General multi-hadron ops and $\pi\pi$ I=2 scattering

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INT Program on Lattice QCD studies of excited resonances and multi-hadron systems
24 August 2012

Hadron Spectrum Collaboration

Outline

- Introduction
- Single and multi-hadron operators
  - Isospin-2 $\pi\pi$ energy levels
- Lüscher method
  - Isospin-2 $\pi\pi$ phase shifts
- Summary

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Very few hadrons stable w.r.t. strong decay

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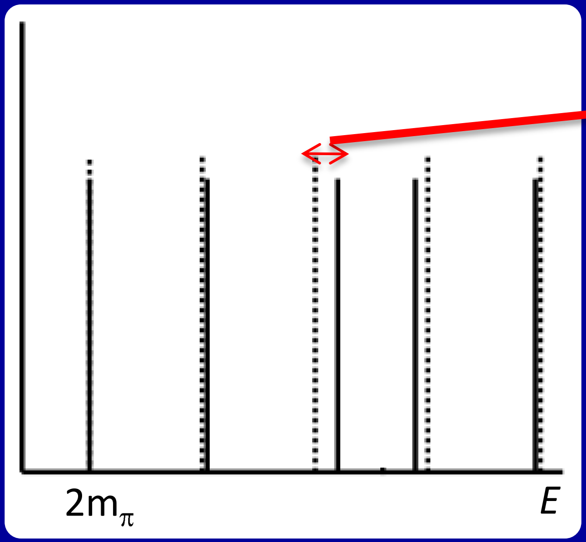

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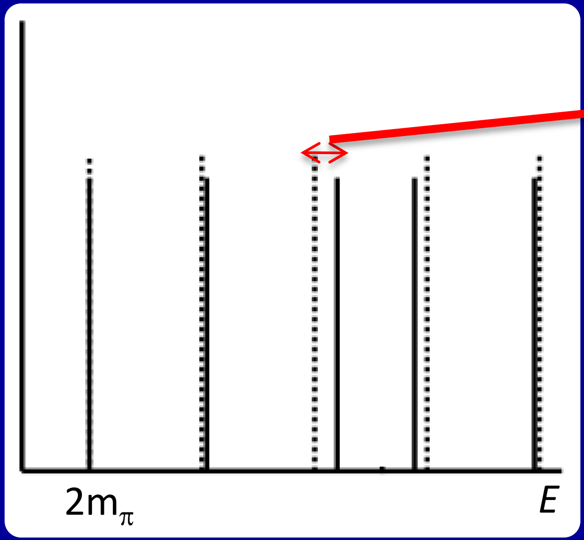
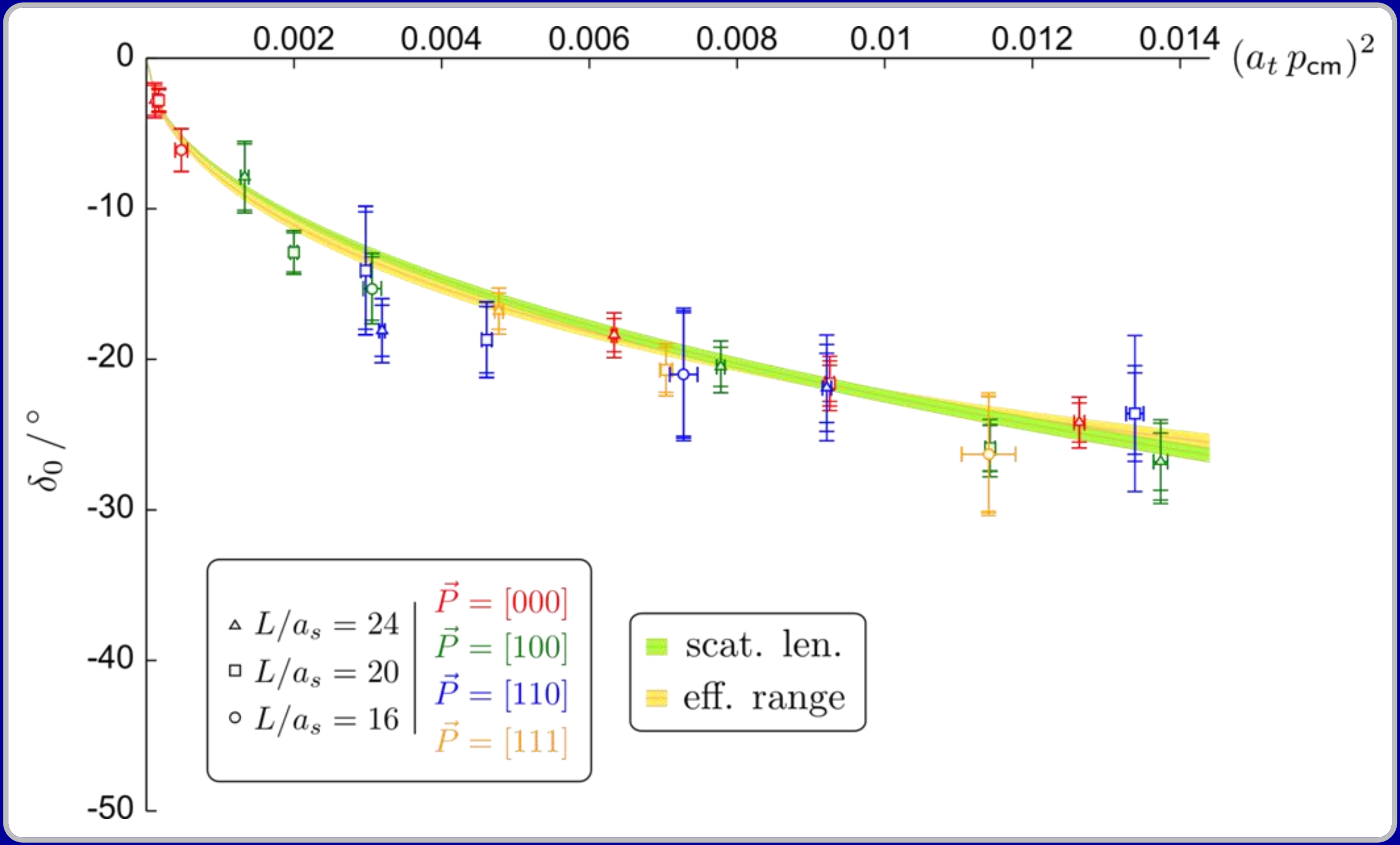


$$\Delta E(L_s) \to \delta(E, L_s)$$

Extract phase shift at discrete p_{cm}

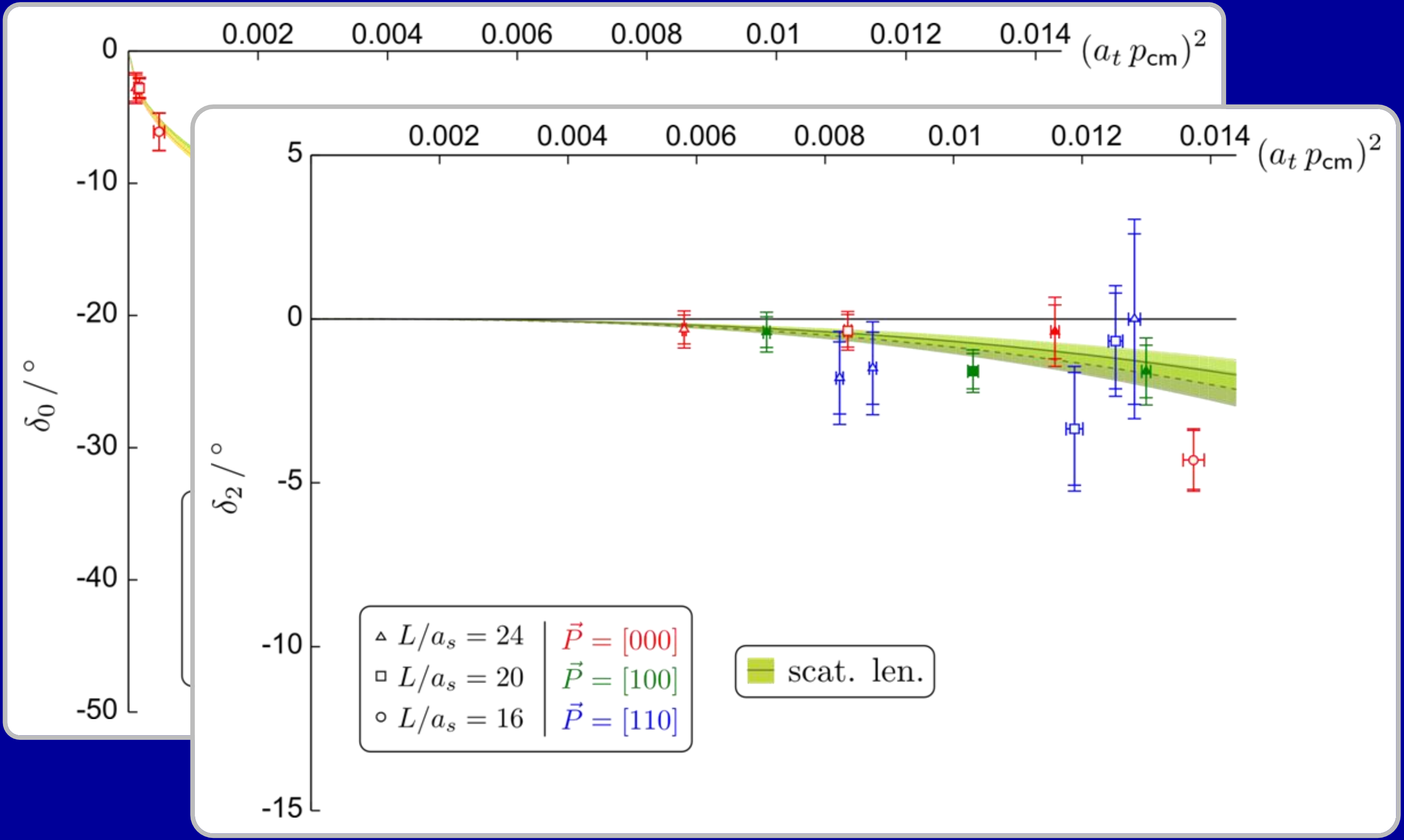
Map out phase shift → resonance parameters (mass, width), decays

$\pi\pi$ I=2 phase shifts



arXiv:1203.6041 [PRD]

ππ I=2 phase shifts



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Single-hadron operators

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Use **distillation**

Peardon et al,
PR D80, 054506 (2009)

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$$\Box_{xy}(t) = \sum_{k=1}^{N} v_x^{(k)}(t) v_y^{(k)\dagger}(t)$$

$$\Box(t) = V(t) V^\dagger(t)$$

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$$C_{AB}^{q'q}(t',t) = \langle 0|\mathcal{O}_A^{q'}(t')\mathcal{O}_B^{q\dagger}(t)|0\rangle$$

$$= \langle 0|\bar{q}'\Box(t')\mathbf{\Gamma}_{t'}^{A}\Box(t')q'(t') \quad \cdot \quad \bar{q}\Box(t)\mathbf{\Gamma}_{t}^{B}\Box(t)q(t)|0\rangle$$

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$$\mathcal{O}(t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \; \bar{\psi}(x)\Gamma_i \overleftrightarrow{D}_j \overleftrightarrow{D}_k \ldots \psi(x)$$

PRL 103 262001 (2009),
PR D82 034508 (2010)

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Circular basis
for D and Γ

Γ x D x D x ...

couple using SU(2) Clebsch Gordans

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Single-hadron operators

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$$\square_{xy}(t) = \sum_{k=1}^{N} v_x^{(k)}(t) v_y^{(k)\dagger}(t)$$

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$\vec{p} = \vec{0}$

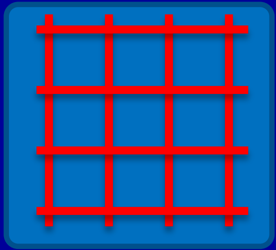
$$\langle 0|\mathcal{O}^{J,M}|J',M'\rangle = Z^{[J]}\delta_{J,J'}\delta_{M,M'}$$

definite J^{PC} and
$M = J_z$ component

Finite volume lattice

Cubic lattice with periodic b.c.s – symmetry group of cube

O_h^D



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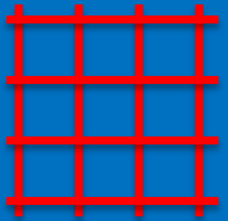

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O_h^D

J	0	1	2	3	4	...
Λ	A_1	T_1	$T_2 + E$	$T_1 + T_2 + A_2$	$A_1 + T_1 + T_2 + E$	...

