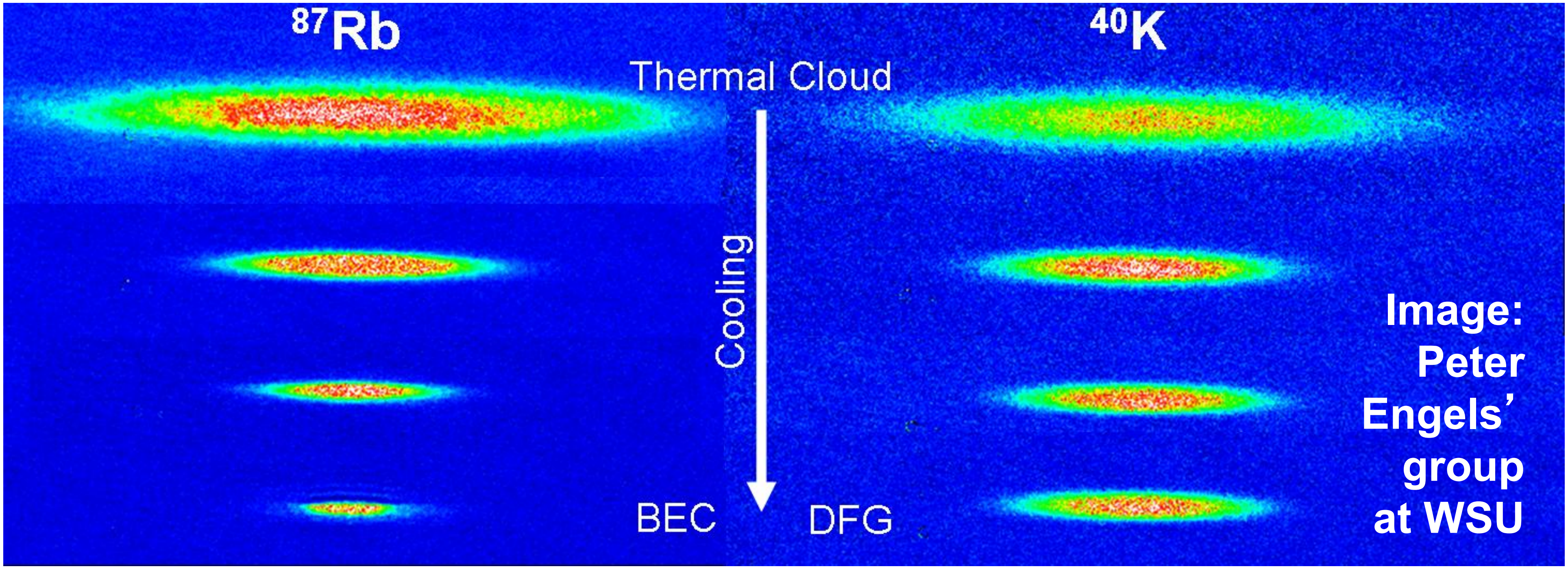

Few-Body Physics of Cold Atoms: Techniques and Results that May be of Interest to Nuclear Physics/Physicists

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In collaboration with graduate students Debraj Rakshit, Xiangyu (Desmond) Yin and Kevin Daily. Trapped boson work with Phil Johnson and Eite Tiesinga.

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Outline of This Talk

- Approach to solving four- and five-particle bound and scattering problems:
  - Combining hyperspherical coordinates, explicitly correlated Gaussian basis functions and stochastic variational technique.
  - Formalism and first results.
- Bosons in a spherically symmetric harmonic trap:
  - Confronting effective field theory results with highly accurate numerical results for finite-range interactions.
- Few-particle system in a box with periodic boundary conditions:
  - A possible path towards obtaining accurate results.

General Consideration

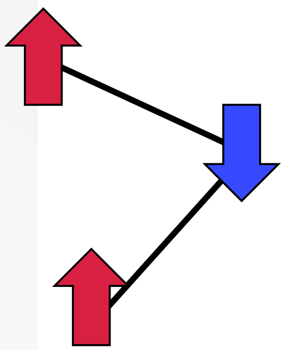
- We are interested in low-energy phenomena (justified for ultracold atomic samples):
  - Physics is governed by just a few (one) partial waves.

- The details of the two-body atom-atom interaction do not matter. Universal if s-wave scattering length a_s >> range of two-body potential r_0.
- Can replace van der Waals (vdW) potential by model potential.
  - vdW potential: hundreds of bound states.
  - model potential: zero or one bound states.

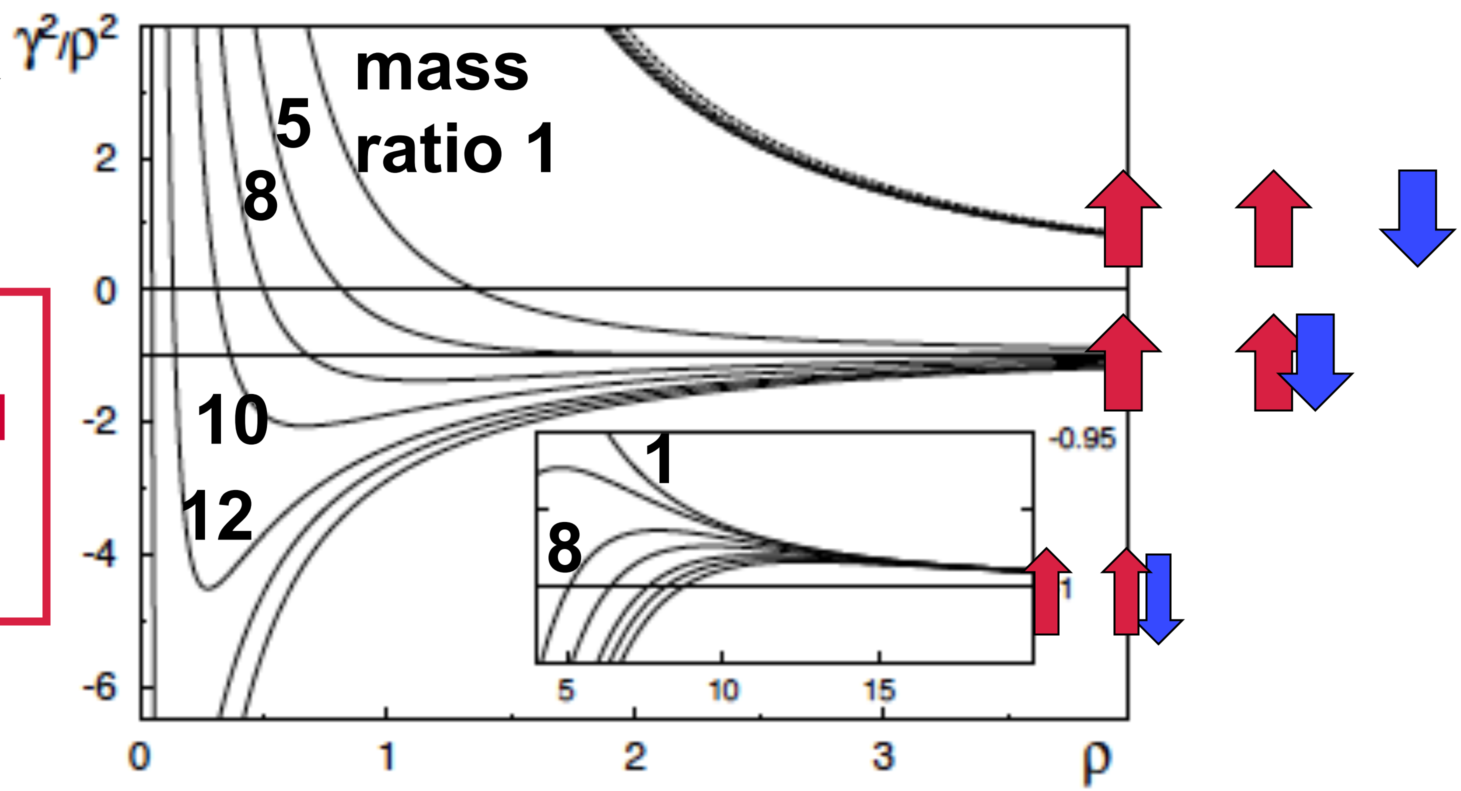
- In nuclear physics: Might be able to get away with "simple soft potentials" that reproduce low-energy phase shifts.

