Causality bounds for neutron-proton scattering

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work done with Dean Lee

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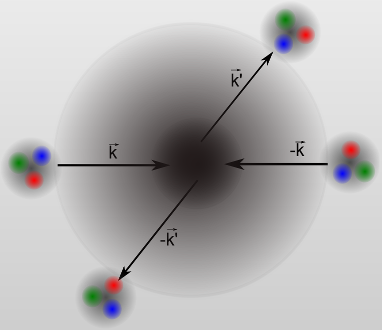
Lee Research Group

Presentation Outline

Outline

Two-body scattering in center of mass frame



Time-independent Schrödinger wave function

$$\psi^{(k)}(\vec{r}) = R_\ell^{(k)}(r)\mathrm{Y}_{\ell,m_\ell}(\theta,\phi)$$

Spherical harmonics

$$\mathrm{Y}_{\ell,m_\ell}(\theta,\phi)$$

Radial wave function

$$R_\ell^{(k)}(r) = \frac{u_\ell^{(k)}(r)}{r}$$

Regularity

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu\, V(r)\, -k^2\right]u_\ell^{(k)}(r)=0$$

$\Rightarrow$ μ is the reduced mass

$\Rightarrow$ Finite range interaction potential

$\Rightarrow$ The interaction is not too singular at short distances.

$$u_\ell^{(k)}(r\to 0)\longrightarrow 0$$

For $V(r)=O(r^{-2+\varepsilon})$ type of potentials with positive ε, the short-distance regularities is satisfied as $r\to 0$.

Phase shift

$$u_\ell^{(k)}(r) \sim \cos\delta_\ell(k)\; S_\ell(kr) - \sin\delta_\ell(k)\; C_\ell(kr) \quad \text{for } \; r > R$$

$S_\ell(kr), C_\ell(kr)$: Riccati-Bessel functions

Phase shift

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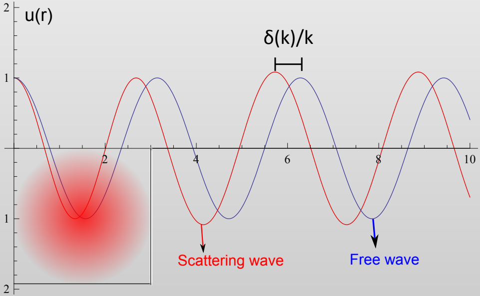


$$u_\ell^{(k)}(r) \sim \sin[kr - \pi\ell/2 + \delta_\ell(k)] \qquad \text{as} \;\; r \to \infty$$

Scattering amplitude

$$u_\ell^{(k)}(r) \sim \frac{e^{-\delta_\ell(k)}}{2i}\left[e^{2i\delta_\ell(k)}\, e^{i(kr-\pi\ell/2)} - e^{-i(kr-\pi\ell/2)}\right]$$

outgoing incoming

$$\sim \frac{ie^{-\delta_\ell(k)}}{2}\left[h_\ell^{(-)}(kr) - \hat{S}\, h_\ell^{(+)}(kr)\right]$$

$\hat{S}$: the scattering matrix

$$\psi_\ell(\vec{r}) \underset{r\to\infty}{\longrightarrow} e^{i\vec{k}\cdot\vec{r}} + f(\vec{p},\vec{p}')\frac{e^{ikr}}{r}$$

$$f(\vec{p},\vec{p}') = \sum_{\ell=0}^{\infty} f_\ell(k) P_\ell(\cos\theta)$$

converges at $k = 0$

$$f_\ell(k) = \frac{e^{2i\delta_\ell(k)} - 1}{2ik} = \frac{k^{2\ell}}{k^{2\ell+1}\cot\delta_\ell(k) - ik^{2\ell+1}}$$

Effective range expansion

$$\frac{k^{2\ell+1}}{\hat{\mathsf{K}}} = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell\, k^2 + \mathcal{O}(k^4) \quad \hat{\mathsf{K}} \text{ : the reaction matrix}$$

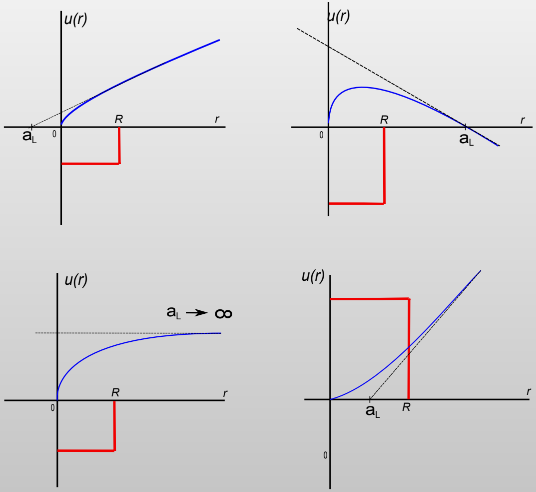
scattering length $[Length]^{2\ell+1}$

effective range $[Length]^{-2\ell+1}$

$$\hat{\mathsf{S}} = (1 + i\hat{\mathsf{K}})(1 - i\hat{\mathsf{K}})^{-1} = e^{2i\delta_\ell(k)} \qquad \hat{\mathsf{K}}^{-1} = \cot\delta_\ell(k)$$

$$k^{2\ell+1}\cot\delta_\ell(k) = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell\, k^2 + \mathcal{O}(k^4)$$

Scattering length



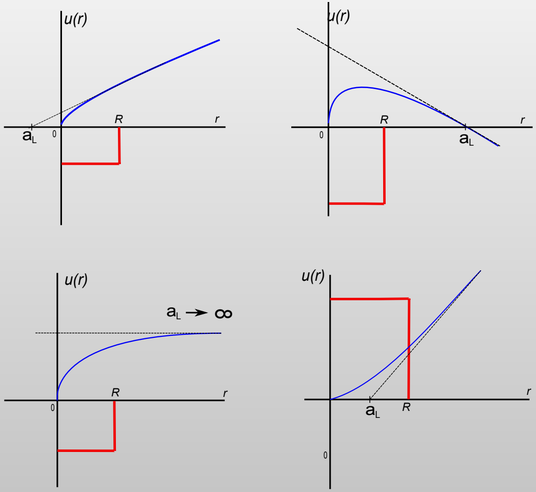
For an attractive potential,

Negative a_ℓ : no bound state

Positive a_ℓ : bound state

$a_\ell \to \infty$: zero energy bound state

Scattering length



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Neutron-proton scattering:

$a_{^1S_0} = -23.727$ [fm]

$a_{^3S_1} = 5.418$ [fm] deuteron !