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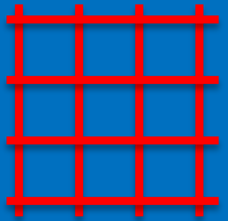

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(+ others for half-integer spin)

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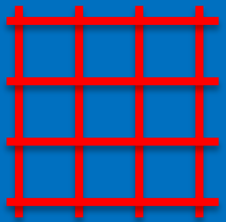

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'Subduce' operators into lattice irreps (J → Λ):

$$\mathcal{O}^{[J]}_{\Lambda,\lambda} = \sum_M \mathcal{S}^{J,M}_{\Lambda,\lambda} \mathcal{O}^{J,M}$$

e.g. $\mathcal{O}^{[2]} \rightarrow T_2$ and $\mathcal{O}^{[2]} \rightarrow E$

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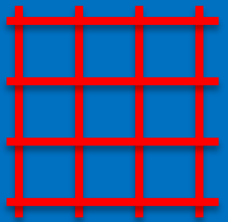

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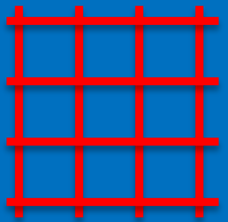

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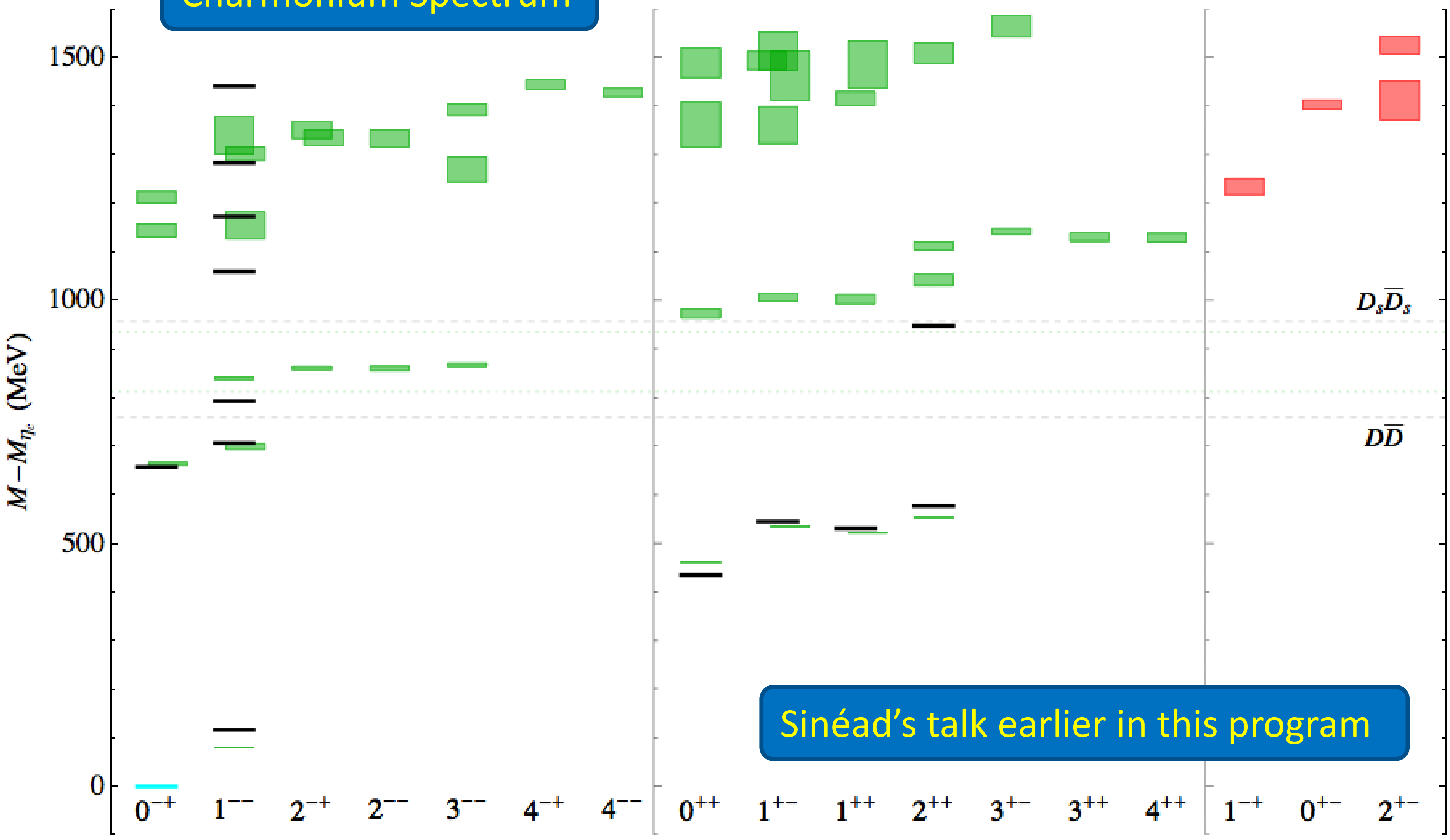
Variational method; up to 26 ops in Λ^{PC} channel

Charmonium Spectrum



Sinéad's talk earlier in this program

24^3, $M_\pi \approx 400$ MeV

JHEP 1207 (2012) 126

Single-hadron operators

$\vec{p} \neq \vec{0}$ Helicity ops

CT, Robert Edwards, Jo Dudek, PR D85 (2012) 014507

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Helicity ops

$$\left[\mathbb{O}^{J^P,\lambda}(\vec{p})\right]^\dagger = \sum_m \mathcal{D}^{(J)}_{m\lambda}(R)\left[\mathcal{O}^{J^P,m}(\vec{p})\right]^\dagger$$

$R: \ (0,0,|\vec{p}|) \to \vec{p}$

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$$\langle 0 | \mathbb{O}^{J^P,\lambda}(\vec{p}) | \vec{p}; J'^{P'}, \lambda \rangle = Z \delta_{\lambda,\lambda'}$$

CT, Robert Edwards, Jo Dudek, PR D85 (2012) 014507

Finite volume lattice

$\vec{p} \neq \vec{0}$

$$\vec{p} = \frac{2\pi}{L_s a_s}(n_x, n_y, n_z)$$

$n_x, n_y, n_z \in \mathbb{Z}$

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Little group

$\mathrm{LG}(\vec{p}) \subset \mathrm{O}_h^{\mathrm{D}}$

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		A_2	$0^-,\ 4,\ \dots$
		E_2	$1,\ 3,\ \dots$
		B_1	$2,\ \dots$
		B_2	$2,\ \dots$
$[n,n,0]$	Dic$_2$	A_1	$0^+,\ 2,\ 4,\ \dots$
		A_2	$0^-,\ 2,\ 4,\ \dots$
		B_1	$1,\ 3,\ \dots$
		B_2	$1,\ 3,\ \dots$
$[n,n,n]$	Dic$_3$	A_1	$0^+,\ 3,\ \dots$
		A_2	$0^-,\ 3,\ \dots$
		E_2	$1,\ 2,\ 4,\ \dots$
$[n,m,0]$ $[n,n,m]$	C$_4$	A_1	$0^+,\ 1,\ 2,\ 3,\ 4,\ \dots$
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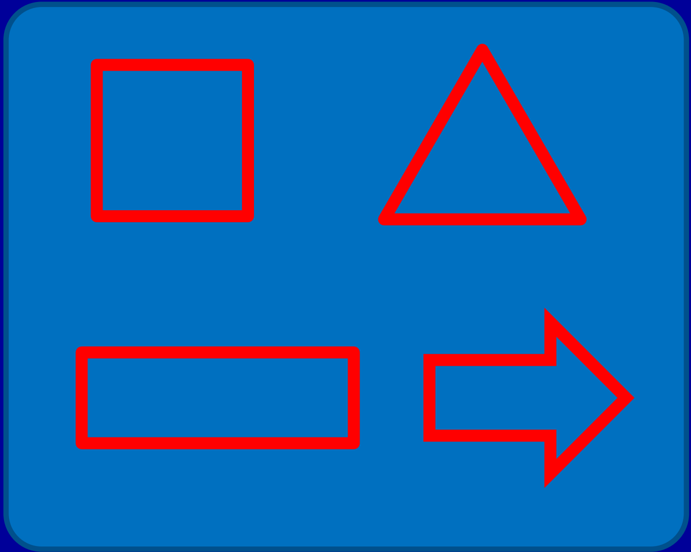

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$$\tilde{\eta} \equiv P(-1)^J$$

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$[n,n,n$			
$[n,m,0$			
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Operator that transforms in **irrep Λ, row μ**

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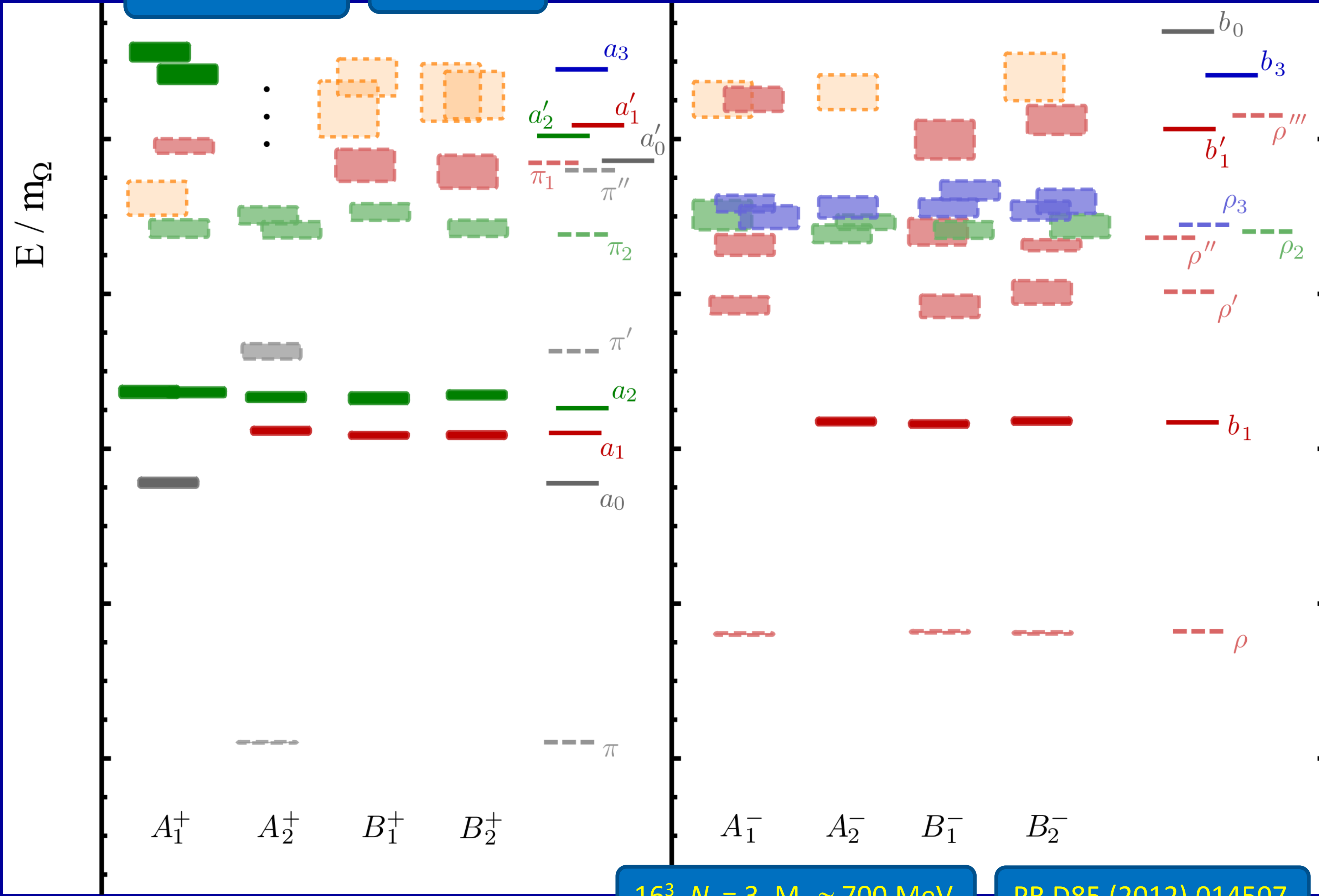
Operator that transforms in **irrep Λ, row μ**

