Universal Relations from few-body physics in ultracold atoms

PRL 2011, PRL 2010, PRA 2008

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With Eric Braaten and Lucas Platter

CTP, MIT

Nucleons

Atoms

Dilute neutron matter

2 spin states

1 scattering length

Fermions with 2 spin states

Universal relations by S. Tan [2005]

Few nucleon systems

Efimov physics

Fermions with >2 spin states

Identical bosons

Universal relations Braaten, DK, Platter

OPE

…

…

EFT with zero-range interaction

Outline

- Strongly interacting ultracold atoms
- Fermions with 2 spin states (2-body physics)
  - Universal relations and Contact
  - Operator Product Expansion (OPE)
- Identical bosons (3-body physics)
  - Efimov physics
  - Universal relations

Puzzle related to renormalization of 3-body coupling

Strongly interacting atoms

- **What are they?**

  Ultracold atoms with large scattering length **(*a*)**

- **Ultracold atoms?**
  - **$T < 10^{-6}$ K while $T_{QGP} > 10^{12}$ K**
  - **Alkali group**

    fermions (even) and bosons (odd)

$$^{1}\mathrm{H},\ ^{6}\mathrm{Li},\ ^{7}\mathrm{Li},\ ^{23}\mathrm{Na},\ ^{39}\mathrm{K},$$
$$^{40}\mathrm{K},\ ^{41}\mathrm{K},\ ^{85}\mathrm{Rb},\ ^{87}\mathrm{Rb},\ ^{133}\mathrm{Cs}$$

Strongly interacting atoms

- **Quantum Mechanics at low energy**

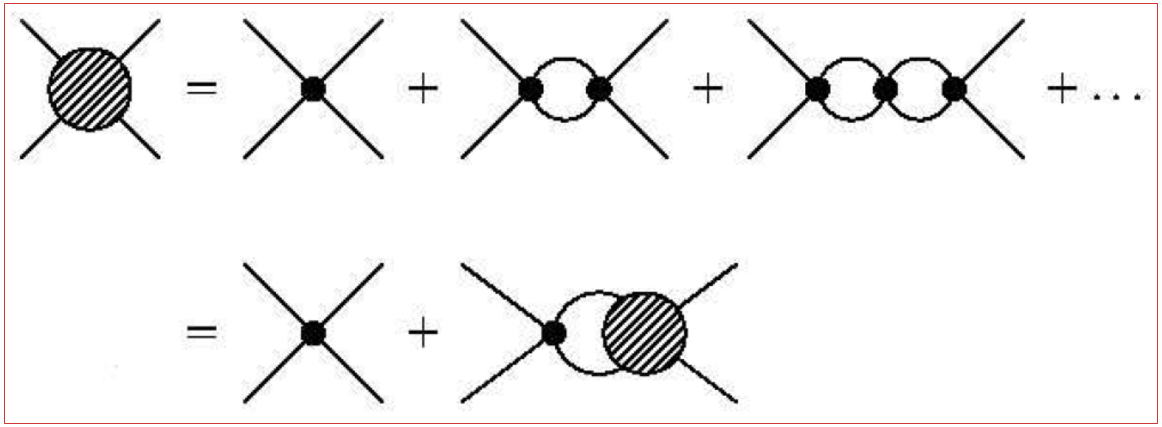
$$f(k) = \frac{1}{-\frac{1}{a} - ik + \frac{r_s}{2}k^2 + \cdots}$$

- **At very low energy (k << 1/range),**
  f(k) depends only on scattering length (*a*)
- **For large *a* (≥ 1/k),**
  interaction is nonperturbative

Effective Field Theory

$$\mathcal{L} = \psi_\sigma^\dagger i \frac{\partial}{\partial t}\psi_\sigma - \mathcal{H} \qquad \sigma = 1,2$$

$$\mathcal{H} = \frac{1}{2}\nabla\psi_\sigma^\dagger \cdot \nabla\psi_\sigma + g\,\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$$



Renormalization

$$f(k) = -\frac{1}{1/a + ik}$$

$$\frac{1}{a} = \frac{4\pi}{g} + \frac{2}{\pi}\Lambda$$

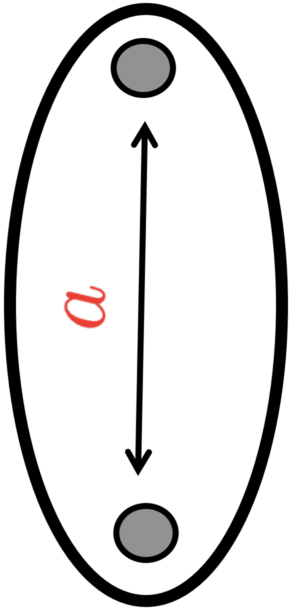
Nonperturbative solution!!

2- body: analytic solution, 3-body: exact numerical solution,

Many-body is challenging : Lattice simulation, ...

Strongly interacting atoms

- **Low energy amplitude** $$f(k) = \frac{1}{-1/a - ik}$$
  - **Cross section** $$\sigma(k) = \frac{4\pi}{1/a^2 + k^2}$$
  - **Molecule (when *a*>0)**
    - **Binding energy** $$E = -\frac{1}{a^2}$$
    - **Size** $$\sqrt{\langle r^2 \rangle} = a/\sqrt{2}$$

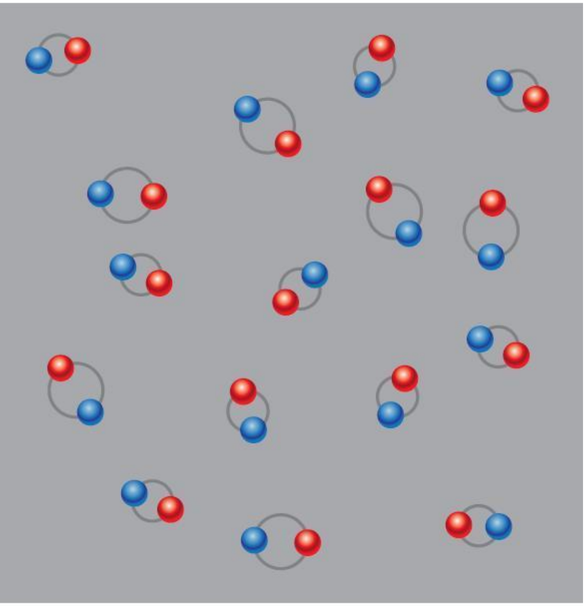


Scale invariance for *a* $\to \pm\infty$

Strongly interacting Atoms

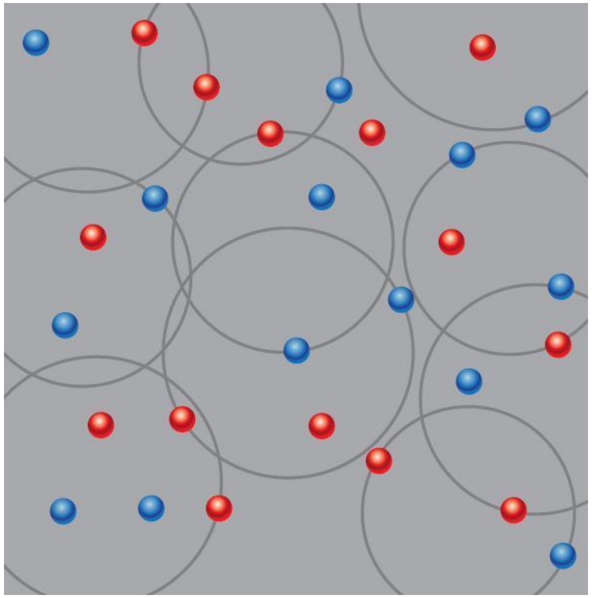
- Many-body physics
  - Identical Bosons : Bose-Einstein Condensate
  - Fermions with 2 spin states