Causality and Unitarity

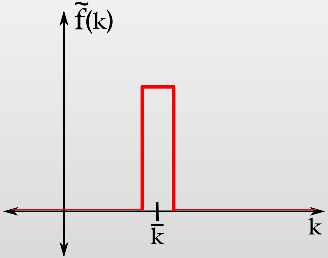
Causality and Unitarity should be preserved !

Unitarity ; the sum of all outcomes probabilities equals one.

Causality; the cause of an event must occur before any resulting consequence are produced.

Causality

Consider a spherical wave packet sharply peaked in momentum space at $k = \bar{k}$.



$$f(r) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikr} \tilde{f}(k)$$

Define

$$\tilde{g}(k) = e^{2i\delta(k)} \tilde{f}(k)$$

Taylor expand the phase shift around $k = \bar{k}$

$$\delta(k) = \delta(\bar{k}) + (k - \bar{k}) \frac{d\delta(\bar{k})}{dk} + \frac{(k-\bar{k})^2}{2!} \frac{d^2\delta(\bar{k})}{dk^2} + \ldots$$

Causality

Fourier transform to the configuration space

$$g(r) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikr} \tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \int dk\, e^{ikr} e^{2i\delta(k)}\, \tilde{f}(k)$$

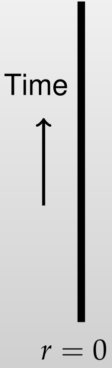
$$\approx \frac{e^{-2i\bar{k}\delta'(\bar{k})} e^{2i\delta(\bar{k})}}{\sqrt{2\pi}} \int dk\, e^{ik[r+2\delta'(k)]}\, \tilde{f}(k)$$

$$\approx e^{-2i\bar{k}\delta'(\bar{k})}\, g\left[r + 2\delta'(k)\right]$$

$\Rightarrow$ An overall phase multiplication $e^{-2i\bar{k}\delta'(\bar{k})}$
$\Rightarrow$ A backward translation in space by $2\delta'(\bar{k})$

A similar relation can be written for the time delay of the scattered wave.

Causality



The wavepacket is shifted by

$$-2\frac{d\delta(k)}{dk}$$

The scattering wave is delayed by

$$2\frac{d\delta(k)}{dE}$$

Wigner:
For finite range potentials

$$\frac{d\delta(k)}{dk} \geq -R$$

Causality Bound !

H. W. Hammer and D. Lee, Annals Phys. 325, 2212 (2010).

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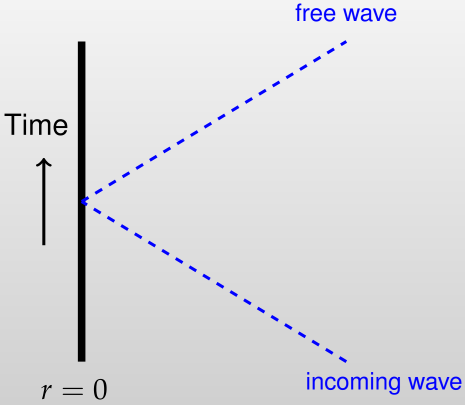
Causality



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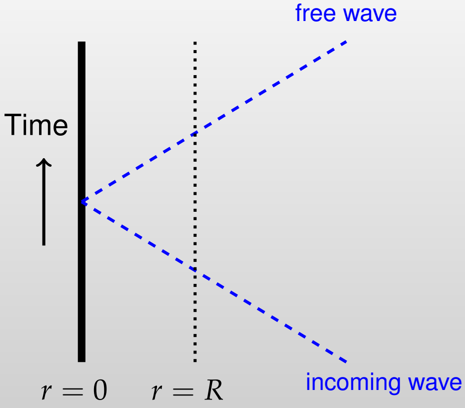

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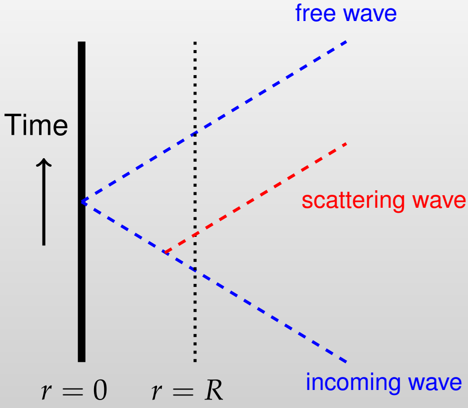

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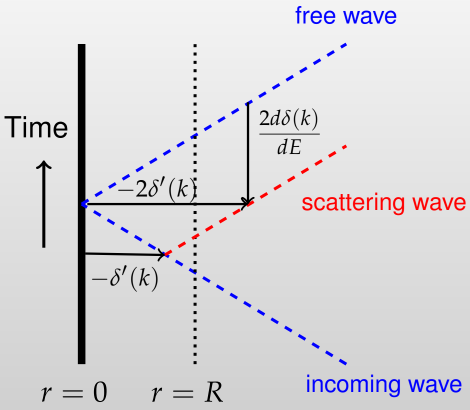

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Outline

Single channel

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + k^2\right] u_L^{(k)}(r) = 2\mu \int_0^R W(r,r') u_L^{(k)}(r') dr'$$

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for $r > R$

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$$= k^{2L+1} \cot \delta_L(k) s_L(k,r) + c_L(k,r)$$

$$S_L(r) = k^{L+1}\, s_L(k,r)$$

$$C_L(r) = k^{-L}\, c_L(k,r)$$

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$s_L(k,r) = s_0(r) + k^2\, s_2(r) + \mathcal{O}(k^4)$

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$$u_L^{(k)}(r) = \frac{-1}{a_L} s_{0,L}(r) + c_{0,L}(r) + k^2 \left\{ \frac{1}{2} r_L s_{0,L}(r) - \frac{1}{a_L} s_{2,L}(r) + c_{2,L}(r) \right\} + \mathcal{O}(k^4)$$

Single channel

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + k_a^2\right] u_{a,L}(r) = 2\mu \int_0^R W(r,r') u_{a,L}(r') dr'$$

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} + k_b^2\right] u_{b,L}(r) = 2\mu \int_0^R W(r,r') u_{b,L}(r') dr'$$

