CORRECTED VERSION

The EM tensor and its relation to angular momentum

PEDAGOGICAL

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1 Definition of the canonical energy-momentum tensor: free field

(Interacting fields more challenging: will be discussed all day Wednesday)

Lagrangian density, function of set of fields ϕ_r and their derivatives $\frac{\partial \phi_r}{\partial x^\mu}$

$$\mathcal{L} = \mathcal{L}\left(\phi_r, \frac{\partial \phi_r}{\partial x^\mu}\right) \tag{1}$$

Here r should be thought of as a spinor index for a Dirac field, a Lorentz index for a vector field, and not there for a scalar field.

1.1 Symmetry under space-time translations

Under an infinitesmal transformation

$$x_\mu \to x_\mu + \epsilon_\mu \tag{2}$$

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \epsilon_\mu \frac{\partial \mathcal{L}}{\partial x_\mu} \tag{3}$$

If $\mathcal{L}$ translationally invariant it does not depend explicitly on x_μ. Thus

$$\delta\mathcal{L} = \sum_r \left[\frac{\partial \mathcal{L}}{\partial \phi_r(x)}\, \delta\phi_r + \frac{\partial \mathcal{L}}{\partial(\partial\phi_r/\partial x_\mu)}\, \delta\left(\frac{\partial \phi_r}{\partial x_\mu}\right)\right] \tag{4}$$

Now

$$\delta\phi_r = \phi_r(x+\epsilon) - \phi_r(x) = \epsilon_\nu \frac{\partial \phi_r}{\partial x_\nu} \tag{5}$$

Also

$$\begin{aligned} \frac{\partial}{\partial x_\mu}\, \delta\phi_r &= \frac{\partial}{\partial x_\mu}\, \phi_r(x+\epsilon) - \frac{\partial}{\partial x_\mu}\, \phi_r(x) \\ &= \frac{\partial}{\partial(x_\mu+\epsilon)}\, \phi_r(x+\epsilon) - \frac{\partial}{\partial x_\mu}\, \phi_r(x) = \delta\left(\frac{\partial \phi_r}{\partial x_\mu}\right) \end{aligned} \tag{6}$$

Now use Euler-Lagrange equations to replace

$$\frac{\partial \mathcal{L}}{\partial \phi_r(x)} = \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial(\partial \phi_r / \partial x_\mu)} \tag{7}$$

Thus

$$\epsilon_\mu \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial}{\partial x_\mu} \left[\sum_r \frac{\partial \mathcal{L}}{\partial(\partial \phi_r / \partial x_\mu)} \epsilon_\nu \frac{\partial \phi_r}{\partial x_\nu} \right] \tag{8}$$

Since ϵ is arbitrary, eventually get

$$\frac{\partial}{\partial x^\mu} t^{\mu\nu} \equiv \partial_\mu \, t^{\mu\nu} = 0 \tag{9}$$

where

$$t^{\mu\nu} = -g^{\mu\nu} \, \mathcal{L} + \sum_r \frac{\partial \mathcal{L}}{\partial(\partial \phi_r / \partial x^\mu)} \frac{\partial \phi_r}{\partial x_\nu} = t^{\mu\nu}_{can} \tag{10}$$

1.2 Conservation of momentum

It follows that

$$P^\nu_{can} \equiv \int d^3x \, t^{0\nu}_{can} \tag{11}$$

is independent of time

$$\frac{\partial P^\nu_{can}}{\partial t} = 0 \tag{12}$$

All above is Classical and P^ν corresponds to the total energy and momentum and also generates space-time translations.

For a quantized free field P^ν_{can} becomes $\hat{P}^\nu_{can}$ and should be the generator of translations. Now a unitary operator produces translations:

$$U(a) \phi_r(x) U^{-1}(a) = \phi_r(x + a) \tag{13}$$

For an infinitesimal translation $x' = x + \epsilon$ we should then have

$$U(\epsilon) = e^{i\epsilon_\mu \, \hat{P}^\mu_{can}} \approx 1 + i\epsilon_\mu \, \hat{P}^\mu_{can} \tag{14}$$

which leads to the requirement

$$i[\hat{P}^\mu_{can} \, , \, \phi_r(x) \,] = \partial^\mu \, \phi_r(x) \tag{15}$$

Note:

1) One has to check that $\hat{P}^{\nu}_{can}$ really does satisfy the correct commutation relation. How can you do this? The procedure of quantization fixes various EQUAL TIME commutation relations. Since $\hat{P}^{\nu}_{can}$ is independent of time, you can choose the time variables in the fields in $\hat{P}^{\nu}_{can}$ to be the same as the time variable in $\phi_r(x)$ and then use the ETCs.

