Effective Field Theory for Bound State Reflection

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In collaboration with Dean Lee

Light Nuclei From First Principles

September 28, 2012

arXiv:1008.5187v2, Eur.Phys.J.A47:41,2011

arXiv:1206.6280

Outline

Effective Field Theory

- Lattice Effective Field Theory

Bound State Reflection

- Motivation
- Reflection Phase Shift
- Alpha Particle in a Box
- Shallow two-body bound states
- Equal masses in one dimension
- Low-energy effective potential
- Numerical Results and comparison

Summary

Still to do

Effective Field Theory

•Define short and long distance scales

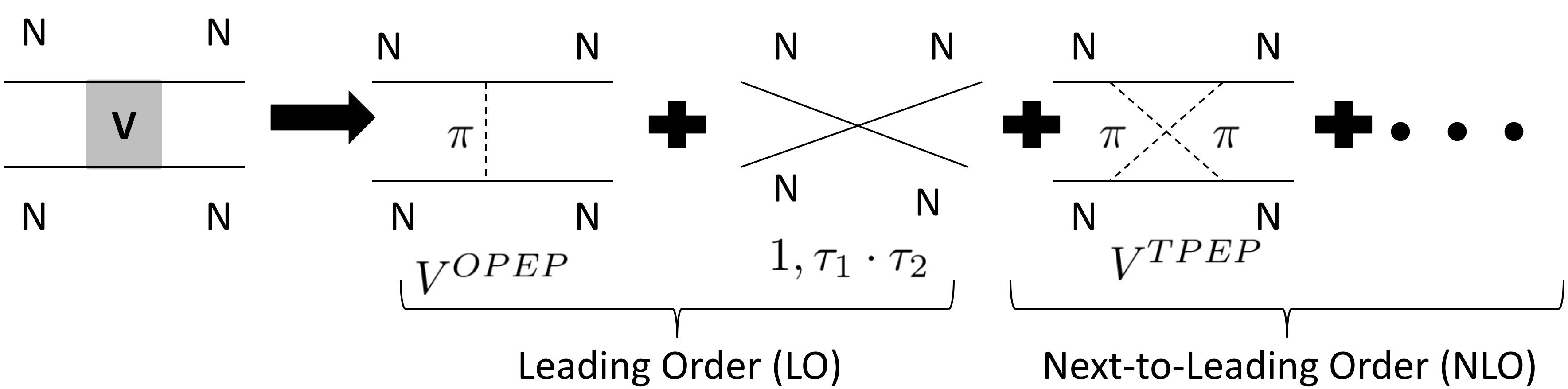
<u>Long distance:</u> $Q_L \ll \Lambda$
Low particle momenta

<u>Short distance:</u> Λ
momentum cutoff

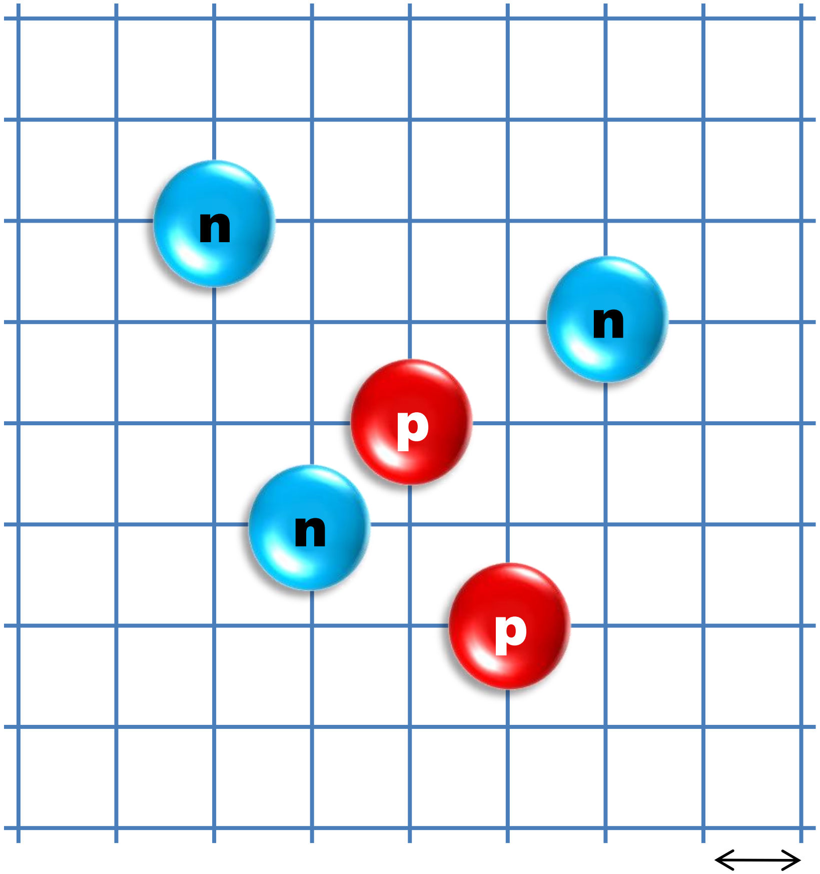
•Integrate out short distance degrees of freedom

$$H_{eff} = H^{(0)} + H^{(1)} + H^{(2)} + \cdots$$

•Derive an effective Hamiltonian as an expansion in some small expansion parameter



Lattice Effective Field Theory



$a \sim 1 - 5$ fm

Lattice Effective Field Theory

•Calculate many body properties of nuclear and neutron matter — *Muller et. al., PRC61(2000)044320*

•Elastic scattering (NN, dimer-fermion…) — *Baur et. al., arXiv:1206.1765v1*

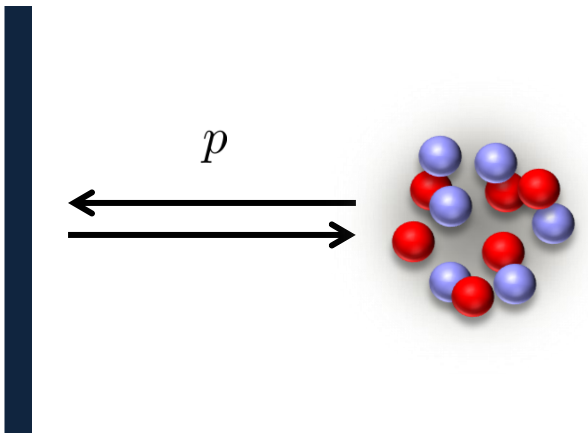
•Energy spectra up to carbon-12 up to NNLO — *Epelbaum et. al., Eur. Phys. J. A45,335-352(2010)*

•Hoyle state calculations — *Epelbaum et. al., arXiv:1208.1328v1*

- •geometric structure of Hoyle state
- •Spin-2 rotational excited state of carbon-12
- •Rms charge radius
- •Quadrupole moments

•Cold atom calculations — *Bulgac et. al., Int J. Mod. Phys. B20, 5165(2006)*

Bound State Reflection



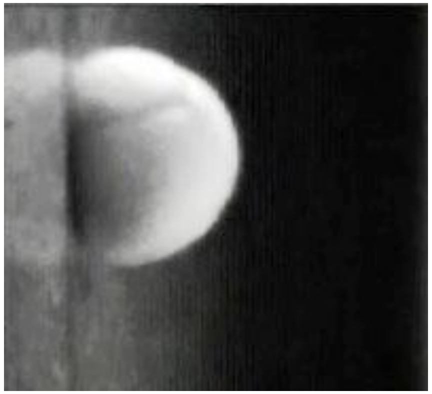
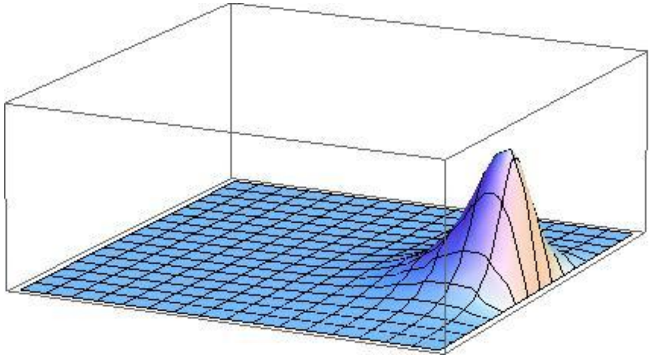
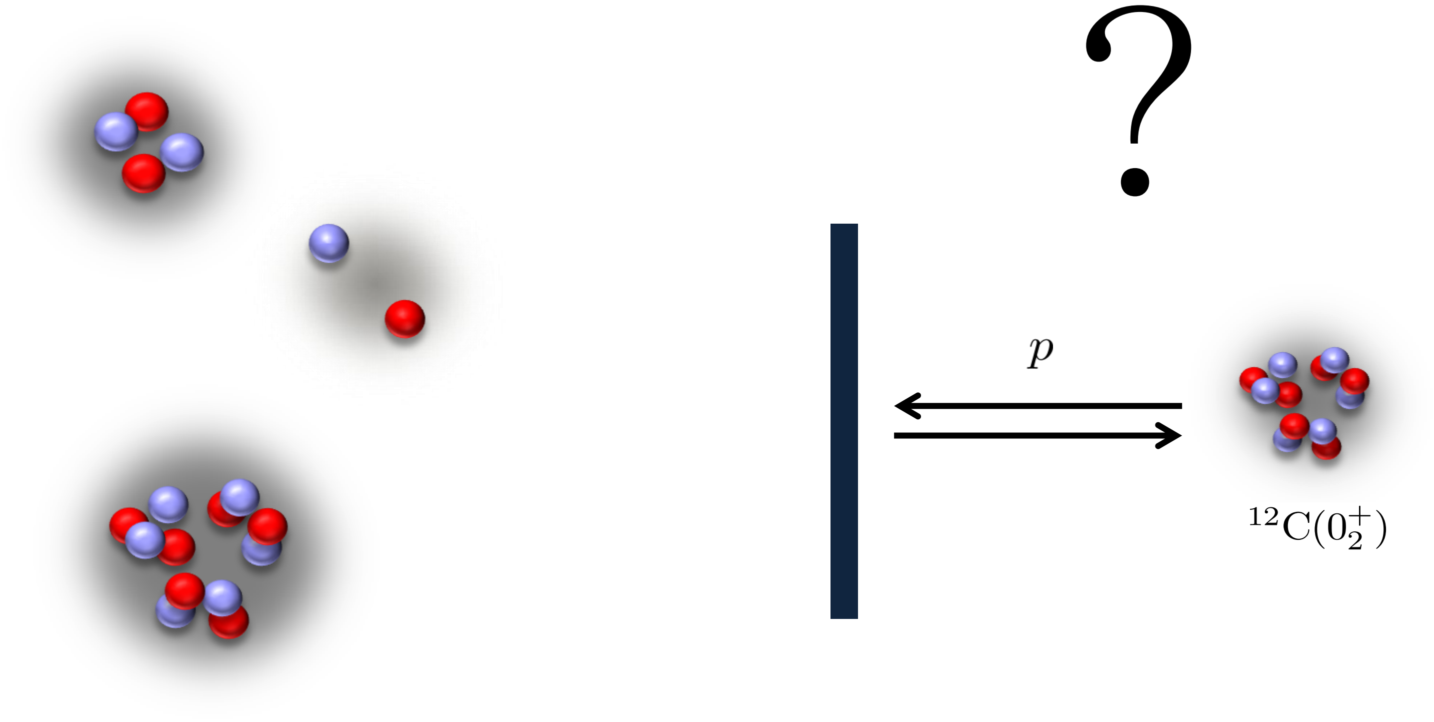


