CORRECTED VERSION

TRANSVERSE ANGULAR MOMENTUM RELATIONS

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1 Expectation value of angular momentum: a reminder

As discussed in my Pedagogical lecture: We need expression for the *non-forward matrix elements* of the energy momentum tensor $t^{\mu\nu}$. Although $t^{\mu\nu}$ transforms as a tensor under Lorentz transformations, its non-forward matrix elements do not. Erroneously assuming that they do led to the incorrect expression for the angular momentum expectation value for a *transversely* polarized nucleon:

$$\langle J_i \rangle_{\text{incorrect}} = \frac{3p_0^2 - m^2}{4mp^0}\, s_i \tag{1}$$

which is quite different from the correct result derived in [1] :

$$\langle J_i \rangle = \frac{1}{2} s_i. \tag{2}$$

Most importantly, if we wish to establish sum rules when $p_0 \to \infty$, then with the incorrect expression you conclude *wrongly* that there cannot be a transverse polarization sum rule. With the correct expression, there is no special difficulty with the transverse case. Note that the error in Eq. (1) is NOT controversial and has been graciously acknowledged by the authors.

2 A philosophy of sum rules

My definition of a sum rule:

- A relation between quantities derived rigorously from the postulates of the theory
- Every quantity in the sum rule can be measured experimentally

This, in principle, allows fundamental tests of the theory.

My definition of a *semi-ideal* sum rule:

One or more terms have to be calculated on the lattice, which I don't consider exactly as an experimental measurement, or using models.

However, I claim that *non-ideal* sum rules, i.e. where not every quantity can be measured experimentally, can sometimes be very useful.

A classic example: In 1988 the EMC published their result that the contribution to the angular momentum of a longitudinally polarized proton arising from the spins of the quarks i.e. the first moment

$$\Delta\Sigma = \int_0^1 dx\, \Delta\Sigma(x) \tag{3}$$

was consistent with zero. This caused a major reaction, including the appearance of a paper entitled: *A crisis in the parton model: where, oh where is the proton spin* [2].

Why the reaction?? Why the expectation that, for a longitudinally polarized proton,

$$\langle J_z \rangle = \frac{1}{2} \approx \Delta\Sigma \quad ? \tag{4}$$

Because :

- Static non-relativistic models of the hadrons have zero orbital angular momentum in the ground state
- The approximate Ellis-Jaffe sum rule which suggested $\Delta\Sigma \approx 0.6$

In retrospect, both these arguments are naive, but the consequences were dramatic. The whole field burst into activity, both theoretical and experimental. The anomalous gluon spin contribution, which does not appear in the operator product expansion, was discovered. People began to worry about orbital angular momentum and Ji [3, 4, 5] eventually produced an extremely interesting longitudinal angular momentum **relation** and a genuine GPD **sum rule**.

3 A new transverse angular momentum relation

In my Pedagogical lecture we saw that the expectation value of $\boldsymbol{J}$ was related to the scalar functions appearing in the expression for the matrix elements of $t^{\mu\nu}$. There we dealt with one single free field. In QCD a similar expression holds for the quark and gluon pieces of $t^{\mu\nu}$.

The only difference is that $t^{\mu\nu}(quark)$ and $t^{\mu\nu}(gluon)$ are not separately conserved.

Also, whereas for the total angular momentum $\boldsymbol{J}_{can} = \boldsymbol{J}_{bel}$, this is not true for the separate quark and gluon pieces.

We shall use the Belinfante version:

$$\begin{aligned}\langle\, P', S' \,|\, t^{\mu\nu}_{q,\,bel}(0) \,|\, P, S\,\rangle &= [\bar{u}'\gamma^{\mu}u\,\bar{P}^{\nu} + (\mu \leftrightarrow \nu)]\mathbb{D}_q(\Delta^2)/2 \\ &+ \frac{1}{2}\left[\frac{i\Delta_\rho}{2M}\,\bar{u}'\sigma^{\mu\rho}u\,\bar{P}^{\nu} + (\mu \leftrightarrow \nu)\right][2\mathbb{S}_q(\Delta^2) - \mathbb{D}_q(\Delta^2)] \\ + \frac{\bar{u}'u}{2M}&\left[\frac{1}{2}[\mathbb{G}_q(\Delta^2) - \mathbb{H}_q(\Delta^2)](\Delta^\mu\Delta^\nu - \Delta^2 g^{\mu\nu}) + M^2\mathbb{R}_q(\Delta^2)g^{\mu\nu}\right] \qquad (5)\end{aligned}$$

where

$$\bar{P} = \frac{1}{2}(P + P') \qquad \Delta = P' - P \qquad u \equiv u(P, S) \qquad u' \equiv u(P', S') \qquad (6)$$

and the spinors are normalized to $\bar{u}u = 2M$. Note that the term $M^2\mathbb{R}g^{\mu\nu}$ is only allowed because we are dealing with a non-conserved density.