2) If one expresses $\hat{P}^{\nu}_{can}$ in terms of the creation and annihilation operators of the free field one sees that it corresponds to the total energy and momentum.

From now on will drop the HAT on operators.

2 Lorentz and rotational symmetry: the canonical angular momentum density

If there is invariance under an infinitesmal Lorentz transformation

$$x_\mu \to x'_\mu = x_\mu + \epsilon_{\mu\nu} x^\nu \qquad \epsilon_{\mu\nu} = -\,\epsilon_{\nu\mu} \tag{16}$$

similar arguments lead to the conservation law

$$\partial_\mu \, \mathcal{M}^{\mu\nu\lambda}_{can} = 0 \tag{17}$$

where

$$\mathcal{M}^{\mu\nu\lambda}_{can} = (x^\nu t^{\mu\lambda}_{can} - x^\lambda t^{\mu\nu}_{can}) + \frac{\partial \mathcal{L}}{\partial(\partial\phi_r/\partial x^\mu)} \left(\Sigma^{\nu\lambda}\right)_r^{\ s} \phi_s \tag{18}$$

The "spin" term $\left(\Sigma^{\nu\lambda}\right)_r^{\ s} \phi_s$ reflects what happens to the field at $x = 0$ i.e. where $0' = 0$

$$\phi_r(0) \to \phi_r(0) - \frac{\epsilon_{\nu\lambda}}{2} \left(\Sigma^{\nu\lambda}\right)_r^{\ s} \phi_s(0) \tag{19}$$

Forms of $\left(\Sigma^{\nu\lambda}\right)_r^{\ s}$ for various fields (note antisymmetry under $\nu \leftrightarrow \lambda$) :

Scalar: 0

Dirac spin 1/2: $\frac{1}{4}[\gamma^\nu \, \gamma^\lambda]_{rs}$ where r, s are spinor indices

Spin 1: $g^{\lambda s} g^\nu_r - g^{\nu s} g^\lambda_r$ where r, s are Lorentz indices.

The conserved angular momentum is

$$
\begin{aligned}
M^{ij}_{can} &\equiv \int d^3x\, \mathcal{M}^{0ij}_{can} \\
&= \underbrace{\int d^3x\,(x^i t^{0j}_{can} - x^j t^{0i}_{can})}_{\text{Orbital}} + \underbrace{\int d^3x\, \frac{\partial \mathcal{L}}{\partial(\partial \phi_r/\partial x^0)}\, \left(\Sigma^{ij}\right)_r^{\ s}\, \phi_s}_{\text{Spin}}
\end{aligned} \tag{20}
$$

Classically M^{ij}_{can} generates rotations in the ij plane and $J^k = 1/2\, \epsilon^{kij} M^{ij}$ is the angular momentum in the k direction.

In the quantized theory the operator M^{ij}_{can} must satisfy

$$
i[M^{ij}_{can}\,, \phi_r(x)] = (x^i\partial^j - x^j\partial^i)\phi_r(x) + (\Sigma^{ij})_r^{\ s}\phi_s(x) \tag{21}
$$

3 Modifications of $t^{\mu\nu}_{can}$: Belinfante version

Generally:

- 1) $t^{\mu\nu}_{can}$ is not symmetric under $\mu \leftrightarrow \nu$, whereas gravity couples to a symmetric tensor.
- 2) $\partial_\nu\, t^{\mu\nu}_{can} \neq 0$

Define

$$
H^{\mu\nu\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial \phi_r/\partial x^\mu)}\, \left(\Sigma^{\nu\lambda}\right)_r^{\ s}\, \phi_s = H^{\mu\lambda\nu} \tag{22}
$$

so that

$$
\mathcal{M}^{\mu\nu\lambda}_{can} = (x^\nu t^{\mu\lambda}_{can} - x^\lambda t^{\mu\nu}_{can}) + H^{\mu\nu\lambda} \tag{23}
$$

Then

$$
0 = \partial_\mu\, \mathcal{M}^{\mu\nu\lambda}_{can} = g^\nu_\mu\, t^{\mu\lambda}_{can} + x^\nu \underbrace{\partial_\mu\, t^{\mu\lambda}_{can}}_{=0} - g^\lambda_\mu\, t^{\mu\nu}_{can} - x^\lambda \underbrace{\partial_\mu\, t^{\mu\lambda}_{can}}_{=0} + \partial_\mu\, H^{\mu\nu\lambda} \tag{24}
$$

so that

$$
t^{\nu\lambda}_{can} - t^{\lambda\nu}_{can} = -\,\partial_\mu\, H^{\mu\nu\lambda} \tag{25}
$$