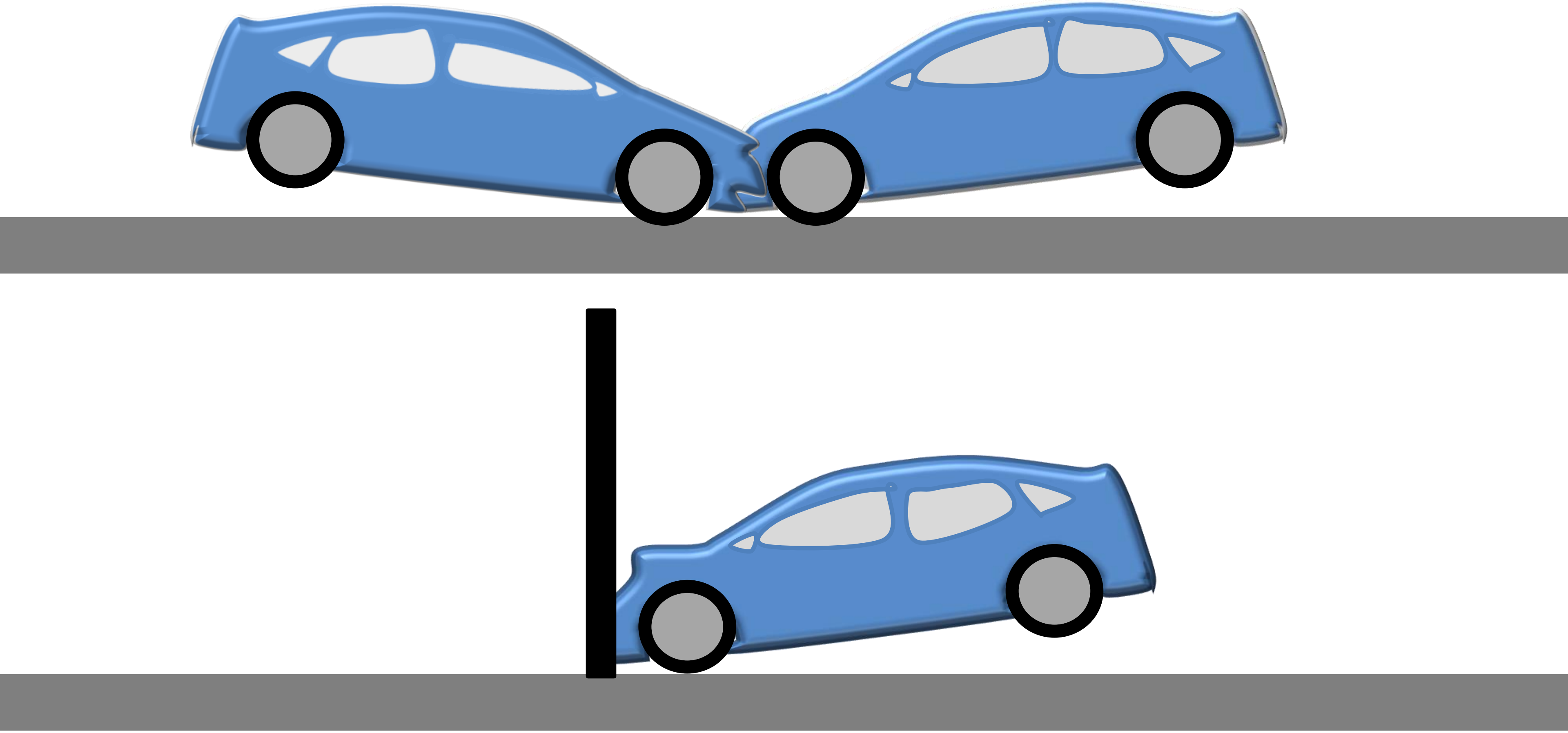
Photo from trevorshp.com

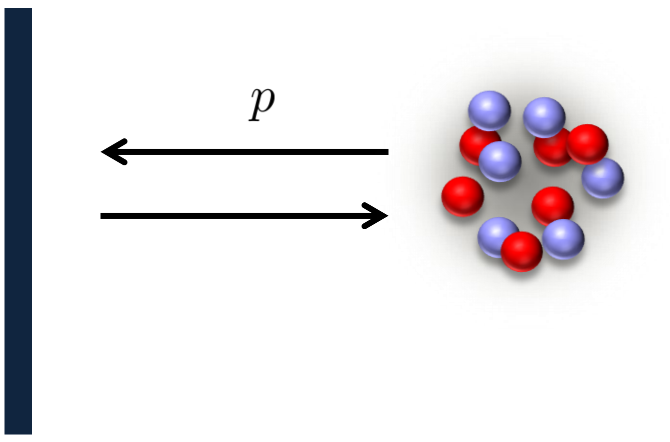


Motivation

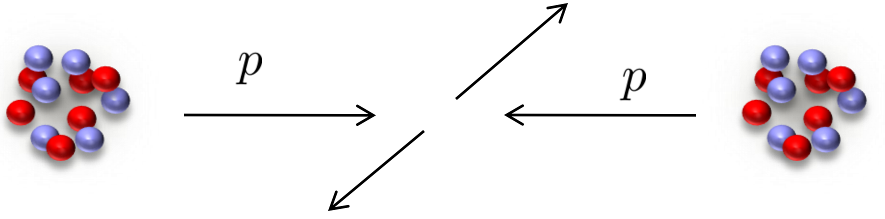
Nuclear Structure



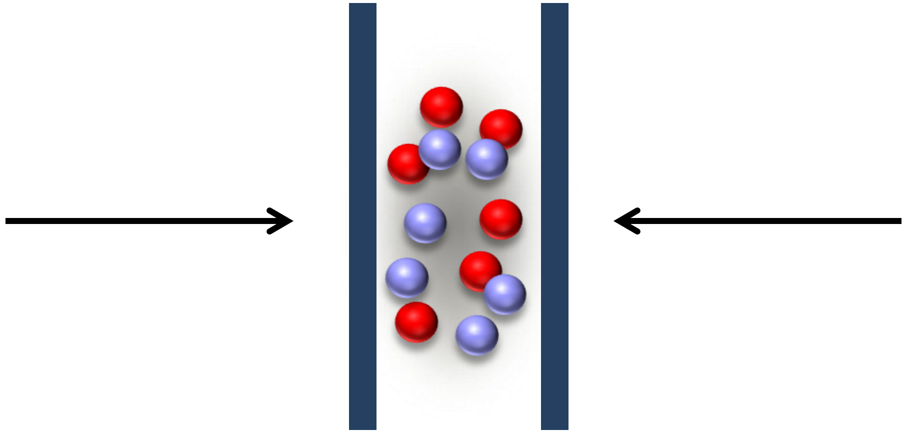




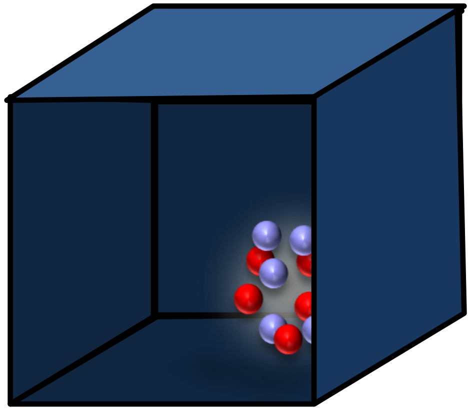
VS



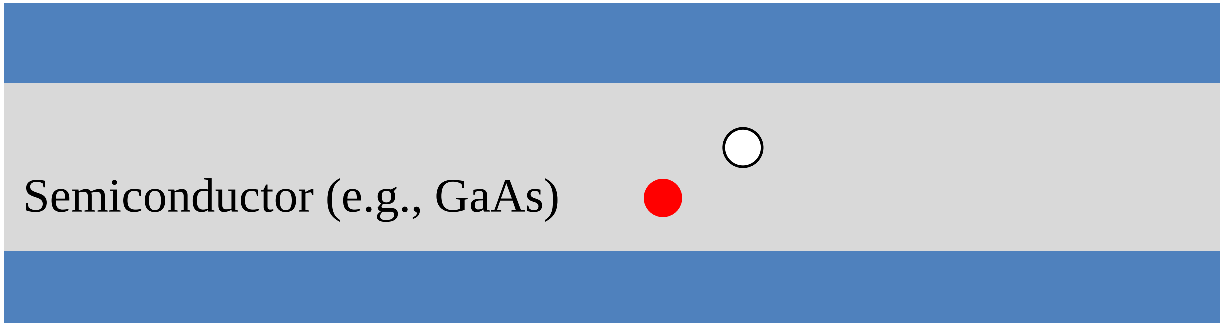