Hyperspherical Coordinate Approach and Effective Potential Curves

Two heavy identical fermions and one light impurity with positive s-wave scattering length (zero-range interactions):



Yields bound and scattering states.



Kartavtsev and Malykh, JPB 40, 1429 (2007).

Hyperradial coordinate measures the overall size of the system (five angles have been integrated out)

General Formalism: Similar to Born-Oppenheimer Approximation

- Hyperspherical coordinate approach: hyperradius R and hyperangles Ω.

- Idea: $H = T_\Omega + T_R + V_{int} = H_{adia}(R) + T_R$

- Step 1: Solve $H_{adia}(R)\ \Phi_\nu(R;\Omega) = U_\nu(R)\ \Phi_\nu(R;\Omega)$ (this is like integrating out the fast electronic degrees of freedom).

- Step 2: Solve $(T_R + U_\nu(R) + \Sigma_{\nu'}\ \text{coupling}_{\nu'\nu})\ F_{\nu q}(R) = E_{\nu q}\ F_{\nu q}(R)$ (this is like solving the nuclear Schroedinger equation).

- For convenience, write $U_\nu(R) = \hbar^2[(s_\nu(R))^2 - ¼] / (2\mu R^2)$

How Do We Solve Hyperangular Schrodinger Equation?

- Use **explicitly correlated Gaussians** to expand hyperangular channel functions [see von Stecher and Greene, PRA (2009) for treatment of 0^+ states].

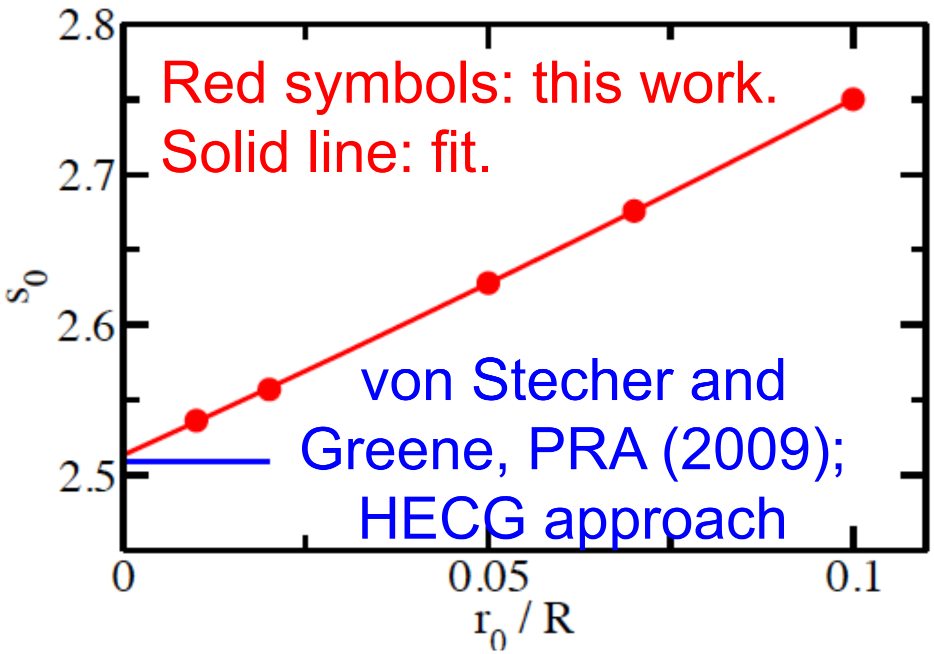
- $\Phi_v(R;\Omega) = \Sigma_k c_k [f(\underline{x}, \underline{u}_1^{(k)}, \underline{u}_2^{(k)})] \exp[- \frac{1}{2} \underline{x}^T \underline{A}^{(k)} \underline{x}] |_R$

- Transform basis functions to hyperspherical coordinates and perform 2N-1 hyperangular integrals analytically.
- For N=4 (N=5), this leave one (two) numerical integration(s).