Define

$$
t^{\nu\lambda}_{bel} = t^{\nu\lambda}_{can} + \frac{1}{2}\,\partial_\mu \left[\overbrace{H^{\mu\nu\lambda} - \underbrace{(H^{\nu\mu\lambda} + H^{\lambda\mu\nu})}_{\text{symmetric under } \nu\leftrightarrow\lambda}}^{\text{antisymm under } \mu\leftrightarrow\nu} \right] \tag{26}
$$

Check:

$$
t^{\nu\lambda}_{bel} - t^{\lambda\nu}_{bel} = t^{\nu\lambda}_{can} - t^{\lambda\nu}_{can} + \partial_\mu\, H^{\mu\nu\lambda} = 0 \qquad \surd \tag{27}
$$

Check:

$$\partial_\nu \, t_{bel}^{\nu\lambda} = \frac{1}{2}\partial_\nu\partial_\mu \left[\overbrace{H^{\mu\nu\lambda} + (H^{\nu\mu\lambda} + H^{\lambda\mu\nu})}^{\text{antisymm under } \mu\leftrightarrow\nu} \right] = 0 \qquad \surd \tag{28}$$

Now note that

$$\begin{aligned} t_{bel}^{0\lambda} &= t_{can}^{0\lambda} + \frac{1}{2}\,\partial_\mu \left[H^{\mu 0\lambda} - (H^{0\mu\lambda} + H^{\lambda\mu 0})\right] \\ &= t_{can}^{0\lambda} + \frac{1}{2}\,\partial_j \left[H^{j 0\lambda} - (H^{0 j\lambda} + H^{\lambda j 0})\right] \end{aligned} \tag{29}$$

Thus

$$t_{bel}^{0\lambda}(x) = t_{can}^{0\lambda}(x) + \text{spatial divergence} \tag{30}$$

It follows that

$$P_{bel}^{\lambda} \equiv \int d^3x \, t_{bel}^{0\lambda}(x) = P_{can}^{\lambda} \tag{31}$$

if the fields vanish at infinity.

4 Modification of the angular momentum density: the Belinfante version

In the expression for $\mathcal{M}_{can}^{\mu\nu\lambda}$ substitute

$$\begin{aligned} t_{can}^{\nu\lambda} &= t_{bel}^{\nu\lambda} - \frac{1}{2}\,\partial_\rho \left[H^{\rho\nu\lambda} - (H^{\nu\rho\lambda} + H^{\lambda\rho\nu})\right] \\ &\equiv t_{bel}^{\nu\lambda} - \frac{1}{2}\,\partial_\rho G^{\rho\nu\lambda} \end{aligned} \tag{32}$$

After some straightforward algebra, find

$$\mathcal{M}_{can}^{\mu\nu\lambda} = (x^\nu t_{bel}^{\mu\lambda} - x^\lambda t_{bel}^{\mu\nu}) + \frac{1}{2}\partial_\rho[x^\lambda G^{\rho\mu\nu} - x^\nu G^{\rho\mu\lambda}] \tag{33}$$

Thus,

$$\begin{aligned} \mathcal{M}_{bel}^{\mu\nu\lambda} &\equiv (x^\nu t_{bel}^{\mu\lambda} - x^\lambda t_{bel}^{\mu\nu}) \\ &= \mathcal{M}_{can}^{\mu\nu\lambda} - \frac{1}{2}\partial_\rho[x^\lambda G^{\rho\mu\nu} - x^\nu G^{\rho\mu\lambda}] \end{aligned} \tag{34}$$

and

$$\mathcal{M}^{0\nu\lambda}_{bel} = \mathcal{M}^{0\nu\lambda}_{can} - \frac{1}{2}\partial_\rho[x^\lambda G^{\rho 0\nu} - x^\nu G^{\rho 0\lambda}] \tag{35}$$

Now note that

$$\partial_0[x^\lambda G^{00\nu} - x^\nu G^{00\lambda}] = 0 \qquad \text{because } G^{00\lambda} = 0 \tag{36}$$

