η and η' mesons from $N_f = 2 + 1 + 1$ flavour lattice QCD

for the ETM collaboration

C. Michael[1], K. Ottnad[2], C. Urbach[2], F. Zimmermann[2]

arXiv:1206.6719

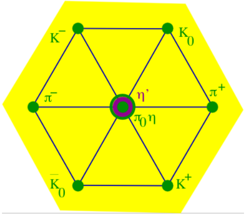
[1] Theoretical Physics Division, Department of Mathematical Sciences, University of Liverpool

[2] Helmholtz - Institut für Strahlen- und Kernphysik (Theorie), Bethe Center for Theoretical Physics, Universität Bonn

INT Workshop 12-2b

Flavour Singlet Pseudo-Scalar Mesons

- nine lightest pseudo-scalar mesons show a peculiar spectrum:
  - 3 very light pions (140 MeV)
  - kaons and the η around 600 MeV
  - η' around 1 GeV



- The large mass of the η' meson is thought to be caused by the QCD vacuum structure and the $U_A(1)$ anomaly

- η' meson is not a (would be) Goldstone Boson

$\Rightarrow$ massive even in the SU(3) chiral limit

Lattice Status

- disconnected contributions significant

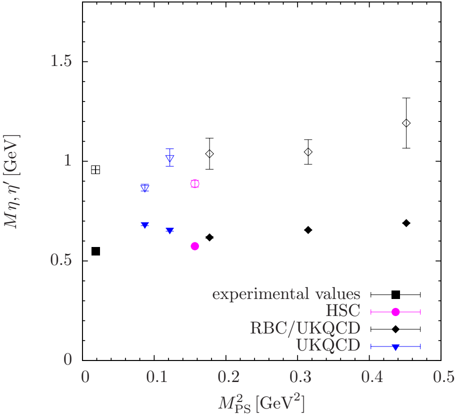
⇒ hard problem

- only a limited amount of lattice results available

- no control of systematics
  - usually only one lattice spacing
  - and/or only one pion mass

⇒ no clear picture

- in particular at light pion masses



filled symbols: η open: η'

[HSC, J. J. Dudek et al., Phys. Rev. D83 (2011)]
[RBC/UKQCD, N. Christ et al., Phys. Rev. Lett. 105 (2010)]
[UKQCD, E. B. Gregory et al., Phys.Rev. D86 (2012)]

$N_f = 2 + 1 + 1$ Wilson Twisted Mass Fermions

- with twisted mass formulation of LQCD
  only *doublets* of quarks can be considered

- light doublet, mass-degenerate:

$$D_\ell = D_W + m_{\text{crit}} + i\mu_\ell\gamma_5\tau^3\,, \qquad \chi_\ell = \begin{pmatrix}\chi_u\\ \chi_d\end{pmatrix}$$

- heavy doublet, mass-split, flavour non-diagonal:

$$D_h = D_W + m_{\text{crit}} + i\mu_\sigma\gamma_5\tau^1 + \mu_\delta\tau^3\,, \qquad \chi_h = \begin{pmatrix}\chi_c\\ \chi_s\end{pmatrix}$$

[Frezzotti, Rossi (2004)]

- rotation from χ- to standard ψ-basis (at maximal twist):

$$\psi_\ell = e^{i\pi\gamma_5\tau^3/4}\chi_\ell\,, \qquad \psi_h = e^{i\pi\gamma_5\tau^1/4}\chi_h$$

$N_f = 2+1+1$ Wilson Twisted Mass Fermions

Pros:

- $\mathcal{O}(a)$ improvement at maximal twist

  [Frezzotti, Rossi; JHEP 0408 (2004)]

$\Leftarrow$ by tuning only one parameter

- excellent scaling behaviour observed
- mixing patterns under renormalisation can be simplified
- note: could easily introduce *u*-*d* mass splitting as well

Cons:

- flavour and parity symmetries broken at finite values of the lattice spacing

$\Rightarrow$ technical complication

$\Rightarrow$ unphysical splittings, mostly in between $m_{\pi^\pm}$ and m_{π^0}

[Dimopoulus, Frezzotti, Michael, Rossi, CU; Phys.Rev. D81 (2010)]