Condensate of molecules



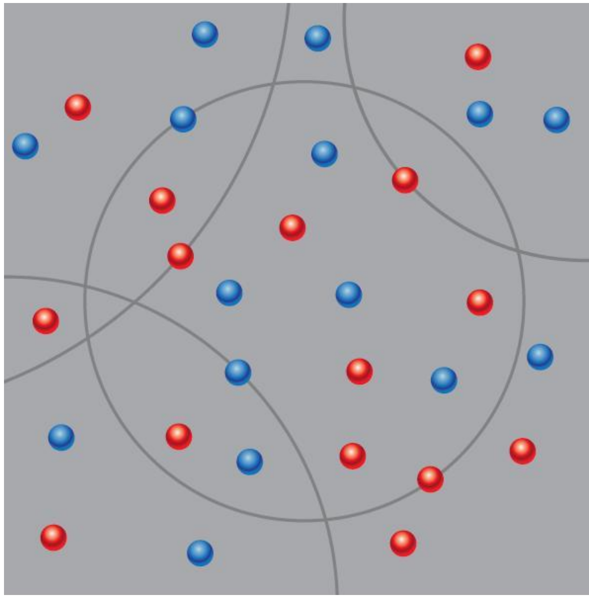
BEC limit ($a \rightarrow 0+$)

Scale invariant matter



unitary limit ($a \rightarrow \pm\infty$)

Fermi gas with Cooper pairing



BCS limit ($a \rightarrow 0-$)

Tan [Annals of Physics 2008]

Universal Relations

for fermions with 2 spin states

- **Hold for any state of the system**

  e.g. few-body/ many-body, homogeneous/trapped,

  normal gas/superfluid, ground state/low temperature, etc

- **Involve an extensive property of the system called the *contact (C)***
- **Determined by 2-body physics**

Universal relations

- **Adiabatic relation:** **Variation of energy with scattering length**

$$\frac{dE}{da} = \frac{C}{4\pi a^2}$$

Tan 2005

- **Tail of the momentum distribution for large k**

$$\rho(k) \to \frac{C}{k^4}$$

Tan 2005

- **Tail of rf transition rate for large *ω***

$$\Gamma(\omega) \to \frac{\Omega^2}{4\pi\omega^{3/2}} C$$

Schneider and Randeria 2010

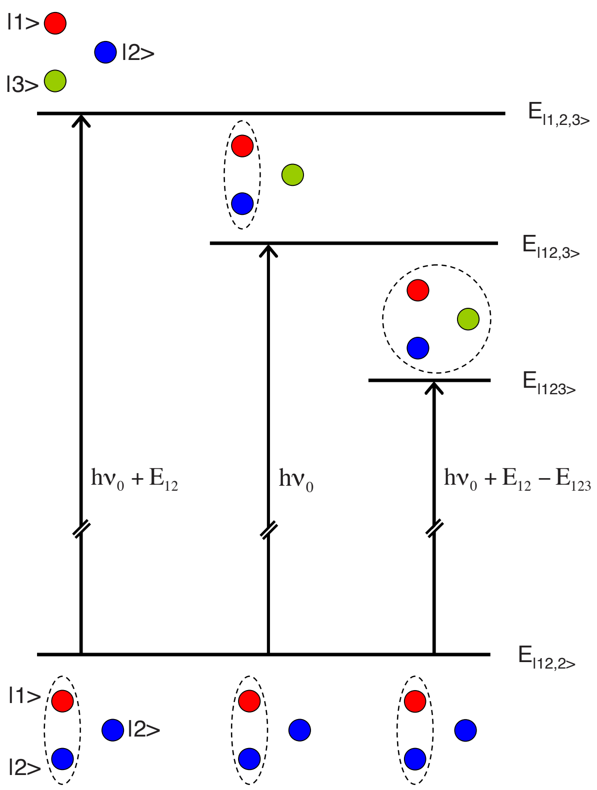
Radio frequency spectroscopy

- **Transition between hyperfine states by a photon with rf frequency**
  - **Association/dissociation of dimer and trimer**
  - **Excite single particle state and probe a spectral function**
- **Tail with final state interaction a'**

Braaten, DK and Lucas 2010

$$\Gamma(\omega) \to \frac{\Omega^2 \left(1/a - 1/a'\right)^2}{4\pi\omega^{3/2}\left(1/a'^2 + \omega\right)} C$$

$\omega^{-3/2}$ and $\omega^{-5/2}$ tails!!



Universal relations

- **Contact as matrix element**

$$\frac{dE}{da} = \frac{C}{4\pi a^2}$$

$$\frac{dE}{da} = \frac{d}{da} \int_R \langle \mathcal{H} \rangle$$

$$= \int_R \langle \frac{d}{da} \mathcal{H} \rangle$$

Feynman-Hellman Theorem

$$\frac{1}{g} = \frac{1}{4\pi a} - \frac{1}{2\pi^2}\Lambda$$

$$\frac{dg}{da} = \frac{g^2}{4\pi a^2}$$

$$= \int_R \langle \frac{dg}{da} \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(R) \rangle$$

$$= \frac{1}{4\pi a^2} \int_R \langle g^2 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1(R) \rangle$$

Universal relations

- **Adiabatic relation** $$\frac{dE}{da} = \frac{C}{4\pi a^2}$$

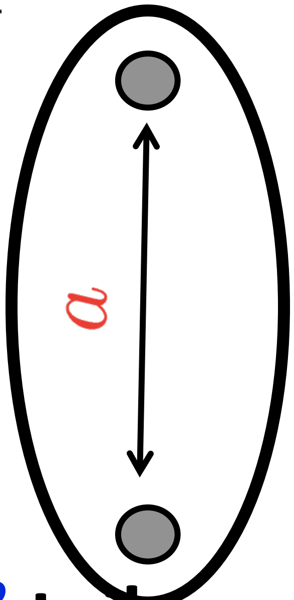
  $\Rightarrow$ **Can be used as an operational definition**

- **The contact *C***
  - ***depends on the states***
  - **extensive thermodynamic quantity depending on *a*, density (n), entropy …**
  - **measures probability for 2 atoms being close together**

Relations for dimer

- **Contact :** $C = 4\pi a^2 \frac{dE}{da} = \frac{8\pi}{a}$ $\quad E = -\frac{1}{a^2}$
- **Exact wavefunction:** $\tilde{\psi}(k) = \frac{\sqrt{8\pi/a}}{k^2+1/a^2}$
- **Momentum distribution:**

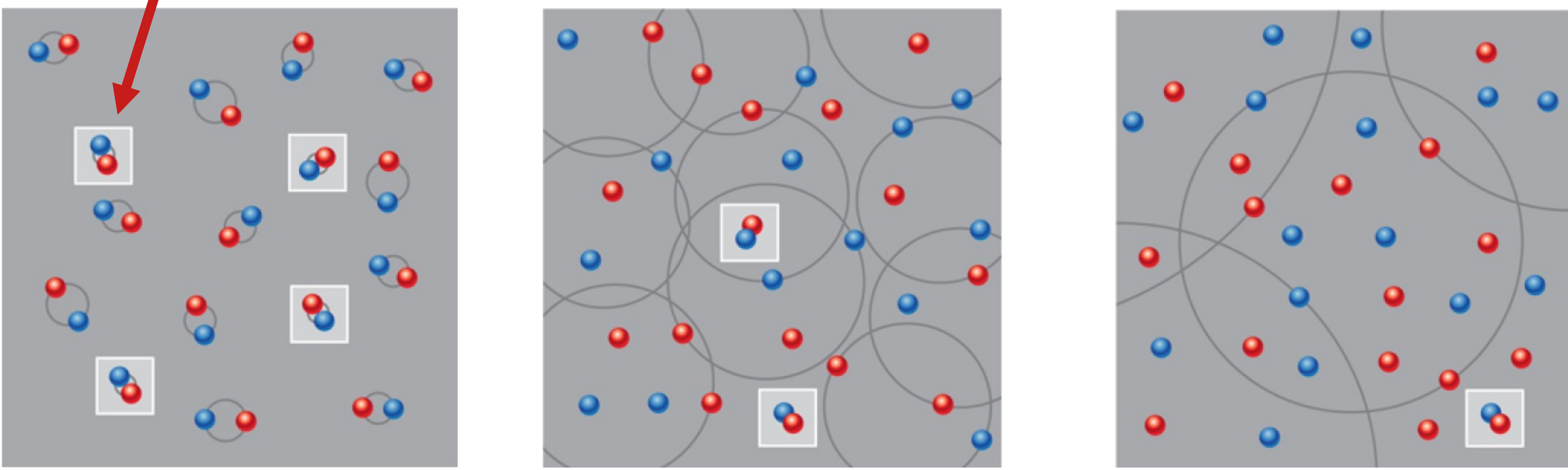
$$\rho(k) = \tilde{\psi}^\dagger \tilde{\psi}(k) \to \frac{8\pi/a}{k^4}$$