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$$\begin{aligned}(k_a^2 - k_b^2) u_{a,L}(r) u_{b,L}(r) = & - u_{b,L}(r) u''_{a,L}(r) + u_{a,L}(r) u''_{b,L}(r) \\ & + 2\mu \int_0^R [u_{b,L}(r) W(r,r') u_{a,L}(r') - u_{a,L}(r) W(r,r') u_{b,L}(r')] dr'\end{aligned}$$

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$$(k_a^2 - k_b^2) \int_\rho^r u_{a,L}(r) u_{b,L}(r) = \left(u_{b,L}(r) u'_{a,L}(r) - u_{a,L}(r) u'_{b,L}(r) \right|_\rho^r + 2\mu \int_\rho^r dr \int_0^R [u_{b,L}(r) W(r,r') u_{a,L}(r') - u_{a,L}(r) W(r,r') u_{b,L}(r')] dr'$$

$$\lim_{\rho\to 0^+} u_{a,L}(\rho) u'_{b,L}(\rho) = \lim_{\rho\to 0^+} u_{b,L}(\rho) u'_{a,L}(\rho) = 0$$

Single channel

Wronskian identity

$$W\left[U_L^{(k_a)}(r), U_L^{(k_b)}(r)\right] = (k_a^2 - k_b^2)\int_0^r U_L^{(k_a)}(r')U_L^{(k_b)}(r')dr'$$

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set $k_b = 0$, and limit $k_a \to 0$

$r \geq R$, interaction range — low evergy limit, $k \to 0$

$$r_L = b_L(r) - 2\int_0^r \left[U_L^{(0)}(r')\right]^2 dr'$$

The generalized form of Bethe's integral formula

Bethe, PR 76 (1949) 38
Hammer and Lee, PLB 681 (2009) 500; Annals Phys. (2010) 2212

$$b_L(r) = -\frac{2\Gamma(L-\frac{1}{2})\Gamma(L+\frac{1}{2})}{\pi}\left(\frac{r}{2}\right)^{-2L+1} - \frac{4}{L+\frac{1}{2}}\frac{1}{a_L}\left(\frac{r}{2}\right)^2 + \frac{2\pi}{\Gamma(L+\frac{3}{2})\Gamma(L+\frac{5}{2})}\frac{1}{a_L^2}\left(\frac{r}{2}\right)^{2L+3}$$

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S-wave $r_0 \leq b_0(r) = \frac{2}{3a_0^2} r^3 - \frac{2}{a_0} r^2 + 2r$

P-wave $r_1 \leq b_1(r) = \frac{2r^5}{45a_1^2} - \frac{2r^2}{3a_1} - \frac{2}{r}$

D-wave $r_2 \leq b_2(r) = \frac{2}{1575a_2^2} r^7 - \frac{2}{5a_2} r^2 - \frac{6}{r^3}$

F-wave $r_3 \leq b_3(r) = \frac{2r^9}{99225a_3^2} - \frac{2r^2}{7a_3} - \frac{90}{r^5}$

G-wave $r_4 \leq b_4(r) = \frac{2r^{11}}{9823275a_4^2} - \frac{2r^2}{9a_4} - \frac{3150}{r^7}$

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$$r_0 = 2.670 \quad \text{fm}$$

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S-wave	$r_0 \leq b_0(r) = \frac{2}{3a_0^2}r^3 - \frac{2}{a_0}r^2 + 2r$	$b_0(r \to 0) = 0$
P-wave	$r_1 \leq b_1(r) = \frac{2r^5}{45a_1^2} - \frac{2r^2}{3a_1} - \frac{2}{r}$	$b_1(r \to 0) = -\infty$
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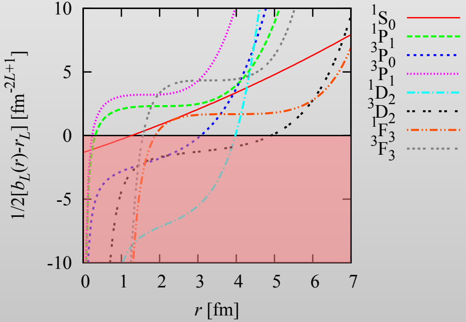


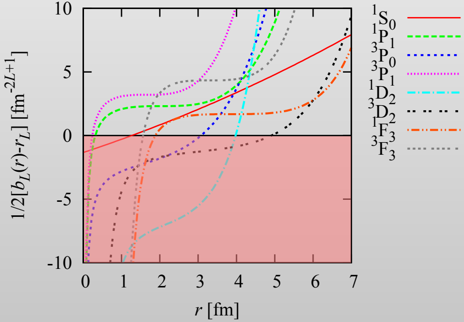
Channel	R^b [fm]
$^1S_0$	1.27
$^1P_1$	0.31
$^3P_0$	3.07
$^3P_1$	0.23
$^1D_2$	3.98
$^3D_2$	4.91
$^1F_3$	1.88
$^3F_3$	1.56

NijmegenII NN scattering data.

Causal range, $0 \le R^b \le R$

Define the causal range, R^b, the minimum range of the interaction that can reproduce physical scattering parameters.

$$b_L(r) - r_L = 2\int_0^r \left[U_L^{(0)}(r')\right]^2 dr' \qquad b_L(R^b) - r_L = 0$$



Channel	R^b [fm]
$^1S_0$	1.27
$^1P_1$	0.31
$^3P_0$	3.07
$^3P_1$	0.23
$^1D_2$	3.98
$^3D_2$	4.91
$^1F_3$	1.88
$^3F_3$	1.56

NijmegenII NN scattering data.

$m_\pi^{-1} \simeq 1.5$ [fm]

Spin-orbital coupling $s = 1, \ell = J \pm 1$

The coupled radial Schrödinger equations

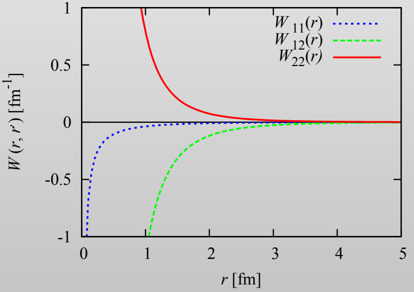
$$\left[-\frac{d^2}{dr^2} - k^2 + \frac{J(J-1)}{r^2}\right] U_{J-1}^{(k)}(r) = -2\mu \int_0^R [W_{11}(r,r')U_{J-1}^{(k)}(r') + W_{12}(r,r')V_{J+1}^{(k)}(r')]\, dr'$$

$$\left[-\frac{d^2}{dr^2} - k^2 + \frac{(J+1)(J+2)}{r^2}\right] V_{J+1}^{(k)}(r) = -2\mu \int_0^R [W_{21}(r,r')U_{J-1}^{(k)}(r') + W_{22}(r,r')V_{J+1}^{(k)}(r')]\, dr'$$

The non-local interaction potentials

$$W(r,r') = \begin{pmatrix} W_{11}(r,r') & W_{12}(r,r') \\ W_{21}(r,r') & W_{22}(r,r') \end{pmatrix}$$

$$W_{21}(r,r') = W_{12}(r,r')$$



Eigenphase parameterization

S-matrix in the BB parameterization.

$$\hat{S}_d = \begin{pmatrix} e^{2i\delta_\alpha} & 0 \\ 0 & e^{2i\delta_\beta} \end{pmatrix}$$

$\hat{S}_d$ is a diagonal matrix.