The $1+1$ Doublet

- heavy doublet:

$$D_h = D_W + m_{\mathrm{crit}} + i\mu_\sigma\gamma_5\tau^1 + \mu_\delta\tau^3\,, \qquad \chi_h = \begin{pmatrix}\chi_c\\ \chi_s\end{pmatrix}$$

- think of μ_σ as the mean s/c mass and μ_δ the splitting
- splitting μ_δ orthogonal to twist μ_σ (τ^3 versus τ^1)
- obtain renormalised quark masses of the doublet

$$\hat{m}_s = Z_{\mathrm{P}}^{-1}\mu_\sigma \; - \; Z_{\mathrm{S}}^{-1}\mu_\delta$$
$$\hat{m}_c = Z_{\mathrm{P}}^{-1}\mu_\sigma \; + \; Z_{\mathrm{S}}^{-1}\mu_\delta$$

- fermion determinant positive and $\mathcal{O}(a)$ improvement remains valid

[Frezzotti, Rossi (2004)]

Flavour Singlet Pseudo-Scalar Mesons

- in the SU(3) symmetric case (sloppy notation)

$$\eta_8 : \qquad \bar{u} i\gamma_5 u + \bar{d} i\gamma_5 d - 2\bar{s} i\gamma_5 s$$

$$\eta_0 : \qquad \bar{u} i\gamma_5 u + \bar{d} i\gamma_5 d + \bar{s} i\gamma_5 s$$

- SU(3) symmetry broken $\Rightarrow$ mixing

$\Rightarrow$ SU(2) plus strange:

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} |\eta_\ell\rangle \\ |\eta_s\rangle \end{pmatrix}$$

- $N_f = 2 + 1 + 1$ possible charm contribution

[Feldmann, Int. J. Mod. Phys. A15, 159 (2000)]

Flavour Singlet Pseudo-Scalar Mesons

- need to estimate correlator matrix

$$\mathcal{C} = \begin{pmatrix} \eta_{\ell\ell} & \eta_{\ell s} & \eta_{\ell c} \\ \eta_{s\ell} & \eta_{ss} & \eta_{sc} \\ \eta_{c\ell} & \eta_{cs} & \eta_{cc} \end{pmatrix}$$

- η_{XY} correlator of appropriate interpolating fields, e.g.

$$\eta_{ss}(t) \quad \equiv \quad \langle \bar{s} i\gamma_5 s(t)\, \bar{s} i\gamma_5 s(0)\rangle$$

  projected to zero momentum

- η: lowest state, η': first state, η_c ...

A Little Twisted Mass Algebra

- rotation to twisted basis

$$\frac{1}{\sqrt{2}}(\bar\psi_u i\gamma_5\psi_u + \bar\psi_d i\gamma_5\psi_d) \quad\to\quad \frac{1}{\sqrt{2}}(\bar\chi_d\chi_d - \bar\chi_u\chi_u) \equiv \mathcal{O}_\ell\,,$$

- and in the heavy sector

$$\begin{pmatrix}\bar\psi_c\\ \bar\psi_s\end{pmatrix}^T i\gamma_5\frac{1\pm\tau^3}{2}\begin{pmatrix}\psi_c\\ \psi_s\end{pmatrix} \quad\to\quad \begin{pmatrix}\bar\chi_c\\ \bar\chi_s\end{pmatrix}^T \frac{-\tau^1\pm i\gamma_5\tau^3}{2}\begin{pmatrix}\chi_c\\ \chi_s\end{pmatrix} \equiv \mathcal{O}_{c,s}\,.$$

- therefore

$$\begin{aligned}\mathcal{O}_c &\equiv Z(\bar\chi_c i\gamma_5\chi_c - \bar\chi_s i\gamma_5\chi_s)/2 - (\bar\chi_s\chi_c + \bar\chi_c\chi_s)/2\,,\\ \mathcal{O}_s &\equiv Z(\bar\chi_s i\gamma_5\chi_s - \bar\chi_c i\gamma_5\chi_c)/2 - (\bar\chi_s\chi_c + \bar\chi_c\chi_s)/2\,.\end{aligned}$$

- with ratio of non-singlet renormalisation constants

$$Z \equiv \frac{Z_P}{Z_S}$$

Estimating Disconnected Diagrams

- fermionic disconnected contributions noisy

$\Rightarrow$ need as many as possible observations

- use R stochastic volume sources ξ^r

$$\lim_{R\to\infty}[\xi_i^*\xi_j]_R = \delta_{ij}\,,\qquad \lim_{R\to\infty}[\xi_i\xi_j]_R = 0$$