N.B. being sloppy with daggers on ops

I=1 mesons

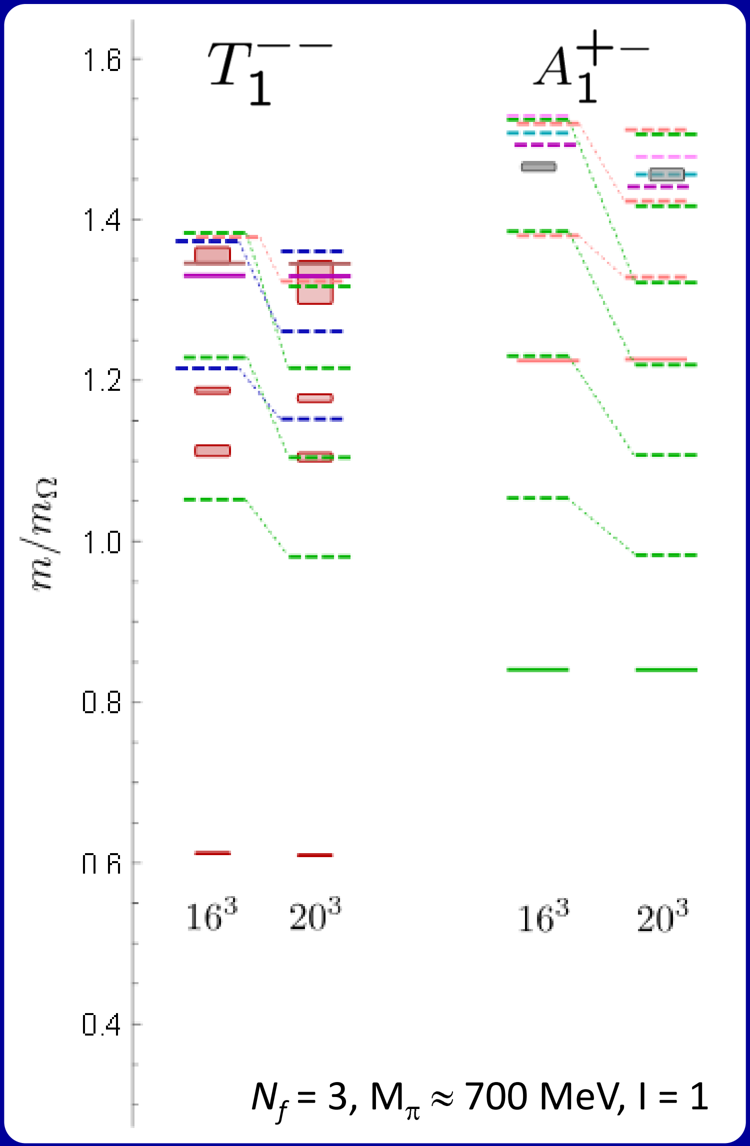
p ~ [0,1,1]



16^3, $N_f = 3$, $M_\pi \approx 700$ MeV

PR D85 (2012) 014507

Multi-hadron states?



Boxes – extracted meson levels

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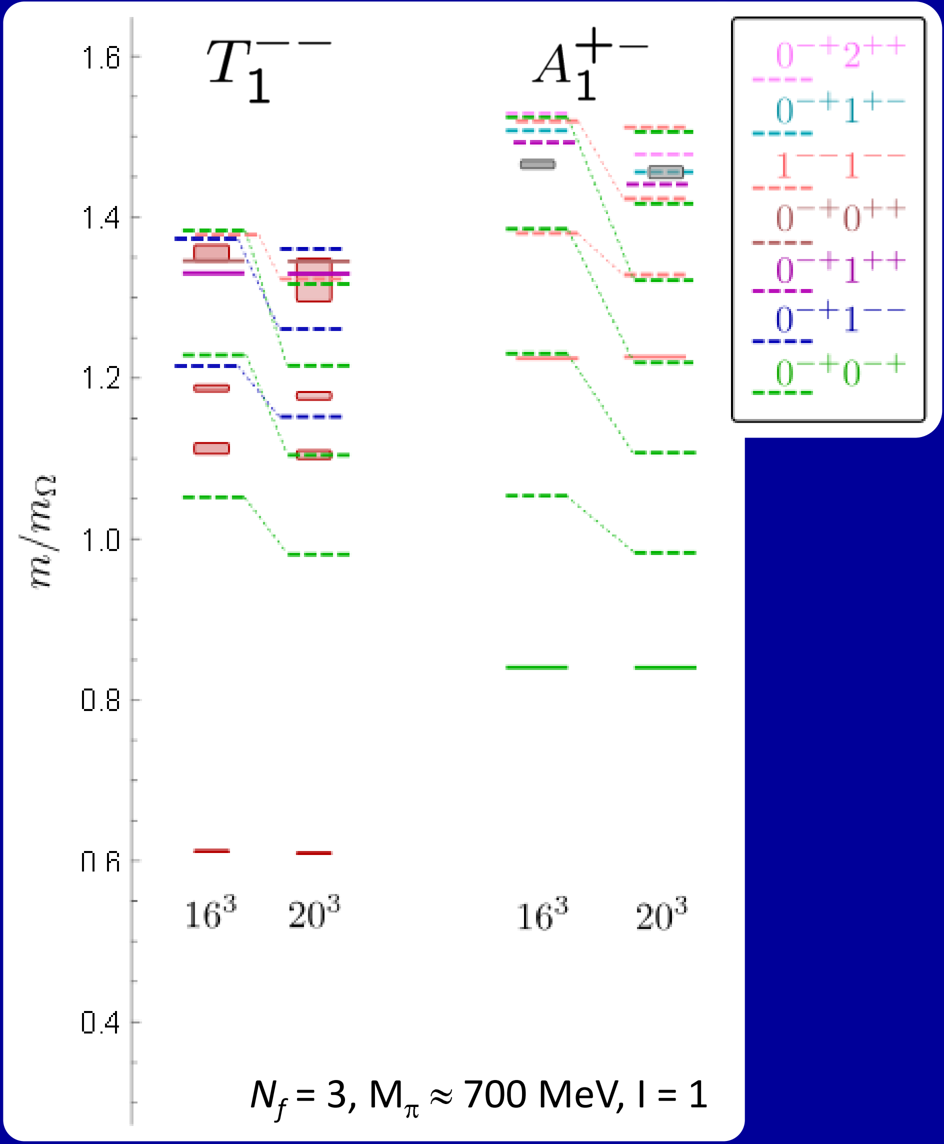

Boxes – extracted meson levels

Dashed lines – non-interacting two-meson levels

$$\vec{p} = \frac{2\pi}{L_s}(n_x, n_y, n_z)$$

$$E_{AB}(p) = \sqrt{m_A^2 + \vec{p}^2} + \sqrt{m_B^2 + \vec{p}^2}$$

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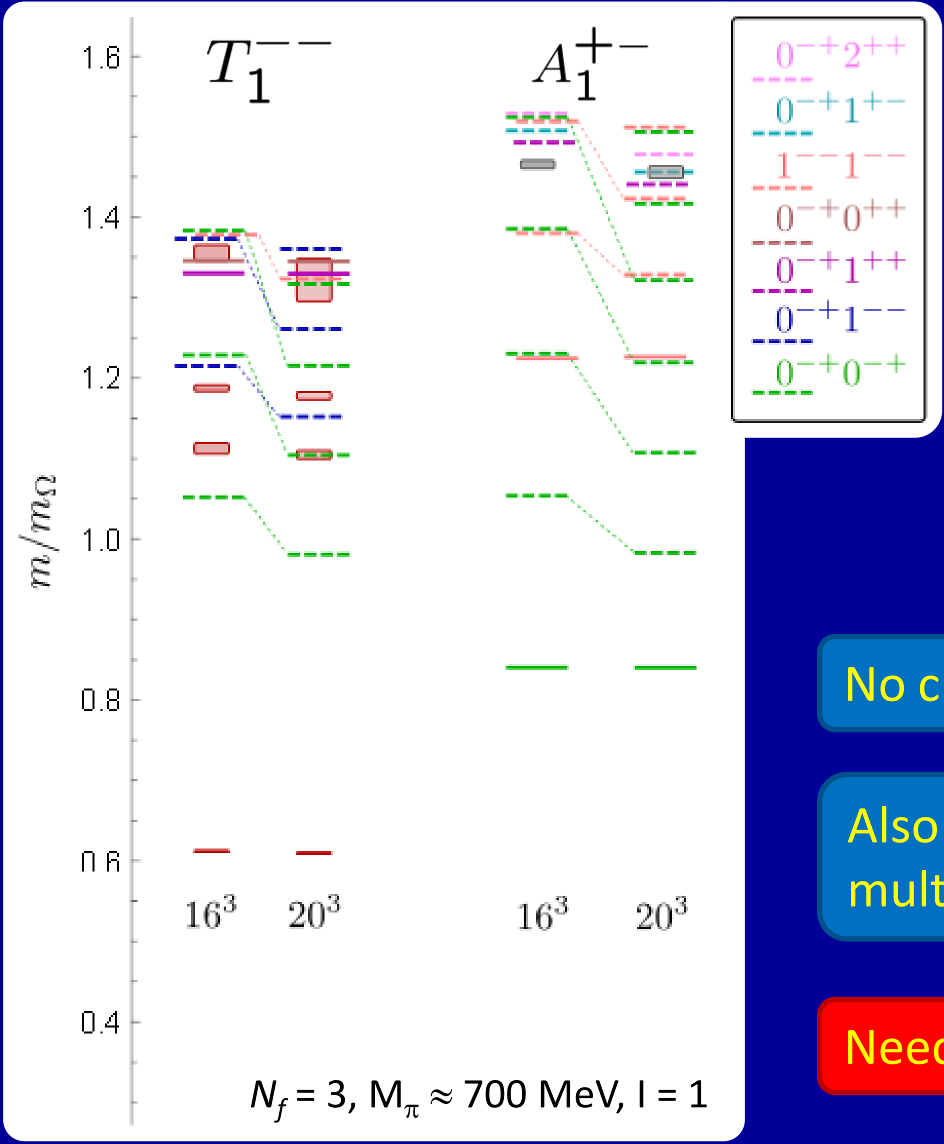

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No clear evidence for two-meson states

Also want to disentangle many multi-meson energy levels

Need ops that 'look like' two mesons

Multi-hadron operators

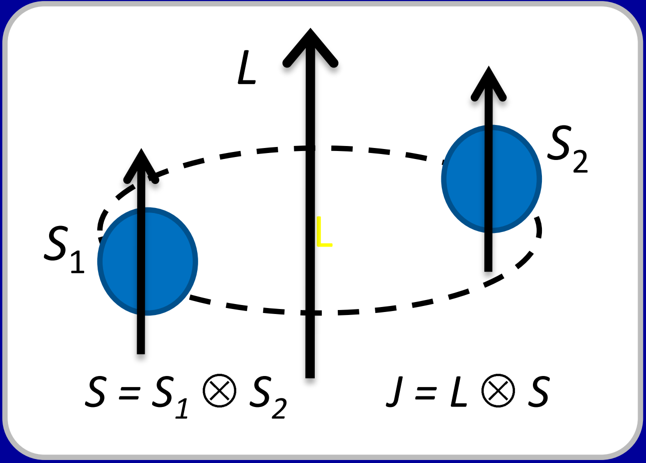
Jo Dudek, Robert Edwards, CT, arXiv:1203.6041 [to appear in PRD]

Multi-hadron operators

Compare continuum formulation

$$\mathcal{O}^{[S,\ell]}_{J,M} \sim \sum_{\lambda_1\lambda_2} \int d\hat{p}\, C(J,\ell,S,M;\vec{p},S_1,\lambda_1;-\vec{p},S_2,\lambda_2)\ \mathcal{O}^{S_1\lambda_1}(\vec{p})\ \mathcal{O}^{S_2\lambda_2}(-\vec{p})$$

$$C = \langle S_1,\lambda_1;S_2,-\lambda_2|S,\lambda\rangle\langle \ell,0;S,\lambda|J,\lambda\rangle\ \mathcal{D}^{(J)*}_{M\lambda}(\hat{p})$$



Jo Dudek, Robert Edwards, CT, arXiv:1203.6041 [to appear in PRD]

Multi-hadron operators

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with

$$\mathbb{O}_{\Lambda,\mu}(\vec{P}) = \sum_{\mu_1,\mu_2}\sum_{\vec{k}_1,\vec{k}_2} C(\vec{P},\Lambda,\mu;\vec{k}_1,\Lambda_1,\mu_1;\vec{k}_2,\Lambda_2,\mu_2)\; \mathbb{O}_{\Lambda_1\mu_1}(\vec{k}_1)\; \mathbb{O}_{\Lambda_2\mu_2}(\vec{k}_2)$$

$$\Lambda_1 \in \mathsf{LG}(\vec{k}_1),\ \Lambda_2 \in \mathsf{LG}(\vec{k}_2),\ \Lambda \in \mathsf{LG}(\vec{P})$$

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Sum over all $\vec{k}_1,\ \vec{k}_2$ related by allowed lattice rot such that $\vec{P} = \vec{k}_1 + \vec{k}_2$

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$$R\,\vec{k}_{1,2}\ \forall R \in \mathrm{LG}(\vec{P})$$

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$R\,\vec{k}_{1,2}\ \forall R \in \mathsf{LG}(\vec{P})$

Sum over all $\vec{k}_1,\ \vec{k}_2$ related by allowed lattice rot such that $\vec{P} = \vec{k}_1 + \vec{k}_2$

Why this approach?