4 Connection with Generalized Parton Distributions

Comparing with the definition of GPDs given by Diehl [6] one finds

$$\int_{-1}^{1} dx x H_q(x, 0, 0) = \mathbb{D}_q = \text{momentum fraction carried by quarks} \qquad (7)$$

Further one sees that

$$\int_{-1}^{1} dx x E_q(x, 0, 0) = (2\,\mathbb{S}_q - \mathbb{D}_q). \qquad (8)$$

From eqs. (7, 8) one has that

$$\int_{-1}^{1} dx x H_q(x, 0, 0) + \int_{-1}^{1} dx x E_q(x, 0, 0) = 2\,\mathbb{S}_q. \qquad (9)$$

5 Connection with angular momentum: old and new results

We now show how the expectation values of $\boldsymbol{J}$ are related to the GPDs.

5.1 Longitudinally polarized nucleon

For the case of a *longitudinally* polarized nucleon moving in the z-direction Bakker, Leader and Trueman (BLT) [7] proved that $\mathbb{S}$ measures the expectation value of the z-component of $\boldsymbol{J}$. Hence eq. (9) can be written

$$\frac{1}{2}\int_{-1}^{1} dx x[H_q(x, 0, 0) + E_q(x, 0, 0)] = \langle\, J_z^{bel}(\text{quark})\,\rangle \qquad (10)$$

which is the relation first derived by Ji [3, 4, 5].
Note that it is NOT a sum rule: we don't know how to measure the RHS of the equation.
Now as mentioned above $\int_{-1}^{1} dxxH_q(x,0,0)$ measures the fraction of the nucleon's momentum carried by quarks and antiquarks of a given flavour, so that adding the gluon contribution[1]

$$\sum_{flavours} \int_{-1}^{1} dxxH_q(x,0,0) + \int_{0}^{1} dxxH_G(x,0,0) = 1. \tag{11}$$

Hence, summing eq. (10) over flavors and adding the analogous equation for gluons, one obtains

$$\begin{aligned} \frac{1}{2} + \sum_{flavors} \int_{-1}^{1} dxxE_q(x,0,0) + \int_{0}^{1} dxxE_G(x,0,0) &= \\ = \sum_{flavours} \langle\, J_z^{bel}(\text{quark})\,\rangle + \langle\, J_z^{bel}(\text{gluon})\,\rangle &= \frac{1}{2} \end{aligned} \tag{12}$$

so that

$$\sum_{flavors} \int_{-1}^{1} dxxE_q(x,0,0) + \int_{0}^{1} dxxE_G(x,0,0) = 0. \tag{13}$$

This is Ji's **sum rule**, a fundamental sum rule that has wide ramifications and can be shown to correspond to the vanishing of the nucleon's anomalous gravitomagnetic moment.

5.2 Transversely polarized nucleon

For the case of a *transversely* polarized nucleon, moving along the positive z-axis, it follows from BLT that

$$\langle\, J_T^{bel}(\text{quark})\,\rangle = \frac{1}{2M} \left[(P_0\,(2\,\mathbb{S}_q - \mathbb{D}_q) + M\,\mathbb{D}_q \right] \tag{14}$$

Substituting eqs. (7, 8) we obtain the new relation

$$\langle\, J_T^{bel}(\text{quark})\,\rangle = \frac{1}{2M} \left[P_0 \int_{-1}^{1} dxxE_q(x,0,0) + M \int_{-1}^{1} dxxH_q(x,0,0) \right] \tag{15}$$

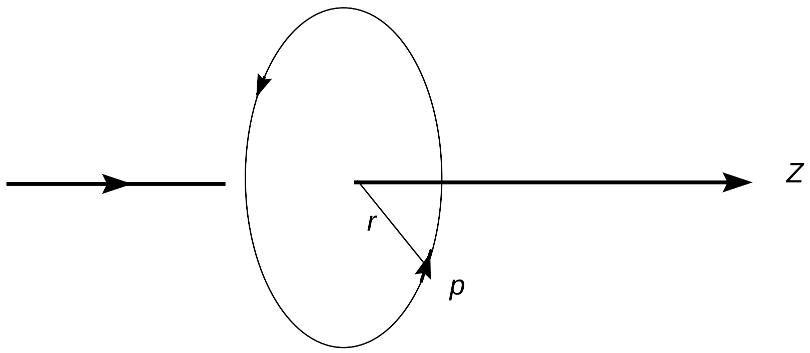
where P_0 is the energy of the nucleon. [See: *A new relation between transverse angular momentum and generalized parton distributions.*(arXiv:1109.1230v2; to appear as a Rapid Communication in PRD)]

The factor P_0 may seem unintuitive. However if we go the rest frame Eq. (15) reduces to the Ji result Eq. (10), as it should, since in the rest frame there is

[1] For gluons the integrals run from 0 to 1.