- then we get

$$[\xi_i^{r*}\phi_j] \;=\; (D^{-1})_{ji} + \mathrm{noise}$$

with

$$\phi_j = (D^{-1})_{jk}\xi_k^r$$

- noise $\propto \sqrt{V_s/R}$ while signal $\mathcal{O}(1)$

A Very Efficient Variance Reduction Method

[Jansen, Michael, CU, Eur.Phys.J. C58 (2008)]

- relation in between up and down Dirac operator

$$D_u - D_d \qquad = \qquad 2\mu_\ell\gamma_5$$

- multiply with $1/D_u$ from left, $1/D_d$ from right

$$\frac{1}{D_d} - \frac{1}{D_u} \qquad = \qquad 2\mu_\ell \frac{1}{D_u}\gamma_5\frac{1}{D_d}$$

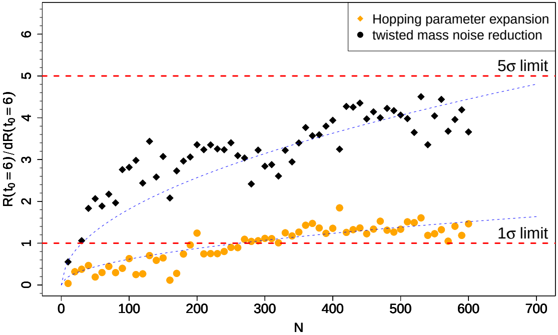
- lhs is what we want, rhs has an extra volume loop

$\Rightarrow$ can be estimated from

$$[\phi^*\phi]_R + \mathrm{noise}$$

- noise $\propto \sqrt{V_s^2/R}$, but signal $\propto V$
- only applicable in the light sector involving τ^3

A Very Efficient Variance Reduction Method: Example



- example for strangeness content of the nucleon

  [ETMC, Dinter et al., JHEP 1208 (2012)]

- easily a factor four improvement

Ensemble-Details

- gauge configurations from ETM Collaboration

  [ETMC, R. Baron et. al., JHEP 06 111 (2010)]

- Iwasaki Gauge action

  [Iwasaki, Nucl. Phys. B258, 141]

- three lattice spacings:
  $a_A = 0.086\ \mathrm{fm}$, $a_B = 0.078\ \mathrm{fm}$ and $a_D = 0.061\ \mathrm{fm}$
- charged pion masses range from $\approx 230\ \mathrm{MeV}$ to $\approx 500\ \mathrm{MeV}$
- $L \geq 3\ \mathrm{fm}$ and $M_\pi \cdot L \geq 3.5$ for most ensembles
- ≈ 600 up to ≈ 2500 gauge configuration per ensemble
- μ_σ, μ_δ fixed for each β
- use $r_0 = 0.45(2)\ \mathrm{fm}$ (from f_π) throughout the talk

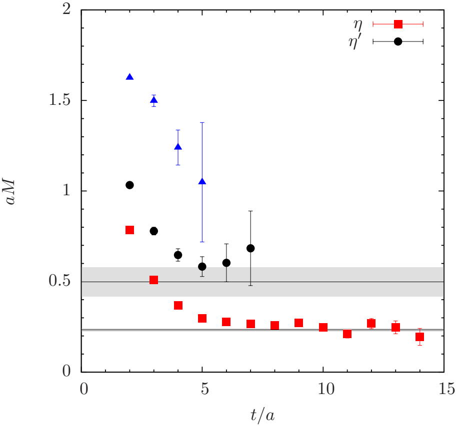
Analysis Procedure

- 24 to 32 volume sources per gauge for disconnecteds
  have tested one ensemble with 64 sources
- local and smeared operators
- two γ-combinations $i\gamma_5$, $i\gamma_0\gamma_5$
- two independent fitting methods (up to 12×12 matrix)
  - solving the GEVP
    [Michael, Teasdale, Nucl.Phys. B215, 433 (1983); Lüscher, Wolff, Nucl.Phys. B339, 222 (1990)]
  - using a factorising fit
- errors computed by bootstrapping and blocking
  1000 bootstrap samples