Jo Dudek, Robert Edwards, CT, arXiv:1203.6041 [to appear in PRD]

Multi-hadron operators

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$$\Lambda_1 \in \mathrm{LG}(\vec{k}_1),\;\; \Lambda_2 \in \mathrm{LG}(\vec{k}_2),\;\; \Lambda \in \mathrm{LG}(\vec{P})$$

Calculate using induced representation

$$R\,\vec{k}_{1,2}\;\; \forall R \in \mathrm{LG}(\vec{P})$$

Sum over all $\vec{k}_1,\;\vec{k}_2$ related by allowed lattice rot such that $\vec{P} = \vec{k}_1 + \vec{k}_2$

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Jo Dudek, Robert Edwards, CT, arXiv:1203.6041 [to appear in PRD]

Isospin-2 $\pi\pi$

Jo Dudek, Robert Edwards, CT, 1203.6041 [PRD]
[supersedes PR D83, 071504(R) (2011)]

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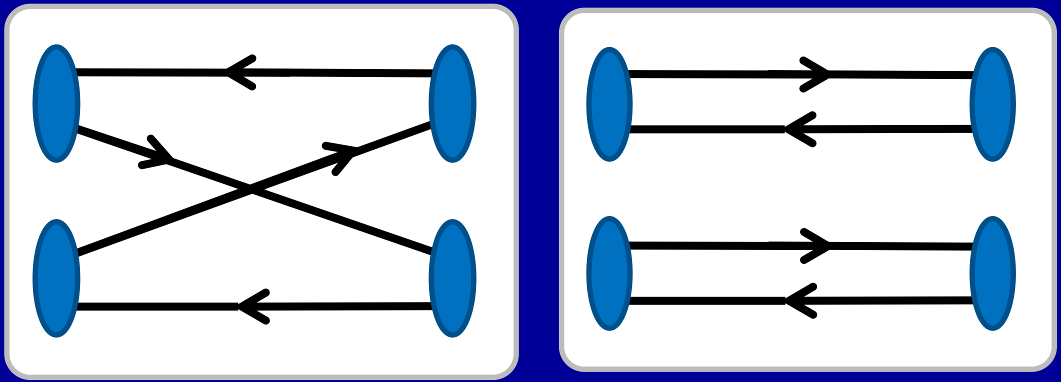

+ similar diagrams

Jo Dudek, Robert Edwards, CT, 1203.6041 [PRD]
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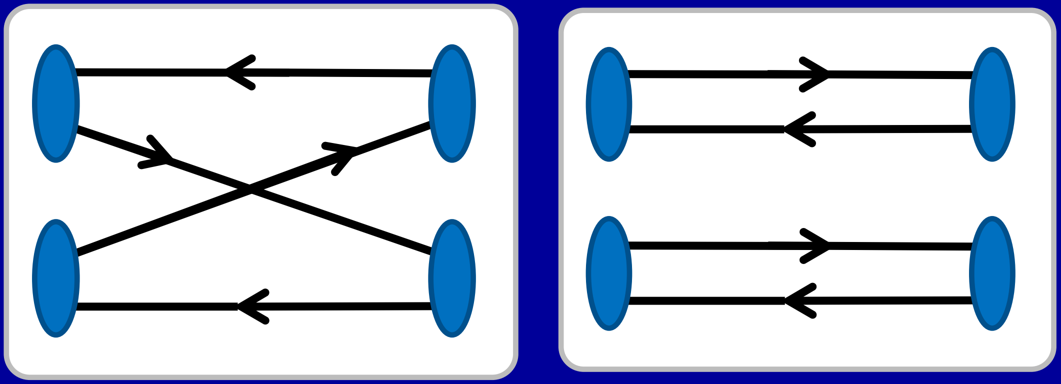
Isospin-2 $\pi\pi$



+ similar diagrams

$$\mathcal{O}(\vec{P}) = \sum_{\vec{p}_1,\, \vec{p}_2} \mathcal{C}_\Lambda(\vec{P}, \vec{p}_1, \vec{p}_2)\ \mathcal{O}_\pi(\vec{p}_1)\ \mathcal{O}_\pi(\vec{p}_2)$$

$$\Lambda_{1,2} = A_1^- \text{ or } A_2$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2$$

$$\vec{P} = [0,0,0],\ [0,0,1],\ [0,1,1],\ [1,1,1]$$

Jo Dudek, Robert Edwards, CT, 1203.6041 [PRD]
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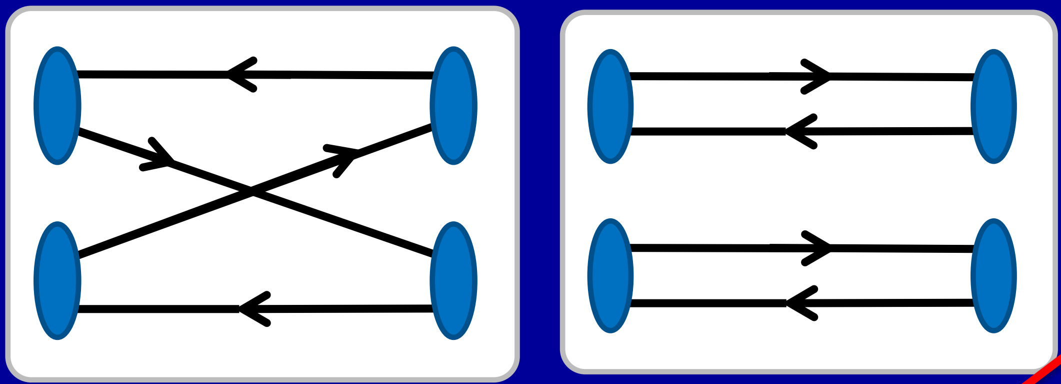

+ similar diagrams

Variationally optimised π ops

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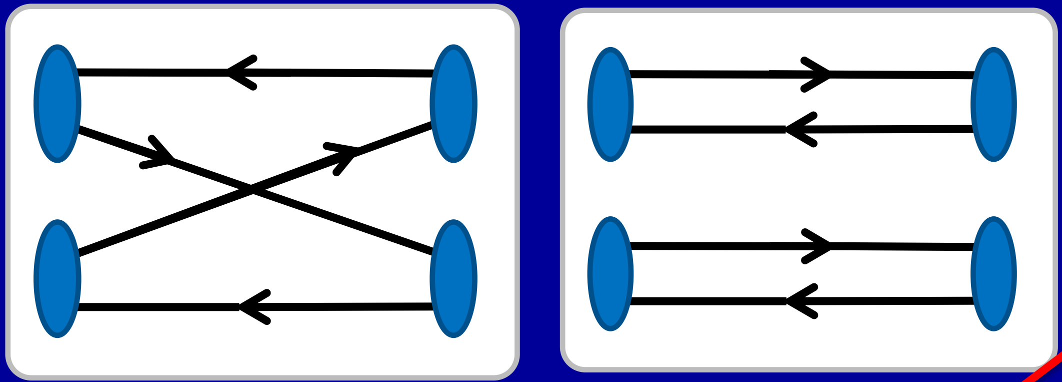

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Dynamical [N_f = 2+1] anisotropic clover

$M_\pi \approx 400$ MeV

Three volumes: 16^3, 20^3, 24^3

($L_s \approx 1.9 – 2.9$ fm, $M_\pi L \sim 4 - 6$)

$a_s \approx 0.12$ fm, finer in t ($a_s/a_t \approx 3.5$)

Jo Dudek, Robert Edwards, CT, 1203.6041 [PRD]
[supersedes PR D83, 071504(R) (2011)]

$\pi\pi$ operators

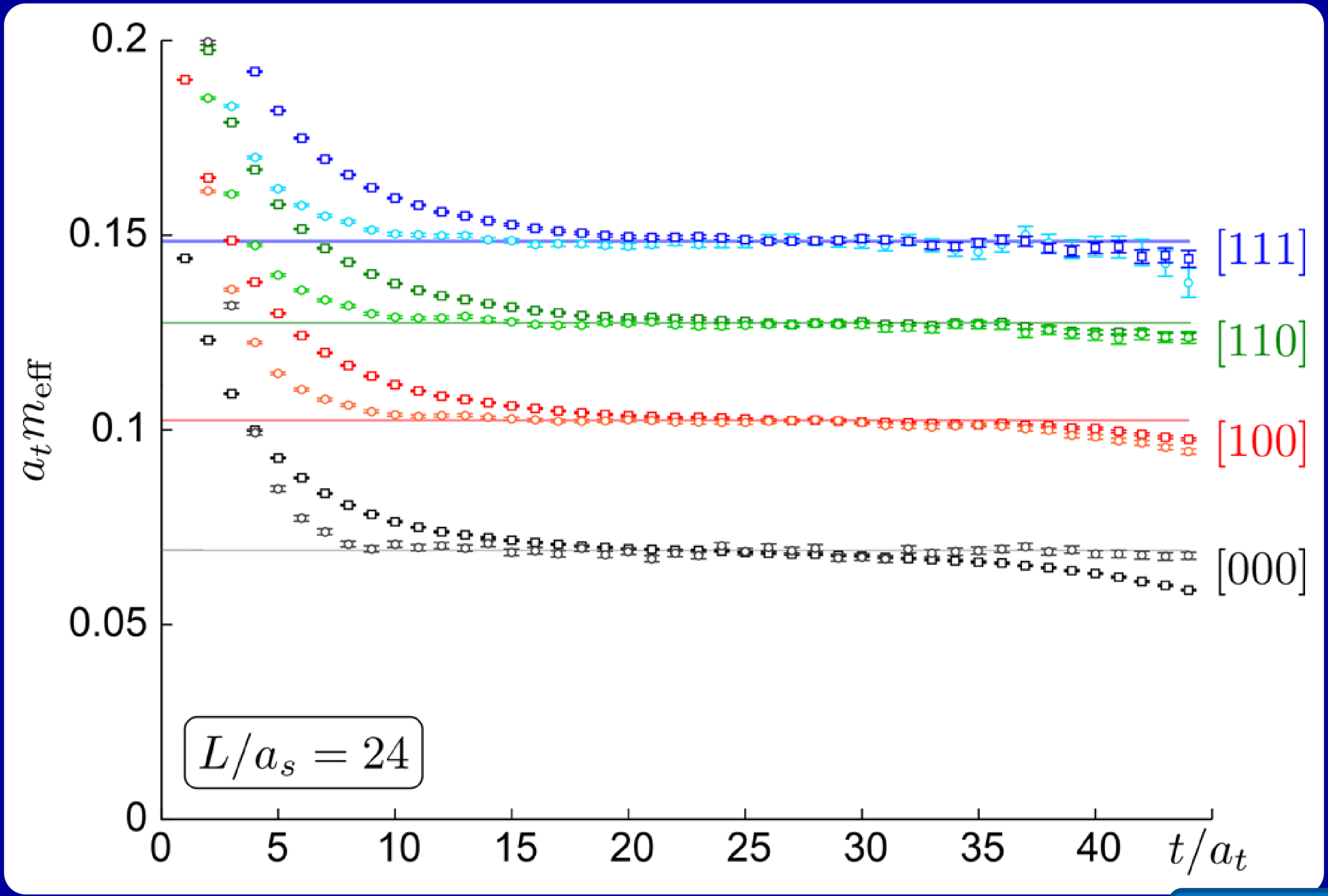
$\vec{P}$	$\vec{k}_1$	$\vec{k}_2$	$\Lambda^{(P)}$
$[0,0,0]$ O_h^D	$[0,0,0]$	$[0,0,0]$	A_1^+
	$[0,0,1]$	$[0,0,-1]$	$A_1^+,\ E^+,\ (T_1^-)$
	$[0,1,1]$	$[0,-1,-1]$	$A_1^+,\ T_2^+,\ E^+,\ (T_1^-,\ T_2^-)$
	$[1,1,1]$	$[-1,-1,-1]$	$A_1^+,\ T_2^+,\ (T_1^-,\ A_2^-)$
	$[0,0,2]$	$[0,0,-2]$	$A_1^+,\ E^+,\ (T_1^-)$
$[0,0,1]$ Dic_4	$[0,0,0]$	$[0,0,1]$	A_1
	$[0,-1,0]$	$[0,1,1]$	$A_1,\ E_2,\ B_1$
	$[-1,-1,0]$	$[1,1,1]$	$A_1,\ E_2,\ B_2$
	$[0,0,-1]$	$[0,0,2]$	A_1
	$[0,-1,-1]$	$[0,1,2]$	$A_1,\ E_2,\ B_1$
	$[-2,0,0]$	$[2,0,1]$	$A_1,\ E_2,\ B_1$
	$[-1,-1,-1]$	$[1,1,2]$	$A_1,\ E_2,\ B_2$
$[0,1,1]$ Dic_2	$[0,0,0]$	$[0,1,1]$	A_1
	$[0,1,0]$	$[0,0,1]$	$A_1,\ (B_1)$
	$[-1,0,0]$	$[1,1,1]$	$A_1,\ B_2$
	$[1,1,0]$	$[-1,0,1]$	$A_1,\ A_2,\ (B_1,\ B_2)$
	$[0,1,-1]$	$[0,0,2]$	$A_1,\ B_1$
	$[0,-1,0]$	$[0,2,1]$	$A_1,\ B_1$
	$[1,-1,1]$	$[-1,2,0]$	$A_1,\ A_2,\ B_1,\ B_2$
	$[1,-1,0]$	$[-1,2,1]$	$A_1,\ A_2,\ B_1,\ B_2$
$[1,1,1]$ Dic_3	$[0,0,0]$	$[1,1,1]$	A_1
	$[1,0,0]$	$[0,1,1]$	$A_1,\ E_2$
	$[2,0,0]$	$[-1,1,1]$	$A_1,\ E_2$
	$[1,-1,0]$	$[0,2,1]$	$A_1,\ A_2,\ 2E_2$
	$[-1,0,0]$	$[2,1,1]$	$A_1,\ E_2$