J. M. Blatt and L. C. Biedenharn, Phys. Rev. 86, 399 (1952).

Eigenphase parameterization

S-matrix in the BB parameterization.

$$\hat{S}_d = \begin{pmatrix} e^{2i\delta_\alpha} & 0 \\ 0 & e^{2i\delta_\beta} \end{pmatrix}$$

$\hat{S}_d$ is a diagonal matrix.

Introduce an orthogonal, unitary matrix, $U = \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix}$, such that

$$\hat{S}_d = U\hat{S}U^{-1}$$

δ_α and δ_β are the eigen-phaseshifts.

ε is the mixing parameter.

J. M. Blatt and L. C. Biedenharn, Phys. Rev. 86, 399 (1952).

Eigenphase parameterization

A unitary scattering matrix is

$$\hat{S} = \begin{pmatrix} e^{2i\delta_\alpha}\cos^2\varepsilon + e^{2i\delta_\beta}\sin^2\varepsilon & \cos\varepsilon\sin\varepsilon\left(e^{2i\delta_\alpha} - e^{2i\delta_\beta}\right) \\ \cos\varepsilon\sin\varepsilon\left(e^{2i\delta_\alpha} - e^{2i\delta_\beta}\right) & e^{2i\delta_\alpha}\sin^2\varepsilon + e^{2i\delta_\beta}\cos^2\varepsilon \end{pmatrix}$$

Eigenphase parameterization

A unitary scattering matrix is

$$\hat{\mathsf{S}} = \begin{pmatrix} e^{2i\delta_\alpha}\cos^2\varepsilon + e^{2i\delta_\beta}\sin^2\varepsilon & \cos\varepsilon\sin\varepsilon\left(e^{2i\delta_\alpha} - e^{2i\delta_\beta}\right) \\ \cos\varepsilon\sin\varepsilon\left(e^{2i\delta_\alpha} - e^{2i\delta_\beta}\right) & e^{2i\delta_\alpha}\sin^2\varepsilon + e^{2i\delta_\beta}\cos^2\varepsilon \end{pmatrix}$$

The coupled channel effective range expansion,

$$k^{L_{ij}+\frac{1}{2}} U \hat{\mathsf{K}}^{-1} U^{-1} k^{L_{ij}+\frac{1}{2}} = -\frac{1}{a_{L_{ij}}} + \frac{1}{2} r_{L_{ij}} k^2 + \mathcal{O}(k^4)$$

Eigenphase parameterization

A unitary scattering matrix is

$$\hat{\mathrm{S}} = \begin{pmatrix} e^{2i\delta_\alpha}\cos^2\varepsilon + e^{2i\delta_\beta}\sin^2\varepsilon & \cos\varepsilon\sin\varepsilon\left(e^{2i\delta_\alpha} - e^{2i\delta_\beta}\right) \\ \cos\varepsilon\sin\varepsilon\left(e^{2i\delta_\alpha} - e^{2i\delta_\beta}\right) & e^{2i\delta_\alpha}\sin^2\varepsilon + e^{2i\delta_\beta}\cos^2\varepsilon \end{pmatrix}$$

The coupled channel effective range expansion,

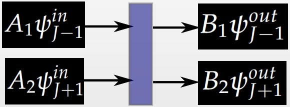
$$k^{L_{ij}+\frac{1}{2}} U \hat{\mathrm{K}}^{-1} U^{-1} k^{L_{ij}+\frac{1}{2}} = -\frac{1}{a_{L_{ij}}} + \frac{1}{2} r_{L_{ij}} k^2 + \mathcal{O}(k^4)$$

$$\begin{array}{lcl} \alpha \text{ - state} & \longrightarrow & k^{2J-1}\cot\delta_\alpha(k) \\ \beta \text{ - state} & \longrightarrow & k^{2J+3}\cot\delta_\beta(k) \end{array}$$

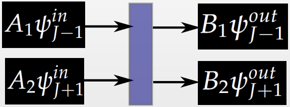
The mixing angle

$$\tan\varepsilon_J(k) = q_0 k^2 + q_1 k^4 + \mathcal{O}\left(k^6\right)$$

Eigenphase-shift

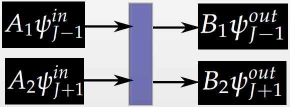


Eigenphase-shift



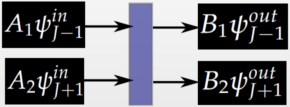
$$S\,|a\rangle = \lambda\,|a\rangle$$

Eigenphase-shift



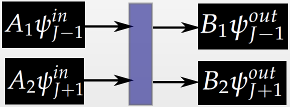
$$\mathsf{S}\,|a\rangle = \lambda\,|a\rangle \;\rightarrow\; \lambda_{\alpha,\beta} = e^{2i\delta_{\alpha,\beta}}$$

Eigenphase-shift

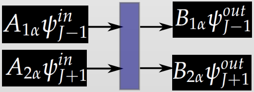


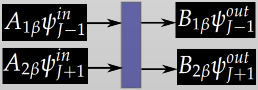
$$S\,|a\rangle = \lambda\,|a\rangle \;\rightarrow\; \lambda_{\alpha,\beta} = e^{2i\delta_{\alpha,\beta}} \;\Rightarrow\; |a_\alpha\rangle = \begin{pmatrix} \cos\varepsilon \\ \sin\varepsilon \end{pmatrix} \text{ and } |a_\beta\rangle = \begin{pmatrix} -\sin\varepsilon \\ \cos\varepsilon \end{pmatrix}$$

