Meaning of gauge symmetry and quantization of gauge field

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Outline

- Non-relativistic QM and gauge potential
- QFT: Gauge choice, quantization, and gauge symmetry
- Gauge invariant extension (GIE)

Non-relativistic QM and gauge potential

- Consider a NR **charged** particle moving in an external gauge potential A^{μ} ,
- Hamiltonian

$$H = \frac{(\vec{P} - e\vec{A})^2}{2m} + e\phi$$

- Eigenvalue problem

$$H\psi_n(\vec{r}) = E_n\psi_n(\vec{r})$$

Wavefunction is gauge-dependent

- If one makes a classical gauge transformation,

$$A(\vec{r}) \rightarrow A(\vec{r}) + \nabla\alpha(\vec{r})$$

  the hamiltonian H changes to H' !

- The eigenfunction also changes,

$$\psi_n(\vec{r}) \rightarrow \psi_n(\vec{r})' = e^{ie\alpha(\vec{r})}\psi(\vec{r})$$

  but the eigen-energy E_n does not

$$H'\psi_n'(\vec{r}) = E_n\psi_n'(\vec{r})$$

- In particular, the wave function of the ground state hydrogen atom is uncertain by a position dependent phase factor

$$e^{ie\alpha(\vec{r})} e^{-ar}$$

Electron momentum distribution

- Electron momentum distribution in the H atom is a physical observable.
- It can be obtained by identifying the physical momentum as

$$-iD = -i\partial - eA$$

Which depends on the gauge choice, just like the hamiltonian!

- However, the expectation value

$$\psi_n^{\ *}\,(-iD)\,\psi_n$$

Is gauge independent!

Coulomb gauge

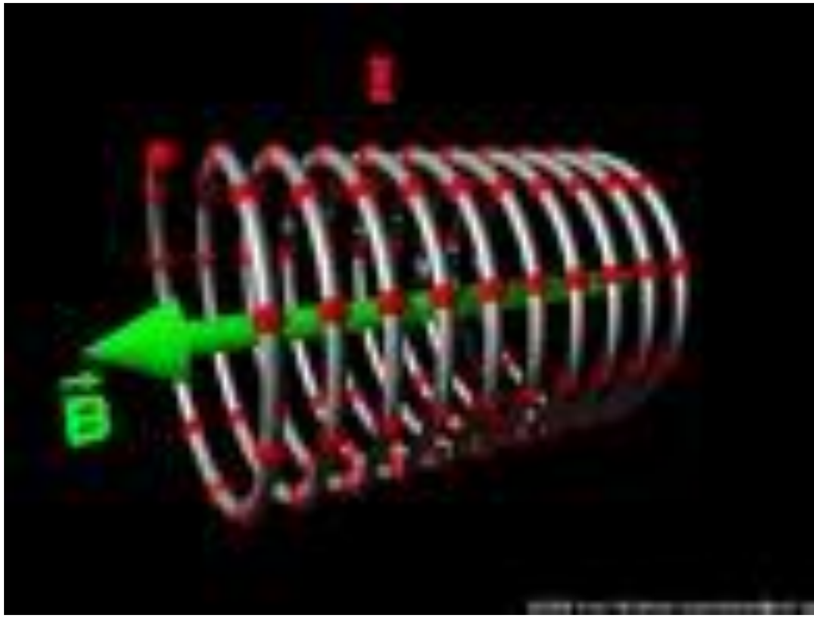
- All calculations for hydrogen atom is done in "Coulomb gauge" in which,

  $A=0$
- Therefore the physical momentum in Coulomb gauge can be calculated with $-i\partial$.
- $-i\partial$ by itself without specifying gauge is not a gauge invariant quantity. It is only physical in $A=0$ gauge.
- The connection between $-i\partial$ and A is an intrinsic feature of the gauge theory.