- **Dimer-dimer transition has $\omega^{-3/2}$ and $\omega^{-5/2}$ tail**

Many-body states

- **Contact density (C/V) for homogeneous gas at T=0**



BEC limit (a→ 0+)	***unitary limit (a→ $\pm\infty$)***	***BCS limit (a→ 0-)***
$4\pi n/a$	$(9.1 \pm 0.3)\, n^{4/3}$	$4\pi^2 a^2 n^2$

Drut, Lähde, Ten [PRL 2011]

Proof of universal relation

- **Operator Product Expansion**

$$\hat{O}_A(\boldsymbol{r})\hat{O}_B(\mathbf{0}) = \sum_i c_i(\boldsymbol{r})\,\hat{O}_i(\mathbf{0})$$

  - ***$\hat{O}_i(0)$*** **with lowest scaling dimension**

$$\psi_1^\dagger\psi_1,\ \psi_1^\dagger\nabla\psi_1,\ g^2\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1,\ \cdots$$

3 4 6-2=4

  - **Matching matrix elements → Wilson coefficients**

    Few-body problem can be solved exactly

  - **Operator identity is valid for any states → Universal relation**

OPE reveals aspects of many-body physics controlled by few-body physics

Operator product expansion

$$\hat{O}_A(\boldsymbol{r})\hat{O}_B(\boldsymbol{0}) = \sum_i c_i(\boldsymbol{r})\,\hat{O}_i(\boldsymbol{0})$$

- **How to determine the Wilson coefficients?**

In QCD

- **Strong coupling constant αs is small at short distance**
- **Perturbation theory in small αs → Wilson coefficients**

In strongly interacting atoms

- **Coupling is large → nonperturbative problem**
- **Few-body problem can be solved exactly.**

→ Wilson coefficients of lowest-dimension operators

Operator product expansion

Braaten and Platter [PRL 2008]

$$\rho(k) = \langle \widetilde{\psi}_1^\dagger \widetilde{\psi}_1(k) \rangle$$
$$= \int_{R,r} e^{-ik\cdot r} \langle \psi_1^\dagger(R - \frac{r}{2}) \psi_1(R + \frac{r}{2}) \rangle$$

After matching for 1 atom and 2 atom state …

$$\psi_1^\dagger(-\tfrac{r}{2})\psi_1(+\tfrac{r}{2}) = 1 \times \psi_1^\dagger\psi_1(0) \quad \to \delta(k)$$
$$+\tfrac{\vec{r}}{2}\cdot[\psi_1^\dagger \nabla\psi_1(0) - \nabla\psi_1^\dagger\psi_1(0)] \quad \to \delta'(k)$$
$$-\tfrac{r}{8\pi} g^2 \psi_1^\dagger\psi_2^\dagger\psi_2\psi_1(0) + \cdots \quad \to 1/k^4$$

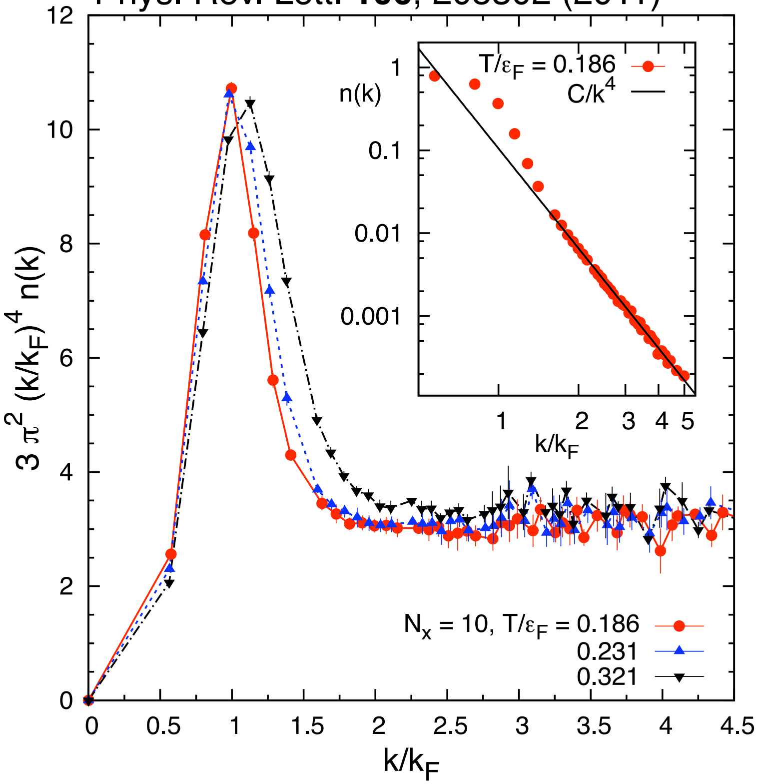
$$\rho(k) \to \frac{C}{k^4}$$

Proof of universal relation

Momentum distribution & Contact

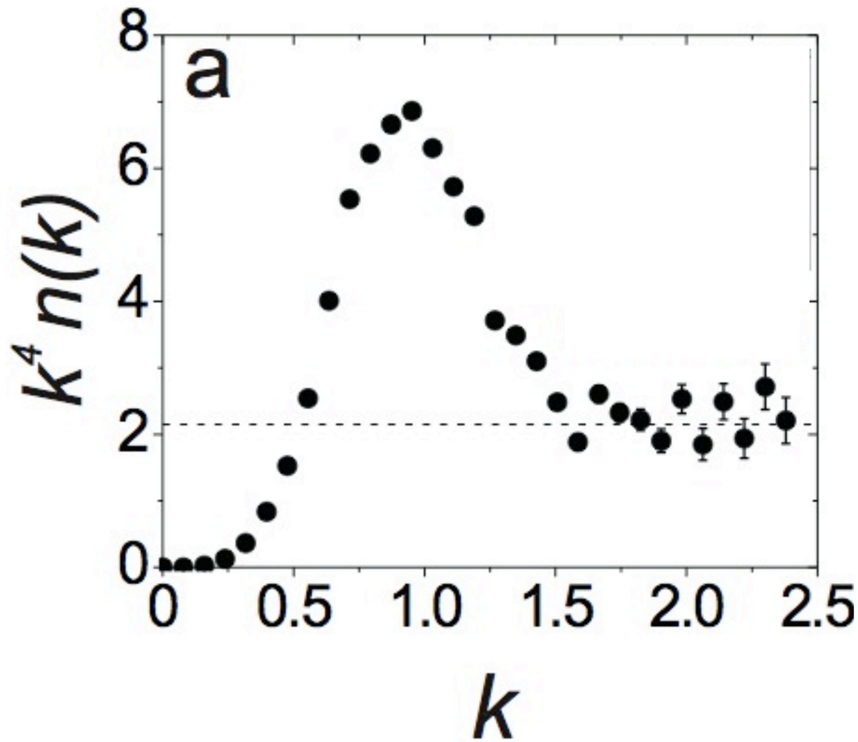
Theory (lattice)

J. E. Drut, T. A. Lähde, T. Ten
Phys. Rev. Lett. **106**, 205302 (2011)



Experiment

J. T. Stewart et al
PRL **104**, 235301 (2010)



- T/T_F = 0 - 0.5 $\quad k_F r_{\rm eff} \simeq 0.3 - 0.5$
- Plateau seen both in **theory** and **experiment**!