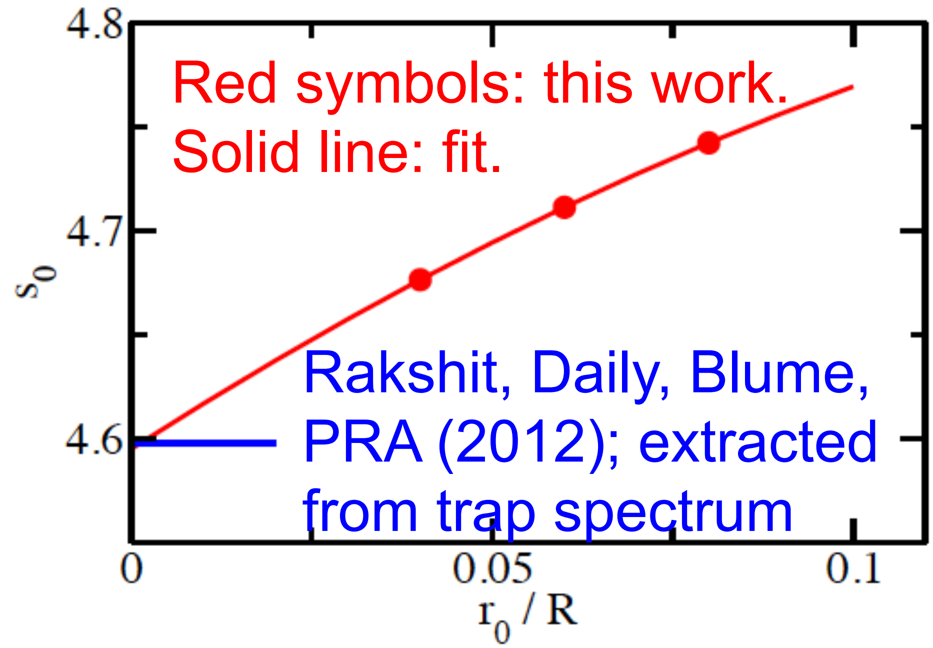
- Explicitly correlated Gaussian depend on non-linear variational parameters: Optimize using stochastic variational "trial and error" approach [minimize $U_v(R)$].

Proof-of-Principle Calculation for Equal-Mass (2,2) System at Unitarity

0^+ symmetry:



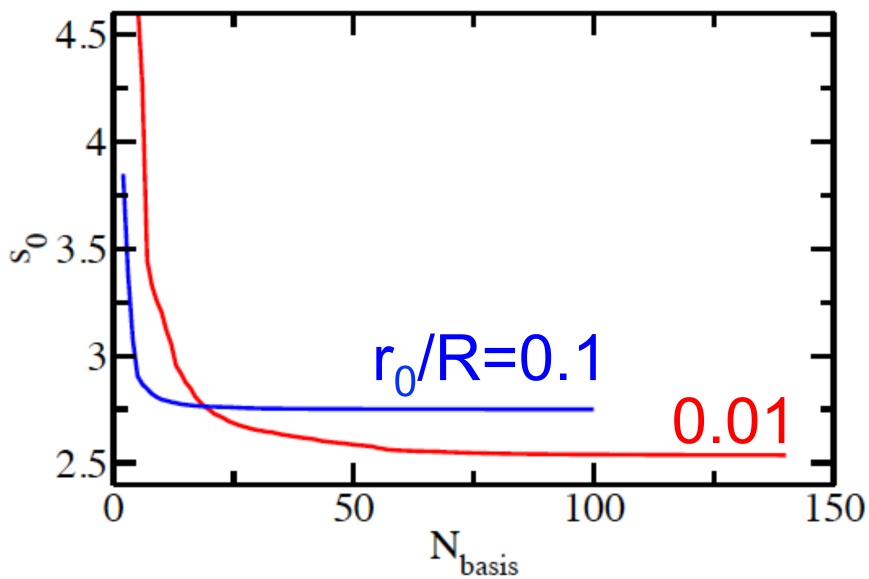
1^- symmetry:

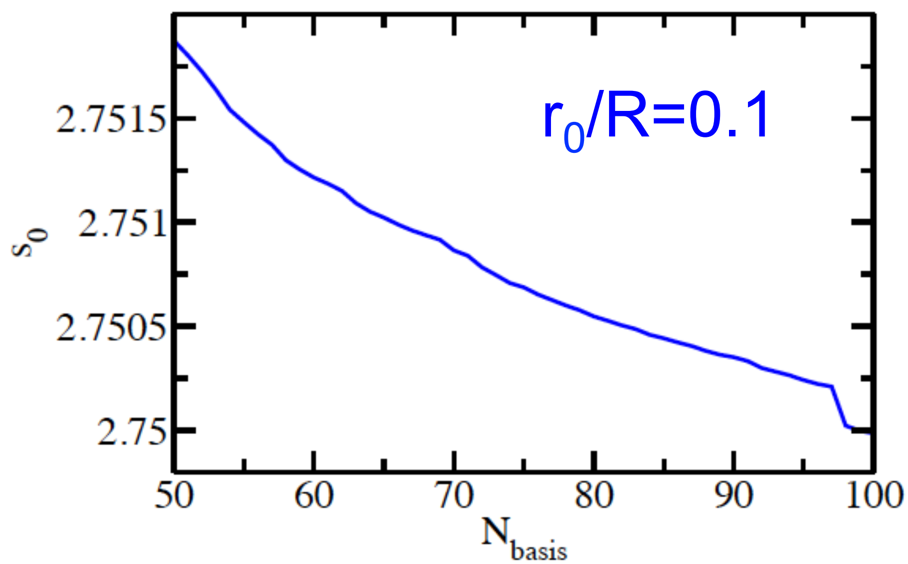


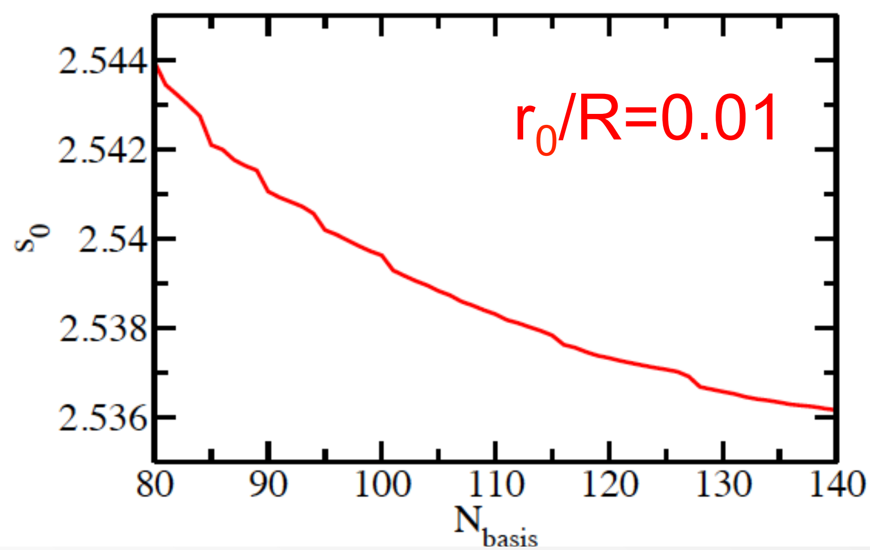
Hyperspherical explicitly correlated Gaussian approach can be applied to states with finite angular momentum.
Finite L matrix elements are tedious to derive (applicable to "any N")… numerics is tractable for N=4, (5)…

Rakshit and Blume, unpublished.