Mechanical Momentum -iD
A la Feynman lecture

- Consider a particle of wave function ψ near a solenoid.
- When the current is turned on suddenly, A is produce and electron gets a kinetic momentum kick
- this cannot be due to $-i\partial$ because the w.f. is continuous in time!
- Therefore, mechanical mom is

  $-iD = -i\partial - eA$



New gauge symmetry?

- In the last few years, "new gauge symmetry" has been proposed by X. S. Chen et al.

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Spin and Orbital Angular Momentum in Gauge Theories: Nucleon Spin Structure and Multipole Radiation Revisited

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Do Gluons Carry Half of the Nucleon Momentum?

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Comments

- X. Ji, Phys. Rev. Lett. 104, 039101 (2010) [1 pages]
- X. Ji, Phys. Rev. Lett. 106, 259101 (2011) [1 pages]

Break the gauge pot apart

$\vec{D}_{\text{pure}} \equiv \vec{\nabla} - ie\vec{A}_{\text{pure}}$, $\vec{A}_{\text{pure}} + \vec{A}_{\text{phys}} \equiv \vec{A}$, and the two are defined via

$$\vec{\nabla} \cdot \vec{A}_{\text{phys}} = 0, \tag{7}$$

$$\vec{\nabla} \times \vec{A}_{\text{pure}} = \vec{0}. \tag{8}$$

$$\vec{A}_{\text{pure}} \rightarrow \vec{A}'_{\text{pure}} = \vec{A}_{\text{pure}} + \vec{\nabla}\Lambda, \tag{9}$$

$$\vec{A}_{\text{phys}} \rightarrow \vec{A}'_{\text{phys}} = \vec{A}_{\text{phys}}, \tag{10}$$

Gauge symmetry "simplified"

- All the gauge-dependent part is in A_{pure}.
- A_{phys} contains purely physical d.o.f's. Thus one can construct all the physical observables using A_{phys}
- Covariant derivative can simply be

$$\vec{D}_{\text{pure}} \equiv \vec{\nabla} - ie\vec{A}_{\text{pure}}$$

This is not unique!

- If that were true, the A_{phys} must be unique! Otherwise, there is no unique physics.
- In fact, there is the condition does not produce a unique A, **it produces infinity number of A_{phys}!**

Proof: A_{phys} is constrained by

$$\nabla \cdot A_{phys} = 0$$

Imagine to make a transformation,

$$A_{phys} \rightarrow A_{phys} + \nabla\alpha$$

With the following condition

$$\nabla^2\alpha = 0$$

There are infinite number of solutions to this equation!

A uniform magnetic field

- Uniform magnetic field in z-direction

$$B = \nabla \times A$$

There are infinite number of choices of A_{phys}, satisfying

$$\nabla \cdot A_{phys} = 0$$

- Two examples

$\mathbf{A}_1 = (-y/2, x/2, 0)$

$\mathbf{A}_2 = (0, x, 0)$

Both satisfying the above condition!

Which one is physical???

Even it is unique…

- One can always add more conditions to make A_{phys} unique.
  - Why these conditions are physical?
  - Why not other conditions?
  - What is the difference from ordinary gauge choices?

Connection to physical E & B

- Forget about the conditions…
- Here is A_{phys} which has already been solved from the gauge field strength E and B

  $$A_{phys} = A_{phys}(E, B)$$

  Therefore, it must be gauge-invariant because it is made of gauge-invariant quantities.
- There are infinite number of relations like this. Again, A_{phys} is not unique!

Degree of freedom

- Consider the simple U(1) theory,

$$\mathcal{L} = \overline{\psi}(i\gamma_\mu D^\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- The vector potential A^μ
  - has two physical polarizations,
  - also mediates the static electric and magnetic interactions
  - has spurious degrees of freedom
- As it is, the dynamics of spurious dof is unspecified, thus, in pert. theory, this leads to a divergent photon propagator (we consider lattice QCD separately)

Gauge choice

- Choosing a gauge to quantize the theory is to constrain the non-physical dof. Therefore the photon propagator becomes finite.
- The electron wave function or propagator depends on gauge choice.
- All green's functions are gauge-dependent.
- S-matrix is gauge invariant, poles of propagators are gauge invariant.