Nuclei under pressure



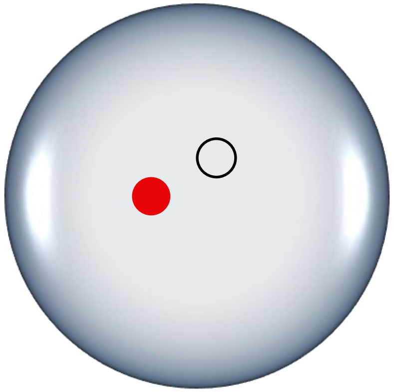
Confined Nuclei



Quantum well



Quantum Dot



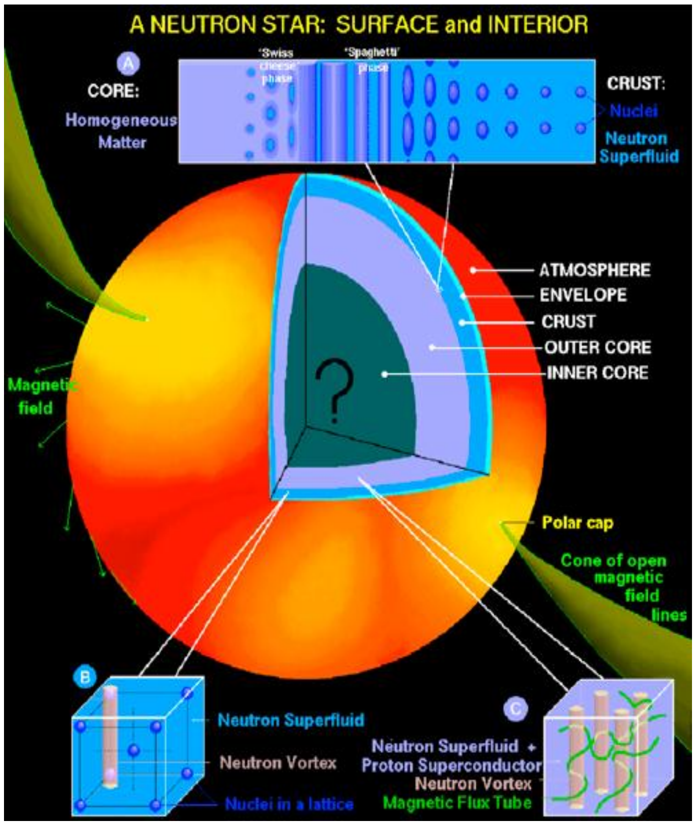
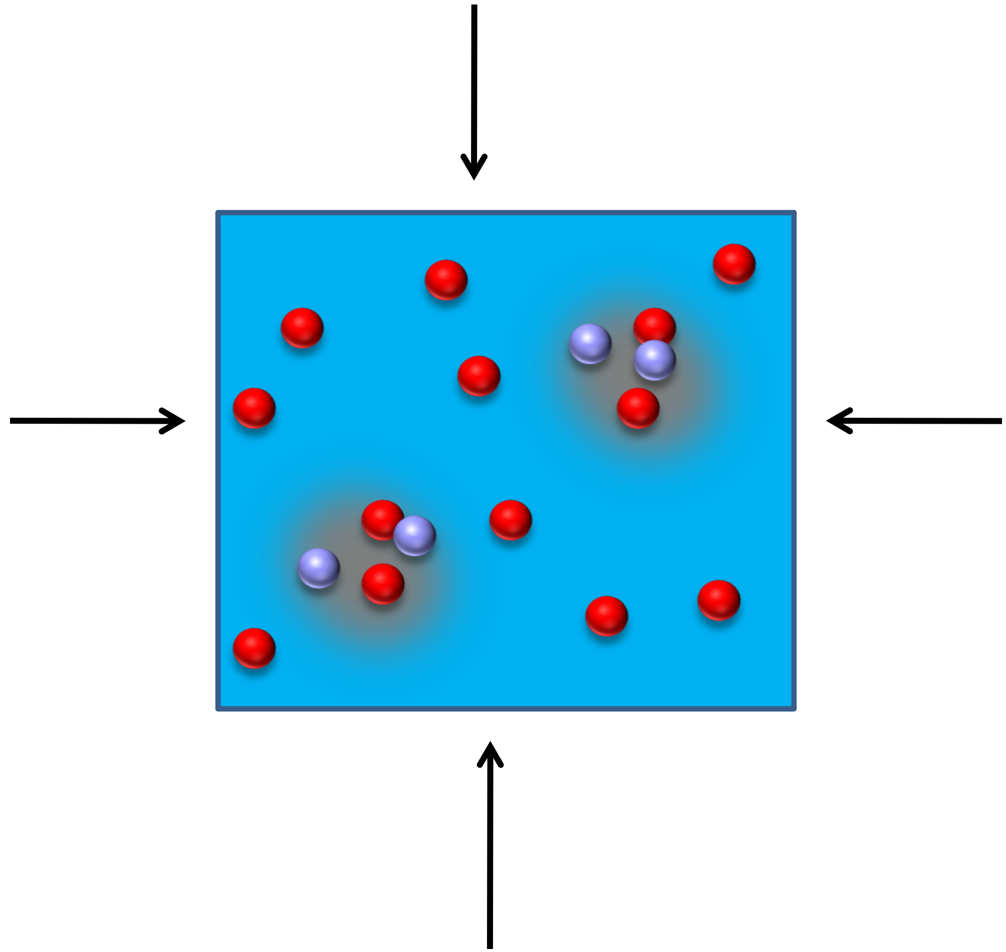
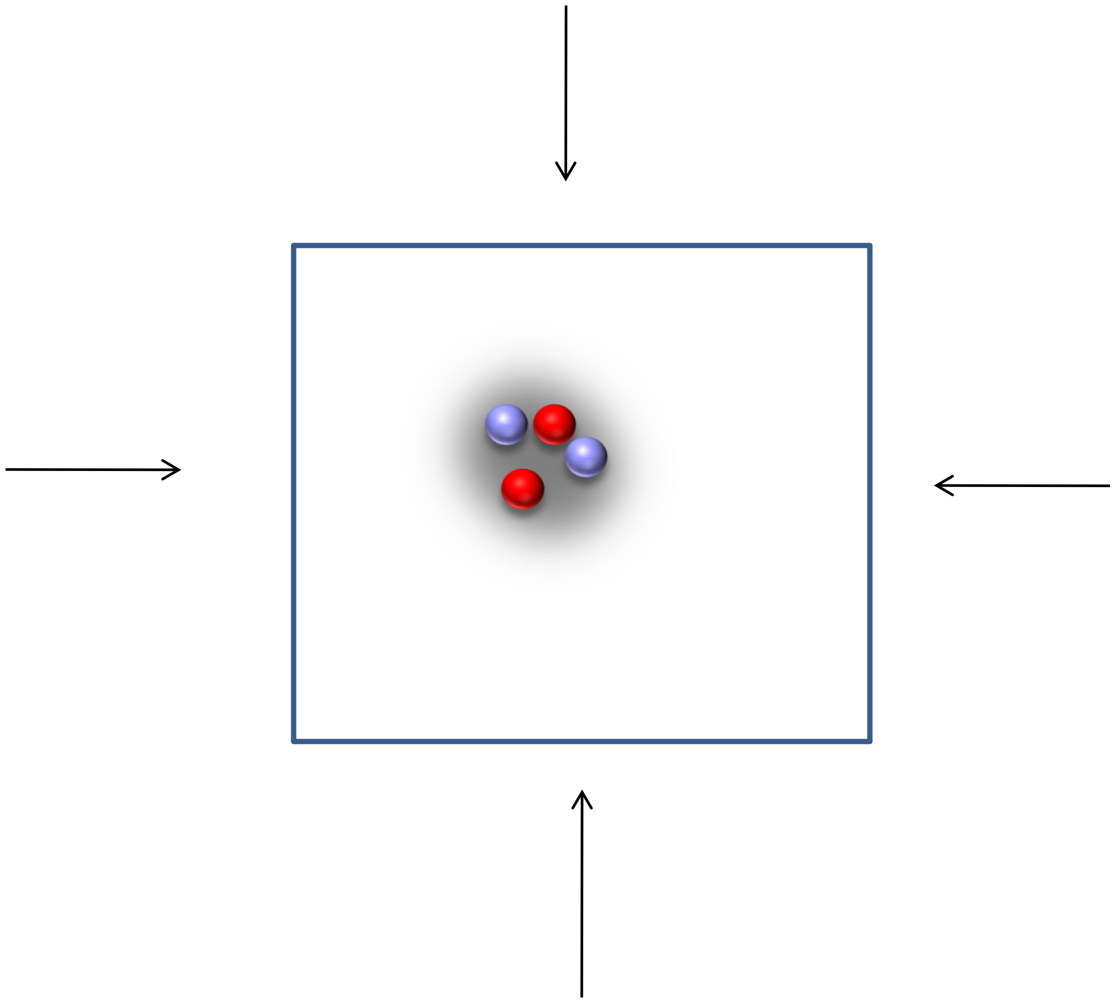
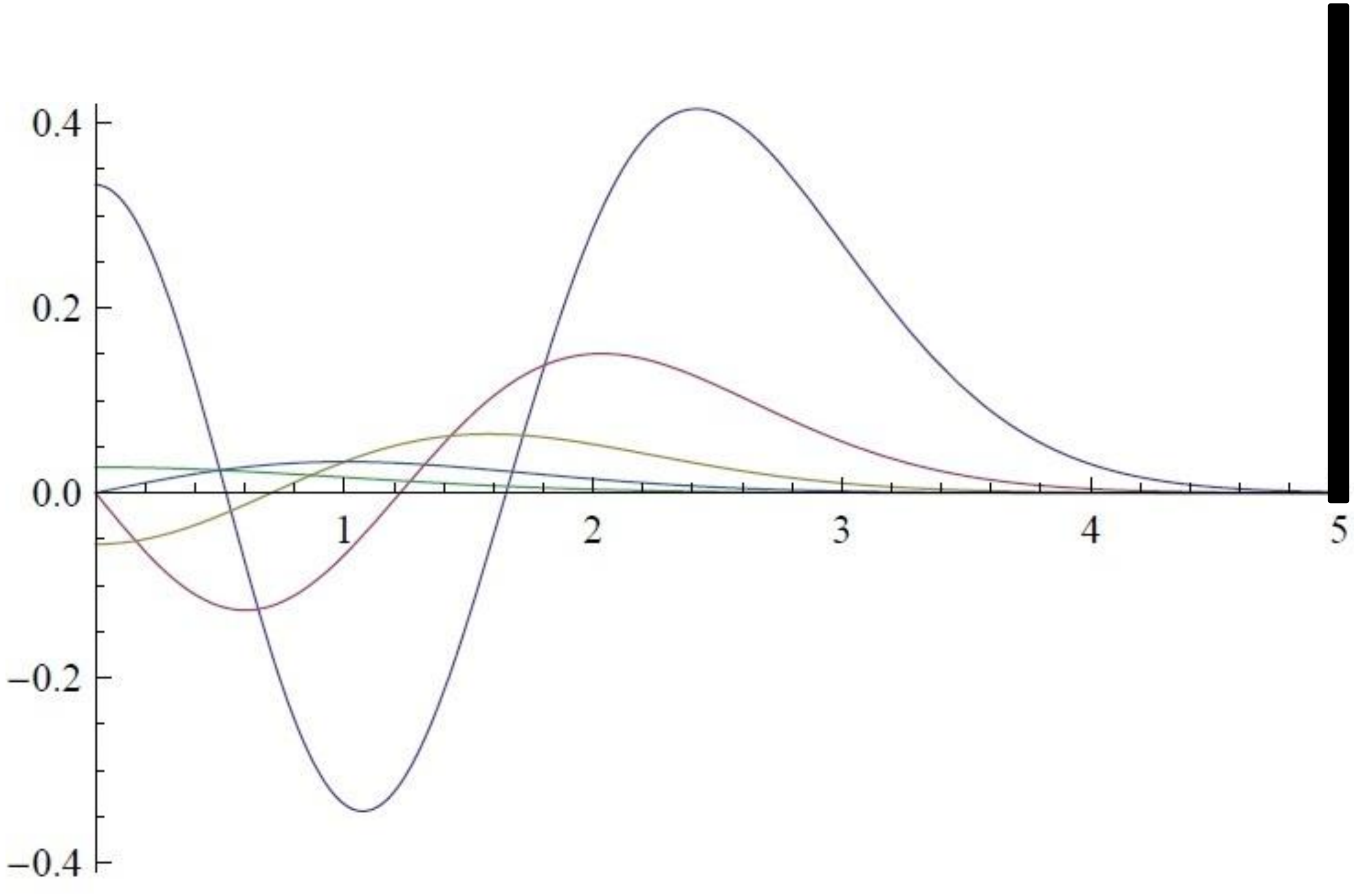