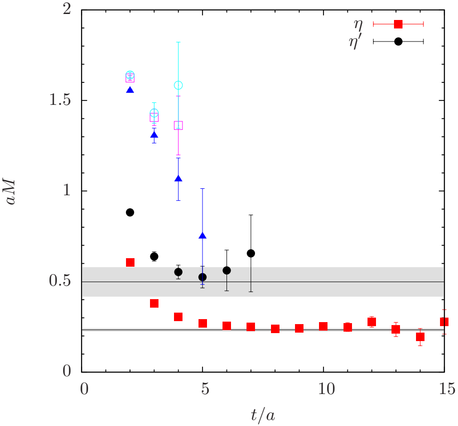
$\Rightarrow$ significant autocorrelation for η': $\sim$ 20 trajectories

Effective Masses $B25$

3 × 3 matrix

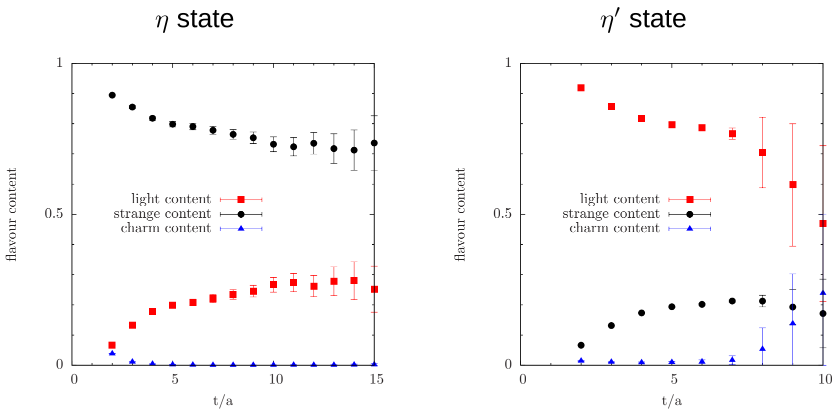


6 × 6 matrix



- ground state η well determined
- next state hardly plateaus

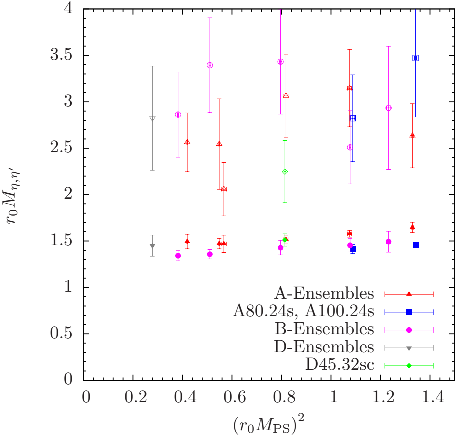
Flavour Content $B25$



- flavour content qualitatively as expected
- no charm contribution to η and η'
- third state (not shown) is charm only (almost)

Summary Masses

- η mass quite precise
- η' rather noisy
  large systematic uncertainties
- mild pion mass dependence in M_η
- physical strange quark mass not fixed for different β values!
- what can we say about lattice artifacts?
- how to perform the chiral extrapolation?

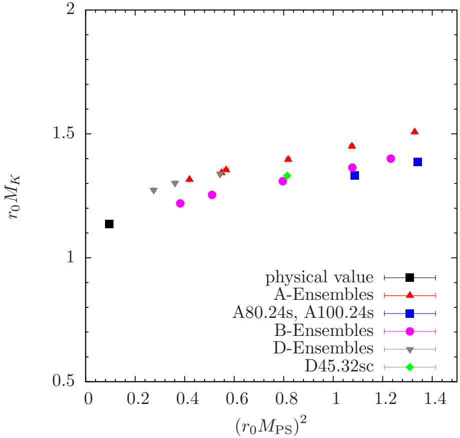


Strange Quark Mass Dependence

- μ_σ and μ_δ fixed for each β
- m_s unfortunately not perfectly tuned to its physical value
- we have two re-tuned ensembles for a_A ($\beta = 1.90$)

$\Rightarrow$ can estimate m_s dependence

$\Rightarrow$ need to come up with a strategy to correct for this effect!