$\pi\pi$ operators

$\vec{P}$	$\vec{k}_1$	$\vec{k}_2$	$\Lambda^{(P)}$
[0, 0, 0] $\mathrm{O}_h^{\mathrm{D}}$	[0, 0, 0]	[0, 0, 0]	A_1^+
	[0, 0, 1]	[0, 0, −1]	A_1^+, E^+, ~~(T_1^-)~~
	[0, 1, 1]	[0, −1, −1]	A_1^+, T_2^+, E^+, ~~(T_1^-, T_2^-)~~
	[1, 1, 1]	[−1, −1, −1]	A_1^+, T_2^+, ~~(T_1^-, A_2^-)~~
	[0, 0, 2]	[0, 0, −2]	A_1^+, E^+, ~~(T_1^-)~~
[0, 0, 1] Dic_4	[0, 0, 0]	[0, 0, 1]	A_1
	[0, −1, 0]	[0, 1, 1]	A_1, E_2, B_1
	[−1, −1, 0]	[1, 1, 1]	A_1, E_2, B_2
	[0, 0, −1]	[0, 0, 2]	A_1
	[0, −1, −1]	[0, 1, 2]	A_1, E_2, B_1
	[−2, 0, 0]	[2, 0, 1]	A_1, E_2, B_1
	[−1, −1, −1]	[1, 1, 2]	A_1, E_2, B_2
[0, 1, 1] Dic_2	[0, 0, 0]	[0, 1, 1]	A_1
	[0, 1, 0]	[0, 0, 1]	A_1, ~~(B_1)~~
	[−1, 0, 0]	[1, 1, 1]	A_1, B_2
	[1, 1, 0]	[−1, 0, 1]	A_1, A_2, ~~(B_1, B_2)~~
	[0, 1, −1]	[0, 0, 2]	A_1, B_1
	[0, −1, 0]	[0, 2, 1]	A_1, B_1
	[1, −1, 1]	[−1, 2, 0]	A_1, A_2, B_1, B_2
	[1, −1, 0]	[−1, 2, 1]	A_1, A_2, B_1, B_2
[1, 1, 1] Dic_3	[0, 0, 0]	[1, 1, 1]	A_1
	[1, 0, 0]	[0, 1, 1]	A_1, ~~E_2~~
	[2, 0, 0]	[−1, 1, 1]	A_1, ~~E_2~~
	[1, −1, 0]	[0, 2, 1]	A_1, ~~A_2, $2E_2$~~
	[−1, 0, 0]	[2, 1, 1]	A_1, ~~E_2~~

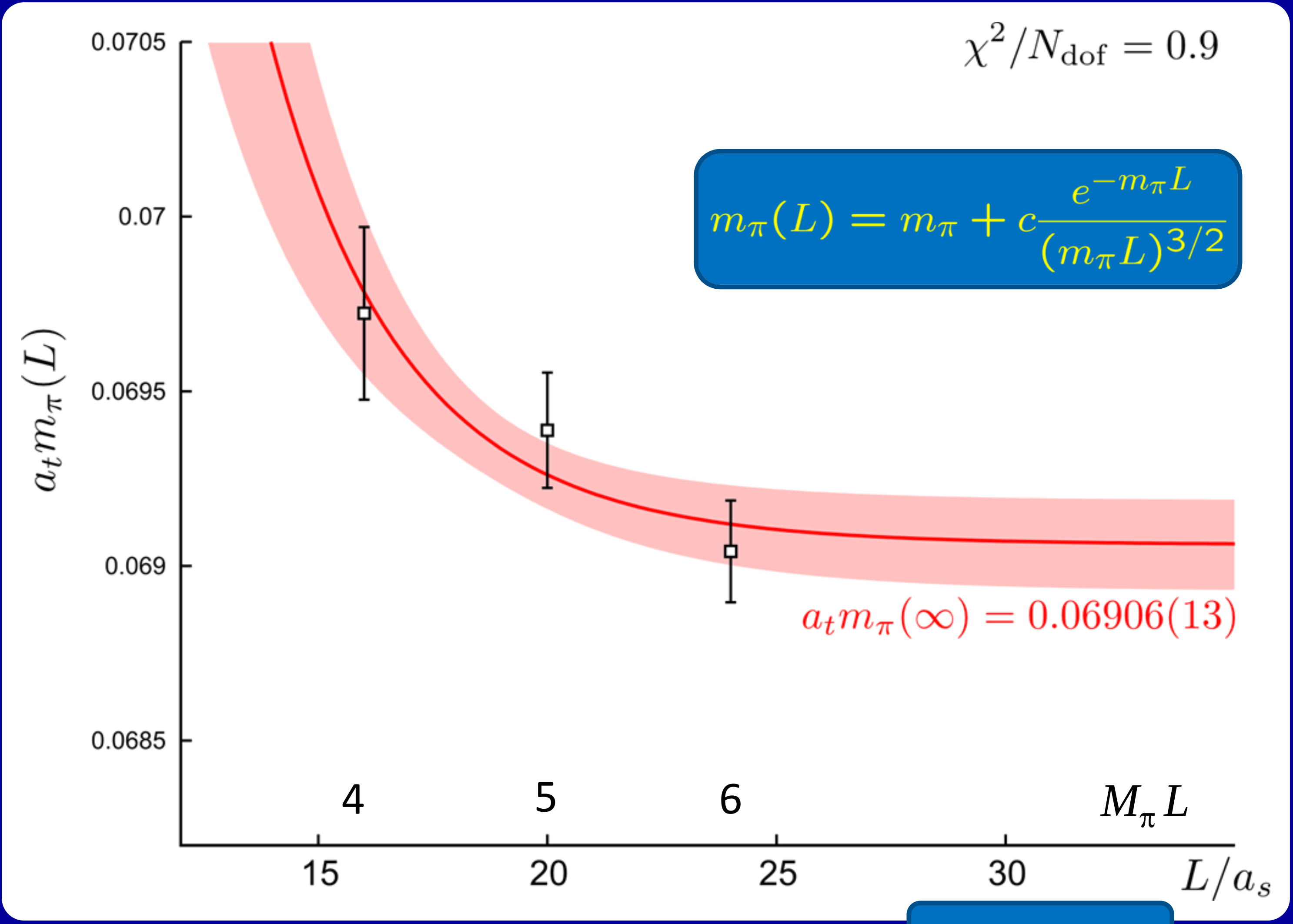
$\vec{P}$	$\Lambda^{(P)}$	16^3	$20^3, 24^3$
[0, 0, 0]	A_1^+	5	5
	E^+	3	3
	T_2^+	2	2
[0, 0, 1]	A_1	4	7
	E_2	2	5
	B_1	1	3
	B_2	1	2
[0, 1, 1]	A_1	5	8
	A_2	1	3
	B_1	1	4
	B_2	1	3
[1, 1, 1]	A_1	3	5

Optimised single-pion ops



$M_\pi \approx 400$ MeV

Pion mass volume dependence



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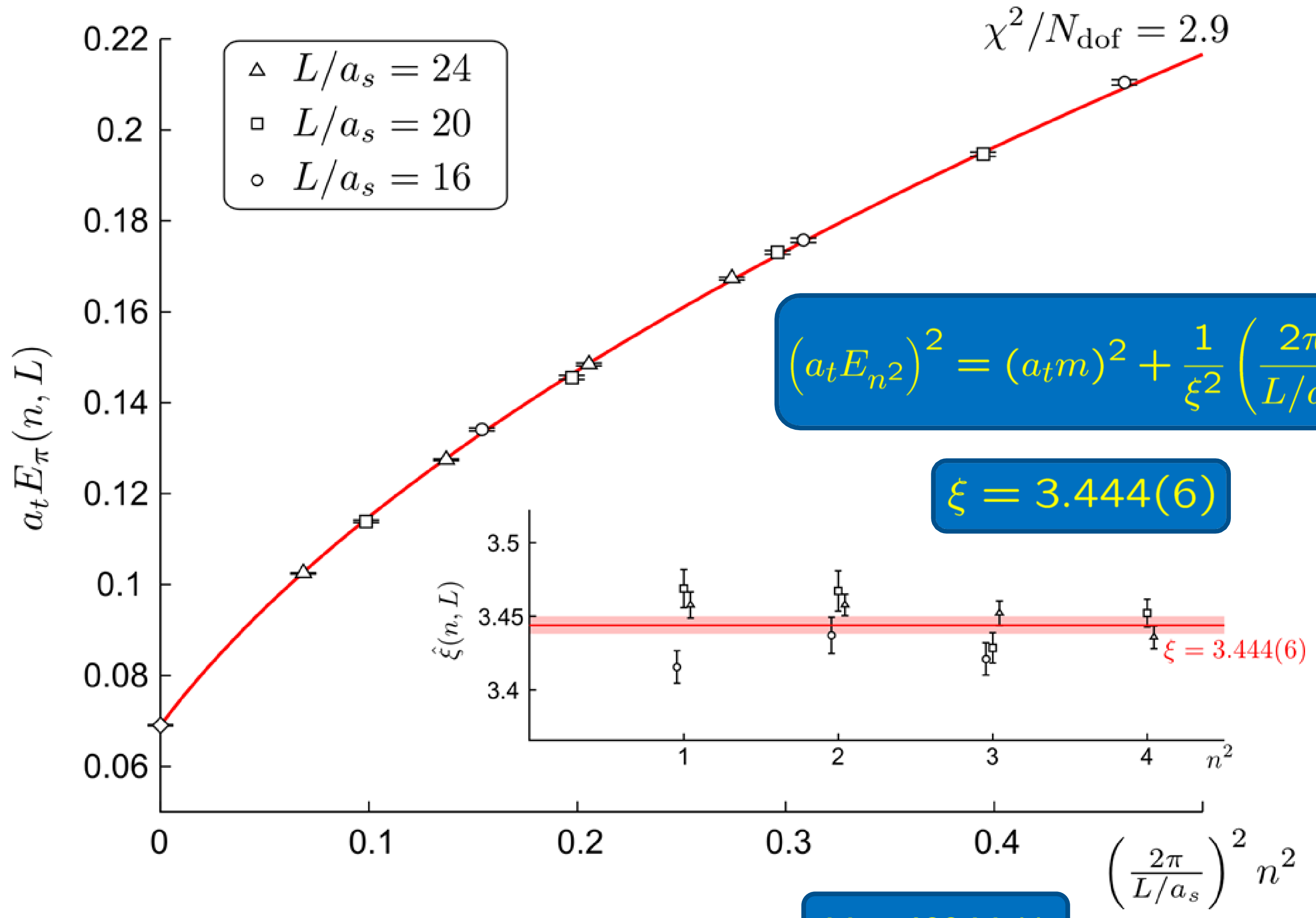
Pion dispersion relation