Image from astro.umd.edu

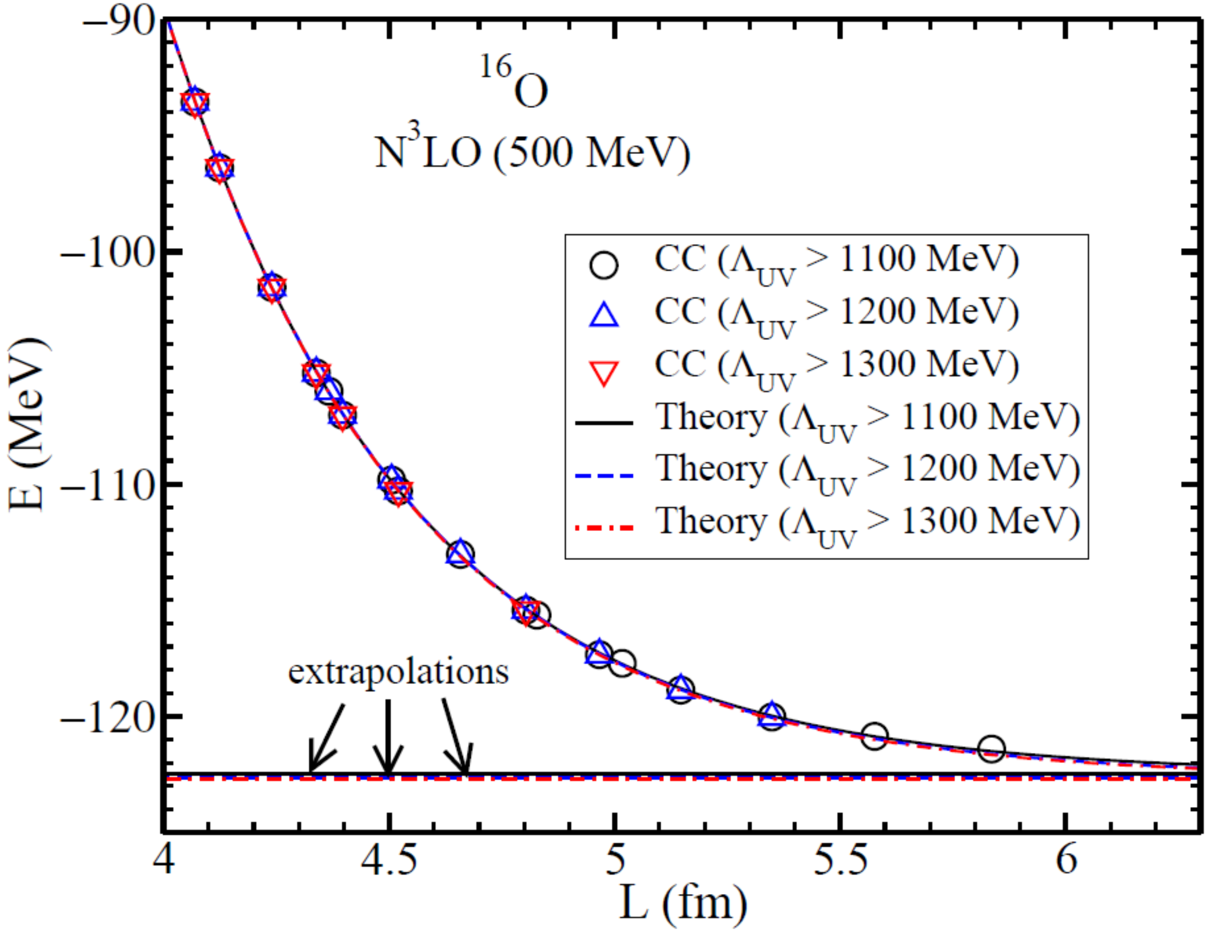




Quantify errors due to finite oscillator space



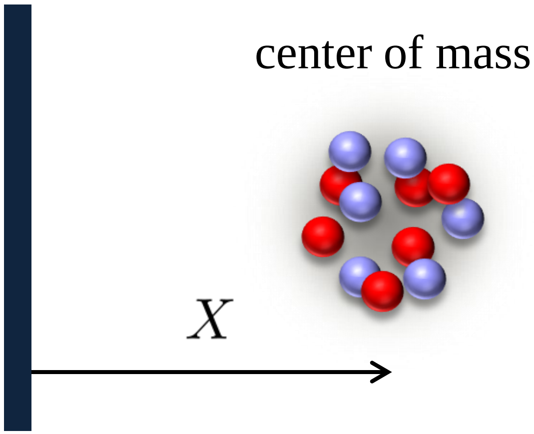
Furnstahl, Hagen, Papenbrock, arXiv:1207.6100v1

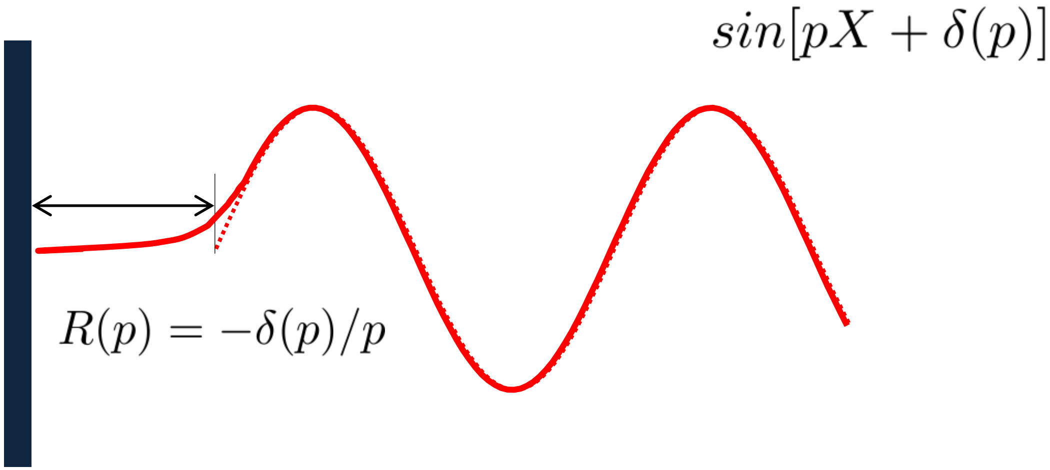


$$E_L = E_\infty + a_0 e^{-2k_\infty L} + \mathcal{O}(e^{-4k_\infty L})$$

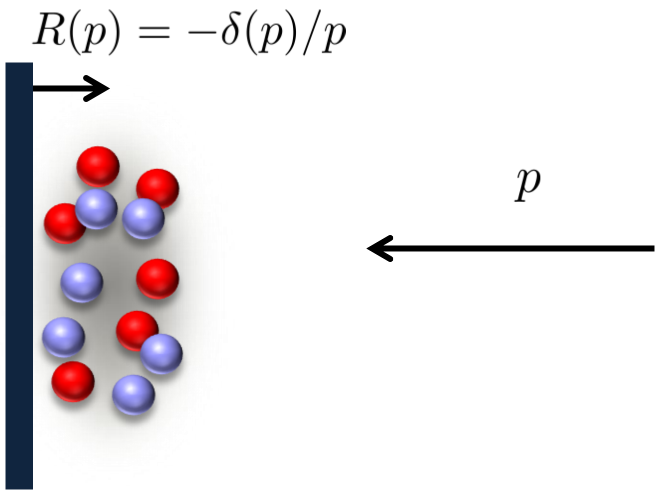
Furnstahl, Hagen, Papenbrock, arXiv:1207.6100v1