Scaling Test for M_η (1)

$\Rightarrow$ fix $r_0 M_{\rm PS}$, $r_0 M_{\rm K}$ and V/r_0

- we don't see a volume dependence in M_η

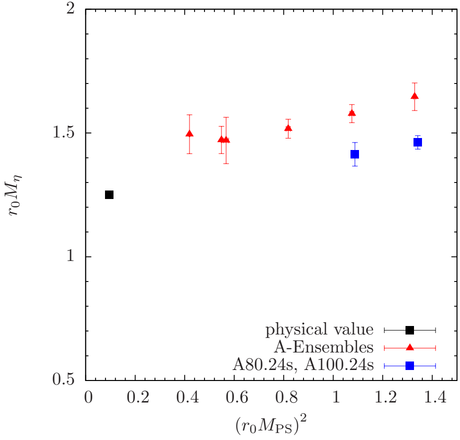
$\Rightarrow$ ignore it

- estimate

$$D_\eta \equiv \frac{d(aM_\eta)^2}{d(aM_{\rm K})2} = 1.6(2)$$

from two A-ensembles

- now assume: D_η independent of $\beta, \mu_\ell, \mu_\sigma, \mu_\delta$



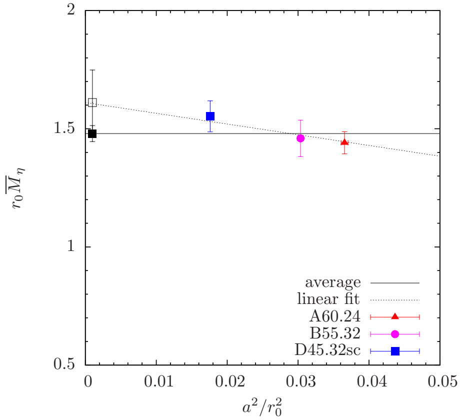
Scaling Test for M_η (2)

- use ensembles $A60$, $B55$, $D45$ with $r_0 M_{\mathrm{PS}} \approx 0.9$
- correct M_η using D_η linearly in M_{K}^2

$\Rightarrow$ $r_0 M_{\mathrm{K}} = 1.34$ fixed

- compatible with both, constant and linear continuum extrapolation

$\Rightarrow$ assign conservative 8% error from difference to all our results



Chiral Extrapolation of M_η

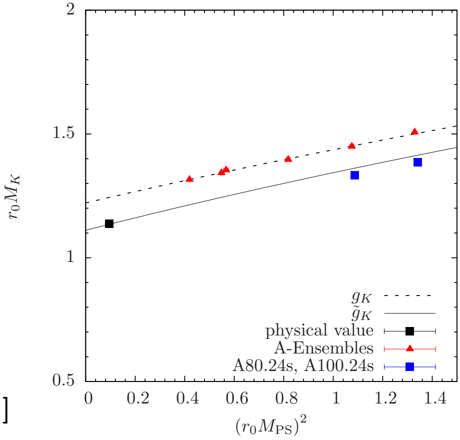
- more ambitious: shift all M_η to physical strange mass
- fit

  $$g_{\mathrm{K}} = a + b(r_0 M_{\mathrm{PS}})^2$$

  to data for $(r_0 M_{\mathrm{K}})^2$ from A ensembles
- adjust a to match physical M_{K} for $M_{\mathrm{PS}} = M_\pi \quad \Rightarrow \tilde{g}_{\mathrm{K}}$
- compute

  $$\delta_{\mathrm{K}}[(r_0 M_{\mathrm{PS}})^2] = (r_0 M_{\mathrm{K}})^2 - \tilde{g}_{\mathrm{K}}[(r_0 M_{\mathrm{PS}})^2]$$

  for all ensembles



Chiral Extrapolation of M_η

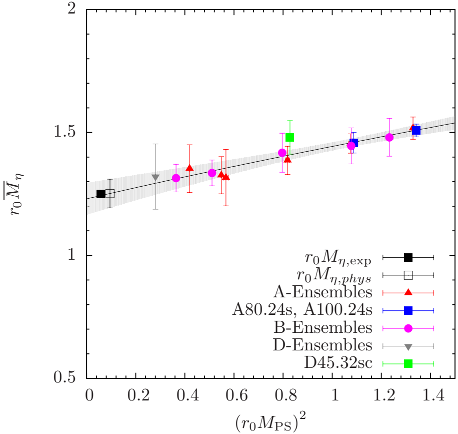
- now correct all $(r_0 M_\eta)^2$ by corresponding