Identical Bosons

2- and 3-body physics

- **2-body : similar to fermions**

  scale invariance when $a \to \pm\infty$

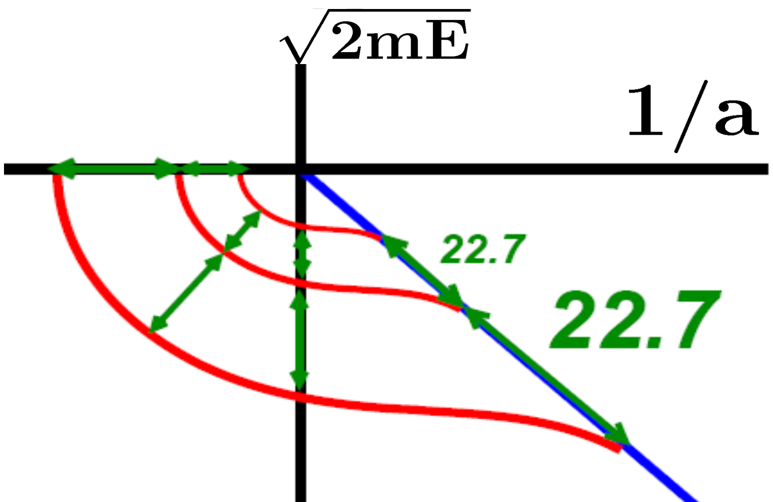
- **3-body :**

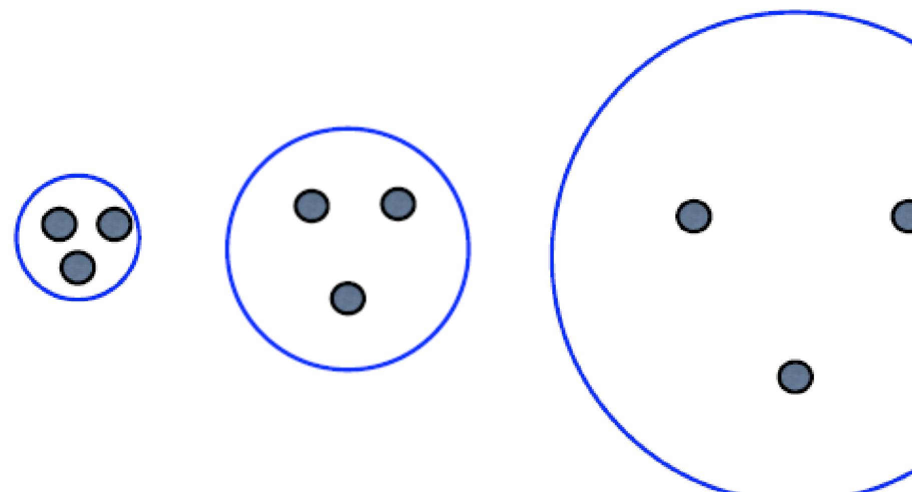
  Log-periodic behavior !!!

  Discrete scale invariance !!!

  Efimov physics

- **Efimov trimers**

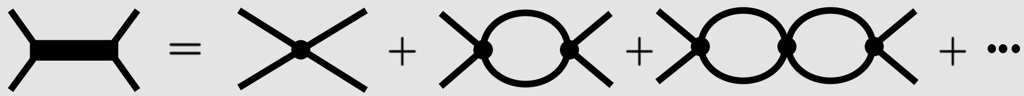


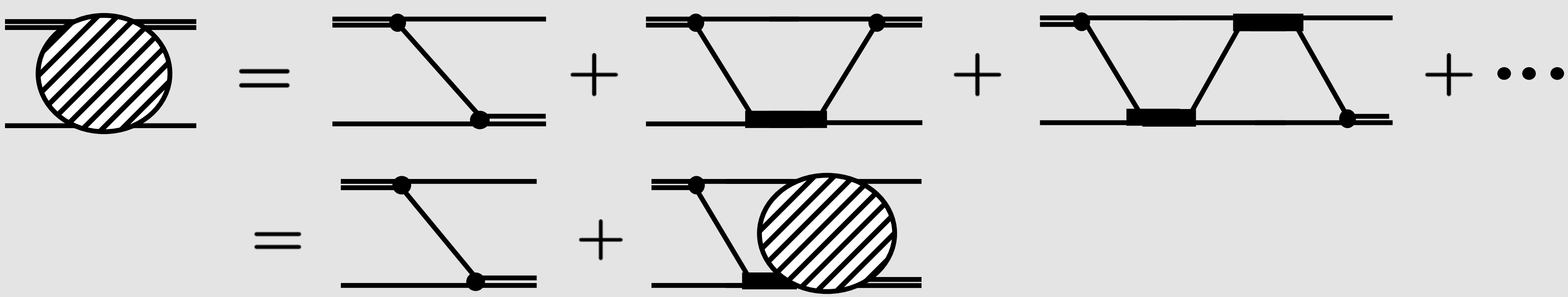


Effective Field Theory for bosons

- **Interaction** $\frac{g_2}{4}\left(\psi^\dagger\psi\right)^2$

- **Integral equation for atom-diatom amplitude**



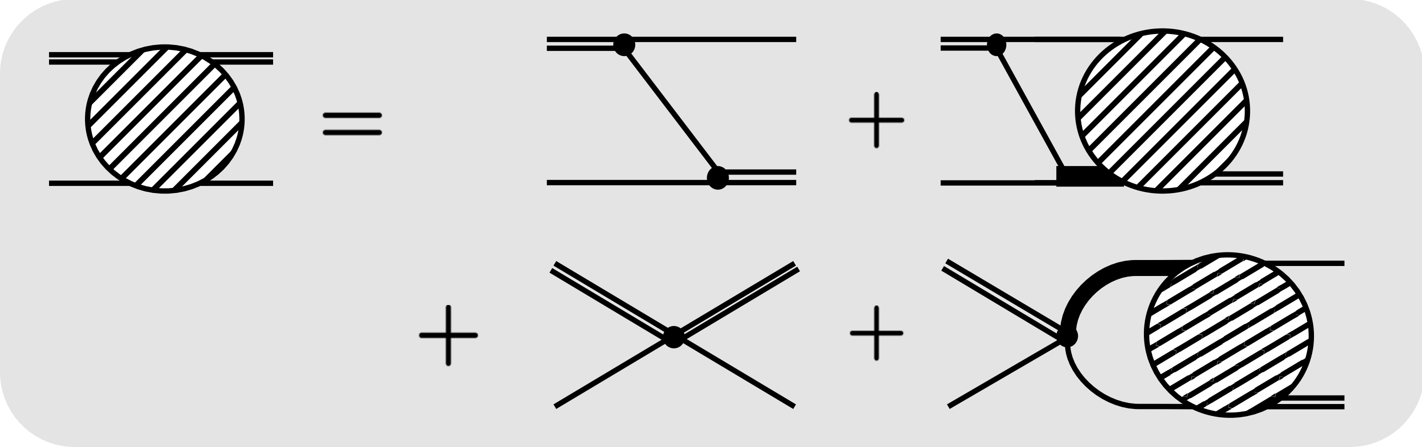


$$\propto \frac{\cos[s_0 \ln(k/\Lambda)]}{k} \qquad s_0 \approx 1.006$$

Effective Field Theory for bosons

Bedaque, Hammer, and van Kolck [PRL 1999]

- **Interactions** $\frac{g_2}{4}\left(\psi^\dagger\psi\right)^2 + \frac{g_3}{36}\left(\psi^\dagger\psi\right)^3$
- **Integral equation for atom-diatom amplitude**



- **approximate solution to renormalization** $s_0 \approx 1.006$

$$H_{\mathrm{BHvK}} \approx -\frac{\sin[s_0 \ln(\Lambda/\Lambda_*) - \arctan(1/s_0)]}{\sin[s_0 \ln(\Lambda/\Lambda_*) + \arctan(1/s_0)]} \quad g_3 = -9\frac{g_2^2}{\Lambda^2}H_{BHvK}$$

Good agreement to numerical solution by BHvK!!

Braaten, DK , Platter [PRL 2011]

Universal Relations

from 3-body physics

- **Hold for any state of the system**
- **Involve 2-body *contact* (*C_2*)**
  and *3-body contact* (*C_3*)
- **Characterized by log-periodic behavior**
  (Efimov physics)

Relations for bosons

- **Adiabatic relations: 2-body and 3-body contacts**

$$a\frac{dE}{da} = \frac{C_2}{8\pi a} \qquad \kappa_*\frac{dE}{d\kappa_*} = -2C_3$$