Quantization: old-fashion approaches

- A^0 has no dynamics because it has no time-derivative, it appears as lagrange multiplier in the lagrangian.
- Gauge theory appears as a constrained system, the constraint is Gauss's law

$$\nabla \cdot E = \rho$$

- Take $A^0 = 0$, temporal or Weyl gauge
- The resulting constraints can be solved e
  - By imposing additional condition on $\nabla \cdot A$ (I)
  - Imposing $\nabla \cdot E = \rho$ as an operator constraint on states (II).

I. Radiation gauge $\nabla \cdot A = 0$

- Detailed discussion can be found in Bjorken and Drell.

$$H = \int d^3x \left[\frac{1}{2}E_\perp^2 + \frac{1}{2}(\nabla \times A_\perp)^2 + \vec{j} \cdot \vec{A}_\perp\right] + \frac{1}{2}\int d^3\vec{x} d^3\vec{y} \frac{j^0(\vec{x}) j^0(\vec{y})}{4\pi|\vec{x} - \vec{y}|} \ .$$

$$[A_\perp^i(\vec{x}), E_\perp^j(\vec{y})] = i\delta_{ij}^{\mathrm{Tr}}(\vec{x} - \vec{y})$$

$$\vec{A} = \int \frac{d^3k}{(2\pi)^3 2k^0} \sum_{\lambda=1}^{2} \left[\vec{\epsilon}(k,\lambda) a(k,\lambda) e^{-ikx} + \vec{\epsilon}^*(k,\lambda) a^\dagger(k,\lambda) e^{ikx}\right] \ .$$

$$D_0^{\mu\nu} = \frac{i}{k^2 + i\epsilon}\left[-g^{\mu\nu} - \frac{k^\mu k^\mu}{(k\cdot\eta)^2 - k^2} + k\cdot\eta \frac{(k^\mu \eta^\nu + k^\nu \eta^\mu)}{(k\cdot\eta)^2 - k^2} - \frac{k^2 \eta^\mu \eta^\nu}{(k\cdot\eta)^2 - k^2}\right] \ .$$

Approach II

- Enlarged Hilbert space

$$H = \int d^3x \left[\frac{1}{2}\vec{E}^2 + \frac{1}{2}(\nabla \times \vec{A})^2 + \vec{j} \cdot \vec{A} \right] \ .$$

$$[A_i(x), E_j(y)] = i\delta_{ij}\delta^3(\vec{x} - \vec{y}) \ .$$

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \sum_{\lambda=1}^{3} \left[\vec{\epsilon}(k,\lambda)a(k,\lambda)e^{-ikx} + \vec{\epsilon}^*(k,\lambda)a^\dagger(k,\lambda)e^{ikx} \right]$$

- Physical space is determined by the constraint

$$(\nabla \cdot \vec{E}^{(+)} - j^0)|\mathrm{phys}\rangle = 0$$

Lorentz gauge $\partial_\mu A^\mu = 0$

- Dynamics is given for the non-physical degrees of freedom

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{\lambda}{2}(\partial \cdot A)^2 .$$

$$A_\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2k^0} \sum_{\lambda=0}^{3} \left[a_{k\lambda}\epsilon_\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger_{k\lambda}\epsilon^*_\mu(k,\lambda)e^{ikx}\right] .$$

$$[a(k\lambda), a^\dagger(k'\lambda')] = -g^{\lambda\lambda'} 2k^0 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

- Selecting physical states

$$\partial^\mu A^{(+)}_\mu |\mathrm{phys}\rangle = 0 .$$

Non-unique physical states

- Lorentz gauge does not fix the gauge completely. Thus, there is an infinite number of physical states, which correspond to additional gauge transformation.