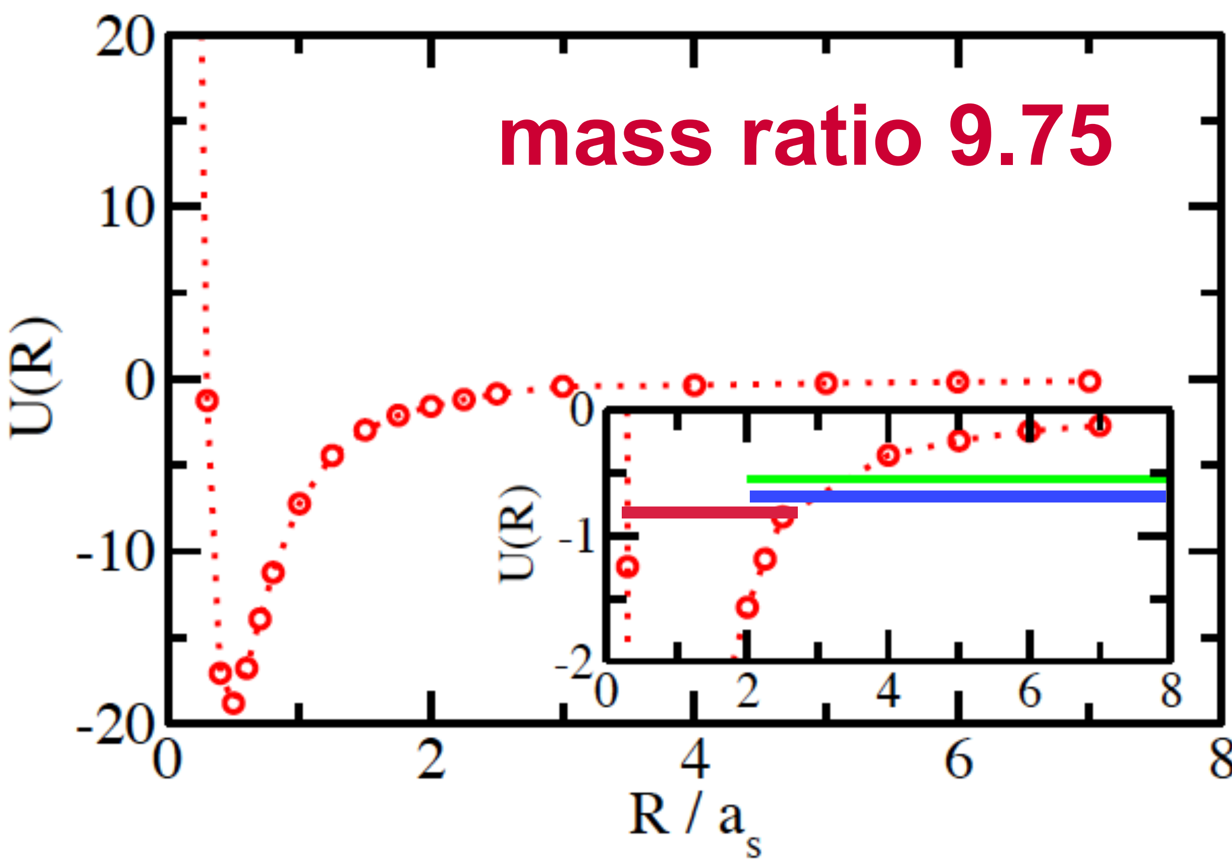
Convergence of Eigenvalue of Hyperangular Schroedinger Equation

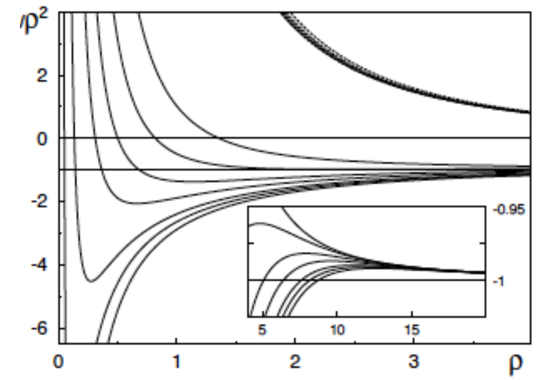






Potential Curve for (3,1) System with 1^+ Symmetry and Positive a_s





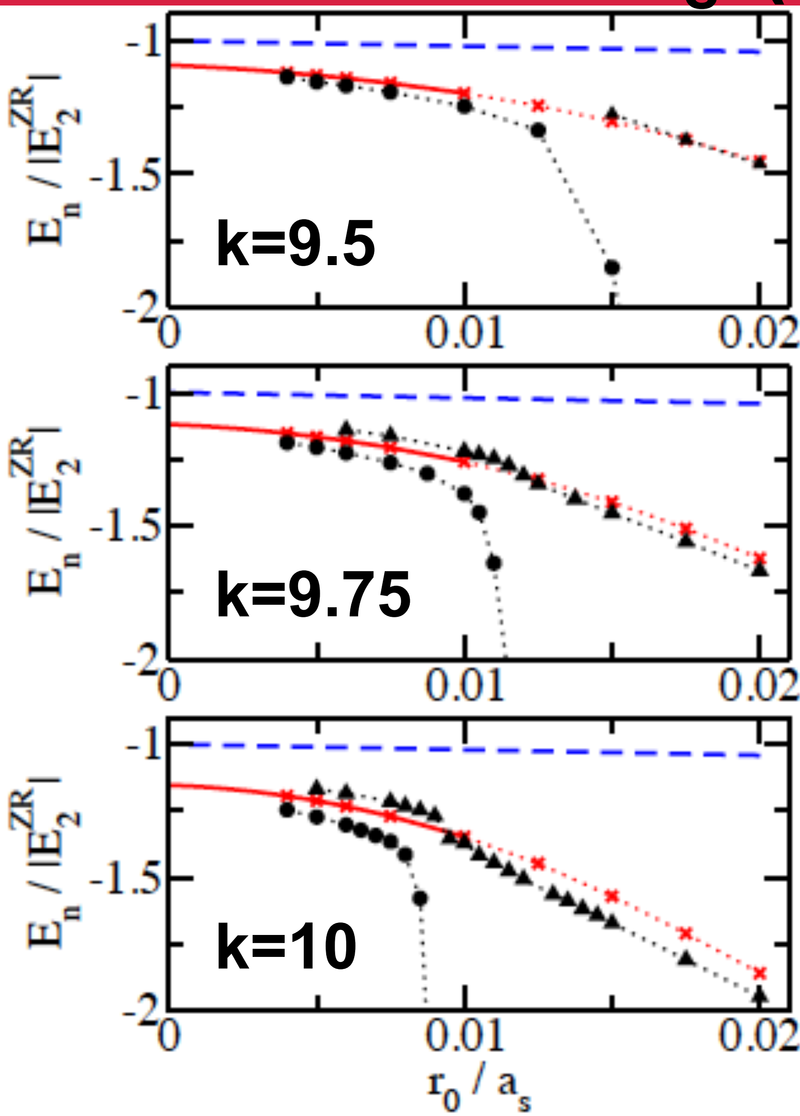
Dimer threshold.