κ_* **= binding momentum of trimer at *a*=** $\pm\infty$

- **Tail of *ρ(k)* is derived by using OPE for** $\psi^\dagger(-\frac{r}{2})\psi(+\frac{r}{2})$

  matching for 1-,2-, and 3- body states

$$\rho(k) \to \frac{C_2}{k^4} + F(k)\frac{C_3}{k^5}$$

$$F(k) = 78.5\,\sin[2s_0\ln(k/\kappa_*) - 1.34]$$

Tail of rf transition for bosons

- **Without final state interaction**

$$\Gamma(\omega) \to \Omega^2 \left[\frac{1}{4\pi\omega^{3/2}} C_2 + \frac{G(\omega)}{2\omega^2} C_3 \right]$$

$$G(\omega) = 9.23 - 13.6 \, \sin[s_0 \ln(\omega/\kappa_*^2) + 2.66]$$

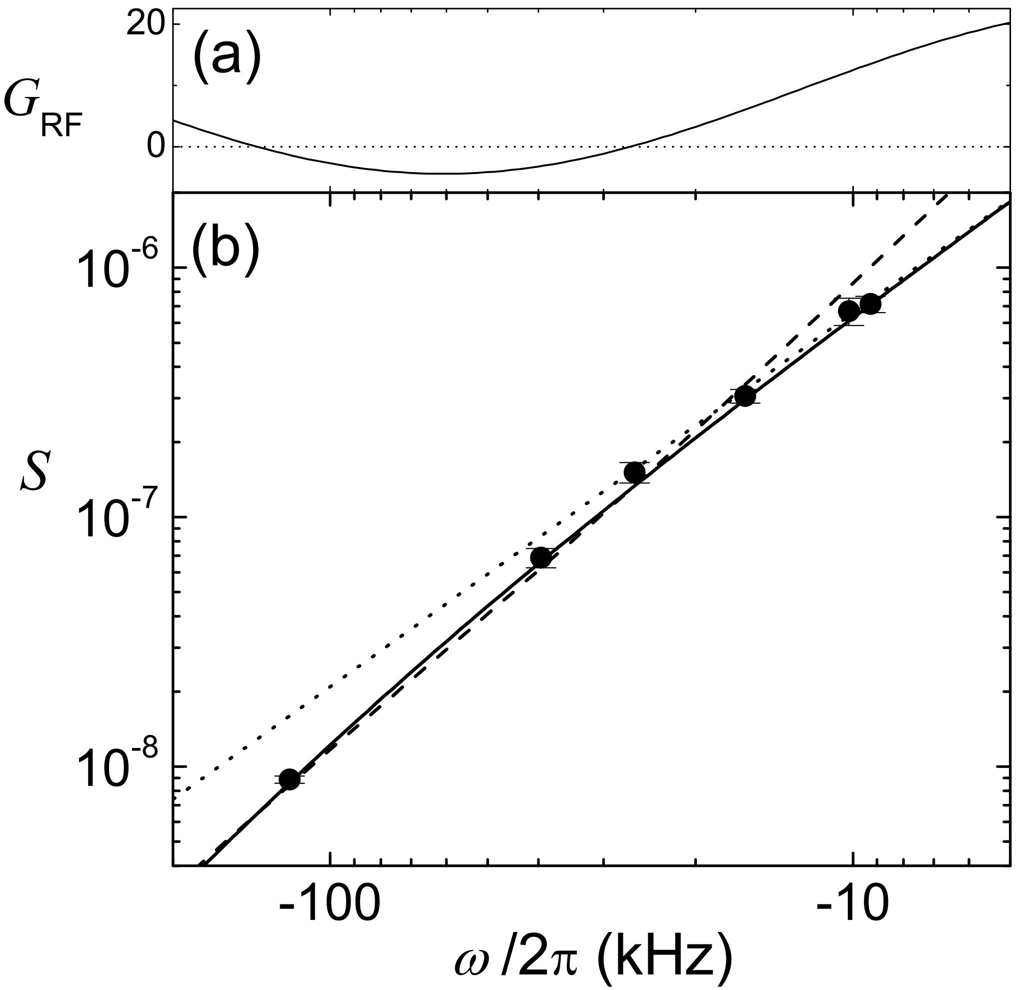
- **With final state interaction**

$$\Gamma(\omega) \to \Omega^2 \left[\frac{(1/a - 1/a')^2}{4\pi\omega^{3/2} \, (1/a'^2 + \omega)} C_2 + ? \, C_3 \right]$$

  - **Coeff. of C_3 needs to be calculated!!**

rf spectroscopy for ^{85}Rb BEC

Wild et al, PRL 108 2012



- **Solid/Dotted line: rate with/without final state interaction**
- **Dashed line: rate for C_3/N_0= 0.1 μm^{-2}**
- **Upper limit of C_3/N_0= 0.07 μm^{-2}**
- **a=982 a0**
- **<n>=10^{13} cm^{-3}**

Relation for trimer state

- **Exact Trimer wavefunction at $a = \pm\infty$ is known**
- **Tail of *ρ(k)* for trimer with** $E = -\kappa_*^2$

$$\rho(k) \to \frac{53.1\,\kappa_*}{k^4} + F(k)\frac{\kappa_*^2}{k^5}$$

Werner and Castin
[arxiv:1001]

$$F(k) = 89.3\,\sin[2s_0 \ln(k/\kappa_*) - 1.34]$$

$$s_0 \approx 1.006$$

- **Contacts for trimer**

$$C_2 = 53.1\kappa_*$$

$$C_3 = \kappa_*^2$$

$$a\frac{dE}{da} = \frac{C_2}{8\pi a}$$

$$\kappa_*\frac{dE}{d\kappa_*} = -2C_3$$

Puzzle

Braaten, DK , Platter 2011

- **From trimer WF**

$$F_{WC}(k) = 89.3 \sin[2 s_0 \ln(k/\kappa_*) - 1.34]$$

- **From the OPE**

12 % smaller

$$F_{OPE}(k) = 78.5 \sin[2 s_0 \ln(k/\kappa_*) - 1.34]$$

$$s_0 \approx 1.006$$

- **Struggling for 3 months**
  - **Missing diagrams in OPE?**
  - **Numerical Error in OPE?**
  - **Error in Wavefunction by *Werner and Castin* ?**
  - **Failure of OPE?**

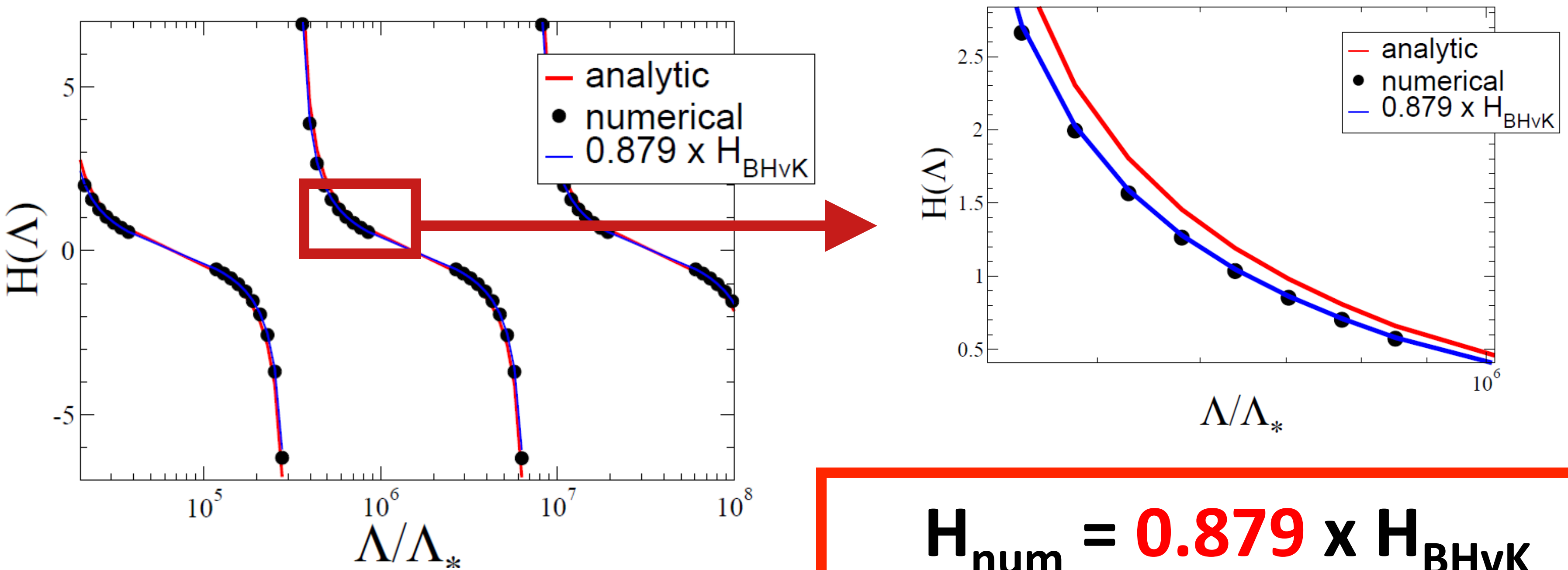
Clue

- **3-body coupling:** $g_3 = -9\frac{g_2^2}{\Lambda^2}H_{BHvK}$

Bedaque, Hammer, van Kolck [PRL 1999]

$$H_{\rm BHvK} \approx -\frac{\sin[s_0 \ln(\Lambda/\Lambda_*) - \arctan(1/s_0)]}{\sin[s_0 \ln(\Lambda/\Lambda_*) + \arctan(1/s_0)]}$$