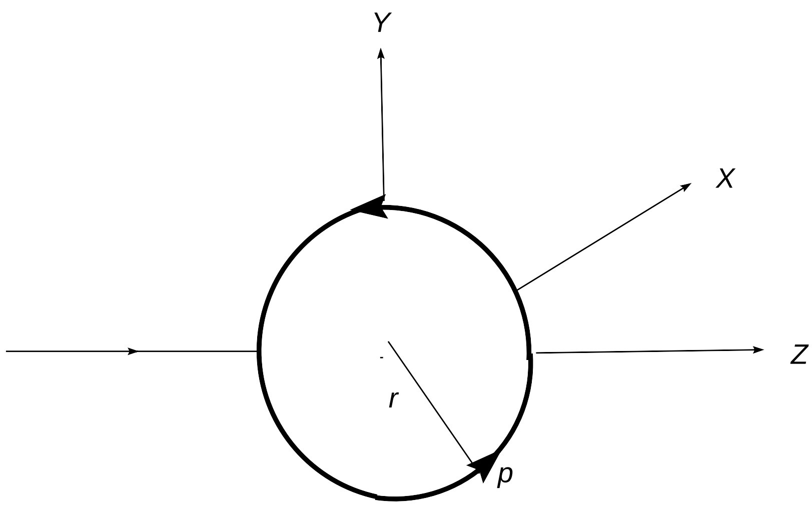
no distinction between X and Z directions. Moreover, for a classical relativistic system of particles, if one calculates the orbital angular momentum about the center of inertia for the system at rest, and then boosts the system one finds that the transverse angular momentum grows like P_0 [8].
In the first diagram the particle is orbiting in the XY plane and the vectors $\boldsymbol{p}$ and $\boldsymbol{r}$ aren't affected by a boost along OZ, so L_z is unchanged.

Figure 1: Boosting L_z



In the second diagram the particle is orbiting in the YZ plane so both $\boldsymbol{p}$ and $\boldsymbol{r}$ are affected by a boost along OZ, so J_x gets boosted.

Figure 2: Boosting L_x



Finally, if one sums Eq. (15) over flavors and adds the analogous gluon equation, one finds that the term proportional to P_0 disappears, as it ought to, as a consequence of Eq. (13), and using Eq. (11), one obtains the correct result for a transversely polarized nucleon

$$\sum_{flavors} \langle\, J_T^{bel}(\text{quark})\,\rangle + \langle\, J_T^{bel}(\text{gluon})\,\rangle = \frac{1}{2}. \tag{16}$$

6 Testing the relations

The relation Eq (15) can be used to test model results and also lattice calculations, since it is possible to treat a moving nucleon on a lattice.
Now BLT derived a sum rule for the *total* angular momentum of a transversely polarized nucleon, namely

$$\frac{1}{2} = \frac{1}{2} \sum_{\text{flavours}} \int dx\, [\Delta_T q(x) + \Delta_T \bar{q}(x)] + \sum_{q,\,\bar{q},\,G} \langle L_T \rangle \tag{17}$$

where $\Delta_T q(x) \equiv h_1(x)$ is the quark transversity distribution [2].
In this context it is important to realize that the quark part of Eq. (17) i.e.

$$\frac{1}{2} \sum_{\text{flavours}} \int dx\, [\Delta_T q(x) + \Delta_T \bar{q}(x)] + \sum_{q,\,\bar{q}} \langle L_T \rangle \tag{18}$$

cannot be identified with $\langle\, J_T^{bel}(\text{quark})\,\rangle$ in Eq. (15). The reason is the following. While for the *total* angular momentum there is no difference between Belinfante and canonical angular momentum, i.e.

$$\langle\, J_T^{bel}(\text{total})\,\rangle = \langle\, J_T^{can}(\text{total})\,\rangle \tag{19}$$

this is not true for the separate quark and gluon pieces, i.e.

$$\langle\, J_T^{bel}(\text{quark})\,\rangle \neq \langle\, J_T^{can}(\text{quark})\,\rangle \tag{20}$$

and in deriving Eq. (17) BLT used the property that $\boldsymbol{J}$ is the generator of rotations. As explained in detail in my Pedagogical talk and in [7] it is the canonical versions of the operators, $\boldsymbol{J}_{can}$, which are the generators of rotations. Thus the expression in Eq. (18) corresponds to $\langle\, J_T^{can}(\text{quark})\,\rangle$ and should not be confused with $\langle\, J_T^{bel}(\text{quark})\,\rangle$.