Eigenphase-shift

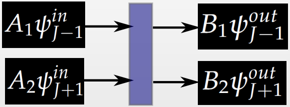


$$\mathrm{S}\,|a\rangle = \lambda\,|a\rangle \;\rightarrow\; \lambda_{\alpha,\beta} = e^{2i\delta_{\alpha,\beta}} \;\Rightarrow\; |a_\alpha\rangle = \begin{pmatrix} \cos\varepsilon \\ \sin\varepsilon \end{pmatrix} \text{ and } |a_\beta\rangle = \begin{pmatrix} -\sin\varepsilon \\ \cos\varepsilon \end{pmatrix}$$

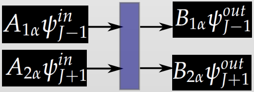




Eigenphase-shift



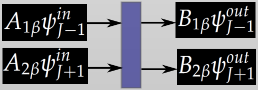
$$\mathrm{S}\,|a\rangle = \lambda\,|a\rangle \;\rightarrow\; \lambda_{\alpha,\beta} = e^{2i\delta_{\alpha,\beta}} \;\Rightarrow\; |a_\alpha\rangle = \begin{pmatrix} \cos\varepsilon \\ \sin\varepsilon \end{pmatrix} \text{ and } |a_\beta\rangle = \begin{pmatrix} -\sin\varepsilon \\ \cos\varepsilon \end{pmatrix}$$



$$\alpha \rightarrow J-1$$

$$U_\alpha(r) = \cos\varepsilon_J(k)\; k^{J-1}[\cot\delta_{J-1}(k) S_{J-1}(kr) + C_{J-1}(kr)]$$

$$V_\alpha(r) = \sin\varepsilon_J(k)\; k^{J-1}[\cot\delta_{J-1}(k) S_{J+1}(kr) + C_{J+1}(kr)]$$



$$\beta \rightarrow J+1$$

$$U_\beta(r) = \sin\varepsilon_J(k)\; k^{J+1}[-\cot\delta_{J+1}(k) S_{J-1}(kr) - C_{J-1}(kr)]$$

$$V_\beta(r) = \cos\varepsilon_J(k)\; k^{J+1}[\cot\delta_{J+1}(k) S_{J+1}(kr) + C_{J+1}(kr)]$$

Causality bound on effective ranges

The Generalized form of Bethe's integral formula to coupled channels;
For $r \geq R$,

$$r_{J-1} = b_{J-1}(r) - 2q_0^2 \frac{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}{\pi} \left(\frac{r}{2}\right)^{-2J-1} - 2\int_0^r \left(\left[U_\alpha^{(0)}(r')\right]^2 + \left[V_\alpha^{(0)}(r')\right]^2 \right) dr'$$

$$r_{J+1} = b_{J+1}(r) + \frac{2q_0^2}{a_{J+1}^2} \frac{\pi}{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})} \left(\frac{r}{2}\right)^{2J+1} - 2\int_0^r \left(\left[U_\beta^{(0)}(r')\right]^2 + \left[V_\beta^{(0)}(r')\right]^2 \right) dr'$$

$$\begin{aligned} b_{J\mp1}(r) = & \frac{1}{a_{J\mp1}^2} \frac{2\pi}{\Gamma\left(J\mp1+\frac{3}{2}\right)\Gamma\left(J\mp1+\frac{5}{2}\right)} \left(\frac{r}{2}\right)^{2(J\mp1)+3} \\ & - \frac{1}{a_{J\mp1}} \frac{4}{J\mp1+\frac{1}{2}} \left(\frac{r}{2}\right)^2 - \frac{2\Gamma\left(J\mp1-\frac{1}{2}\right)\Gamma\left(J\mp1+\frac{1}{2}\right)}{\pi} \left(\frac{r}{2}\right)^{-2(J\mp1)+1} \end{aligned}$$

Causality bounds

The lower partial wave

$$r_{J-1} \leq b_{J-1}(r) - 2q_0^2 \frac{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}{\pi} \left(\frac{r}{2}\right)^{-2J-1}$$

the higher partial wave

$$r_{J+1} \leq b_{J+1}(r) + \frac{2q_0^2}{a_{J+1}^2} \frac{\pi}{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})} \left(\frac{r}{2}\right)^{2J+1}$$

Causality bounds

The lower partial wave

$$r_{J-1} \le b_{J-1}(r) - 2q_0^2 \frac{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}{\pi} \left(\frac{r}{2}\right)^{-2J-1}$$

$$r_{J-1} \to -\infty \quad \text{as } r \to 0$$

the higher partial wave

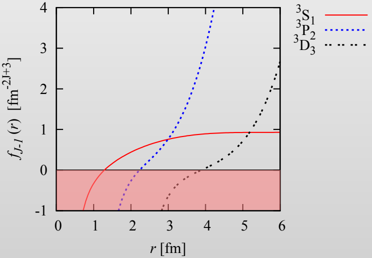
$$r_{J+1} \le b_{J+1}(r) + \frac{2q_0^2}{a_{J+1}^2} \frac{\pi}{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})} \left(\frac{r}{2}\right)^{2J+1}$$

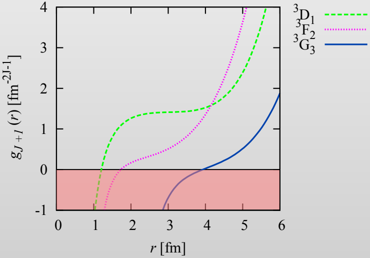
$$r_{J+1} \to -\infty \quad \text{as } r \to 0$$

Causal range

$$b_{J-1}(R^b) - 2q_0^2 \frac{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}{\pi}\left(\frac{R^b}{2}\right)^{-2J-1} - r_{J-1} = 0$$

$$b_{J+1}(R^b) + \frac{2q_0^2}{a_{J+1}^2}\frac{\pi}{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}\left(\frac{R^b}{2}\right)^{2J+1} - r_{J+1} = 0$$





Channels	$^3S_1$	$^3P_2$	$^3D_3$	$^3D_1$	$^3F_2$	$^3G_3$
R^b [fm]	1.29	2.23	4.03	1.20	1.73	3.92

Cauchy-Schwarz range

The Cauchy-Schwarz inequality

$$\left(\int [f_1(r)\ f_2(r)] \left[\begin{array}{c} f_1(r) \\ f_2(r) \end{array}\right] dr\right)\left(\int [g_1(r)\ g_2(r)] \left[\begin{array}{c} g_1(r) \\ g_2(r) \end{array}\right] dr\right) \geq \left|\int [f_1(r)g_1(r) + f_2(r)g_2(r)]\, dr\right|^2$$