$$D_\eta \cdot \delta_{\mathrm{K}}[(r_0 M_{\mathrm{PS}})^2]$$

$\Rightarrow$ $(r_0 \bar{M}_\eta)^2[(r_0 M_{\mathrm{PS}})^2]$

- all β-values fall on the same curve!
- extrapolate $(r_0 \bar{M}_\eta)^2$ linearly in $(r_0 M_{\mathrm{PS}})^2$ to $M_{\mathrm{PS}} = M_\pi$
- result

$$M_\eta = 549(33)_{\mathrm{stat}}(44)_{\mathrm{sys}}\ \mathrm{MeV}$$



Chiral Extrapolation of M_η

- alternatives to avoid D_η

$\Rightarrow$ use the ratio $(M_\eta/M_\mathrm{K})^2$

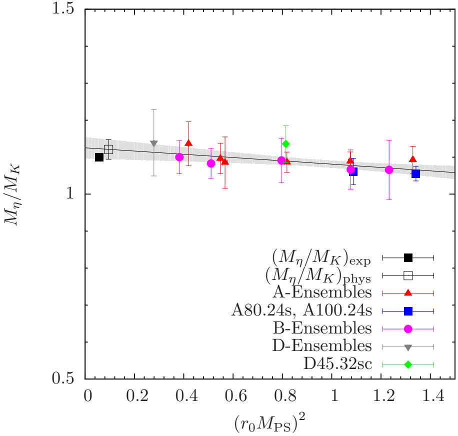
- all β-values on a single curve

$\Rightarrow$ in particular: m_s dependence seems to cancel

- extrapolate linearly in $(r_0 M_\mathrm{PS})^2 2$ to physical point

- result

$$M_\eta = 558(13)_\mathrm{stat}(45)_\mathrm{sys}\ \mathrm{MeV}$$



Chiral Extrapolation of M_η

- or use GMO relation

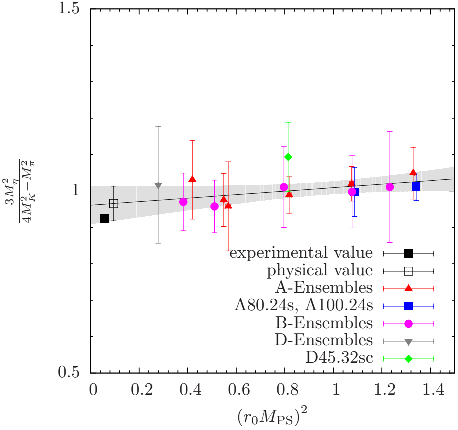
$$3M_\eta^2 = 4M_K^2 - M_\pi^2$$

valid for SU(3)

- experimentally ~ 0.925

- result

$$M_\eta = 559(14)_{\mathrm{stat}}(45)_{\mathrm{sys}}\ \mathrm{MeV}$$

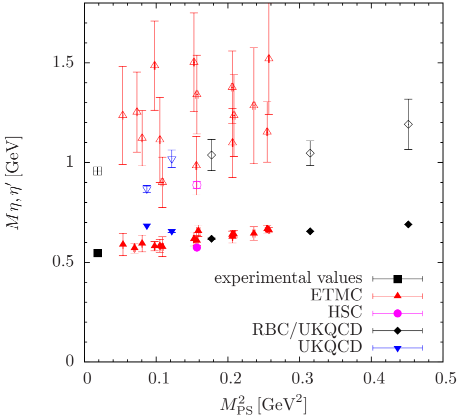


- weighted average over three methods

$$M_\eta = 557(15)_{\mathrm{stat}}(45)_{\mathrm{sys}}\ \mathrm{MeV}$$

Comparing to Other Lattice Results

- overall mutual agreement
- apart from maybe UKQCD at smallest M_{PS}
- we have pushed significantly more chiral
- ETMCs η' is much noisier despite
    - similar number of independent gauges (RBC, HSC, ETMC)
    - similar number of inversions (RBC, ETMC)