Assuming the photon momentum is along the z direction, we can express the above condition in terms of photon annihilation operators

$$[a(k,0) - a(k,3)]|\text{phys}\rangle = 0 \ . \qquad (50)$$

Define $a(k\pm) = (a(k0) \pm a(k3))/\sqrt{2}$, the commutation relations $[a(k\pm), a^\dagger(k\pm)] = 0$ follow and the photons created by $a^\dagger(k\pm)$ have zero norm! On the other hand, $[a(k\pm), a^\dagger(k\mp)] \neq 0$. The constraint in Eq. (50) implies that the physical states contain no $a^\dagger(k+)$ type of photons, but may contain arbitrary linear combinations of $a^\dagger(k-)$ type of photons. This is allowed because of the zero-norm property. Therefore, the "physical" states defined by Eq. (50) are not completely physical!

Gauge symmetry

- How to prove gauge invariance of a matrix element?
  $< \psi|O|\psi >$
- The physical state is gauge-dependent. The states in the different gauges are not even in the same Hilbert spaces. Therefore, there is a super G which transforms from one gauge to another,
  $$|\psi' > = G|\psi >$$
- The only way to guarantee gauge symmetry is that the operator must be gauge invariant
  $$O = GOG^{-1}$$

Path integral method

- Non-operator, state space formulation
- Any gauge conditions work so long as FP determinant is non-vanishing
- Easy to prove if the operator is gauge invariant the matrix element is also invariant.

$$\langle \hat{O}(A) \rangle = \frac{1}{\mathcal{N}} \int [DA_\mu] O(A) \Delta_F(A) \delta(F(A)) \exp[iS] \ .$$

[Does the gluon spin contribute in a gauge invariant way to nucleon spin?.](#)
[Pervez Hoodbhoy](#), [Xiang-Dong Ji](#) ([Maryland U.](#)). UMD-PP-00-008, DOE-ER-40762-190. Aug 1999. 7 pp.
Published in **Phys.Rev. D60 (1999) 114042**
e-Print: **hep-ph/9908275**

How to quantize $A = A_{phys} + A_{pure}$?

- There is no known way to quantize the theory with both A_{pure}, and A_{phys} as quantum mechanical degrees of freedom.
- Quantization only makes sense in Coulomb/radiation gauge where one can choose $A_{pure} = 0$
- However, in any other gauge, no one knows how to make the quantization either in canonical way or in path integral formulation.

Gauge invariant extention (GIE)

- Consider a gauge variant operator O
- Define its matrix element in a particular gauge
- Require that in any other gauge, the matrix element has to be the same,
- This is called GIE.

Gluon spin

- Defined as in any gauge

$$S_g = (\vec{E} \times \vec{A})^3$$

- Assume its matrix element in A+ =0 gauge as physical, then in any other gauge, the operator becomes

$$S_g^{\rm inv} = \frac{i}{2}\int \frac{dx}{x}\int d\xi^- e^{ix\xi^- P^+} F^{+\alpha}(0) L_{[0,\xi^-]} \tilde{F}^{+}{}_{\alpha}(\xi^-) \ ,$$

- Obviously gauge invariant, in any gauge, it generates the same answer.

Comments

- S_g(inv) can be measured in exp. through the integral of the gluon distribution.
- One can define GIE of spin in any other gauge, one gets a different answer.
- S_g (inv) has no spin interpretation in any other gauge.
- S_g (inv) is non-local and has not proper Lorentz symmetry properties.

Locality and Lorentz Symmetry

- "physical quantities" become **non-local**

$$\vec{A}_{\text{pure}} = \frac{\vec{\nabla}}{\nabla^2}\vec{\nabla}\cdot\vec{A}$$

$$\vec{A}_{\text{phys}} = \vec{A} - \frac{\vec{\nabla}}{\nabla^2}\vec{\nabla}\cdot\vec{A}$$

- **Lorentz transformation** of A_{phys} becomes a mess.
- All physical theory and observables must obey Lorentz symmetry, even when constructing with only physical degrees of freedoms.
- In fact, in any effective field theory with correct physical degrees of freedoms, one must go through constructions will all symmetry taken into account properly!