Thus finally

$$\mathcal{M}^{0\nu\lambda}_{bel} = \mathcal{M}^{0\nu\lambda}_{can} - \frac{1}{2}\partial_j[x^\lambda G^{j0\nu} - x^\nu G^{j0\lambda}] \tag{37}$$

so that $\mathcal{M}^{0\nu\lambda}_{bel}$ and $\mathcal{M}^{0\nu\lambda}_{can}$ differ by a spatial divergence.
Hence,

$$\begin{aligned} M^{ij}_{bel} &\equiv \int d^3x\, \mathcal{M}^{0ij}_{bel} = \int d^3x\, (x^i t^{0j}_{bel} - x^j t^{0i}_{bel}) \\ &= M^{ij}_{can} \end{aligned} \tag{38}$$

if the fields vanish fast enough at infinity

NOTE: No spin term in M^{ij}_{bel} ! Somewhat unintuitive. Looks purely orbital.

5 Some food for thought

There are several delicate questions involved in the above, both at classical and quantum level.

5.1 Classical: a circularly polarized light beam

Applying the above to a free classical electromagnetic field, one gets

$$\boldsymbol{J}_{can} = \underbrace{\int d^3x\, (\boldsymbol{E} \times \boldsymbol{A})}_{\text{spin term}} + \underbrace{\int d^3x\, E^i(\boldsymbol{x} \times \boldsymbol{\nabla} A^i)}_{\text{orbital term}} \tag{39}$$

and

$$\boldsymbol{J}_{bel} = \int d^3x\, [\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B})] \tag{40}$$

Consider a left-circularly polarized (= positive helicity) beam propagating along OZ i.e. along $\boldsymbol{e}_{(z)}$:

$$A^\mu = \left(0,\, \frac{E_0}{\omega}\cos(kz - \omega t),\, \frac{E_0}{\omega}\sin(kz - \omega t),\, 0\right) \tag{41}$$

gives correct $\boldsymbol{E}$ and $\boldsymbol{B}$.
$\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{A}$ all rotate in the XY plane.
Consider the component of $\boldsymbol{J}_{can}$ along OZ. Note that

$$\boldsymbol{\nabla}\, A_{x,y} \propto \boldsymbol{e}_{(z)} \quad \text{so that} \quad (\boldsymbol{x}\times\boldsymbol{\nabla}\, A_{x,y})_z = 0 \tag{42}$$

so only the spin term contributes to $J_{can,\,z}$. Substituting for $\boldsymbol{E}$ and $\boldsymbol{A}$ one finds

$$J_{can,\,z} \text{ per unit volume} = \frac{E_0^2}{\omega} \tag{43}$$

The energy density is E_0^2, so for one photon per unit volume $E_0^2 = \hbar\omega$ so that

$$J_{can,\,z} \text{ per photon} = \hbar \qquad \surd \tag{44}$$

For Belinfante case

$$\boldsymbol{E}\times\boldsymbol{B} \propto \boldsymbol{e}_z \tag{45}$$

so that

$$J_{bel,\,z} \quad \text{per unit volume} = [\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})]_z = 0 \qquad \times \tag{46}$$

5.2 Quantum: what does it mean to say an operator vanishes at infinity?

The equivalence of canonical and Belinfante momentum and angular momentum depended on being able to neglect integrals of spatial divergences. For a classical field, which has numerical values, this is clear. But what about quantum operators?

Usually we are interested in expectation values of these operators i.e their forward matrix elements. For these it is usually possible to justify neglecting the contribution at infinity.

5.2.1 Spatial divergence of a local operator

A local operator $O(x)$ is defined at one space-time point and must satisfy the law of translation i.e.

$$e^{ia\cdot P}O(x)e^{-ia\cdot P} = O(x+a) \tag{47}$$

For the spatial divergence of a local operator we have

$$\begin{aligned}
\langle \boldsymbol{p}' \,|\, \partial_j O(x)\,|\boldsymbol{p}\,\rangle \quad &= \quad \frac{\partial}{\partial x^j}\,\langle \boldsymbol{p}' \,|\, O(x)\,|\boldsymbol{p}\,\rangle \\
= \frac{\partial}{\partial x^j}\,\langle \boldsymbol{p}'|e^{-ix\cdot P}O(0)\,e^{ix\cdot P}|\boldsymbol{p}\,\rangle &= \left[\frac{\partial}{\partial x^j}\,e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')}\right]\langle \boldsymbol{p}'|O(0)|\boldsymbol{p}\,\rangle \\
&= \quad i(p'^j - p^j)\langle \boldsymbol{p}' \,|\, O(0)\,|\boldsymbol{p}\,\rangle\, e^{-i\boldsymbol{x}\cdot(\boldsymbol{p}-\boldsymbol{p}')}
\end{aligned} \tag{48}$$

so that as $\boldsymbol{p}' \to \boldsymbol{p}$

$$\langle \boldsymbol{p} \,|\, \partial_j O(x) \,|\boldsymbol{p}\,\rangle = 0 \quad \text{if} \quad \langle \boldsymbol{p} \,|\, O(0) \,|\boldsymbol{p}\,\rangle \quad \text{is non-singular} \tag{49}$$