η and η' Mixing

- write correlator matrix

$$\mathcal{C}_{qq'}(t) = \sum_n \frac{A_{q,n}A_{q',n}}{2m^{(n)}} \left[\exp(-m^{(n)}t) + \exp(-m^{(n)}(T-t))\right]$$

with amplitudes $A_{q,n}$ corresponding to $\langle 0|\bar{q}q|n\rangle$
($n \equiv \eta, \eta', \dots$ and $q = \ell, s, c$)

- define mixing angles via (ignoring charm)

$$\begin{pmatrix} A_{\ell,\eta} & A_{s,\eta} \\ A_{\ell,\eta'} & A_{s,\eta'} \end{pmatrix} = \begin{pmatrix} f_\ell \cos\phi_\ell & -f_s \sin\phi_s \\ f_\ell \sin\phi_\ell & f_s \cos\phi_s \end{pmatrix}$$

- for $\phi_s \approx \phi_\ell$

$$\tan^2\phi = -\frac{A_{\ell,\eta'}A_{s,\eta}}{A_{\ell,\eta}A_{s,\eta'}}$$

η and η' Mixing

- ϕ_ℓ and ϕ_s are too noisy separately

- single mixing angle ϕ can be determined

- linear fit in $(r_0 M_{\mathrm{PS}})^2$

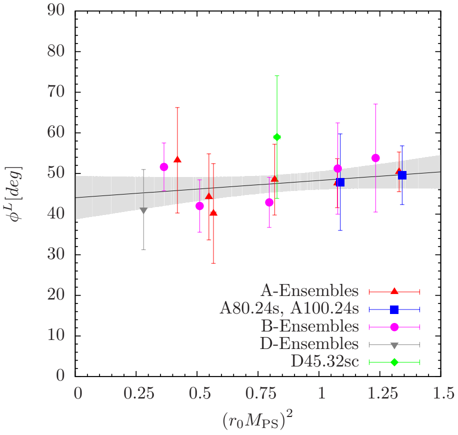
$$\phi = 44(5)^\circ$$

or in singlet/octet basis

$$\theta = -10(5)^\circ$$

$\Rightarrow$ good agreement with other determinations

- note: statistical error only



Mixed Action Approach

- idea: use different valence action for s and c
  - $\Rightarrow$ to avoid s/c flavour mixing
  - $\Rightarrow$ to vary strange and charm quark masses
- up/down stay unitary
- add two valence strange quarks s and s'

$$\begin{aligned} s \equiv s(+):& \quad D_W + m_{\mathrm{crit}} + \mu_s i\gamma_5 \\ s' \equiv s(-):& \quad D_W + m_{\mathrm{crit}} - \mu_s i\gamma_5 \end{aligned}$$

- can proof that continuum limit is correct

  [Frezzotti, Rossi, JHEP 08, 007 (2004)]
- the different signs allow to use the variance reduction trick

$$\bar\psi_s\psi_s = \frac{1}{2}(\bar\psi_s\psi_s + \bar\psi_s\psi_s) = \frac{1}{2}(\bar\chi_s\chi_s - \bar\chi_{s'}\chi_{s'})$$

Matching Unitary and Mixed Actions

- there are different matching quantities possible
  - match μ_s with $\mu_\sigma - Z\mu_\delta$
  - unitary with mixed $\bar{s}(+)d(-)$ kaon (denote M_{K+})

    $\Rightarrow$ smallest lattice artifacts for f_{PS} and M_{PS}

    [Sharpe, Wu,Phys. Rev. D 71 (2005), Frezzotti et al., JHEP 0604 (2006)]
  - unitary with mixed $\bar{s}(-)d(-)$ kaon (denote M_{KOS})

    $\Rightarrow$ usually larger lattice artifacts
  - and of course many other quantities

$\Rightarrow$ tried first to match with M_{K+}

Matching Unitary and Mixed Actions

- there are different matching quantities possible
  - match μ_s with $\mu_\sigma - Z\mu_\delta$
  - unitary with mixed $\bar{s}(+)d(-)$ kaon (denote M_{K^+})
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    [Sharpe, Wu,Phys. Rev. D 71 (2005), Frezzotti et al., JHEP 0604 (2006)]
  - unitary with mixed $\bar{s}(-)d(-)$ kaon (denote $M_{\mathrm{K^{OS}}}$)
    $\Rightarrow$ usually larger lattice artifacts
  - and of course many other quantities

$\Rightarrow$ tried first to match with M_{K^+}

- ... and got pathological results
- so, why is that?

