~~renormalization?~~
- ~~finite volume?~~
- sea quarks?
- fermion action?
- excited states?

- *perturbative* vs. *non-perturbative* renormalization
- agreement between $(2.5\text{ fm})^3$ and $(3.5\text{ fm})^3$ at $m_\pi = 350$ MeV
- qualitative agreement with phenomenology (no disc. contractions!)

Quarks Angular Momentum (1): J^{u+d}



Following [X. Ji PRL '97],

$$J_q^3 = \langle N| \int d^3x \, M^{012} |N\rangle,$$

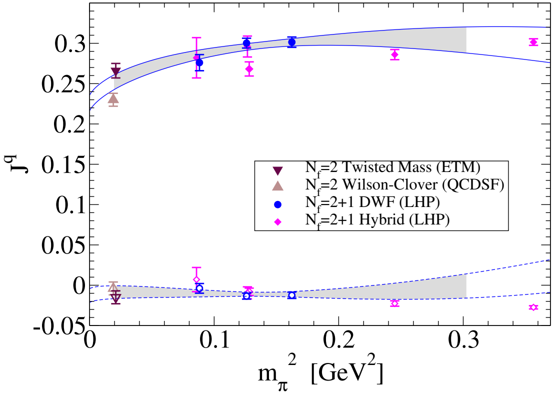
$$M^{\alpha\mu\nu} = x^\mu T_q^{\alpha\nu} - x^\nu T_q^{\alpha\mu}$$

and

$$\boxed{J^q = \frac{1}{2}\left[A_{20}^q(0) + B_{20}^q(0)\right]}$$

- Gluon contribution $J^g = \frac{1}{2} - J^q \sim 52\%$ of the nucleon spin
- result agrees with QCD sum rule estimations [Balitsky, Ji (1997)]

Quarks Angular Momentum (2): J^u, J^d



Following [X. Ji PRL '97],

$$J_q^3 = \langle N| \int d^3x\ M^{012} |N\rangle,$$

$$M^{\alpha\mu\nu} = x^\mu T_q^{\alpha\nu} - x^\nu T_q^{\alpha\mu}$$

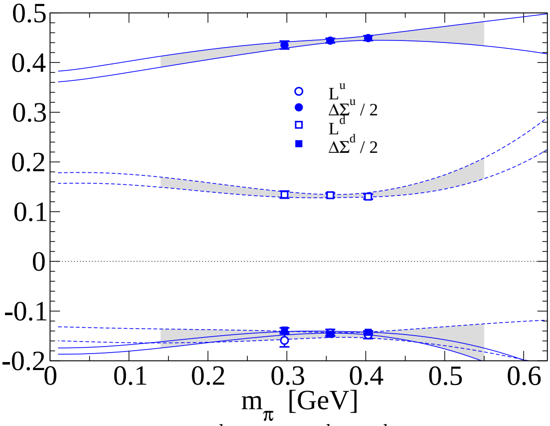
and

$$\boxed{J^q = \frac{1}{2}\left[A_{20}^q(0) + B_{20}^q(0)\right]}$$

Most contribution to the nucleon spine comes from u-quarks:

$$|J^d| \ll |J^u|$$

Quark Spin and OAM



$$L^q = J^q - S^q,$$

$$S_q = \frac{1}{2}\Delta\Sigma_q$$

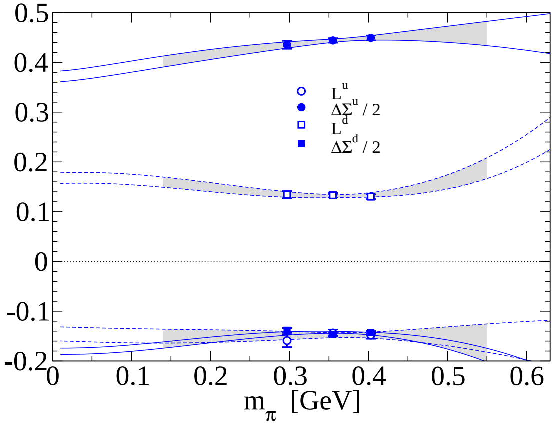
$$= \int dx\, (\Delta q(x) + \Delta\bar{q}(x))$$

$$|J^d| \ll \quad |S^d|, |L^d|$$

$$|L^{u+d}| \ll \quad |L^u|, |L^d|$$

In agreement with [Hägler *et al* (2007)]