- **Small discrepancy with numerical result (H_{num})**



$$H_{num} = 0.879 \times H_{BHvK}$$

Solution

- H_{BHvK} **is an approximate solution and has 12% correction in prefactor**

$$H_{\rm BHvK} \approx -h_0 \frac{\sin[s_0 \ln(\Lambda/\Lambda_*) - \arctan(1/s_0)]}{\sin[s_0 \ln(\Lambda/\Lambda_*) + \arctan(1/s_0)]}$$

$$h_0 \approx 0.879$$

- $F_{OPE}(k)$ **also changes by** h_0

$$F_{OPE}(k) = \frac{78.5}{h_0} \sin[2 s_0 \ln(k/\kappa_*) - 1.34]$$

Agrees with F(k) from trimer wavefunction!!

$$\rho(k) \to \frac{C_2}{k^4} + F_{OPE}(k) \frac{C_3}{k^5}$$

Summary

- Universal relations for strongly interacting atoms
- OPE is a powerful tool

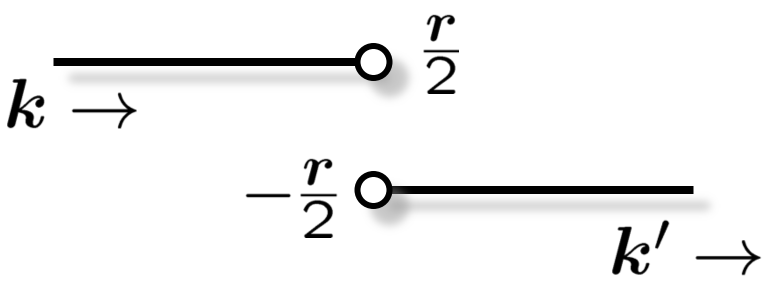
  many-body physics controlled by few-body physics
- Contact is central quantity
  - 2-body contact for fermions with 2 spin states
  - 2- and 3-body contacts for identical bosons

    (and for fermions with >2 spin states)
- Efimov feature in the tail of many-body physics!!

Back up

Matching for 1-atom State

$$\psi_1^\dagger(-\tfrac{r}{2})\psi_1(+\tfrac{r}{2})$$



$$ie^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}/2}$$
$$= i - (\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}/2 + \cdots$$

$$\psi_1^\dagger\psi_1(0)$$

$$ie^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{0}}$$

Wilson Coefficient -> 1

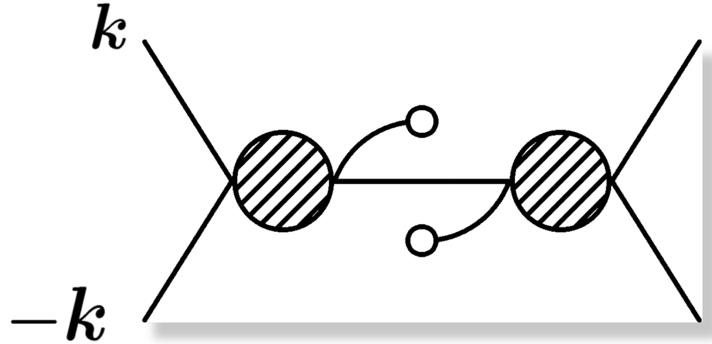
$$\psi_1^\dagger\nabla^j\psi_1(0) - \nabla^j\psi_1^\dagger\psi_1(0)$$

$$i^2(k_j + k'_j)e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{0}}$$

Wilson Coefficient -> $r_j/2$

Matching for 2-atom State

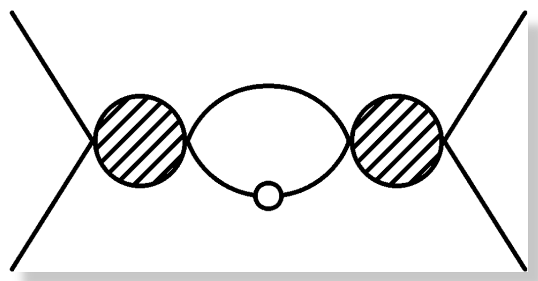
$\psi_1^\dagger(-\frac{r}{2})\psi_1(+\frac{r}{2})$



$i2\pi f(k)^2 \frac{e^{ikr}}{k}$

$\psi_1^\dagger\psi_1(0)$

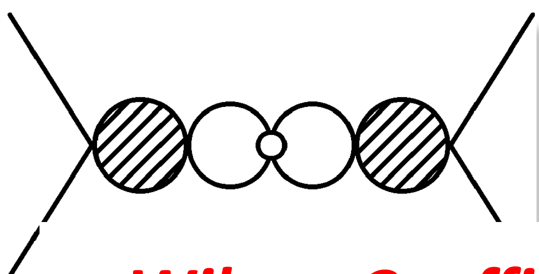
$\psi_1^\dagger\nabla^j\psi_1(0) - \nabla^j\psi_1^\dagger\psi_1(0)$



$i2\pi f(k)^2 \frac{1}{k}$

0

$g^2\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1(0)$



$16\pi^2 f(k)^2$

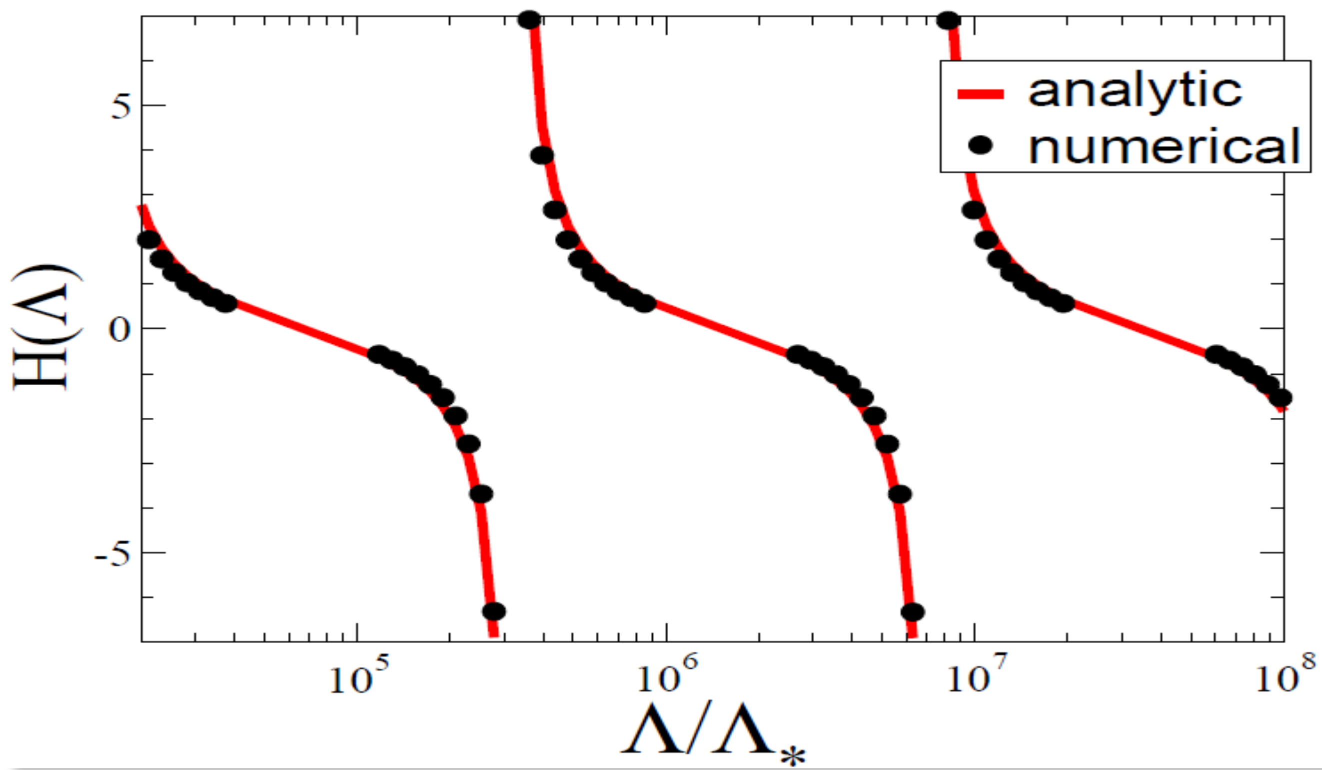
Wilson Coefficient -> -r /(8π)