Matching Unitary and Mixed Actions

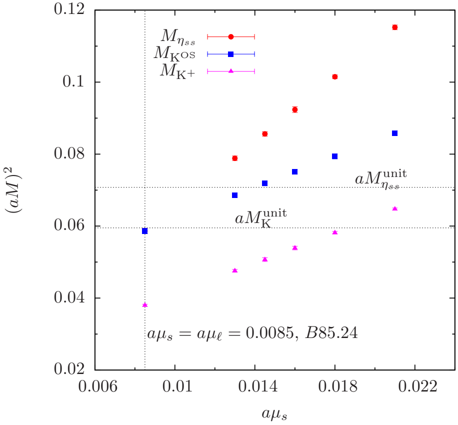
- η, η' have significant disconnected contributions

$\Rightarrow$ which don't know about μ_s!

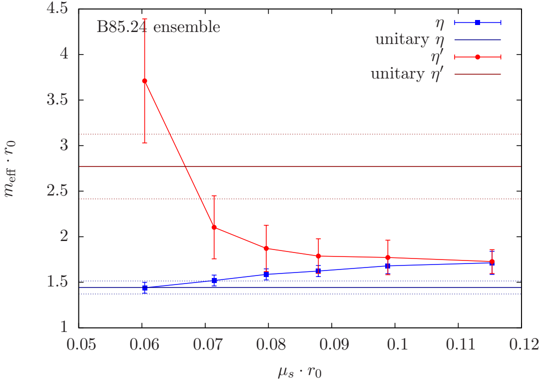
- connected and disconnected have to match to produce the correct correlation matrix

$\Rightarrow$ match connected only $M_{\eta_{ss}}$

- due to $M_{\mathrm{PS}^\pm}$ - M_{PS^0} splitting: significantly smaller matching μ_s-value



μ_s-Dependence



$\Rightarrow$ at $M_{\eta_{ss}}$ matching point we find reasonable agreement in between unitary and mixed approach

- currently exploring this further

Summary

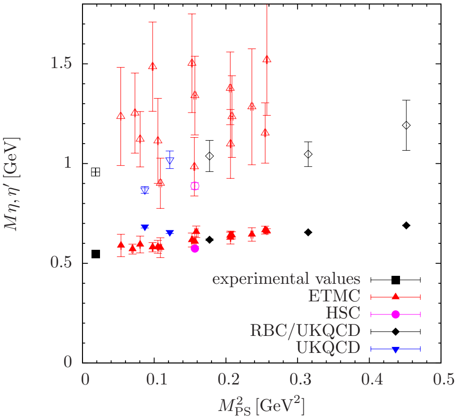
- η and η' for three lattice spacings and various quark mass values
- η can be extracted precisely

$$M_\eta = 557(15)_{\rm stat}(45)_{\rm sys}\ \mathrm{MeV}$$

- η' noisy, significant systematics
- single mixing angle

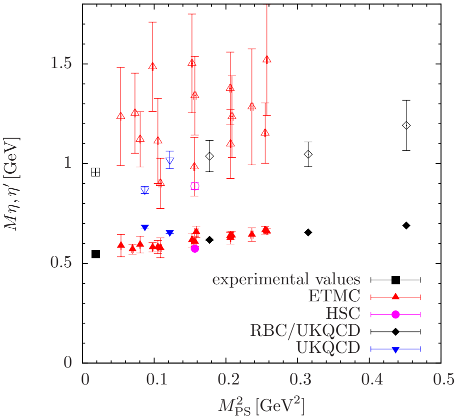
$$\phi = 44(5)^\circ (\ \text{or} \quad \theta = 10(5)^\circ\ )$$

- small lattice artifacts in M_η



Outlook

- noise reduction techniques for η'
- larger operator basis
- $\eta\pi$ scattering length
- $\eta \to \gamma\gamma$



Acknowledgements

- computing time from
  - FZ Jülich on JUGENE and JUDGE
  - on local GPU cluster founded by DFG
- financial support from DFG in CRC 16
- supported by the Bonn-Cologne Graduate School (BCGS)
- thanks to all members of ETMC

ϕ_ℓ and ϕ_s