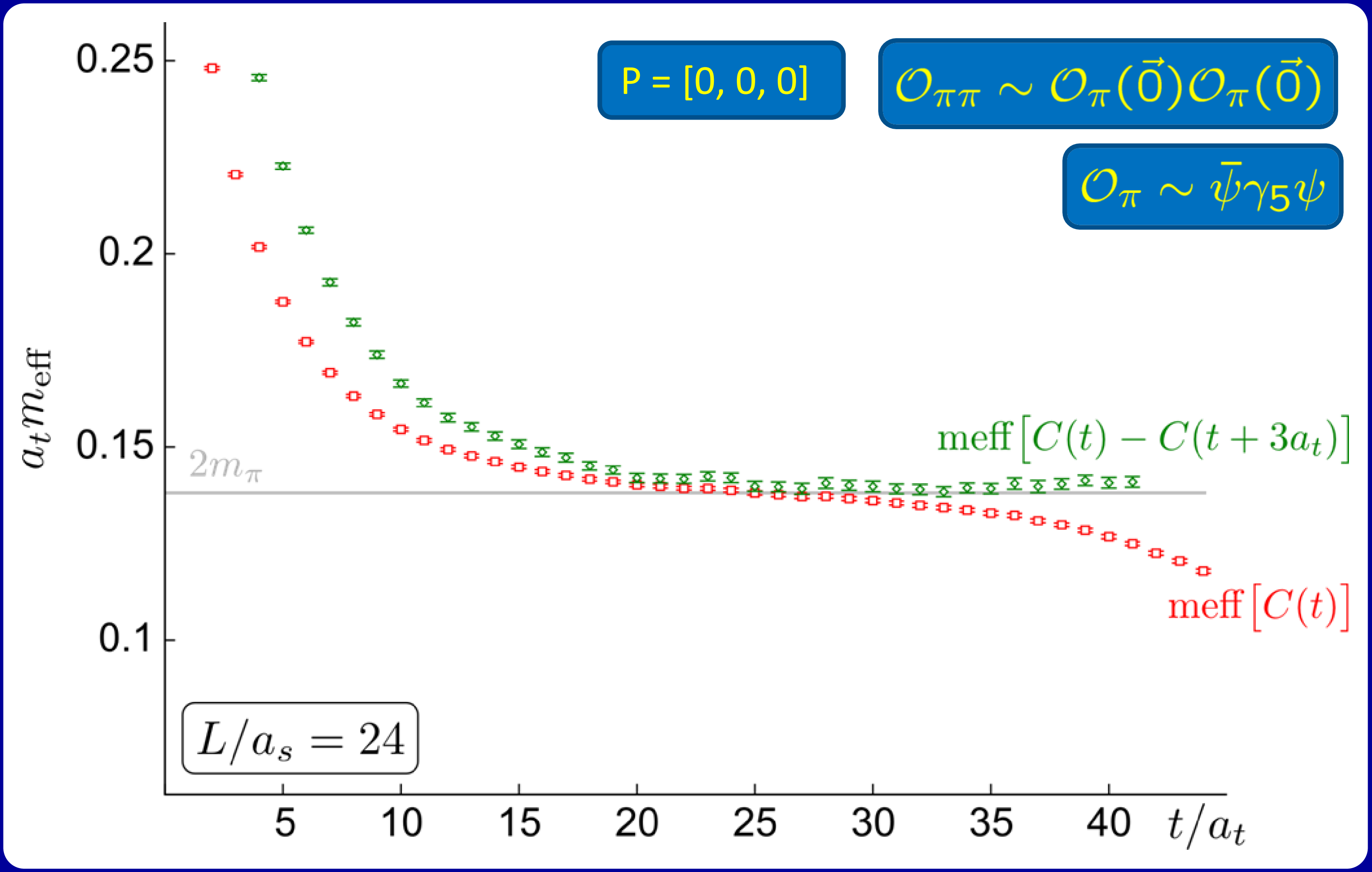


$$\left(a_t E_{n^2}\right)^2 = (a_t m)^2 + \frac{1}{\xi^2}\left(\frac{2\pi}{L/a_s}\right)^2 n^2$$

$\xi = 3.444(6)$

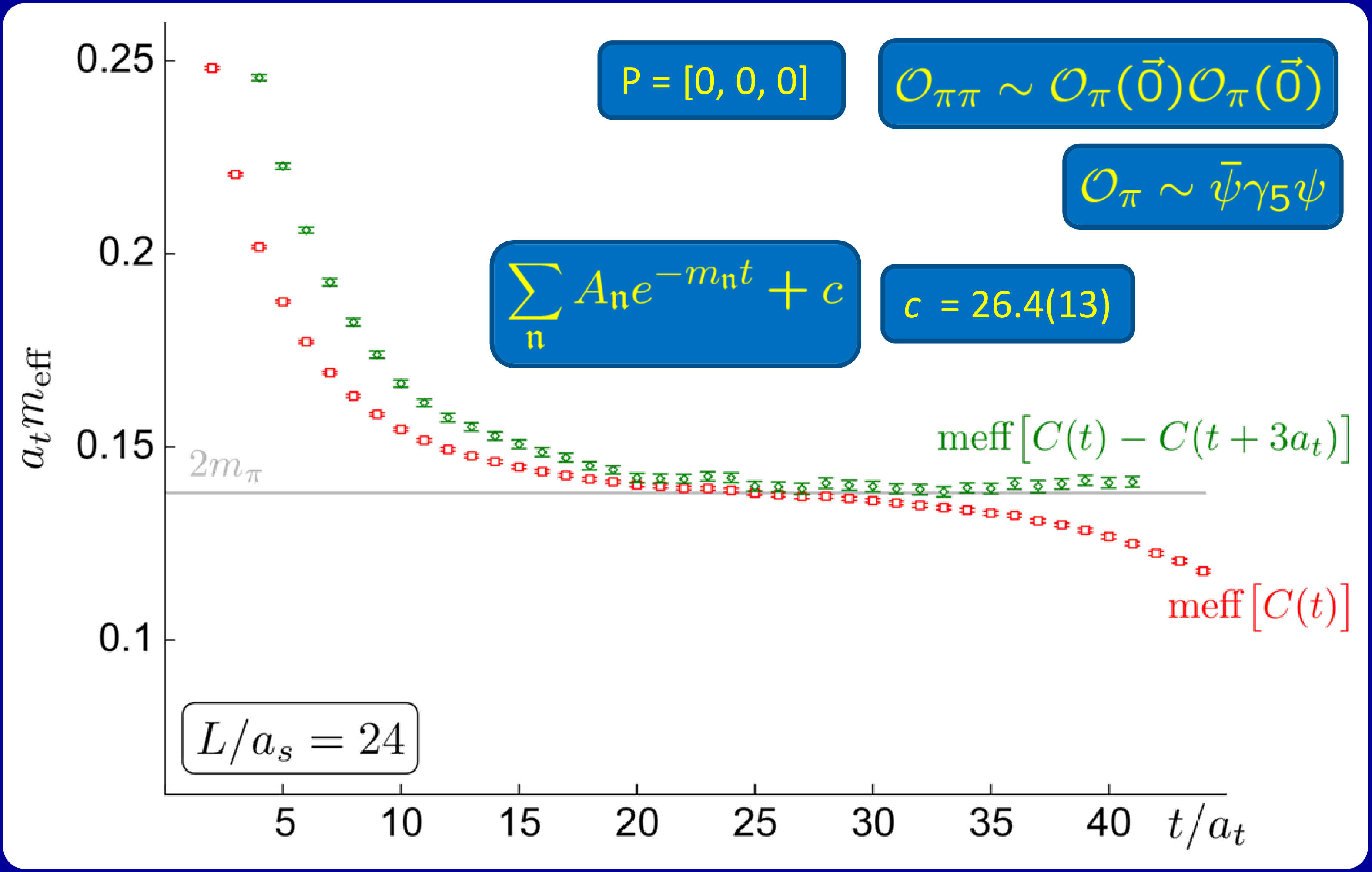
$M_\pi \approx 400$ MeV

$\pi\pi$ at finite T



24^3, $M_\pi \approx 400$ MeV

$\pi\pi$ at finite T



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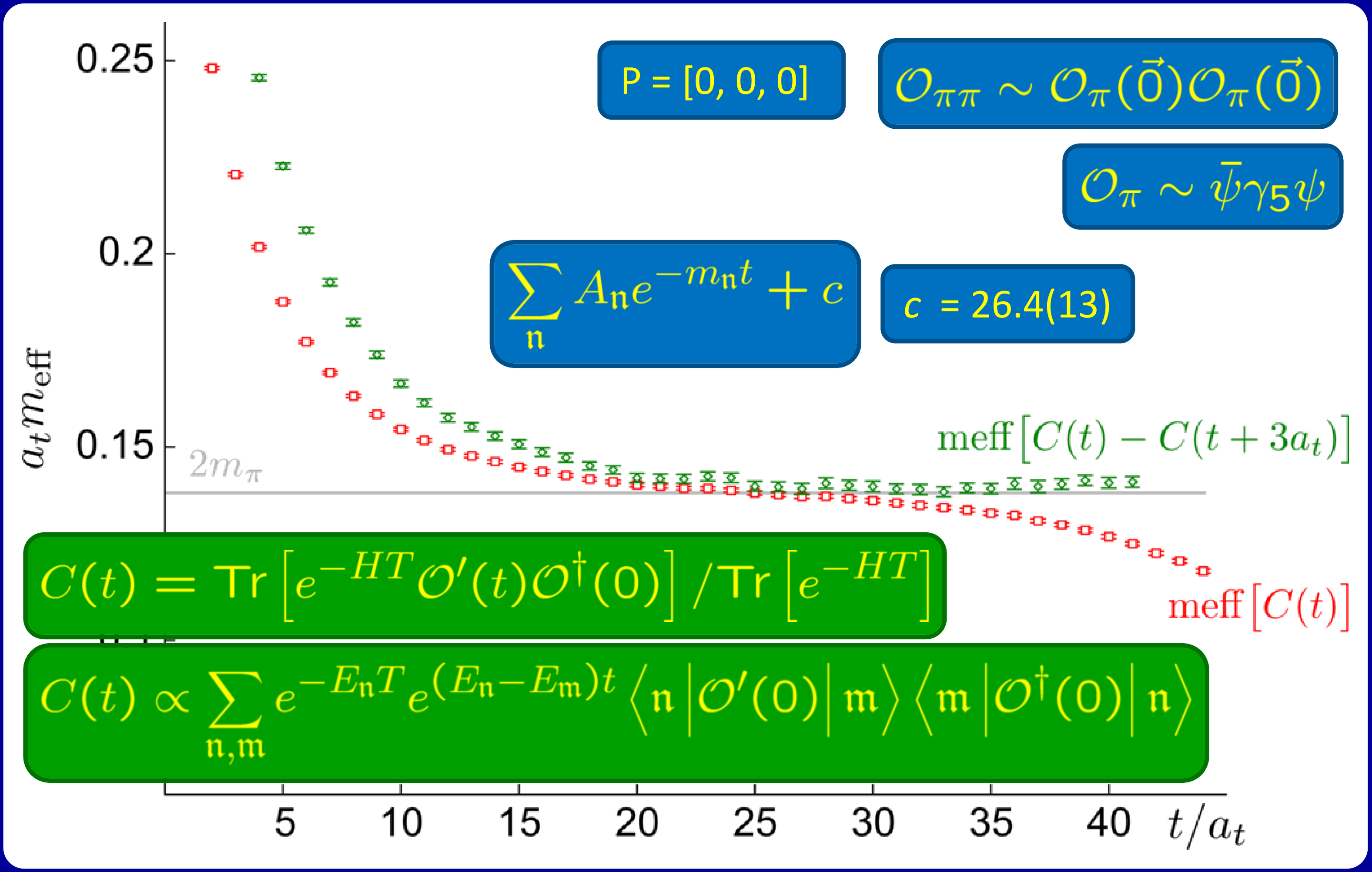
$\pi\pi$ at finite T