Effective Field Theory for bosons

$$g_3 = -9\frac{g_2^2}{\Lambda^2}H_{BHvK}$$

Bedaque, Hammer, van Kolck [PRL 1999]

$$H_{BHvK} = -\frac{\sin[s_0 \ln(\Lambda/\Lambda_*) - \arctan(1/s_0)]}{\sin[s_0 \ln(\Lambda/\Lambda_*) + \arctan(1/s_0)]}$$

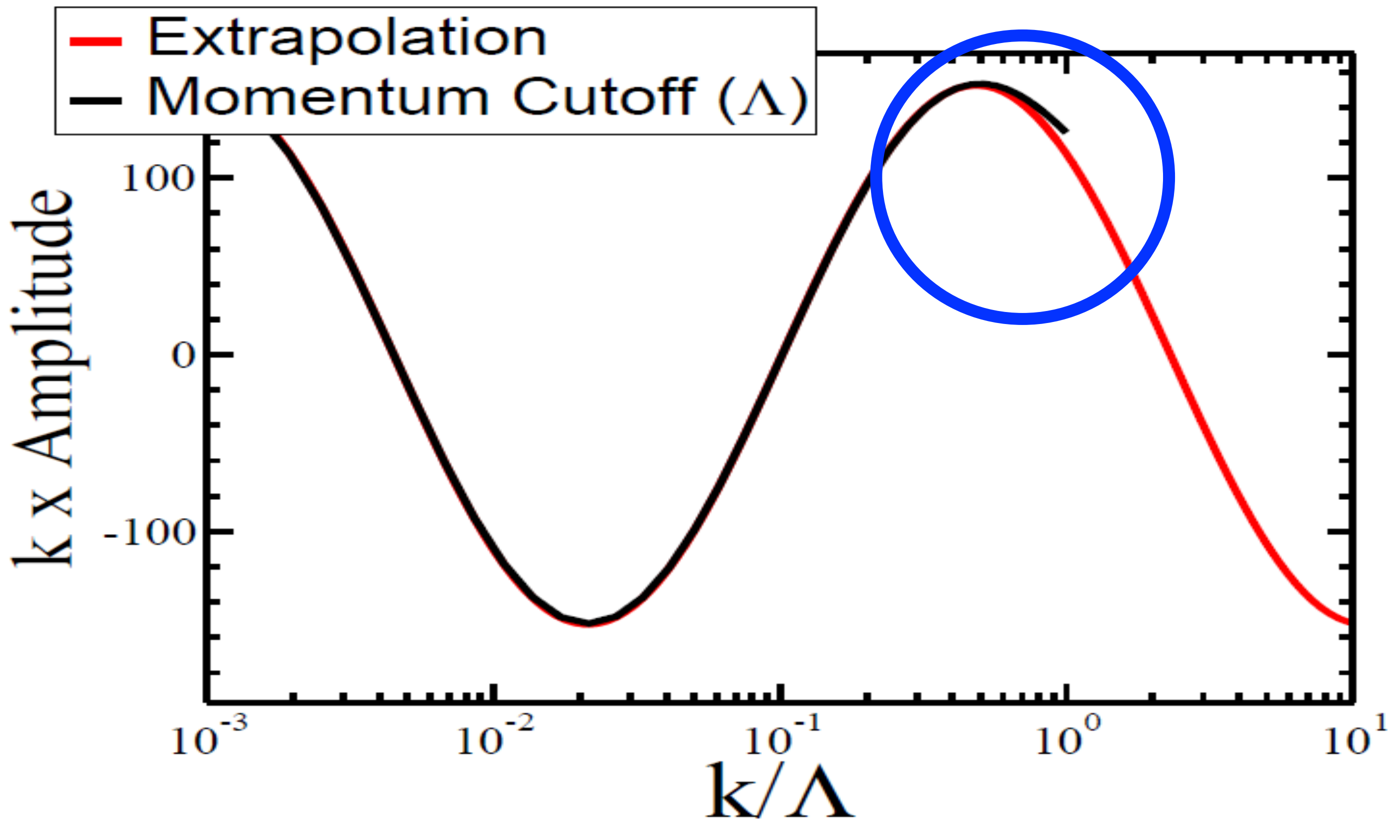


$$s_0 \approx 1.006$$

3-body problem is renormalized!

Diagnostic

- **Assumption in derivation of H_{HBvK} :**
  log-periodic scaling behavior $\cos\left[s_0 \ln\left(k/\Lambda_*\right)\right]/k$
  is extrapolated up to cutoff (Λ)



- **Many more universal relations**
  - **Pressure relation, Virial theorem, Energy relation : Tan [2005]**
  - **Photoassociation: Werner, Carruel + Castin [EPJB 2009]; Zhang + Leggett [PRA 2009]**
  - **Structure factors: Son + Thompson [PRA 2010]; Hu, Liu + Drummond [EPL 2010]; Goldberger + Rothstein[arXiv:1012]**
  - **Correlation for viscosity: Taylor + Randeria [PRA2010]; Enss, Haussmann + Zwerger [Annals Phys. 2011]**
  - **Hard probe: Nishida [arXiv:1110]**

- **Cold atom experiments : measurement of contact**
  - **Vale group at Swinburne PRL 2010**
  - **Jin group at JILA PRL 2010, 2012**

More Universal relations

- **Homogeneous system**

Pressure relation

$$\mathcal{P} = \frac{2}{3}\frac{E}{V} + \frac{1}{12\pi a}\frac{C}{V}$$

Tan 2005

- **Trapped in harmonic potential** $\frac{1}{2}\omega^2 r^2$

Virial theorem

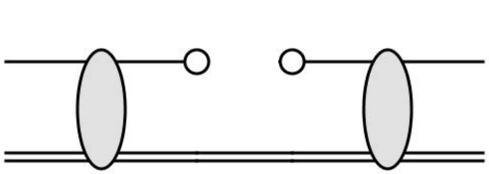
$$E = 2V_{HO} - \frac{1}{8\pi a}C$$

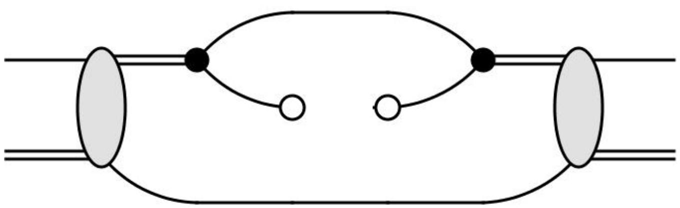
Tan 2005

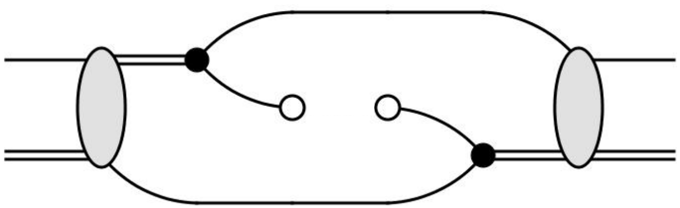
Proof using OPE

- **Matching for atom-diatom State**

$$\psi^\dagger(-\tfrac{r}{2})\psi(+\tfrac{r}{2})$$





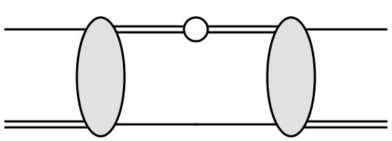


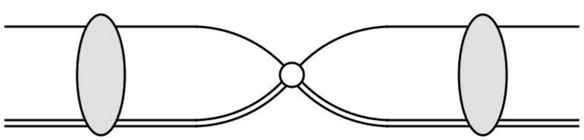
$$\psi^\dagger\psi(0),\ \psi^\dagger\nabla\psi(0)$$

$$g^2\psi^\dagger\psi^\dagger\psi\psi(0)$$

$$\frac{-g^2H'}{8\Lambda^2}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi(0)$$