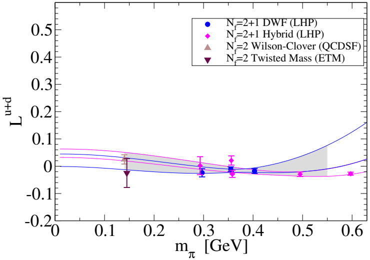
Quark Spin and OAM



$$L^q = J^q - S^q,$$

$$S_q = \frac{1}{2}\Delta\Sigma_q$$

$$= \int dx\,(\Delta q(x) + \Delta\bar{q}(x))$$



$$|J^d| \ll |S^d|, |L^d|$$

$$|L^{u+d}| \ll |L^u|, |L^d|$$

$$|L^{u+d}| \ll \frac{1}{2}$$

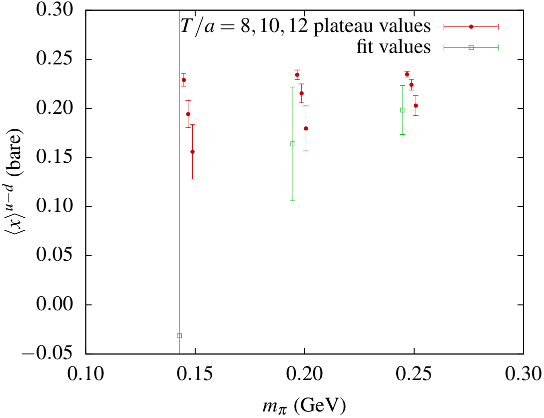
In agreement with [Hägler *et al* (2007)]

Mass Gap $\Delta M = M_{\text{exc}} - M_N$

Wilson-Clover $32^3 \times 48$, $m_\pi = 200$ MeV

	M[MeV]	aM	$a\Delta M$
π	200	0.118	
$2\pi/L$	334	0.196	
N	1021	0.600	
$N\pi\pi$	1421	0.835	0.235
$N\pi$	1463	0.860	0.260
"$N(1440)$"	(?)1520	0.894	0.294
$N\pi$	1637	0.962	0.362
$N\pi$	1786	1.050	0.450
"$N(1710)$"	(?)1790	1.052	0.452
	...		
C_{2pt} best fit	≈ 2130	1.25(19)	0.65(19)

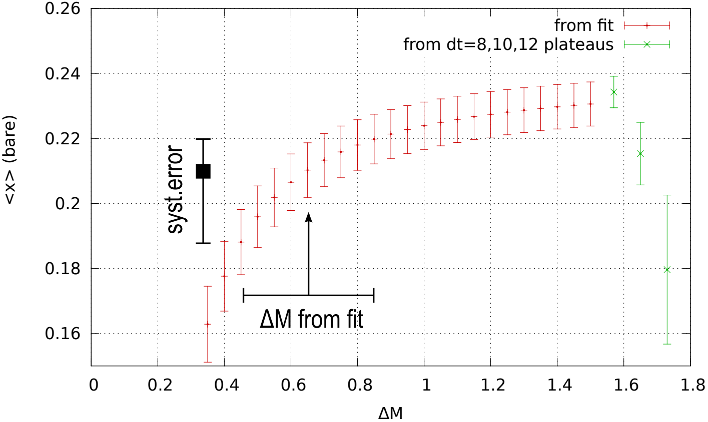
Momentum fraction $\langle x \rangle^{u-d}$: vary Δt



- Separate $\bar{N}$ and N with $\Delta t = 0.93,\ 1.16,\ 1.39$ fm
- $m_\pi \approx 150,\ 200$ and 250 MeV, $m_N \approx 0.97 \ldots 1.06$ GeV
- Fit to a 2-state model with *fixed* ΔM