P = [0, 0, 0]

$\mathcal{O}_{\pi\pi} \sim \mathcal{O}_\pi(\vec{0})\mathcal{O}_\pi(\vec{0})$

$\mathcal{O}_\pi \sim \bar{\psi}\gamma_5\psi$

$\sum_{\mathfrak{n}} A_{\mathfrak{n}} e^{-m_{\mathfrak{n}} t} + c$

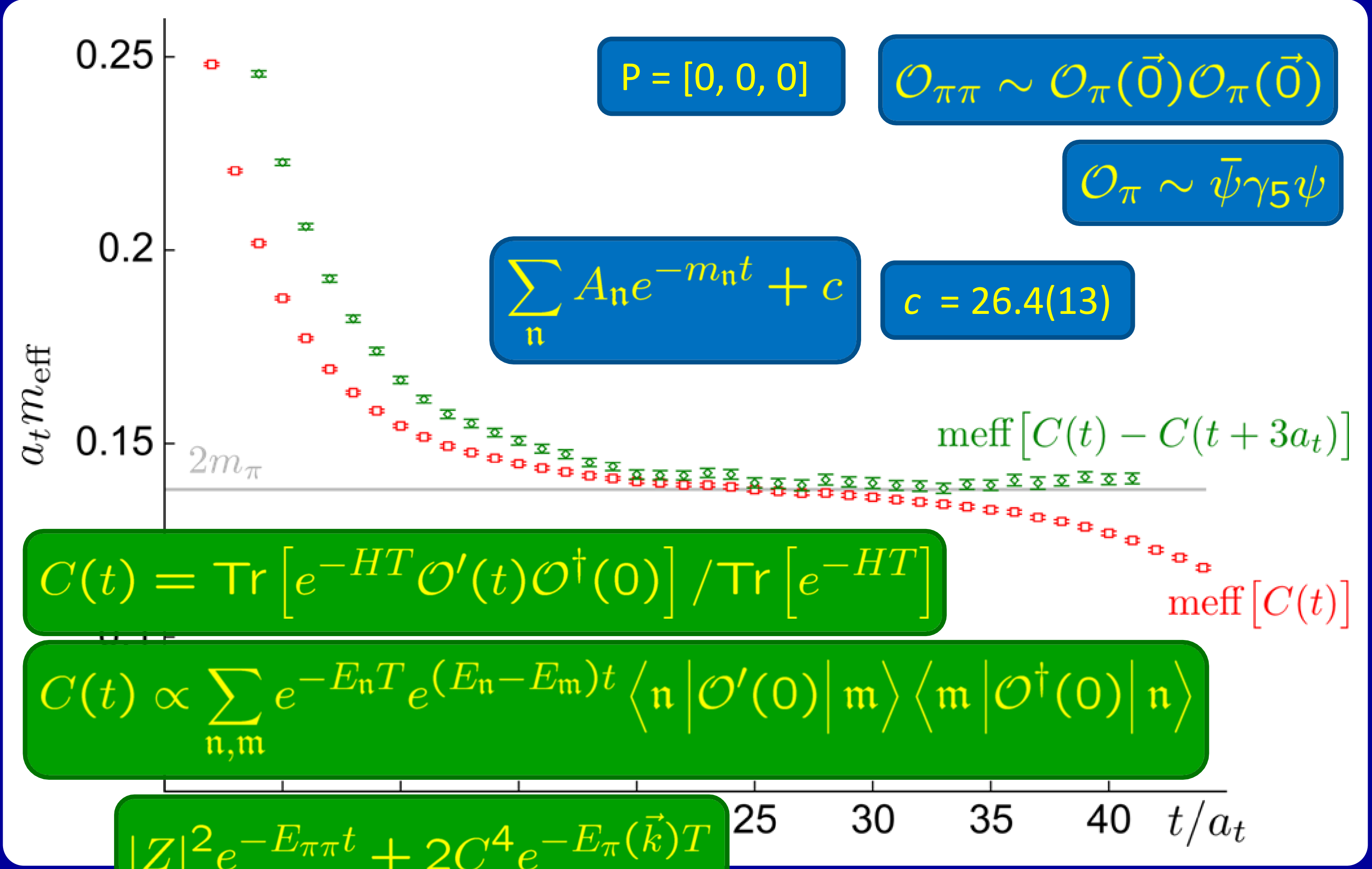
c = 26.4(13)

$$C(t) = \mathrm{Tr}\left[e^{-HT}\mathcal{O}'(t)\mathcal{O}^\dagger(0)\right] / \mathrm{Tr}\left[e^{-HT}\right]$$

$$C(t) \propto \sum_{\mathfrak{n},\mathfrak{m}} e^{-E_{\mathfrak{n}}T} e^{(E_{\mathfrak{n}}-E_{\mathfrak{m}})t} \left\langle \mathfrak{n} \middle| \mathcal{O}'(0) \middle| \mathfrak{m} \right\rangle \left\langle \mathfrak{m} \middle| \mathcal{O}^\dagger(0) \middle| \mathfrak{n} \right\rangle$$

24^3, $M_\pi \approx 400$ MeV

$\pi\pi$ at finite T



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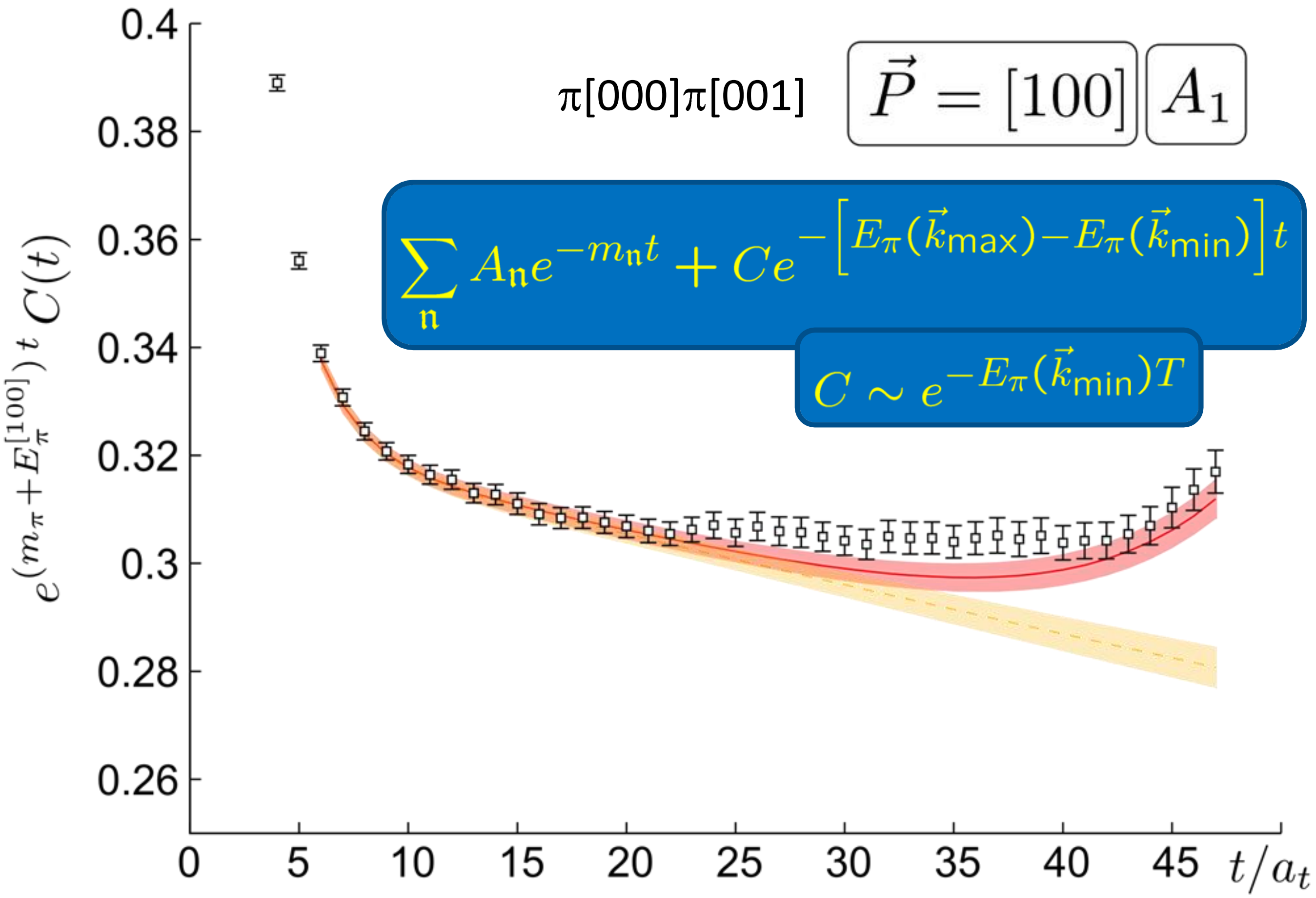
$$|Z|^2 e^{-E_{\pi\pi}t} + 2C^4 e^{-E_\pi(\vec{k})T}$$