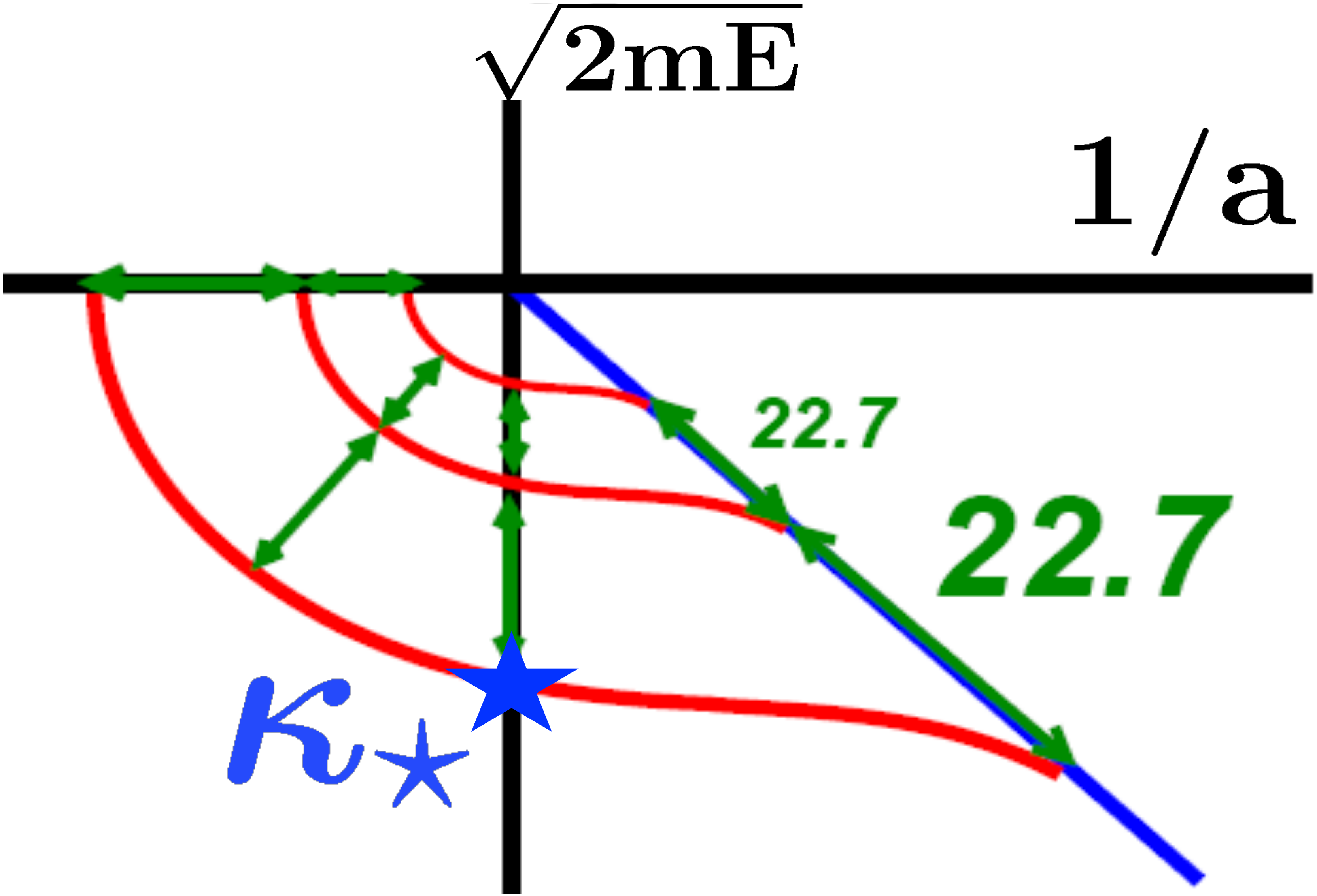
No contribution since vanishing Wilson coeff.





Trimer Spectrum

Vitaly Efimov [1970]



2- and 3-body physics

- **2-body : similar to fermions**

  scale invariance when $a \to \pm\infty$

- **3-body :**

  Log-periodic behavior !!!

  Discrete scale invariance !!!

Efimov physics

Due to imaginary anomalous dimension s_0=1.006

$$(k/\Lambda)^{is_0} = e^{is_0 \ln(k/\Lambda)} \to \cos[s_0 \ln(k/\Lambda)]$$

Invariant under $k \to e^{2\pi/s_0}k = 22.7^2 \times k$

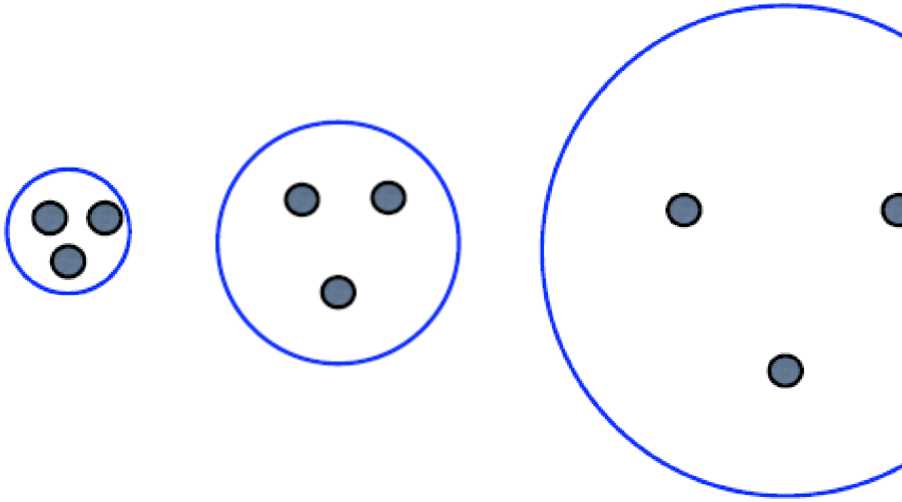
Efimov Trimers *Vitaly Efimov [1970]*

- **Infinitely many tri-atomic molecules with accumulation point at 0 binding energy at *a*= $\pm\infty$**
- **Trimer Energies differ by 22.7^2 = 515**
- **Trimer Sizes differ by 22.7**

$$e^{\pi/s_0} \approx 22.7$$

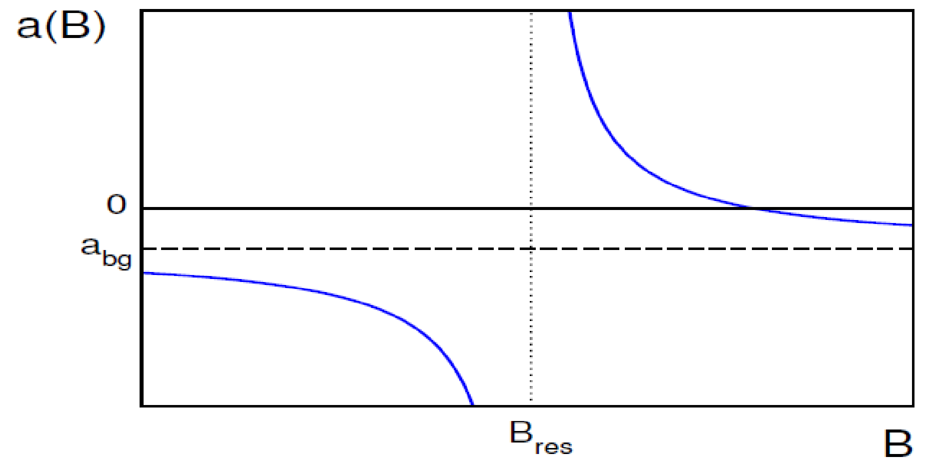
$$s_0 \approx 1.006$$

- **Trimer Structure**



Strongly interacting particles

- **For atoms,**
  - **Near Feshbach resonance,**
    ***a* varies with the B field !**



- **For nucleons,**
  - ***a* = -19 fm (n-n) and *a* = +5.3 fm (n-p spin-triplet)**

$\ell_\pi \approx 1.4$ fm

  - ***a* varies with quark masses**
    - ***Tuning u* and *d* masses → *a* = $\pm\infty$ *for the 2 channels***

      Braaten, Hammer [PRL 2003]

    - **Constraint on quark mass variation from BBN**

      quark mass → a → binding energies → BBN

      Bedaque, Luu, Platter [PRC 2011]