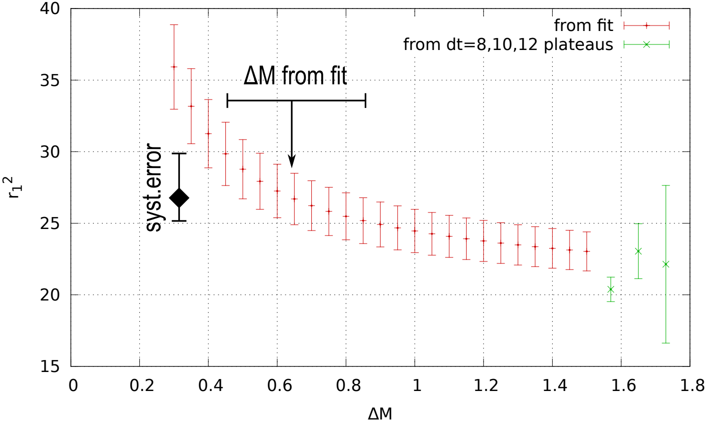
Also reported in [Renner et al (ETMC Collab)]

Momentum fraction $\langle x \rangle^{u-d}$: fix ΔM



- Separate $\bar{N}$ and N with $\Delta t = 0.93,\ 1.16,\ 1.39$ fm
- One point $m_\pi \approx 200$ MeV, $m_N \approx 1.00$ GeV
- Fit to a 2-state model with fixed ΔM

Dirac radius $\langle r_1^2 \rangle^{u-d}$: fix ΔM



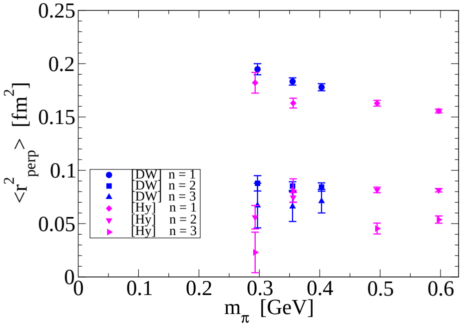
- $m_\pi \approx 200$ MeV, $m_N \approx 1.00$ GeV
- Separate $\bar{N}$ and N with $\Delta t = 0.93,\ 1.16,\ 1.39$ fm
- Fit to a 2-state model with fixed ΔM

Summary

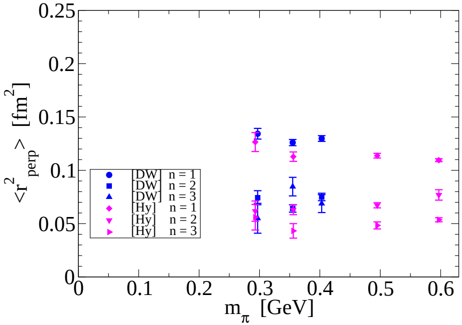
- Extrapolated results from different LQCD actions/volumes/discretizations agree near the physical point;
  - disagreement with phenomenology for $\langle x \rangle_{u-d}$
  - qualitative agreement with phenomenology for quark angular momentum
- Calculations at m_π^{phys} are necessary
- Systematic bias due to excited states is likely to increase towards m_π^{phys}
- Need a systematic "overhaul" of hadron structure calculations:
  - better control of excited states
  - additional statistics
  - disconnected contractions

BACKUP SLIDES

Generalized Radii

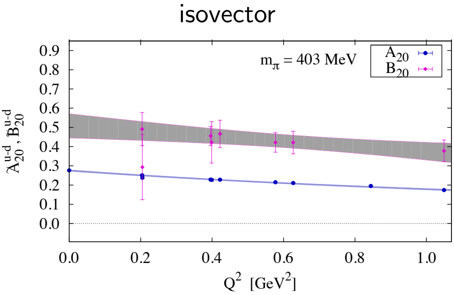


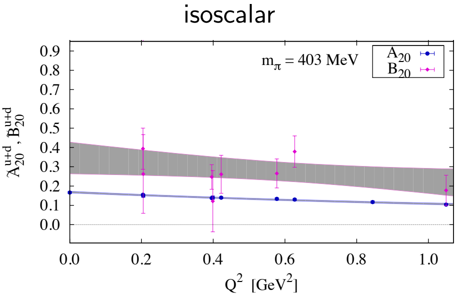
(spin-independent)