5.2.2 Spatial divergence of a compound operator

In the angular momentum case the spatial divergence involves an operator of the form $x\,O(x)$. While this is defined at one space-time point it is *not* a local operator.
To see this suppose that $Q(x) = x\,O(x)$ is a local operator. Then

$$Q(x) = e^{-ix\cdot P} Q(0)\, e^{ix\cdot P} = 0 \quad \text{for all x, since} \quad Q(0) = 0 \tag{50}$$

It is then much more difficult to show that one can neglect the expectation value of the spatial divergence of a compound operator. It can be done, but requires use of wave packets, as demonstrated by Shore and White [1].

6 The expectation value of the angular momentum

For illustration we will use $\boldsymbol{J}_{bel}$.

Consider its matrix elements for a Dirac particle whose state is specified by momentum $\boldsymbol{p}$ and covariant spin (pseudo)vector S i.e $|\boldsymbol{p},\, S\,\rangle$:

$$S^\mu = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{s}}{m},\, \boldsymbol{s} + \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{m(p_0+m)}\,\boldsymbol{p} \right) \tag{51}$$

where $\boldsymbol{s}$ is the rest-frame spin vector and the normalization is $S^2 = -1$.
Recall that $p\cdot S = 0$ for a physical particle.

We would like to evaluate $\langle \boldsymbol{p},\, S \,|\, M^{ij}_{bel} \,|\boldsymbol{p},\, S\,\rangle$. So, for example, we have

$$\langle \boldsymbol{p},\, S \,|\, J_{bel,\,z} \,|\boldsymbol{p},\, S\,\rangle = \int d^3x\, \langle \boldsymbol{p},\, S \,|\, [x\, t^{02}_{bel}(t,\boldsymbol{x}) - y\, t^{01}_{bel}(t,\boldsymbol{x})] \,|\boldsymbol{p},\, S\,\rangle \tag{52}$$

Consider the first term:
Use translational invariance to shift $t^{02}_{bel}(t,\boldsymbol{x}) = e^{-ix\cdot P} t^{02}_{bel}(0)\, e^{ix\cdot P}$.
Thus evaluate

$$\int d^3x\, x\, \underbrace{\langle \boldsymbol{p},\, S \,|\; t^{02}_{bel}(0) \,|\boldsymbol{p},\, S\,\rangle}_{\text{independent of } \boldsymbol{x}} \tag{53}$$

$$= \infty \qquad \text{or} \qquad = 0 \qquad ???? \qquad \text{totally ambiguous!} \tag{54}$$

The problem is an old one: In ordinary QM plane wave states give infinities.

The solution is an old one: Build a wave packet, a superposition of physical plane wave states.

Problem: How do you build a physical wave packet for a spin 1/2 particle?

Cannot use

$$\int d^3p\, \psi(\boldsymbol{p}) |\boldsymbol{p},\, S\,\rangle \quad \text{with } \textit{fixed} \quad S \tag{55}$$

since need $p \cdot S = 0$ i.e. $S = S(\boldsymbol{p})$.
(If you could have fixed S, it would have made life much simpler.)

So you are forced to consider

$$\int d^3p\, d^3p'\, \psi^*(\boldsymbol{p'})\, \psi(\boldsymbol{p})\, \langle \boldsymbol{p'},\, S' \,|\, M^{ij}_{bel}\, |\boldsymbol{p},\, S\,\rangle \quad =$$
$$= \int d^3p\, d^3p'\, \psi^*(\boldsymbol{p'})\, \psi(\boldsymbol{p})\, \langle \boldsymbol{p'},\, S' \,|\, (x^i t^{0j}_{bel} - x^j t^{0i}_{bel})\, |\boldsymbol{p},\, S\,\rangle \tag{56}$$

where $S' = S(\boldsymbol{p'})$.