Cauchy-Schwarz range

The Cauchy-Schwarz inequality

$$\left(\int [f_1(r)\ f_2(r)] \begin{bmatrix} f_1(r) \\ f_2(r) \end{bmatrix} dr\right)\left(\int [g_1(r)\ g_2(r)] \begin{bmatrix} g_1(r) \\ g_2(r) \end{bmatrix} dr\right) \geq \left| \int [f_1(r)g_1(r) + f_2(r)g_2(r)]\, dr \right|^2$$

$$\left[\int \left(\left[U_\alpha^{(0)}(r')\right]^2 + \left[V_\alpha^{(0)}(r')\right]^2\right) dr'\right]\left[\int \left(\left[U_\beta^{(0)}(r')\right]^2 + \left[V_\beta^{(0)}(r')\right]^2\right) dr'\right]$$
$$\geq \int \left(U_\alpha^{(0)}(r')U_\beta^{(0)}(r') + V_\alpha^{(0)}(r')V_\beta^{(0)}(r')\right) dr'$$

Cauchy-Schwarz range

The Cauchy-Schwarz inequality

$$\left(\int [f_1(r)\; f_2(r)] \begin{bmatrix} f_1(r) \\ f_2(r) \end{bmatrix} dr \right) \left(\int [g_1(r)\; g_2(r)] \begin{bmatrix} g_1(r) \\ g_2(r) \end{bmatrix} dr \right) \geq \left| \int [f_1(r) g_1(r) + f_2(r) g_2(r)]\, dr \right|^2$$

$$\left[\int \left(\left[U_\alpha^{(0)}(r') \right]^2 + \left[V_\alpha^{(0)}(r') \right]^2 \right) dr' \right] \left[\int \left(\left[U_\beta^{(0)}(r') \right]^2 + \left[V_\beta^{(0)}(r') \right]^2 \right) dr' \right] \geq \int \left(U_\alpha^{(0)}(r') U_\beta^{(0)}(r') + V_\alpha^{(0)}(r') V_\beta^{(0)}(r') \right) dr'$$

for $r \geq R$ (interaction range)

$$\left[b_{J-1}(r) - 2q_0^2 \frac{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}{\pi} \left(\frac{r}{2} \right)^{-2J-1} - r_{J-1} \right] \times \left[b_{J+1}(r) + \frac{2q_0^2}{a_{J+1}^2} \frac{\pi}{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})} \left(\frac{r}{2} \right)^{2J+1} - r_{J+1} \right] \geq \left| d_J(r) - q_1 \frac{2}{a_{J+1}} \right|^2$$

at $r = R^{C-S}$ the equality holds.

Cauchy-Schwarz range

The Cauchy-Schwarz inequality

$$\left(\int [f_1(r)\ f_2(r)] \begin{bmatrix} f_1(r) \\ f_2(r) \end{bmatrix} dr\right)\left(\int [g_1(r)\ g_2(r)] \begin{bmatrix} g_1(r) \\ g_2(r) \end{bmatrix} dr\right) \geq \left| \int [f_1(r)g_1(r) + f_2(r)g_2(r)]\, dr \right|^2$$

$$\left[\int \left(\left[U_\alpha^{(0)}(r')\right]^2 + \left[V_\alpha^{(0)}(r')\right]^2\right) dr'\right]\left[\int \left(\left[U_\beta^{(0)}(r')\right]^2 + \left[V_\beta^{(0)}(r')\right]^2\right) dr'\right] \geq \int \left(U_\alpha^{(0)}(r')U_\beta^{(0)}(r') + V_\alpha^{(0)}(r')V_\beta^{(0)}(r')\right) dr'$$

for $r \geq R$ (interaction range)

$$\left[b_{J-1}(r) - 2q_0^2 \frac{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}{\pi}\left(\frac{r}{2}\right)^{-2J-1} - r_{J-1}\right] \times \left[b_{J+1}(r) + \frac{2q_0^2}{a_{J+1}^2} \frac{\pi}{\Gamma(J+\frac{1}{2})\Gamma(J+\frac{3}{2})}\left(\frac{r}{2}\right)^{2J+1} - r_{J+1}\right] \geq \left| d_J(r) - q_1 \frac{2}{a_{J+1}} \right|^2$$

at $r = R^{C-S}$ the equality holds.

Channels	${}^3S_1$-${}^3D_1$	${}^3P_2$-${}^3F_2$	${}^3D_3$-${}^3G_3$
$R^{\text{C-S}}$ [fm]	1.297	4.656	5.680

Integral formula for the mixing parameter

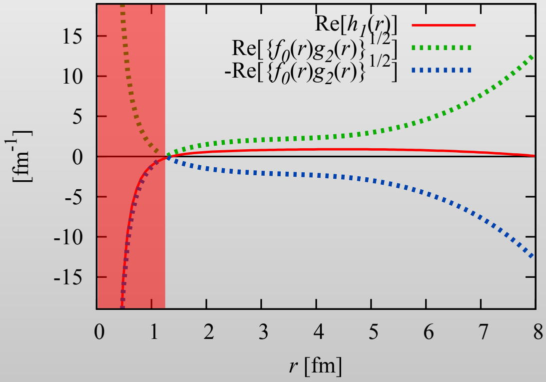
For $r \geq R$,

$$q_1 \frac{2}{a_{J+1}} = d_J(r) - 2\int_0^r \left[U_\alpha^{(0)}(r') U_\beta^{(0)}(r') + V_\alpha^{(0)}(r') V_\beta^{(0)}(r') \right] dr'$$

$$d_J(r) = \frac{-q_0}{a_{J-1}a_{J+1}} \frac{2\pi}{\Gamma\left(\frac{1}{2}+J\right)\Gamma\left(\frac{3}{2}+J\right)} \left(\frac{r}{2}\right)^{2J+1}$$
$$+ \frac{q_0}{a_{J+1}} \frac{4}{(2J-1)(2J+3)} r^2 - 2q_0 \frac{\Gamma\left(J+\frac{1}{2}\right)\Gamma\left(J+\frac{3}{2}\right)}{\pi} \left(\frac{r}{2}\right)^{-2J-1}$$