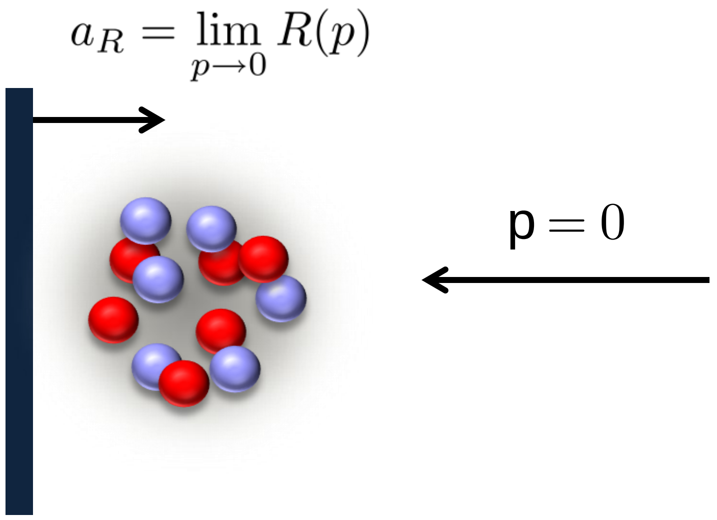
Reflection phase shift





Reflection Radius



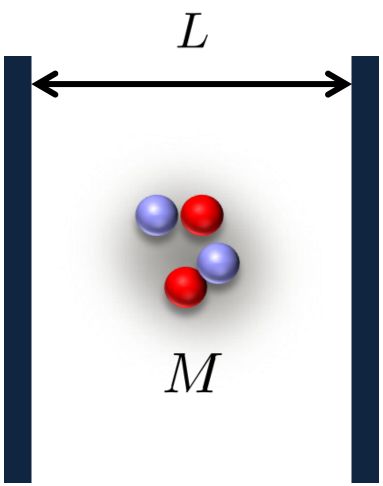


Effective range expansion

$$p\cot\delta(p) = -\frac{1}{a_R} + \frac{1}{2}r_R p^2 - \mathcal{P}_R p^4 + \cdots$$

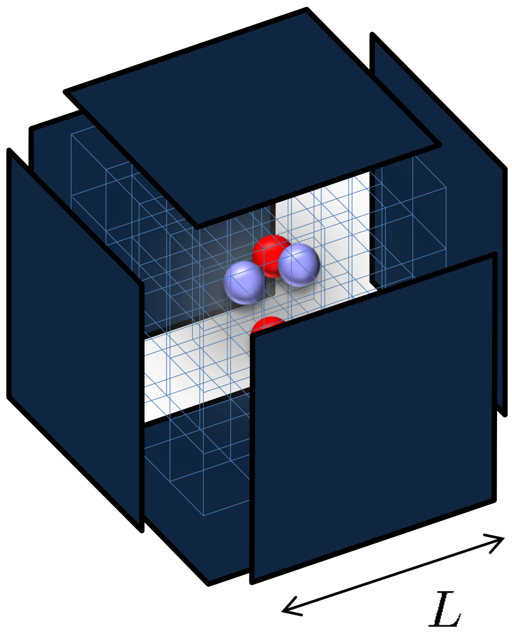
a_R is the scattering length,
r_R is the effective range, and
$\mathcal{P}_R$ is the shape parameter

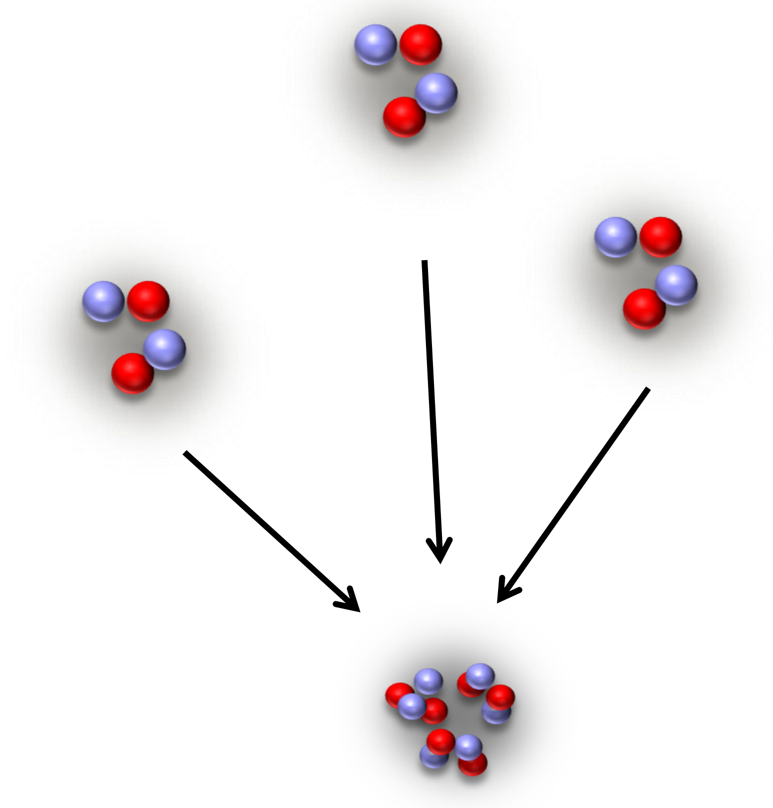
Linear confinement energy



$$E_K \approx \frac{d}{2M}\frac{\pi^2}{[L-2R(E_K)]^2}$$

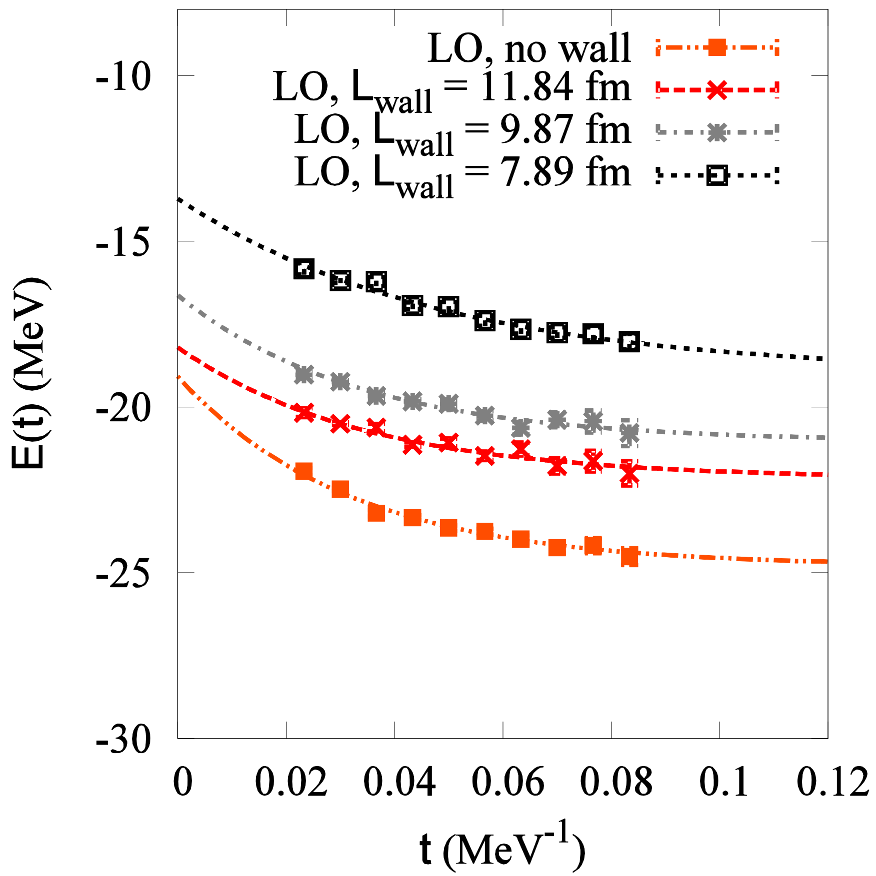
Alpha particle in a hard-wall box





Compressed alpha clusters within carbon-12

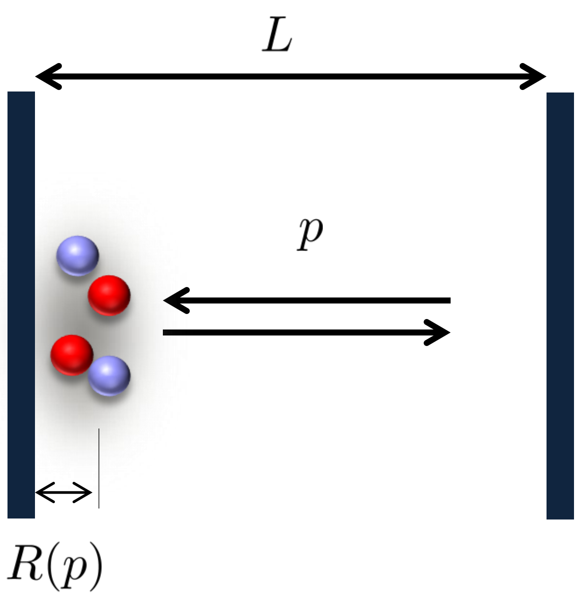
Alpha particle energy (LO)



Alpha particle reflection radius (LO)

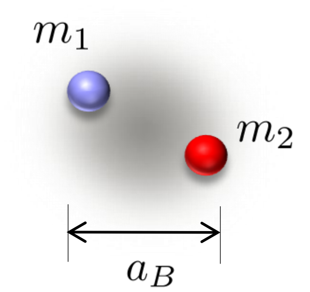
At leading order the alpha particle matter radius is 1.53fm

L	$p(L)$	$R[p(L)]$
11.8 fm	81(9) MeV	2.1(4) fm
9.9 fm	97(10) MeV	1.6(3) fm
7.9 fm	118(10) MeV	1.3(2) fm