Hence we need expressions for the non-forward matrix elements $\langle \boldsymbol{p'},\, S' \,|\, t^{\mu\nu}_{bel}\, |\boldsymbol{p},\, S\,\rangle$.

7 The matrix elements of the energy momentum tensor

In order to discuss angular momentum we will need to understand the structure of the matrix elements of $t^{\mu\nu}$.

First consider the matrix elements of something like the electromagnetic current j^{μ}_{em}.

Under Lorentz transformations j^{μ}_{em} transforms like a 4-vector. Consider its matrix elements for a Dirac particle whose state is specified by momentum $\boldsymbol{p}$ and covariant spin vector S i.e $|\boldsymbol{p},\, S\,\rangle$:

It would be *wrong* to write

$$\langle \boldsymbol{p'}\, S'\, | j^{\mu}_{em}(0)\, |\boldsymbol{p}\, S\,\rangle = Ap^{\mu} + Bp'^{\mu} + C(p \cdot S')S^{\mu} + D(p' \cdot S)S'^{\mu} + \ldots\ldots \tag{57}$$

because the *non-forward* matrix element does *not* transform as a 4-vector.

The reason is the Wigner (or Wick) rotation.

Let $U(\Lambda)$ be the unitary operator corresponding to a Lorentz transformation $p \to \Lambda\, p$. Then

$$U(\Lambda)\,|\boldsymbol{p},\, S\,\rangle \neq |\boldsymbol{\Lambda p},\, \Lambda S\,\rangle \tag{58}$$

.

There is also a Wigner rotation of the spin. For *forward*, and only for forward, matrix elements do the Wigner rotations cancel out. (For a pedagogical discussion see Chapter 2 of [2]).

Of course for the em current we never try to do something like Eq. (57). We write

$$\langle \boldsymbol{p'}\, S'\,|j^{\mu}_{em}(0)\;|\boldsymbol{p}\, S\,\rangle = \bar{u}(\boldsymbol{p'}\, S')\left[F_1(q^2)\,\gamma^{\mu} + \frac{\kappa}{2m}\,F_2(q^2)\,i\sigma^{\mu\nu}\,q_{\nu}\right]u(\boldsymbol{p},\, S) \tag{59}$$

and the spinors effectively absorb the Wigner rotations.

Similarly, we *cannot* say that the non-forward matrix element of $t^{\mu\nu}$ transforms like a second rank tensor. Doing so has led to some errors in the literature. (See [3] for a detailed explanation).

For any particle with spin, one must factor out the wave-functions (the analogues of the Dirac spinors), and what is left then transforms like a tensor.

Finally, the simplest way to handle the wave packets is to fix the rest frame spin vector $\boldsymbol{s}$, and to study

$$\int d^3p\, d^3p'\, \psi^*_{\boldsymbol{p_0},\,\boldsymbol{s}}(\boldsymbol{p'})\, \psi_{\boldsymbol{p_0},\,\boldsymbol{s}}(\boldsymbol{p})\, \langle \boldsymbol{p'},\, \boldsymbol{s}\,|\, M^{ij}_{bel}\,|\boldsymbol{p},\, \boldsymbol{s}\,\rangle \tag{60}$$

where $\psi_{\boldsymbol{p_0},\,\boldsymbol{s}}(\boldsymbol{p})$ is sharply peaked at $\boldsymbol{p} = \boldsymbol{p_0}$ so that $\boldsymbol{p'} - \boldsymbol{p}$ is small, and ultimately we take the limit $\boldsymbol{p'} - \boldsymbol{p} \to 0$.

The most general structure of the matrix elements of the conserved hermitian operator $t^{\mu\nu}(0)$, with no special symmetry under $\mu \leftrightarrow \nu$ is

$$\begin{aligned}
\langle \boldsymbol{p'},\, \boldsymbol{s}|t^{\mu\nu}(0)|\boldsymbol{p},\, \boldsymbol{s}\rangle \;\; = \;\; & \bar{u}(\boldsymbol{p'}, \boldsymbol{s})\{\mathbb{G}(p^{\mu}p^{\nu} + p'^{\mu}p'^{\nu}) \\
& + \mathbb{H}(p^{\mu}p'^{\nu} + p^{\nu}p'^{\mu}) + m\mathbb{S}[(p+p')^{\mu}\gamma^{\nu} + (p+p')^{\nu}\gamma^{\mu}] \\
& + (p\cdot p' - m^2)(\mathbb{G} - \mathbb{H})g^{\mu\nu} \\
& + m\mathbb{A}[(p+p')^{\mu}\gamma^{\nu} - (p+p')^{\nu}\gamma^{\mu}]\}u(\boldsymbol{p}, \boldsymbol{s})
\end{aligned} \tag{61}$$

where the $u(\boldsymbol{p},\, \boldsymbol{s}), u(\boldsymbol{p'},\, \boldsymbol{s})$ are the usual canonical Dirac spinors normalized to $\bar{u}u = 1$ and $\mathbb{G}, \mathbb{H}, \mathbb{S}$ and $\mathbb{A}$ are Lorentz scalars. Note that all terms, except the $\mathbb{A}$-term, are symmetric in $\mu \leftrightarrow \nu$.