Trimer threshold.

Expect: Four-body bound state.

This is work in progress: Go to larger hyperradii; calculate excited curves and coupling elements; and solve coupled channel equations in R.

Heavy-Light (3,1) System with $L^{\Pi}=1^{+}$ and Positive a_s (no fixed R!)



Dashed blue: Two-body state.

Red: Universal three-body states [see Kartavtsev and Malykh, JPB 40, 1429 (2007)].

Black: Away from resonance-like feature, universal four-body states.

- Universal four-body bound states exist for mass ratio >9.5.
- Properties are fully determined by a_s.
- Four-body states are tied to three-body states.

D. Blume, arXiv:1208.2907.

Under Which Conditions Do Universal States Exist?

- $(T_R + U_v(R) + \Sigma_{v'} \text{coupling}_{v'v})\, F_{vq}(R) = E_{vq}\, F_{vq}(R)$

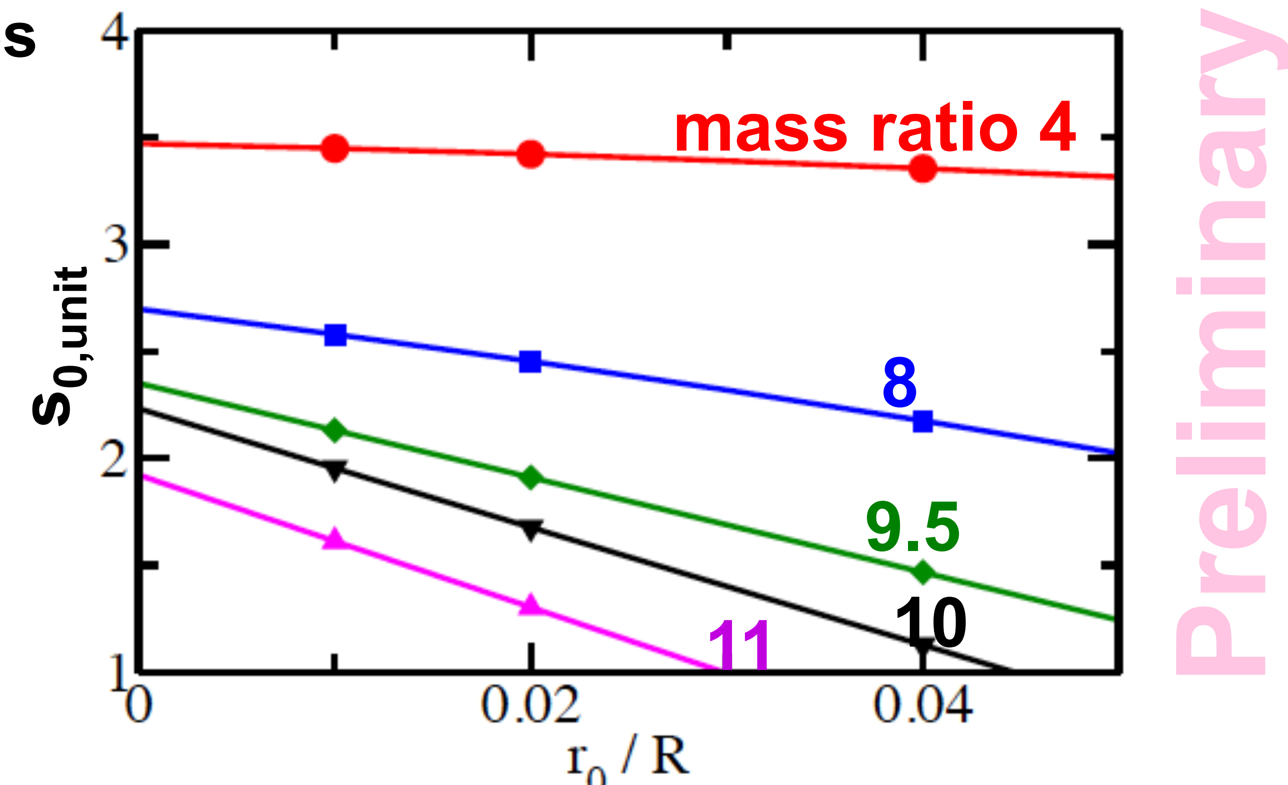
- For finite a_s, small R and r_0=0:
- $U_v(R) + \Sigma_{v'} \text{coupling}_{v'v} \sim \hbar^2[(s_{0,unit})^2 - ¼] / (2\mu R^2)$

- $S_{0,unit}$>1 [for (2,1), mass ratio κ<8.6]: Only "regular" solution contributes (wave fct. vanishes at small R).
- 1>$s_{0,unit}$>0 (8.6<κ<13.607): In principle, "regular" and "irregular" solutions can contribute (depends on two-body potential). See work by Petrov, Nishida, Tan, Son, Werner, Castin,…

- Our earlier work on (2,1) systems with Gaussian interactions at unitarity shows that regular solution dominates for mass ratios considered here [see Blume and Daily, PRL and PRA (2010)].

Hyperangular Eigenvalue for Various Mass Ratios: (3,1) System with $L^{\Pi}=1^{+}$

Interspecies s-wave scattering length is infinitely large.