Cauchy-Schwarz range

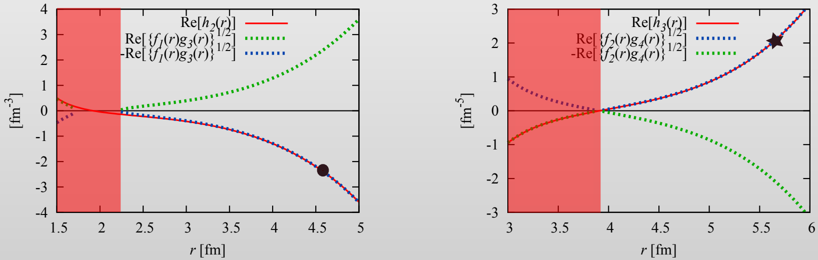
$$\left[b_0(r) - q_0^2 \frac{6}{r^3} - r_0\right]\left[b_2(r) + q_0^2 \frac{2r^3}{3a_2^2} - r_2\right] \geq \left|d_1(r) - q_1 \frac{2}{a_2}\right|^2$$



The equality holds at $r = 1.297$ fm

Cauchy-Schwarz range

$$\left[b_1(r) - q_0^2 \frac{90}{r^5} - r_1\right]\left[b_3(r) + q_0^2 \frac{1}{a_3^2}\frac{2r^5}{45} - r_3\right] \geq \left|d_2(r) - q_1 \frac{2}{a_3}\right|^2$$



$$\left[b_2(r) - q_0^2 \frac{3150}{r^7} - r_2\right]\left[b_4(r) + \frac{q_0^2}{a_4^2}\frac{2r^7}{1575} - r_4\right] \geq \left|d_3(r) - q_1 \frac{2}{a_4}\right|^2$$

Summary

Single Channel

Channel	$^1S_0$	$^1P_1$	$^3P_0$	$^3P_1$	$^1D_2$	$^3D_2$	$^1F_3$	$^3F_3$
R^b [fm]	1.27	0.31	3.07	0.23	3.98	4.91	1.88	1.56

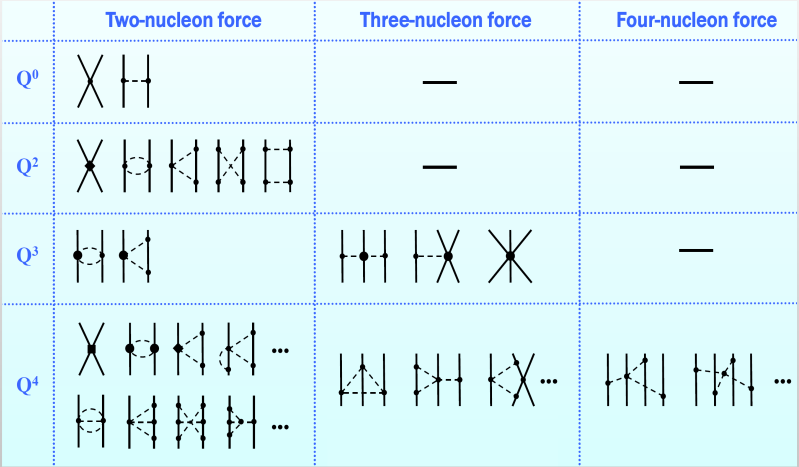
Coupled channels

Channels	$^3S_1$	$^3P_2$	$^3D_3$	$^3D_1$	$^3F_2$	$^3G_3$
R^b [fm]	1.29	2.23	4.03	1.20	1.73	3.92

Channels	$^3S_1$-$^3D_1$	$^3P_2$-$^3F_2$	$^3D_3$-$^3G_3$
$R^{\text{C-S}}$ [fm]	1.297	4.656	5.680

Chiral effective field theory

At low energy, relevant Feynman diagrams

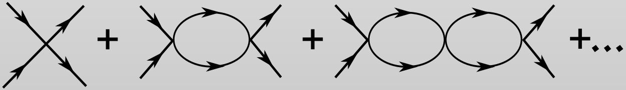


Pionless effective field theory

At lower energy,

pion is integrated out

The leading order contribution to the NN interaction is only NN contact interactions



Outline

Nuclear potential tail

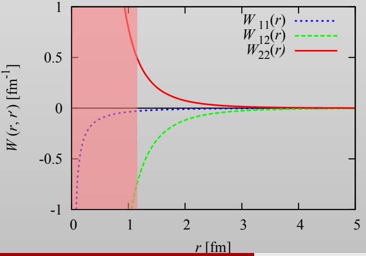
$$V_{OPE}(r) = V_C(r) + S_{12} V_T(r)$$

$$V_C(r) = f_\pi \, (\vec{\tau}_1 \cdot \vec{\tau}_2)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) \frac{e^{-m_\pi r}}{r}$$

$$V_T(r) = f_\pi \, (\vec{\tau}_1 \cdot \vec{\tau}_2) \left(1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right) \frac{e^{-m_\pi r}}{r}$$

$$S_{12} = 3(\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

$$f_\pi = \frac{g_{\pi N}^2}{12\pi} \left(\frac{m_\pi}{2M_N}\right)^2$$

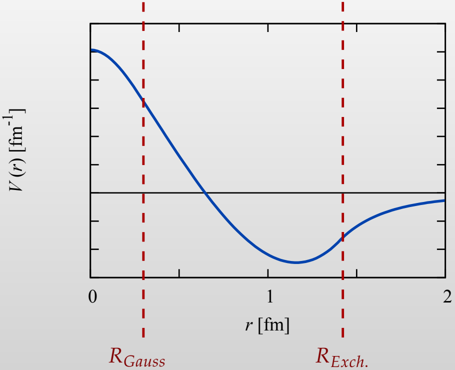


$J = 1$

$L_1 = 0, \; L_2 = 2$

$T = 0, \; S_{1,2} = \frac{1}{2}$

Nuclear potential model



$$V(r) = V_{\text{Gauss}}(r)\theta(R_{\text{Gauss}} - r) + V_{\text{Spline}}(r)\theta(r - R_{\text{Gauss}})\theta(R_{\text{Exch.}} - r) + V_{\text{Exch.}}(r)\theta(r - R_{\text{Exch.}})$$