Expanding this to first order in $\Delta^{\mu} = p'^{\mu} - p^{\mu}$, yields

$$
\begin{aligned}
\langle p+\Delta/2,\, \boldsymbol{s}\,|\, t^{\mu\nu}(0)\,|\, p-\Delta/2,\, \boldsymbol{s}\rangle \;&=\; 2\mathbb{D}p^\mu p^\nu - \frac{i\Delta_\rho}{M}\big\{\mathbb{S}(p^\mu \epsilon^{\rho\nu\alpha\beta} + p^\nu \epsilon^{\rho\mu\alpha\beta}) \\
&+\mathbb{A}(p^\mu \epsilon^{\rho\nu\alpha\beta} - p^\nu \epsilon^{\rho\mu\alpha\beta}) + \frac{\mathbb{D}}{M(p_0+M)} p^\mu p^\nu \epsilon^{0\rho\alpha\beta}\big\} S_\alpha p_\beta \qquad (62)
\end{aligned}
$$

where $\mathbb{D} = \mathbb{G} + \mathbb{H} + 2\mathbb{S}$, and we have ignored a term in $g^{\mu\nu}$, irrelevant for the angular momentum analysis.

Note that this is *not* a tensor in its indices μ, ν. Also note that for the symmetric $t^{\mu\nu}_{bel}(0)$ we simply put $\mathbb{A} = 0$.

Using this in the wave packet expression, relabeling the central momentum by $\boldsymbol{p}$ instead of $\boldsymbol{p_0}$, we find, after much labour,

$$
\begin{aligned}
\langle\, \psi_{\boldsymbol{p},\boldsymbol{s}}\,|\, M^{ij}_{bel}\,|\, \psi_{\boldsymbol{p},\boldsymbol{s}}\,\rangle \;&=\; \frac{1}{m}\big\{\frac{\mathbb{D}}{2(p_0+m)}\,(p^j\,\epsilon^{0i\alpha\beta} - p^i\,\epsilon^{0j\alpha\beta}) \\
&+\mathbb{S}\,\epsilon^{ij\alpha\beta}\big\}\, S_\alpha\, p_\beta \qquad (63)
\end{aligned}
$$

Here the normalization is

$$\langle\, \psi_{\boldsymbol{p},\boldsymbol{s}}\,|\, \psi_{\boldsymbol{p},\boldsymbol{s}}\,\rangle = 1 \qquad (64)$$

so this is the expectation value.

The energy of the state is related to the *forward* matrix element of t^{00} and leads to

$$\mathbb{D} = 1. \qquad (65)$$

We obtain the value of $\mathbb{S}$ by choosing a wave-packet with $\boldsymbol{p} = (0,0,p)$ and $\boldsymbol{s} = (0,0,1)$. This is then a helicity state $|\psi_{1/2}\rangle$ and should be an eigenstate of J_z with eigenvalue $1/2$. Hence for this state

$$\langle\psi_{1/2}|J_z|\psi_{1/2}\rangle = \langle\psi_{1/2}|M^{12}|\psi_{1/2}\rangle = 1/2 \qquad (66)$$

which implies

$$\mathbb{S} = 1/2 \qquad (67)$$

Finally we get a surprisingly simple result

$$\langle \boldsymbol{J}_{bel}\rangle = \frac{1}{2}\boldsymbol{s} \qquad (68)$$

Note:

- This holds both for longitudinal *and* transverse polarization
- Ge the same result for $\boldsymbol{J}_{can}$

The result is not at all intuitive!

So check this by a completely independent approach.

7.1 Approach via rotational properties of states

This is the simplest most direct approach.

a) It does not need wave packets because it does not use the energy momentum tensor.

b) It works for arbitrary spin, and equally well for *helicity states* or standard *canonical or boost states*.

Let $|p, m\rangle$ be a state with momemtum p which has spin projection m in the rest system.