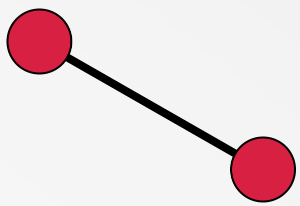


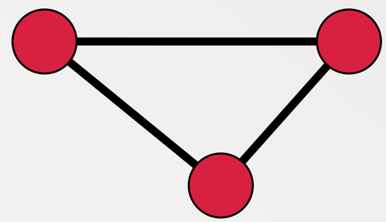
Conclusion: **Universal bound states in heavy light mixtures with positive interspecies s-wave sc. length exist for mass ratio > 9.5.**

Conclusion supported by (i) close link between 3- and 4-body free-space energy, (ii) hyperradial densities, (iii) s_0 value at unitarity.

Harmonically Trapped Bose Gas with Small s-wave Scattering Length

Well known:
Interaction energy (IE) of N bosons ≠ N(N-1)/2 × IE of 2 bosons





boson 3 affects how bosons 1 and 2 interact (boson 3 is not a spectator).

Question:
How to "divide" IE of N identical bosons in an isotropic harmonic trap into two-body, three-body, four-body,… contributions?

Our approach:
Apply perturbation theory for small a_s (this is conveniently done by applying formalism of second quantization to Hamiltonian with zero-range interactions); renormalization via effective field theory ideas.

Series in a_s and Effective Range r_{eff}: Effective N-Body Interactions

$E(N) = E^{NI} + U_2 N_{pair} + U_3 N_{trimer} + U_4 N_{tetramer} + \ldots,$

where U_N are effective N-body interactions:

1st order PT
2nd order PT
3rd order PT

$$U_2 = c_{2,(1)}\, a_s/a_{ho} + c_{2,(2)}\, (a_s/a_{ho})^2 + c_{2,(3)}\, (a_s/a_{ho})^3 + \ldots$$
$$+\, d_{2,(1,2)}(r_{eff} a_s^2)/a_{ho}^3 + \ldots$$

$$U_3 = c_{3,(2)}\, (a_s/a_{ho})^2 + c_{3,(3)}\, (a_s/a_{ho})^3 + \ldots$$

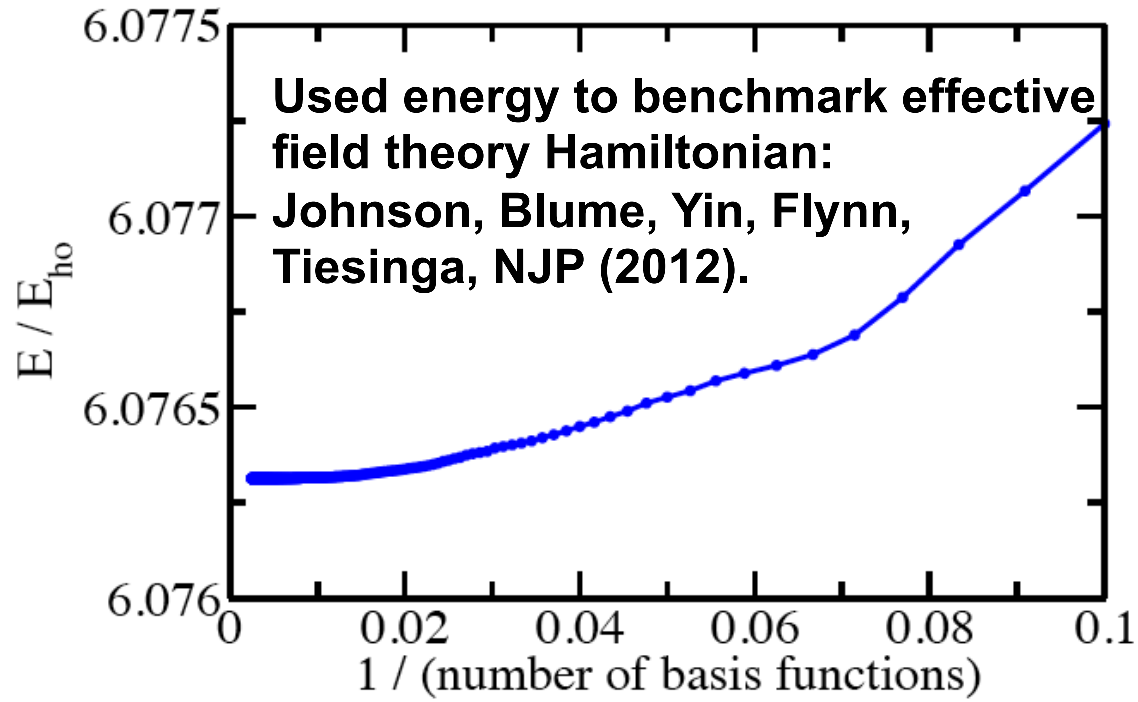
$$U_4 = c_{4,(3)}\, (a_s/a_{ho})^3 + \ldots$$

Leading-order effective four-body interaction “competes” with effective range term: $k\cot(\delta(k)) = -1/a_s + r_{eff}\, k^2/2$.