Consider now the Belinfante quark angular momentum operator which consists of a spin and an orbital term (the spin term is the same in Belinfante and canonical versions)

$$\boldsymbol{J}^{bel}(\text{quark}) = \boldsymbol{S}(\text{quark}) + \boldsymbol{L}^{bel}(\text{quark}) \tag{21}$$

[2] Note that the sum of quark and antiquark transversity densities does not correspond to the hadronic matrix element of a local operator and is unrelated to the chiral-odd tensor charge of the nucleon

where

$$\boldsymbol{S}(\text{quark}) = \int d^3x\, \bar{\psi}(x)\boldsymbol{\gamma}\gamma_5\psi(x). \tag{22}$$

For a nucleon in a state with covariant spin vector $\mathcal{S}$

$$\mathcal{S}^\mu = (\mathcal{S}_0, \boldsymbol{\mathcal{S}}) = \left(\frac{\boldsymbol{P}\cdot\boldsymbol{s}}{M},\ \boldsymbol{s} + \frac{\boldsymbol{P}\cdot\boldsymbol{s}}{M(P_0+M}\boldsymbol{P}\right) \tag{23}$$

where $\boldsymbol{s}$ is the rest frame spin vector, one has, for the expectation value

$$\begin{aligned}\langle \boldsymbol{S}(\text{quark})\,\rangle &= \frac{1}{2P_0}\langle P,\, \mathcal{S}\,|\,\bar{\psi}(0)\boldsymbol{\gamma}\gamma_5\psi(0)\,|\,P,\,\mathcal{S}\,\rangle \\ = \frac{a_0^f M}{P_0}\boldsymbol{\mathcal{S}} &= \frac{M}{P_0}\boldsymbol{\mathcal{S}}\,\frac{1}{2}\int_0^1 dx\,[\Delta q(x)+\Delta\bar{q}(x)]_{\overline{MS}}\end{aligned} \tag{24}$$

where a_0^f is the contribution to the nucleon axial charge a_0 from a quark plus antiquark of flavor f. Its expression in terms of the longitudinal polarized parton densities is scheme dependent [9]. For longitudinal polarization $\mathcal{S}_z = P_0/M$, yielding the usual result

$$\langle S_L(\text{quark})\,\rangle = \frac{1}{2}\int_0^1 dx\,[\Delta q(x)+\Delta\bar{q}(x)]_{\overline{MS}} \tag{25}$$

For transverse polarization, say in the X direction, $\mathcal{S}_x = 1$ so that

$$\langle S_T(\text{quark})\,\rangle = \frac{M}{2P_0}\int_0^1 dx\,[\Delta q(x)+\Delta\bar{q}(x)]_{\overline{MS}}. \tag{26}$$

In the case of longitudinal polarization, use of the Ji relation to estimate

$$\begin{aligned}\langle\, L_z^{bel}(\text{quark})\,\rangle &= \frac{1}{2}\left[\int_{-1}^1 dxxE_q(x,0,0) + \int_{-1}^1 dxxH_q(x,0,0)\right] \\ &- \frac{1}{2}\int_0^1 dx\,[\Delta q(x)+\Delta\bar{q}(x)]_{\overline{MS}}\end{aligned} \tag{27}$$

has proved extremely interesting in testing models, and in comparing different definitions of quark and gluon angular momentum.

The transverse relation can offer additional insights into **all of these**, by extending them to the new domain of transversely polarized and moving nucleons. In particular it would be very interesting to see how model calculations reproduce the P_0-dependence[3] in

$$\begin{aligned}\langle\, L_x^{bel}(\text{quark})\,\rangle &= \frac{1}{2M}\left[P_0\int_{-1}^1 dxxE_q(x,0,0) + M\int_{-1}^1 dxxH_q(x,0,0)\right] \\ &- \frac{M}{2P_0}\int_0^1 dx\,[\Delta q(x)+\Delta\bar{q}(x)]_{\overline{MS}}.\end{aligned} \tag{28}$$

[3] As expected Eq. (28) agrees with Eq. (27)in the rest frame where $P_0 = M$.

7 Summary

There exists a perfectly good transverse angular momentum relation connecting the transverse angular momentum of the quarks to GPDs, rather similar to Ji's relation for longitudinal angular momentum. It would be interesting to use the relations for testing models, but it should not be forgotten that all the terms in these relations depend on the renormalization scale μ.

References

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