$$E_K \approx \frac{d}{2M} \frac{\pi^2}{[L - 2R(E_K)]^2}$$

Shallow two-body bound state



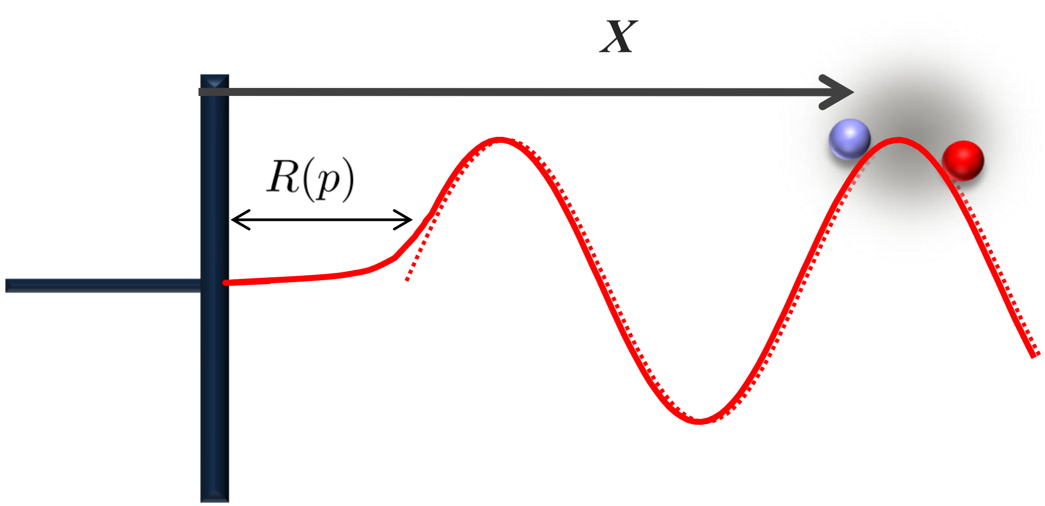
scattering length

$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$$ reduced mass

$$M = m_1 + m_2$$ total mass

$$E_B = -\frac{\kappa_B^2}{2\mu}$$ energy due to binding

$$\kappa_B = \frac{1}{a_B}$$ binding momentum



$$E_K = \frac{p^2}{2M}$$ kinetic energy

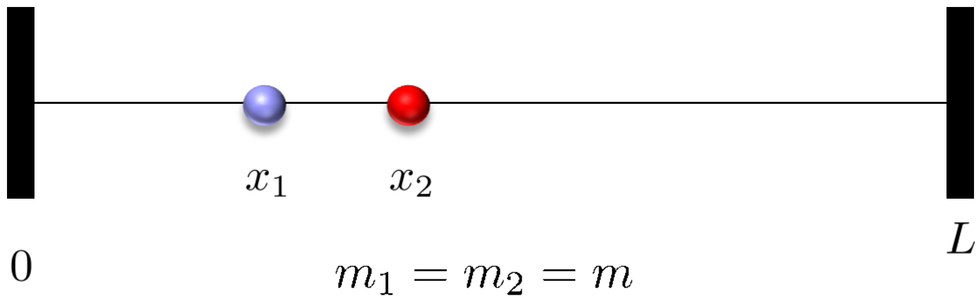
Effective range expansion

$$p \cot \delta(p) = -\frac{1}{a_R} + \frac{1}{2} r_R p^2 - \mathcal{P}_R p^4 + \cdots$$

Scattering parameters described by universal dimensionless constants

$$\kappa_B a_R,\ \kappa_B r_R,\ \kappa_B^3 \mathcal{P}_R,\ \cdots$$

Equal masses in one dimension



$$m_1 = m_2 = m$$

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{1}{2m}\frac{\partial^2}{\partial x_2^2} + C_B\,\delta(x_1 - x_2)$$

<u>Bethe ansatz with hard-wall boundaries</u>

$$\begin{aligned}\psi(x_1, x_2) \propto \\ &\theta(x_2 - x_1)\left\{\sin(k_a x_1)\sin\left[k_b(x_2 - L)\right] + D_I \sin(k_b x_1)\sin\left[k_a(x_2 - L)\right]\right\} \\ &+ \theta(x_1 - x_2)\left\{D_{II}\sin(k_a x_2)\sin\left[k_b(x_1 - L)\right] + D_{III}\sin(k_b x_2)\sin\left[k_a(x_1 - L)\right]\right\}\end{aligned}$$

$$E = \frac{k_a^2}{2m} + \frac{k_b^2}{2m}$$

Symmetric under parity

$$x_1 \to L - x_1,\ x_2 \to L - x_2$$
$$D_{III} = 1,\ D_{II} = D_I$$

Symmetric under particle exchange

$$x_1 \leftrightarrow x_2$$
$$D_{II} = 1,\ D_{III} = D_I$$

$$\psi(x_1, x_2) \propto$$
$$\theta(x_2 - x_1)\left\{\sin(k_a x_1)\sin\left[k_b(x_2 - L)\right] + \sin(k_b x_1)\sin\left[k_a(x_2 - L)\right]\right\}$$
$$+\,\theta(x_1 - x_2)\left\{\sin(k_a x_2)\sin\left[k_b(x_1 - L)\right] + \sin(k_b x_2)\sin\left[k_a(x_1 - L)\right]\right\}$$

Contact interaction

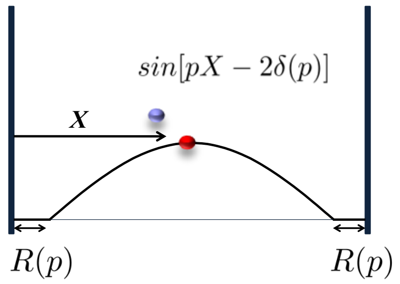
$$\lim_{x_1 \to x_2^+}\left[\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\psi(x_1, x_2)\right] = 2C_B\mu\ \psi(x_1, x_2)\big|_{x_1 = x_2}$$
$$\mu = m/2$$

Solution

$$(k_a + k_b)\sin\left[(k_a + k_b)\,L/2\right] = -2C_B\mu\cos\left[(k_a + k_b)\,L/2\right]$$

$$p = k_a + k_b$$

Phase shift and Reflection Radius



$$p\cot\delta(p) = -2\kappa_B$$

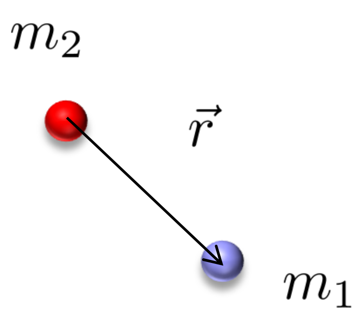
$$\kappa_B R(E_K) = \frac{1}{2}\sqrt{\frac{|E_B|}{E_K}}\tan^{-1}\sqrt{\frac{E_K}{|E_B|}}$$

Exact solution is analytic for all energies. No inelastic break-up…
consequence of exact integrability.

Low-energy effective potential

Two particles in *d* dimensions with attractive zero range interactions. The coefficient of the regulated delta function interaction is tuned to produce the desired binding energy

$$H = -\frac{1}{2m_1}\vec{\nabla}^2_{r_1} - \frac{1}{2m_2}\vec{\nabla}^2_{r_2} + C_B \delta^{(d)}(\vec{r}_1 - \vec{r}_2)$$



$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$$ reduced mass

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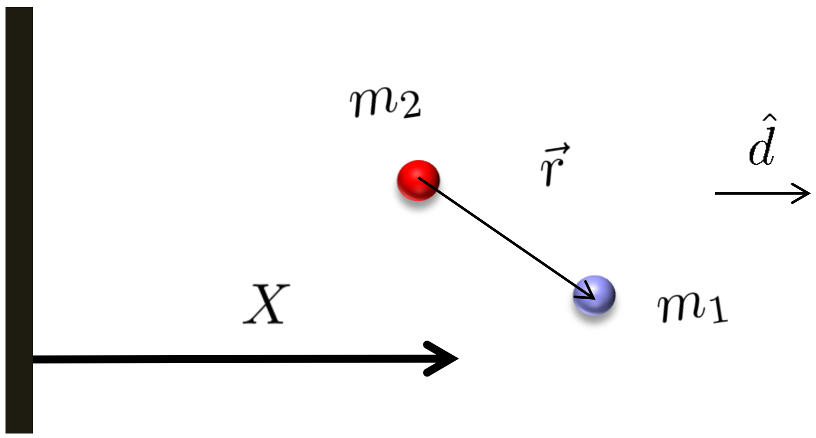


$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M} \qquad X = (\vec{R})_d$$

$$H = -\frac{1}{2M} \vec{\nabla}_R^2 + H_{\mathrm{rel}}$$

$$H_{\mathrm{rel}} = -\frac{1}{2\mu} \vec{\nabla}_r^2 + C_B \delta^{(d)}(\vec{r})$$

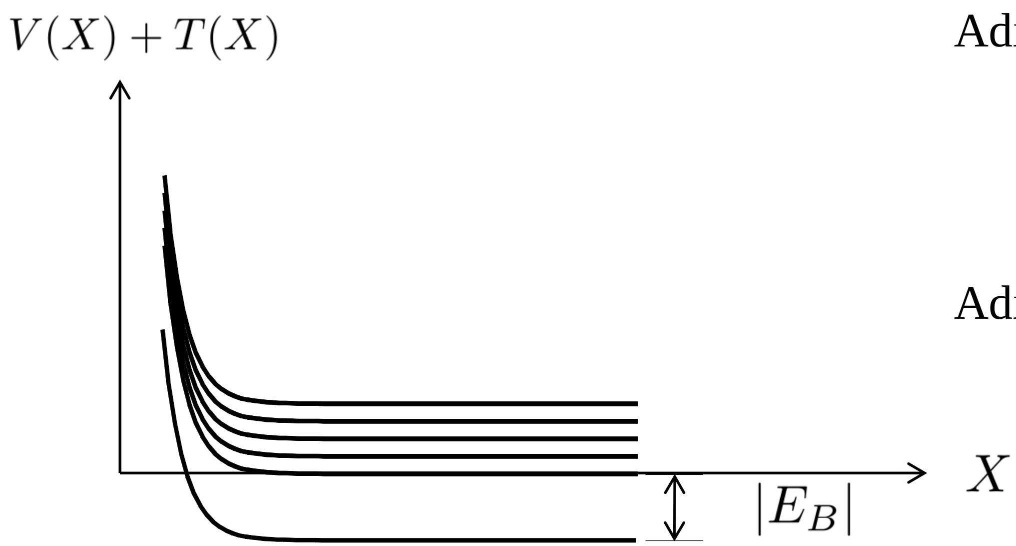
Choose inertial frame where center-of-mass motion parallel to the wall is zero.

Adiabatic expansion in the soft scattering limit

$$p\cot\delta(p) = -1/a_R + O(E_K/|E_B|)$$

Eigenstates of H_{rel} $\left|\psi_X^j(\vec{r})\right\rangle$

To compute a_R we keep only the lowest eigenstate for any fixed X, and calculate the adiabatic potential $V(X)$ and adiabatic diagonal correction $T(X)$.