$$V(r) \to V(r)\theta(R - r)$$

Nuclear potential model

$$V_{\text{Gauss}}(r) = C_G e^{-m_G^2 r^2}$$

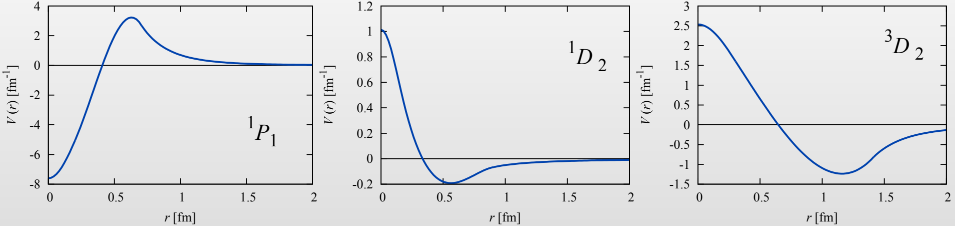
$$V_{\text{Spline}}(r) = C_1 + C_2 r + C_3 r^2 + C_4 r^3$$

$$V_{\text{Exch.}}(r) = V_C^{\pi,A,B}(r) + S_{12} V_T^{\pi,D,F}(r)$$

$$V_C^{\pi,A,B}(r) = \frac{g_{\pi N}^2}{12\pi}\left(\frac{m_\pi}{2M_N}\right)^2 \left\{ C_\pi \frac{e^{-m_\pi r}}{r} + C_A \frac{e^{-m_A r}}{r} + C_B \frac{e^{-m_B r}}{r} \right\}$$

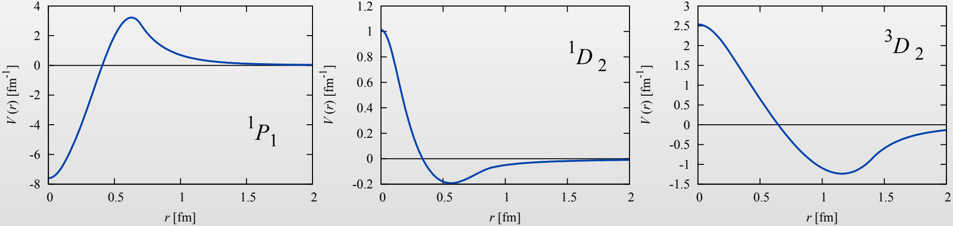
$$V_T^{\pi,D,F}(r) = \frac{g_{\pi N}^2}{12\pi}\left(\frac{m_\pi}{2M_N}\right)^2 \left\{ C_\Pi \left[1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2}\right] \frac{e^{-m_\pi r}}{r} + C_D \left[1 + \frac{3}{m_D r} + \frac{3}{(m_D r)^2}\right] \frac{e^{-m_D r}}{r} + C_F \left[1 + \frac{3}{m_F r} + \frac{3}{(m_F r)^2}\right] \frac{e^{-m_F r}}{r} \right\}$$

Results



4
2
0
-2
-4
-6
-8
V (r) [fm-1]
1P1
0
0.5
1
1.5
2
r [fm]
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
V (r) [fm-1]
1D 2
0
0.5
1
1.5
2
r [fm]
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
V (r) [fm-1]
3D 2
0
0.5
1
1.5
2
r [fm]

Results



Causal range / Potential range	$R^b_{^1P_1}$ fm	$R^b_{^1D_2}$ fm	$R^b_{^3D_2}$ fm
$R = 2$ fm	0.4	2.0	1.2
$R = 5$ fm	0.4	2.3	2.6
$R = 12$ fm	0.3	3.8	4.6
$R = 15$ fm	0.3	4.0	4.8
$R = 50$ fm	0.3	4.0	4.9

Outline

Summary

- the constraints of causality and unitarity for neutron-proton scattering for all spin channels up to $J = 3$,
- the causal range R^b and the Cauchy-Schwarz range R^{C-S} are as larger as 5 fm in some channel,
- if one reproduces the physical scattering data using strictly finite range interactions, then the range of these interactions must be larger than R^b and R^{C-S},
- the causality and unitarity bounds derived here are physical constraints for the convergence of perturbative calculations in pionless effective field theory,
- for pionless effective field theory, if the cutoff momentum is too high, it is not possible to obtain the correct threshold physics in coupled channels without violating causality and unitarity.

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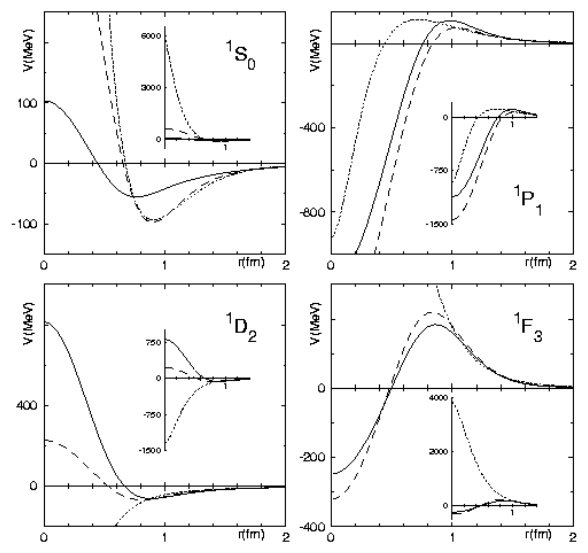
Summary

$^1S_0$	1.27
$^1P_1$	0.31
$^3P_0$	3.07
$^3P_1$	0.23
$^1D_2$	3.98
$^3D_2$	4.91
$^1F_3$	1.88
$^3F_3$	1.56



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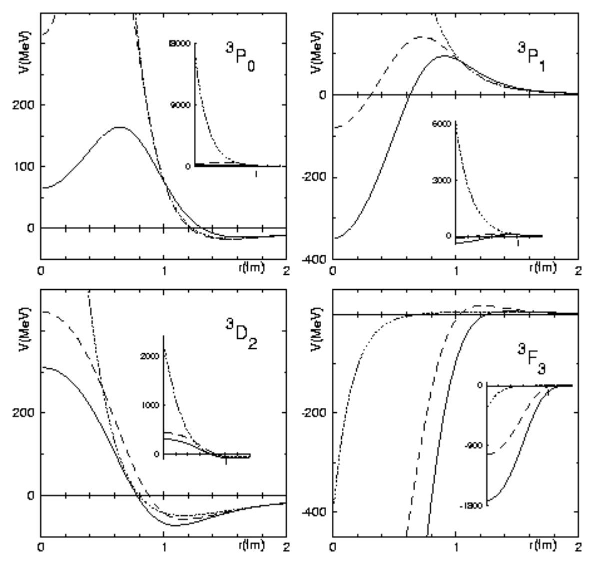

Summary

$^3S_1$	1.29
$^3D_1$	1.20
$^3P_2$	2.23
$^3F_2$	1.73
$^3D_3$	4.03
$^3G_3$	3.92