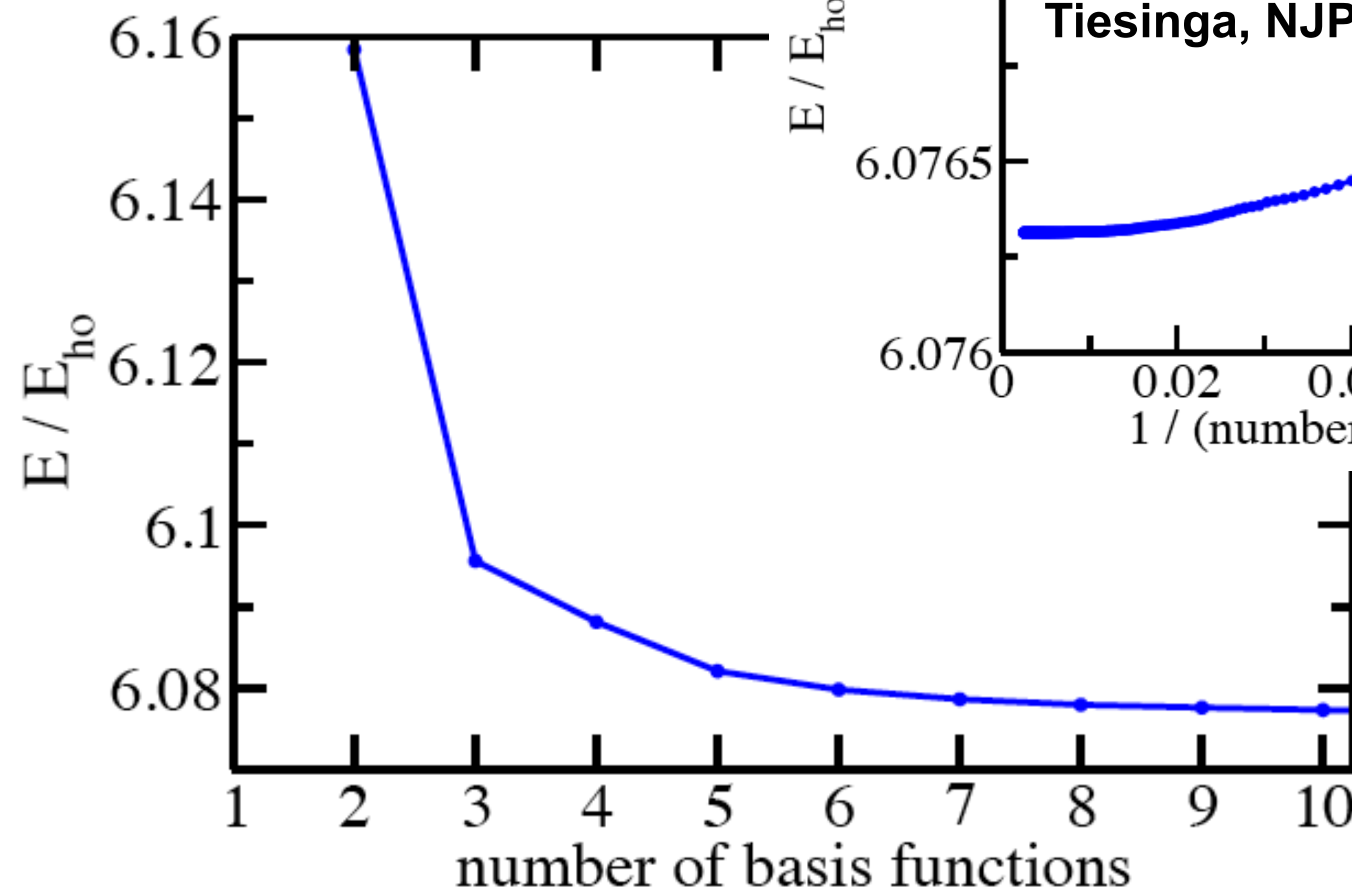
Johnson, Tiesinga, Porto, Williams, New J. Phys. 11, 093022 (2009);
Johnson, Blume, Yin, Flynn, Tiesinga, New J. Phys. 14, 053037 (2012).

Harmonically Trapped Five-Boson System: Illustration of Convergence

$r_0 = 0.01a_{ho}$
$a_s = 0.0096a_{ho}$

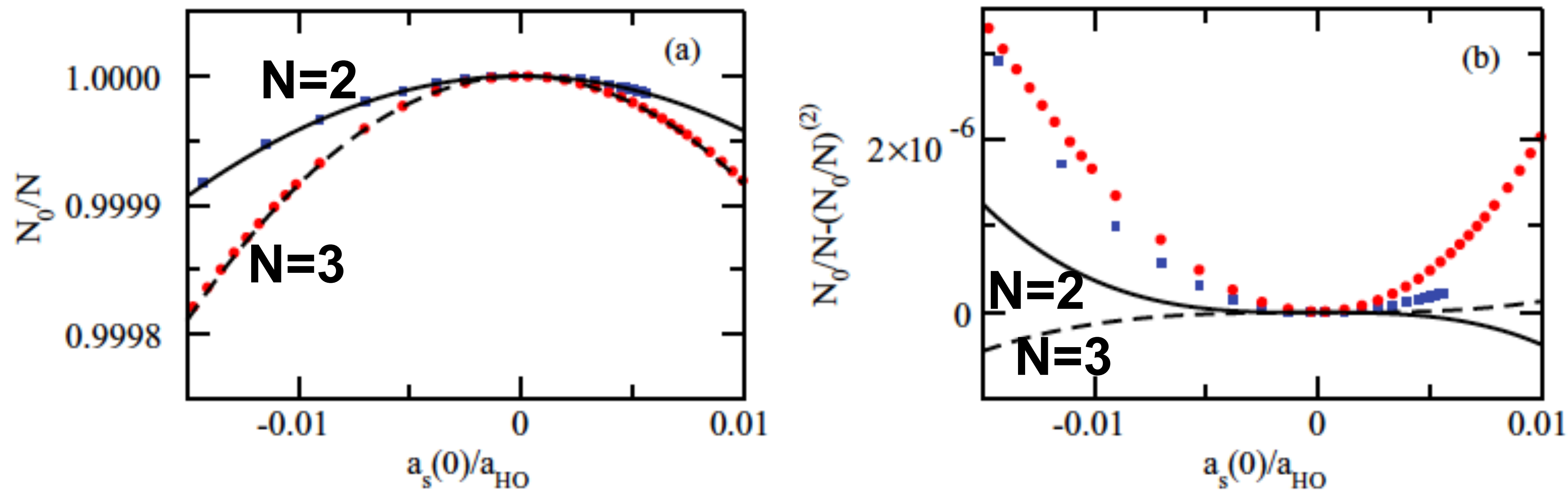


Used energy to benchmark effective field theory Hamiltonian: Johnson, Blume, Yin, Flynn, Tiesinga, NJP (2012).



For each N_b, try a few 1000 and keep the best.
$\Delta E \sim 2\text{x}10^{-8} h\nu$.
$(a_s/a_{ho})^4 \sim 10^{-8}$.