Adiabatic potential

$$V^j(X) = \left\langle \psi_X^j(\vec{r}) \right| H_{\rm rel} \left| \psi_X^j(\vec{r}) \right\rangle$$

Adiabatic diagonal correction

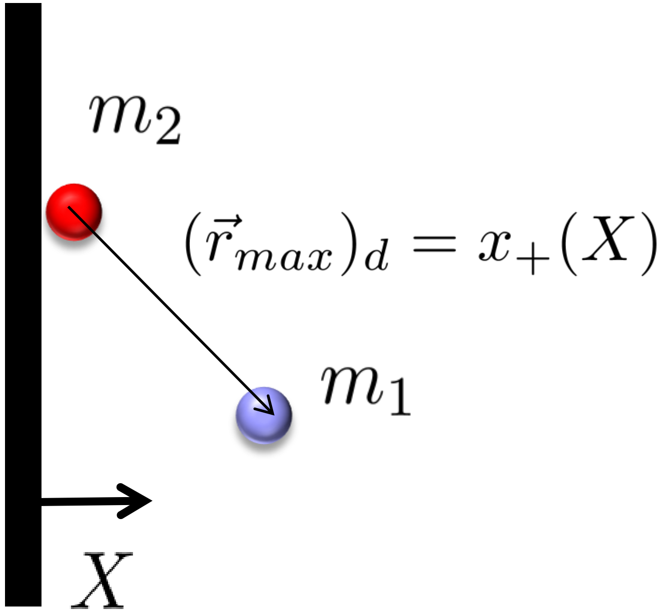
$$T^j(X) = \left\langle \psi_X^j(\vec{r}) \right| \frac{-\partial_X^2}{2M} \left| \psi_X^j(\vec{r}) \right\rangle$$

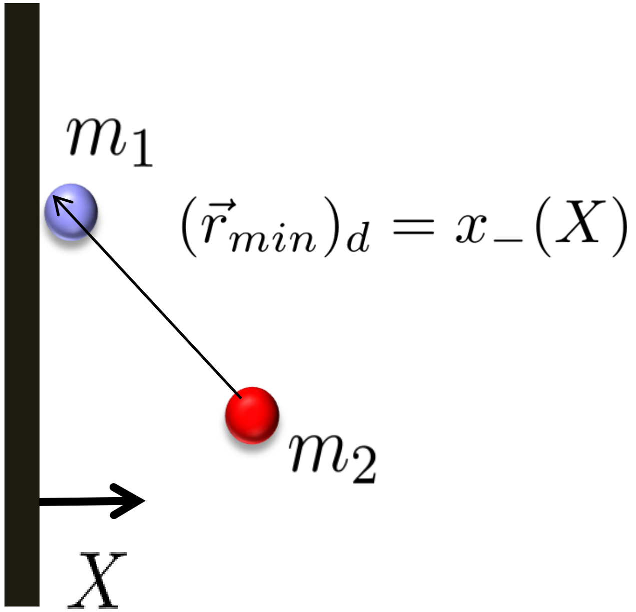
Relative coordinate boundary conditions

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \qquad x_-(X)/2 \le r_d \le x_+(X)/2$$

$$x_+(X) = \frac{2M}{m_1}X$$

$$x_-(X) = -\frac{2M}{m_2}X$$

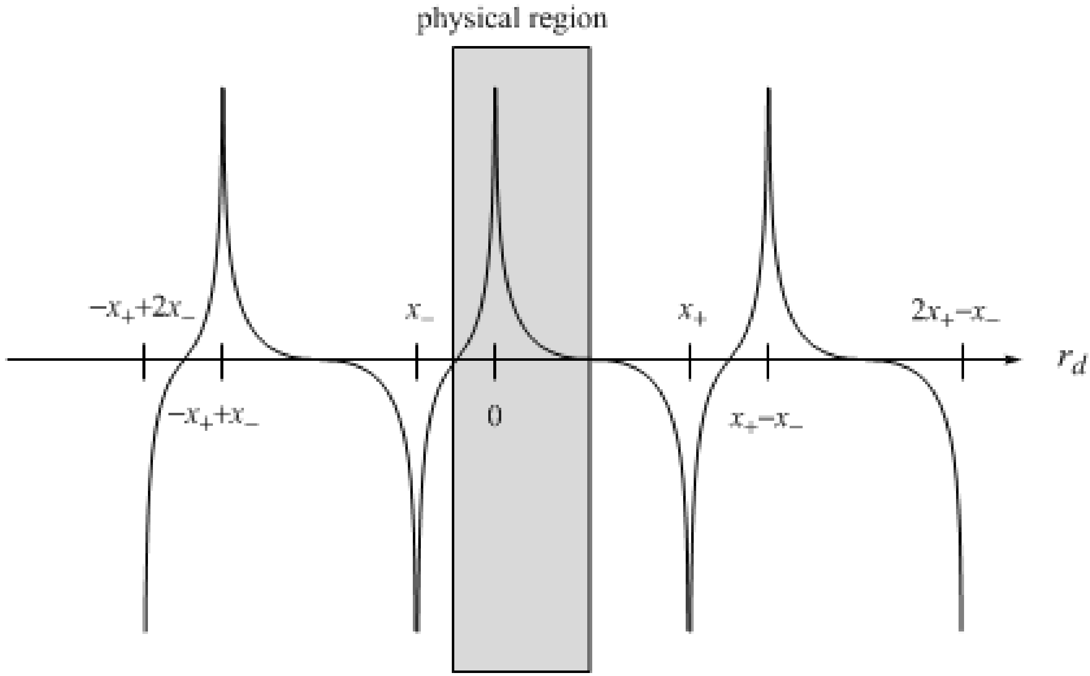




The ground state of H_{rel} at infinite volume is a d-dimensional Green's function or Yukawa function,

$$\int \frac{d^d\vec{p}}{(2\pi)^d} \frac{e^{-i\vec{p}\cdot\vec{r}}}{p^2+\kappa_B^2}.$$

To construct the solution with hard-wall boundaries we use the method of images with alternating signs.



If we keep all images the answer is exact. But this is analytically tractable only for the one-dimensional case. Instead we keep a finite number of images organized as an asymptotic expansion in powers of

$$e^{-\kappa_B x_+(X)} = e^{-\frac{2M}{m_1}\kappa_B X}$$
$$e^{-\kappa_B |x_-(X)|} = e^{-\frac{2M}{m_2}\kappa_B X}$$

The final result is an expansion for the reflection scattering length in powers of

$$e^{-\kappa_B x_+(a_R)}, e^{-\kappa_B |x_-(a_R)|} \leq e^{-2\kappa_B a_R}$$

At zeroth order we recover the infinite volume result

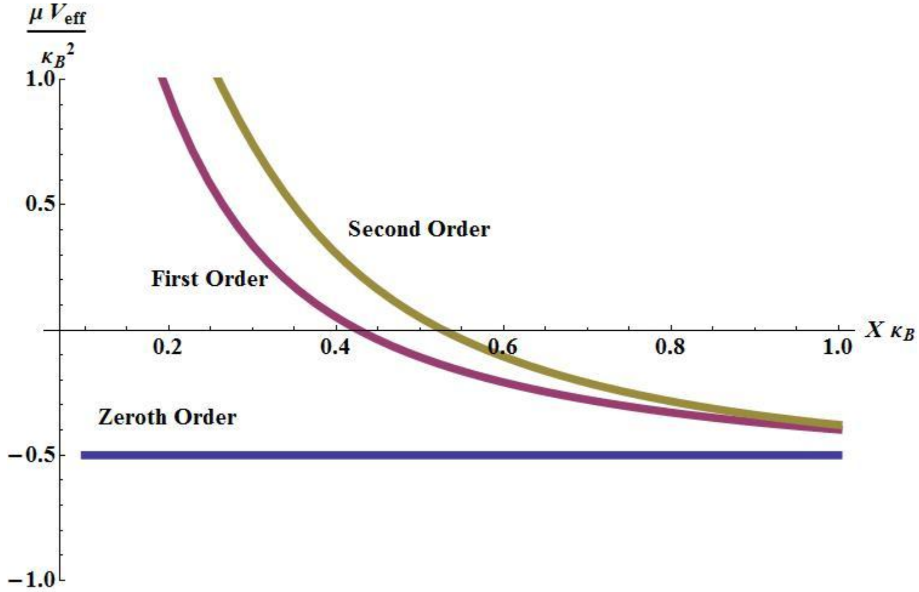
$$V^{(0)}(X) + T^{(0)}(X) = E_B$$

$1D$

For one dimension the first-order correction to the effective potential is

$$T^{(1)}(X) + V^{(1)}(X)$$

$$= \frac{\kappa_B^2 M^2}{m_1 m_2}\left[\frac{e^{-\kappa_B x_+(X)}}{m_1} + \frac{e^{-\kappa_B |x_-(X)|}}{m_2}\right]$$



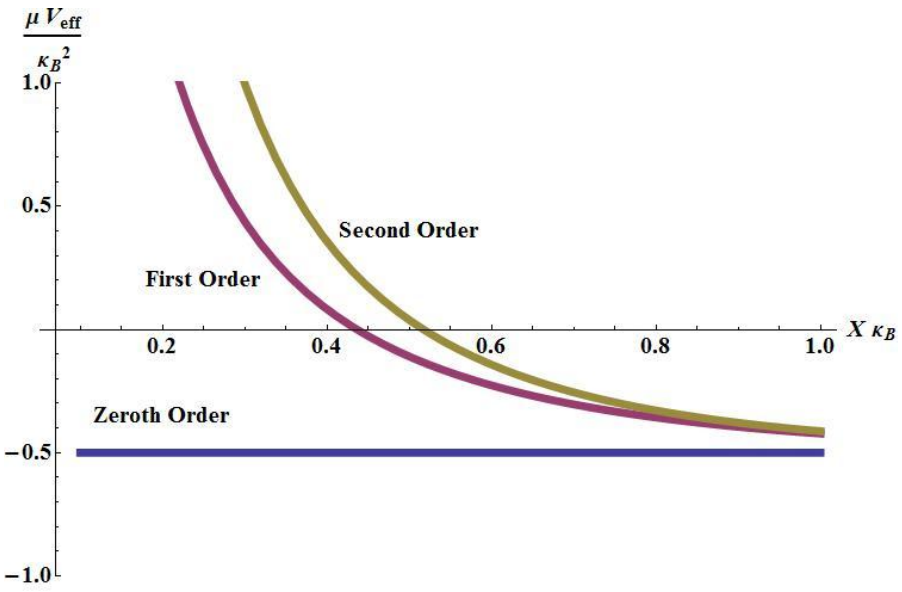
Plot showing the zeroth, first, and second order potentials in one dimension

2D

For two dimensions the first-order correction to the effective potential is

$$T^{(1)}(X) + V^{(1)}(X) =$$

$$\left(\frac{1}{\mu} + \frac{M}{m_1^2}\right) \kappa_B^2 K_0[\kappa_B r_+(X)] + \left(\frac{1}{\mu} + \frac{M}{m_2^2}\right) \kappa_B^2 K_0[\kappa_B r_-(X)]$$