Under a rotation about axis-i through an angle β:

$$U[R_i(\beta)]\,|p, m\rangle = |R_i(\beta)p, n\rangle\,\mathcal{D}^s_{nm}(R_W(p, \beta)). \tag{69}$$

where $U[R_i(\beta)]$ is the unitary operator effecting the rotation and $R_W(p, \beta)$ is the Wigner rotation.

For a pure rotation the Wigner rotation is very simple

$$R_W(p, \beta) = R_i(\beta). \tag{70}$$

Since the angular momentum operators are the generators of rotations

$$U[R_i(\beta)] = \exp(-i\beta J_i) \tag{71}$$

We have

$$\frac{\partial}{\partial\beta}U[R_i(\beta)] = -iJ_i\,\exp(-i\beta J_i) \tag{72}$$

so that

$$J_i = i\lim_{\beta\to 0}\frac{\partial}{\partial\beta}U[R_i(\beta)] \tag{73}$$

Thus

$$\begin{aligned}\langle p', m'|J_i|p, m\rangle &= i\,\frac{\partial}{\partial\beta}\langle p', m'|\,U[R_i(\beta)]\,|p, m\rangle\Big|_{\beta=0}\\ &= i\,\frac{\partial}{\partial\beta}\,\{\langle p', m'\,|\,R_i(\beta)p,\,n\rangle\mathcal{D}^s_{nm}[R_i(\beta))]\}_{\beta=0}\end{aligned} \tag{74}$$

Leads to exactly the same result!

$$\langle \boldsymbol{J}_{bel}\rangle = \frac{1}{2}\boldsymbol{s}. \tag{75}$$

What's more, it works for *any* spin ! Previous approach was strictly for spin 1/2.

7.2 Consequence of incorrect treatment of matrix element of $t^{\mu\nu}$ as a tensor

For standard (e.g. Bjorken-Drell) canonical spin states, as derived by Jaffe-Manohar,

$$\langle J_i \rangle_{JM} = \frac{1}{4Mp^0} \left\{ (3p_0^2 - M^2)s_i - \frac{3p_0 + M}{p_0 + M}(\boldsymbol{p} \cdot \boldsymbol{s})p_i \right\} \tag{76}$$

to be compared with

$$\langle J_i \rangle = \frac{1}{2} s_i \tag{77}$$

In general these are different. However, one may easily check that, for longitudinal polarization, i.e. when

$$\boldsymbol{s} = \hat{\boldsymbol{p}} \qquad \boldsymbol{p} = (0, 0, p) \qquad \boldsymbol{s} = (0, 0, 1) \tag{78}$$

$$\begin{aligned} \langle J_z \rangle_{JM} &= \frac{1}{4Mp^0} \left\{ (3p_0^2 - M^2) - \frac{3p_0 + M}{p_0 + M}\, p^2 \right] \\ &= \frac{1}{4Mp^0} \left\{ (3p_0^2 - M^2) - \frac{3p_0 + M}{p_0 + M}\, (p_0^2 - m^2) \right] \\ &= \frac{1}{4Mp^0} \left\{ (3p_0^2 - M^2) - (3p_0 + M)\, (p_0 - m) \right] \\ &= \frac{1}{2} = \frac{1}{2}\, s_z \end{aligned} \tag{79}$$

in agreement with Eq.(77).

Of particular importance is the case of transverse polarization $\boldsymbol{s} \perp \boldsymbol{p}$:

$$\langle J_i \rangle_{JM} = \frac{3p_0^2 - m^2}{4mp^0}\, s_i \tag{80}$$

which is quite different from Eq. (77)!

Moreover, if you assume that sum rules should only apply to the "infinite momentum frame" i.e. to the limit $p^0 \to \infty$, then you conclude, *wrongly*, that you cannot have a transverse polarization sum rule.
Note that the fact that Eq. (80) is incorrect is NOT controversial. It has been graciously acknowledged by the authors.

8 Concluding: beyond the single free field case

In the single free field case, aside from the incorrect conclusion about transverse sum rules, things are fairly straightforward and non-controversial.

However, when dealing with a system of interacting fields, all sorts of *new* problems and ambiguities arise. These will form the subject matter of Wednesday's talks.

It is also interesting to ask if you really need to consider only $p^0 \to \infty$ in order to derive a sum rule. This will be discussed in my talk on Thursday: *A new relation between transverse angular momentum and generalized parton distributions.*(arXiv:1109.1230v2; to appear as a Rapid Communication in PRD)

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