(spin-dependent)

$n = 2$ Spin-dependent GFFs $\tilde{A}_{20}$, $\tilde{B}_{20}$



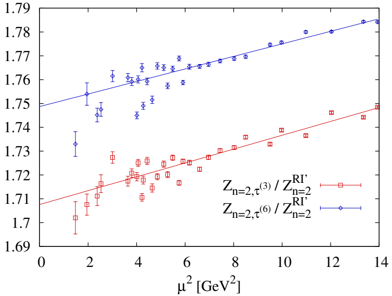


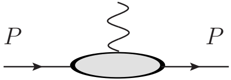
- On a lattice, rotational symmetry is broken $O(4) \to H(4)$
- Tensors split into irred. reps. of $H(4)$

For energy-momentum tensor there are two irred.' of H(4):

$$(\mathbf{4}_1)^{\otimes 2} = \mathbf{1}_1 \oplus \mathbf{6}_1 \oplus \underbrace{\mathbf{6}_3 \oplus \mathbf{3}_1}_{\text{trace=0, symm.}}$$

[Göckeler et al, hep-lat/9602029]
(both $\mathbf{6}_3$ and $\mathbf{3}_1$ are needed to extract gen. form factors)
Use non-perturbative renormalization (RI/MOM) for quark-bilinear $T_q^{\mu\nu}$:





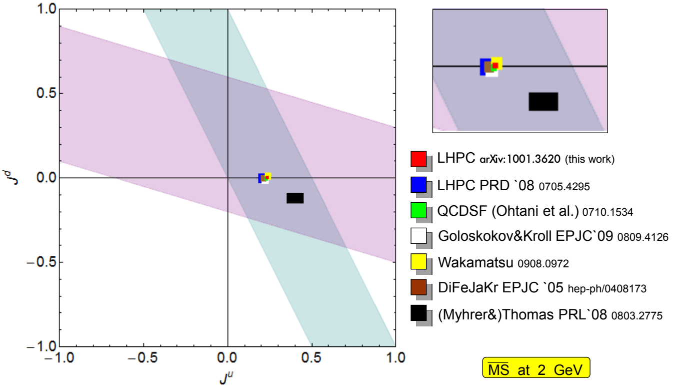
$$Z_{\mathcal{O}}^{\text{scale-indep.}} = \frac{Z_{\mathcal{O}}^{\text{lat}}(a\mu)}{Z_{\mathcal{O}}^{\text{pert}}(\mu)}$$

$$Z_{\mathcal{O}}^{\mathbf{3}_1} / Z_{\mathcal{O}}^{\mathbf{6}_3} \approx 0.98$$

Normalize to $\overline{\text{MS}}$ at $\mu^2 = (2\ \text{GeV})^2$

Quark Angular Momentum: p,n-DVCS

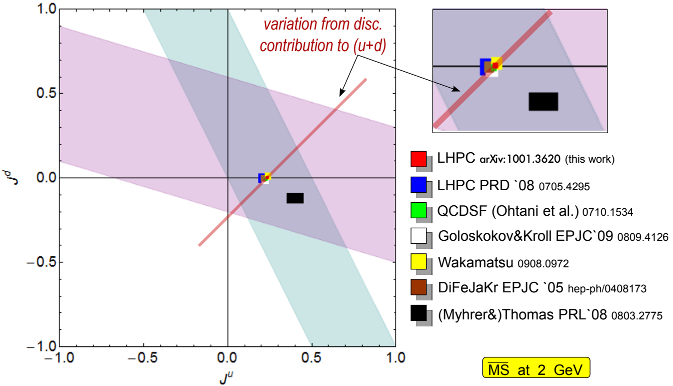
[JLab Hall A PRL`07; HERMES JHEP`08]



Picture: Ph. Hägler, MENU 2010, W&M, Virginia, USA; J.Phys.Conf.Ser.295:012009,2011

Quark Angular Momentum: p,n-DVCS

[JLab Hall A PRL`07; HERMES JHEP`08]



Picture: Ph. Hägler, MENU 2010, W&M, Virginia, USA; J.Phys.Conf.Ser.295:012009,2011