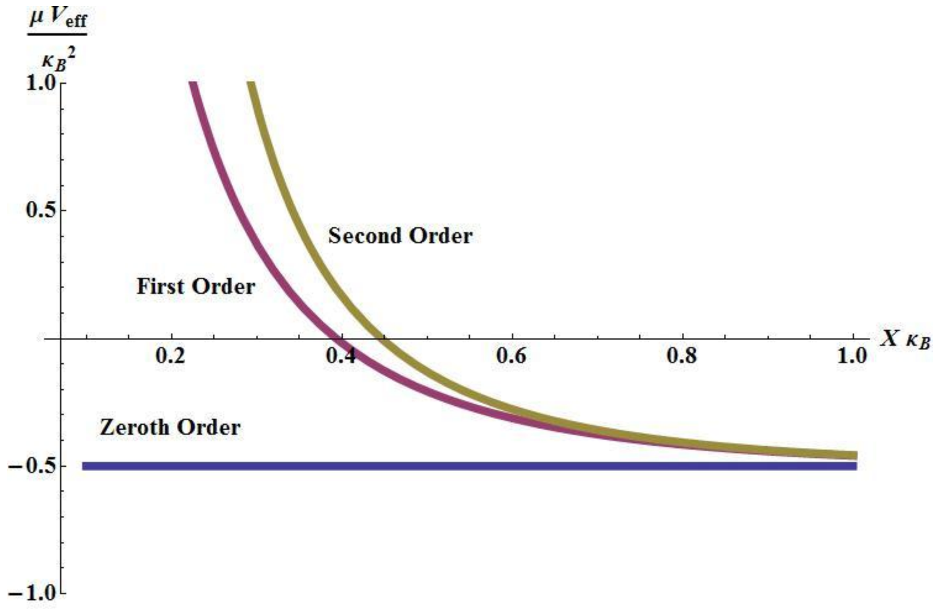


Plot showing the zeroth, first, and second order potentials in two dimensions

$3D$

For three dimensions the first-order correction to the effective potential is

$$T^{(1)}(X) + V^{(1)}(X) = \frac{\kappa_B M}{2m_1 m_2 X}\left[e^{-\kappa_B x_+(X)} + e^{-\kappa_B |x_-(X)|}\right]$$



Plot showing the zeroth, first, and second order potentials in three dimensions

Numerical results and comparison

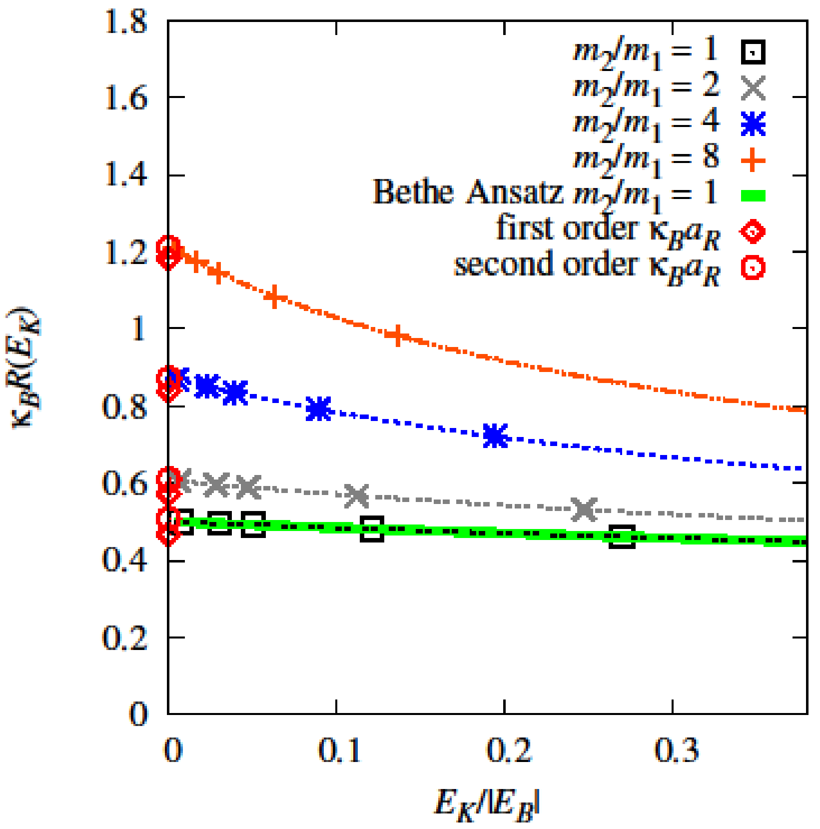
Using a simple Hubbard lattice model with attractive on-site interactions, we have computed the reflection radius $R(p)$ for the shallow two-body bound state. We have considered both one, two, and three spatial dimensions as well as mass ratios

$$\frac{m_2}{m_1} = 1,\ 2,\ 4,\ 8$$

To take the zero range limit, we have extrapolated the lattice results to zero lattice spacing. For the two- and three-dimensional systems we have also extrapolated to infinite volume for directions perpendicular to the wall.

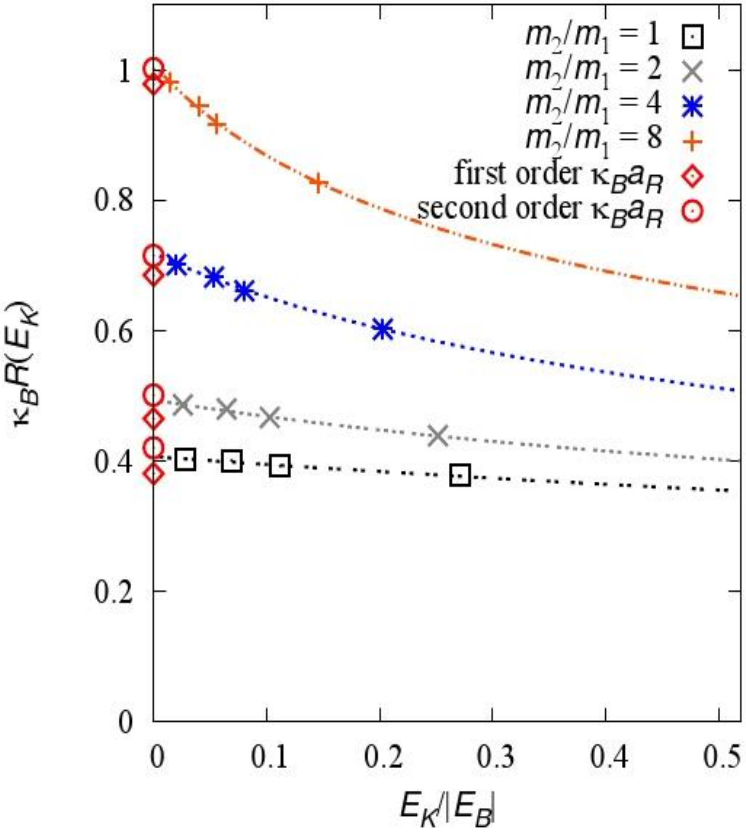
1D

Comparing lattice, exact, and EFT results



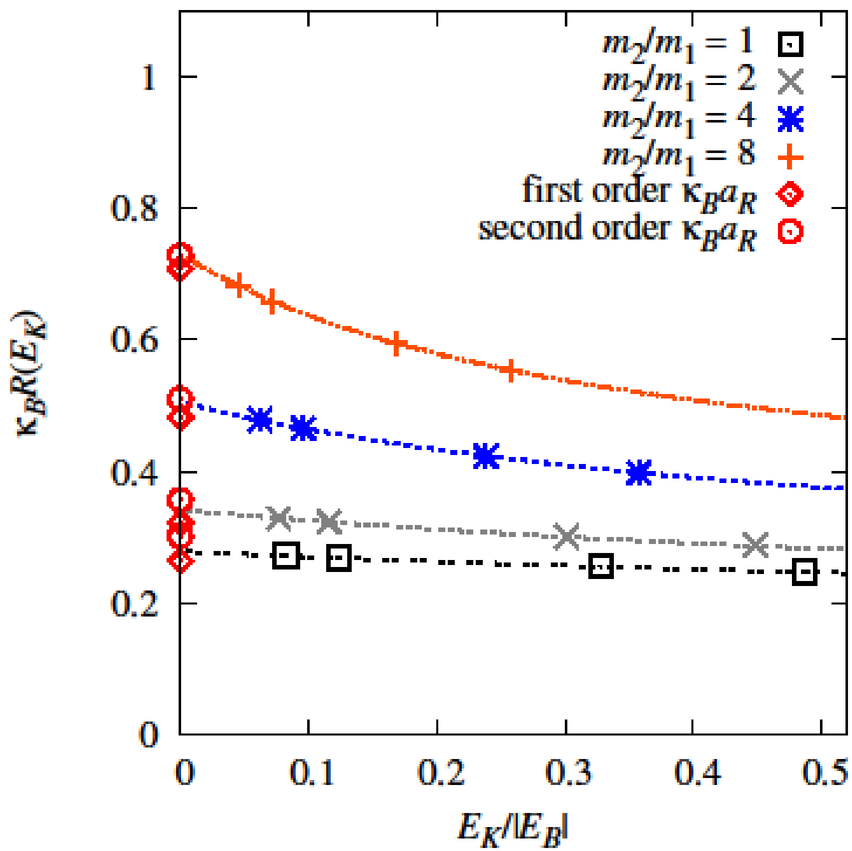
2D

Comparing lattice and EFT results



3D

Comparing lattice and EFT results



Summary

We have presented a new tool for probing the structure of quantum bound states by studying elastic scattering off of hard-wall boundaries including:

- Universal effective potential for shallow two body bound states
- Exact solution for equal mass two body bound states in one dimension
- Numeric Simulations in one, two, and three dimensions for shallow two body bound states as well as the alpha particle
- Comparison of the above three methods

Still To Do

- Varying the boundary conditions

- Verifying universal constants with systems such as the deuteron or ultra-cold atomic systems

- Further numeric simulations of realistic composite bodies such as the deuteron and triton as well as larger systems such as carbon-12 to probe characteristics of nuclear structure

- Applications to ultra-cold atomic systems as well as quantum dots and wells

- Investigate inelastic scattering and the inelastic threshold