Condensate Fraction N/N_0 of Weakly-Interacting Trapped Bose Gas

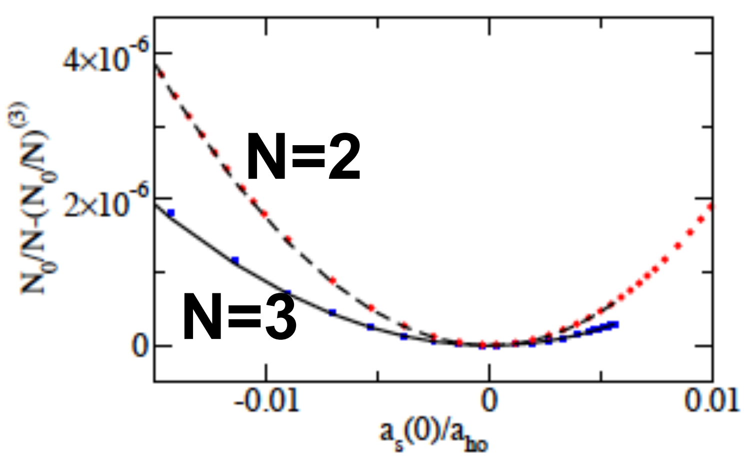


$$\langle \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} \rangle = \frac{\langle \psi_0^{(k)} | \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} | \psi_0^{(k)} \rangle}{\langle \psi_0^{(k)} | \psi_0^{(k)} \rangle}$$

$$N_0/N = 1 - 0.420004(N-1)\left[\frac{a_s(0)}{a_{\text{HO}}}\right]^2 + \left[-0.373241(N-1) + 0.439464(N-1)(N-2)\right]\left[\frac{a_s(0)}{a_{\text{HO}}}\right]^3 + \cdots$$

Condensate Fraction N/N_0 of Weakly-Interacting Trapped Bose Gas

- Expect: N/N_0 is determined by a_s and r_{eff}. But new parameter…



$$
\begin{aligned}
N_0/N = 1 &- 0.420004(N-1)\left[\frac{a_s(0)}{a_{ho}}\right]^2 \\
&+ \Big[-0.373241(N-1) \\
&\quad + 0.439464(N-1)(N-2)\Big]\left[\frac{a_s(0)}{a_{ho}}\right]^3 \\
&+ \Big[0.406786(N-1) + \gamma_3^{(4)}(N-1)(N-2) \\
&\quad + \gamma_4^{(4)}(N-1)(N-2)(N-3)\Big]\left[\frac{a_s(0)}{a_{ho}}\right]^4 \\
&+ 2(N-1)\mathrm{Re}(D_0) \\
&- (3/2)\times 0.420004(N-1)\frac{r_e[a_s(0)]^3}{a_{ho}^4}
\end{aligned}
$$

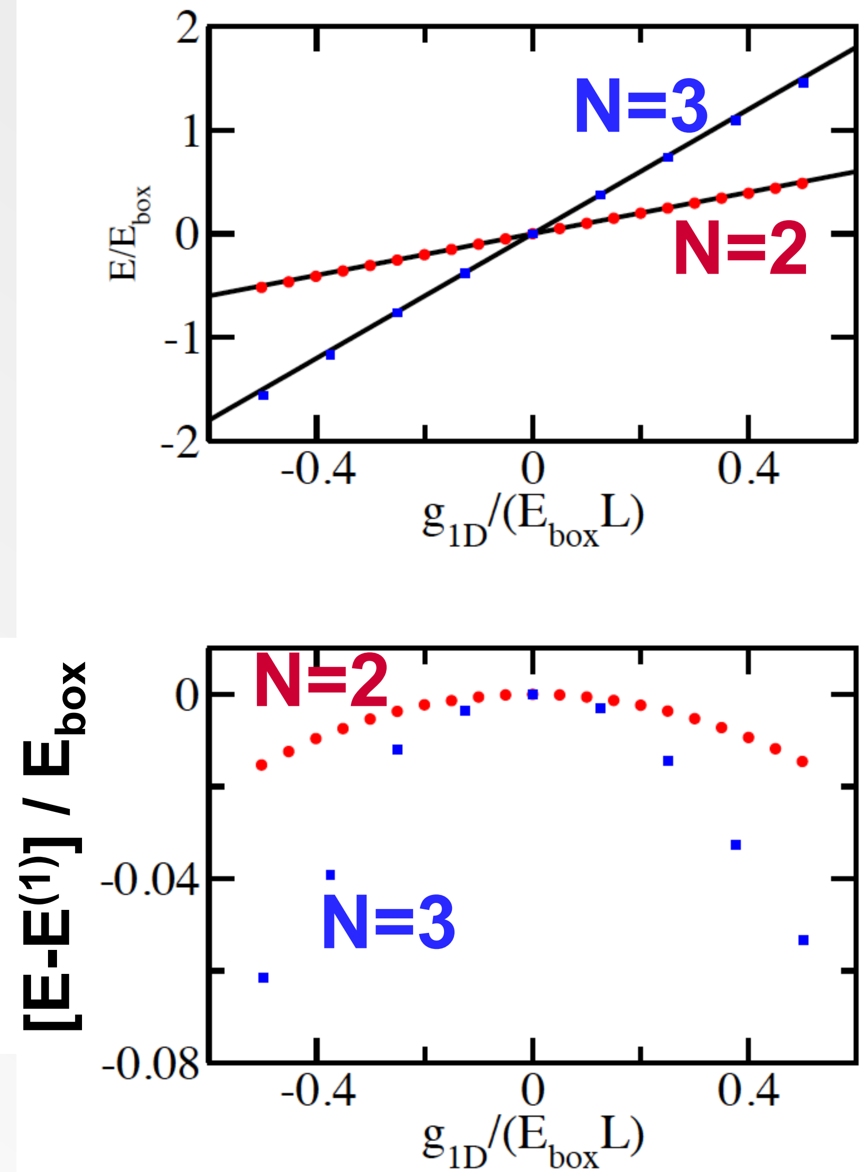
not needed to describe energy (two-body parameter) → $+ 2(N-1)\mathrm{Re}(D_0)$

- **Broader implication:** Two low-energy Hamiltonian that yield the same energy do not necessarily yield the same condensate fraction, momentum distribution,…

Daily, Yin, Blume, PRA 85, 053614 (2012).

Few-Particle System in a Box with Periodic Boundary Conditions

- Application of explicitly correlated Gaussian to periodic systems.
- Few-boson system in a box (so far, 1D).
- Extension to 3D is possible.
- Weakly-interacting 3D Bose gas studied by Savage et al. and Tan, motivated by lattice calculations.



Summary

- Explicitly correlated Gaussian evaluated at fixed hyperradius R provide promising basis to be used in hyperspherical framework:
  - Provides access to bound state and scattering continuum (N=4 and 5).

- Construction and testing of effective low-energy Hamiltonian:
  - Observables besides the energy.
  - Highly accurate benchmark calculations.

- Few-body systems with periodic boundary conditions:
  - A potential alternative approach...