24^3, $M_\pi \approx 400$ MeV

$\pi\pi$ at finite T



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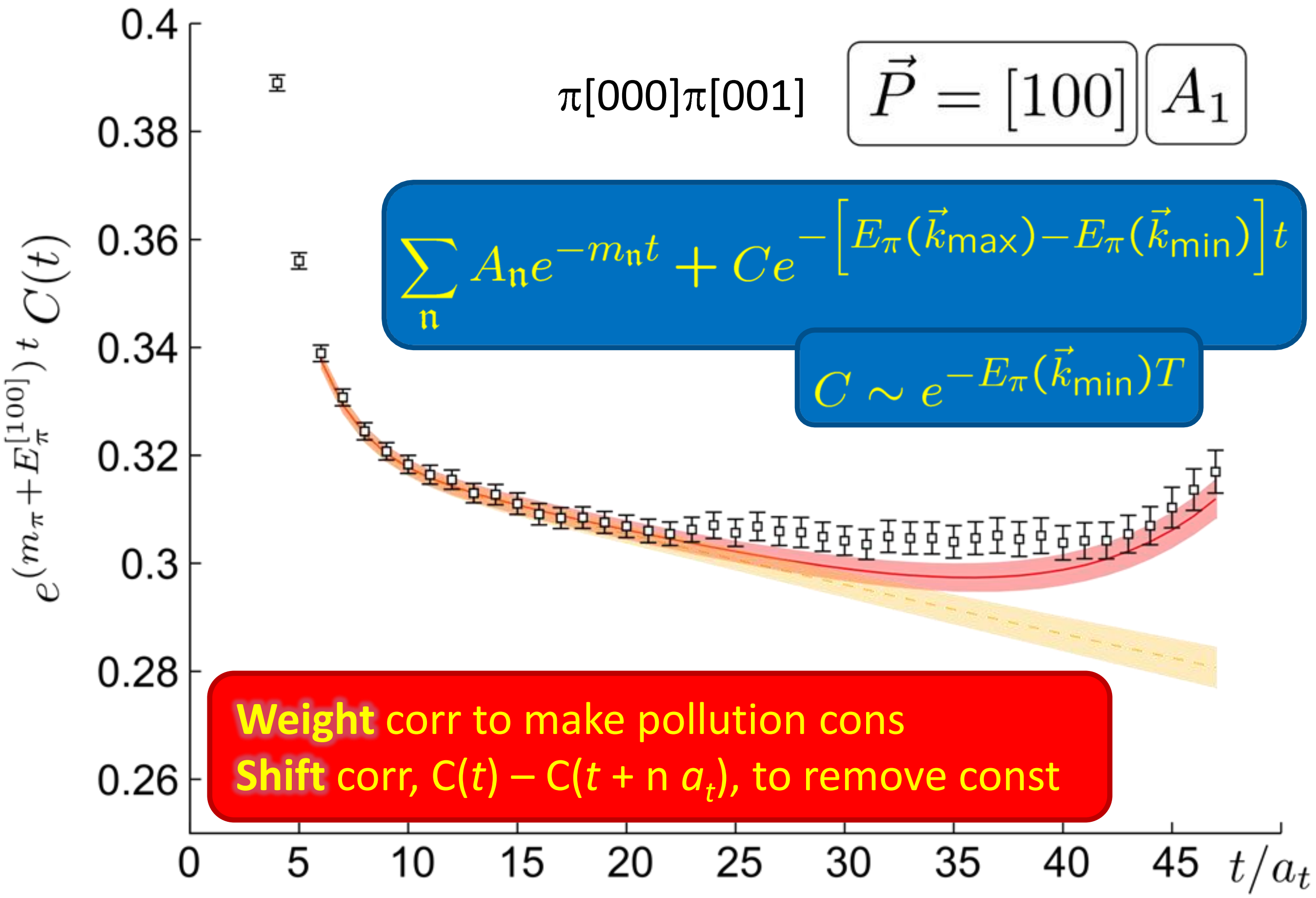
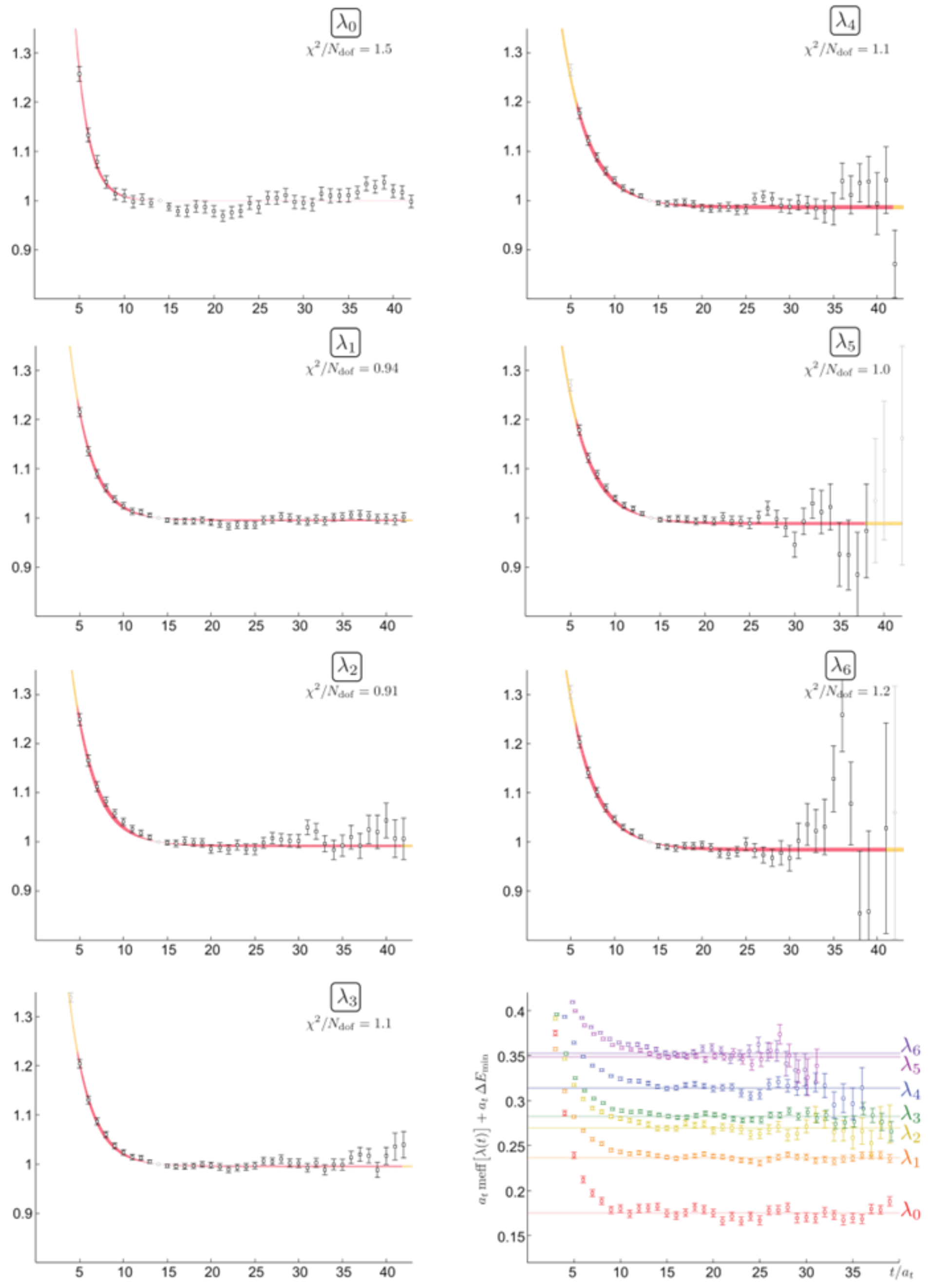


Weight corr to make pollution cons

Shift corr, C(t) – C(t + n a_t), to remove const

24^3, $M_\pi \approx 400$ MeV



Principal Correlators

$$\lambda(t) \cdot e^{m(t-t_0)}$$

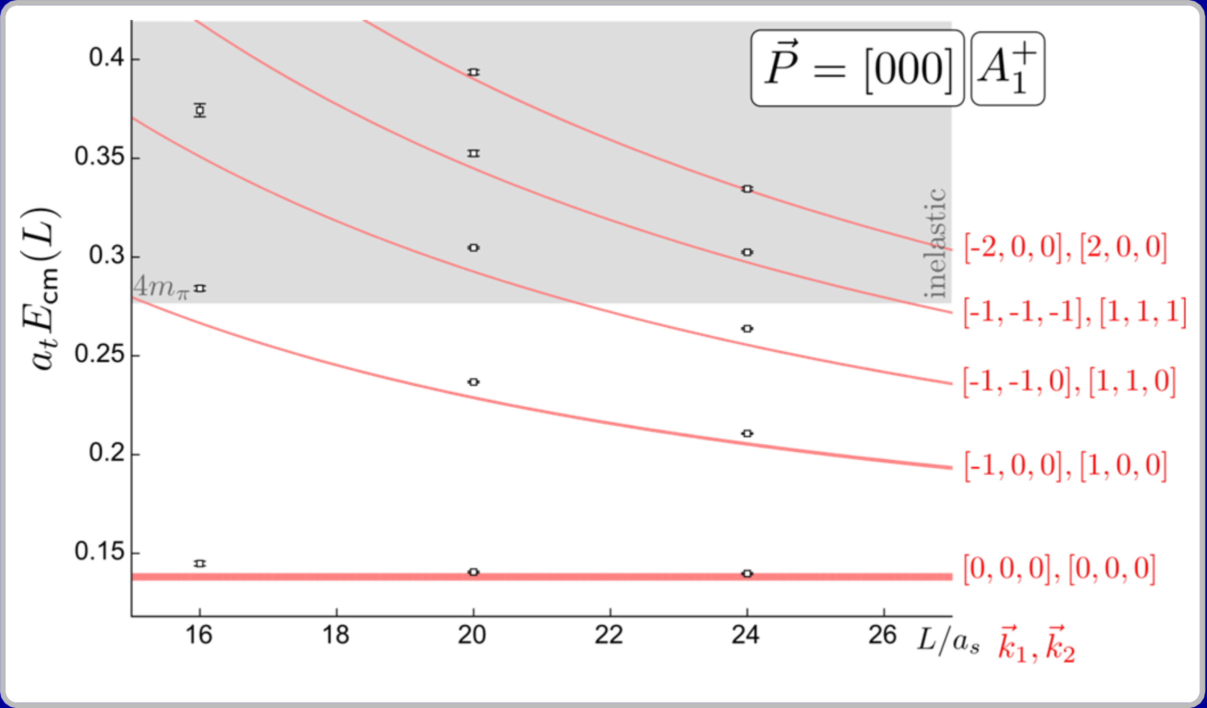
From variational method with weighted-shifted correlators

P = [0, 0, 1] A_1

24^3, $M_\pi \approx 400$ MeV

ππ I=2 spectra

arXiv:1203.6041 [PRD]



$M_\pi \approx 400$ MeV

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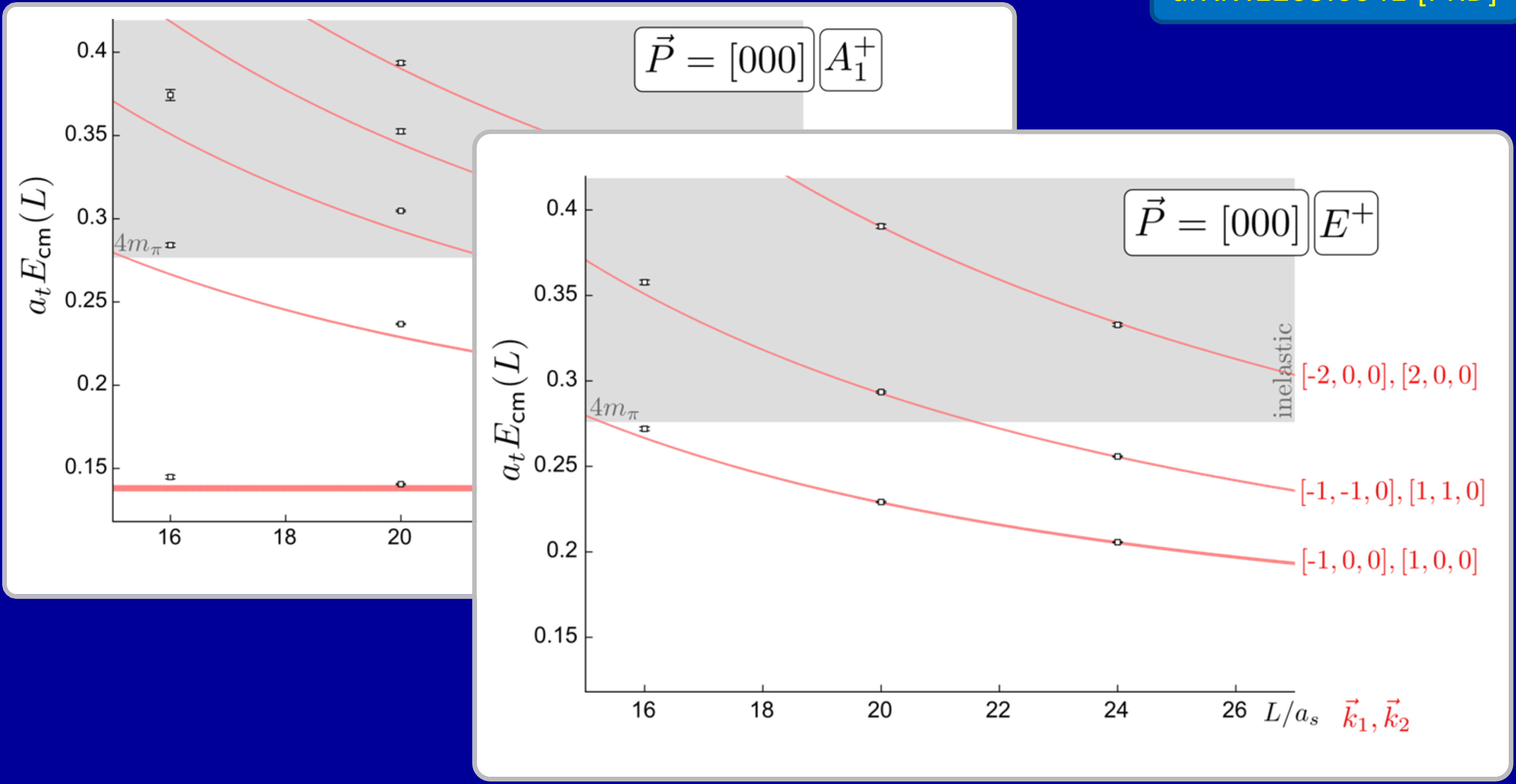

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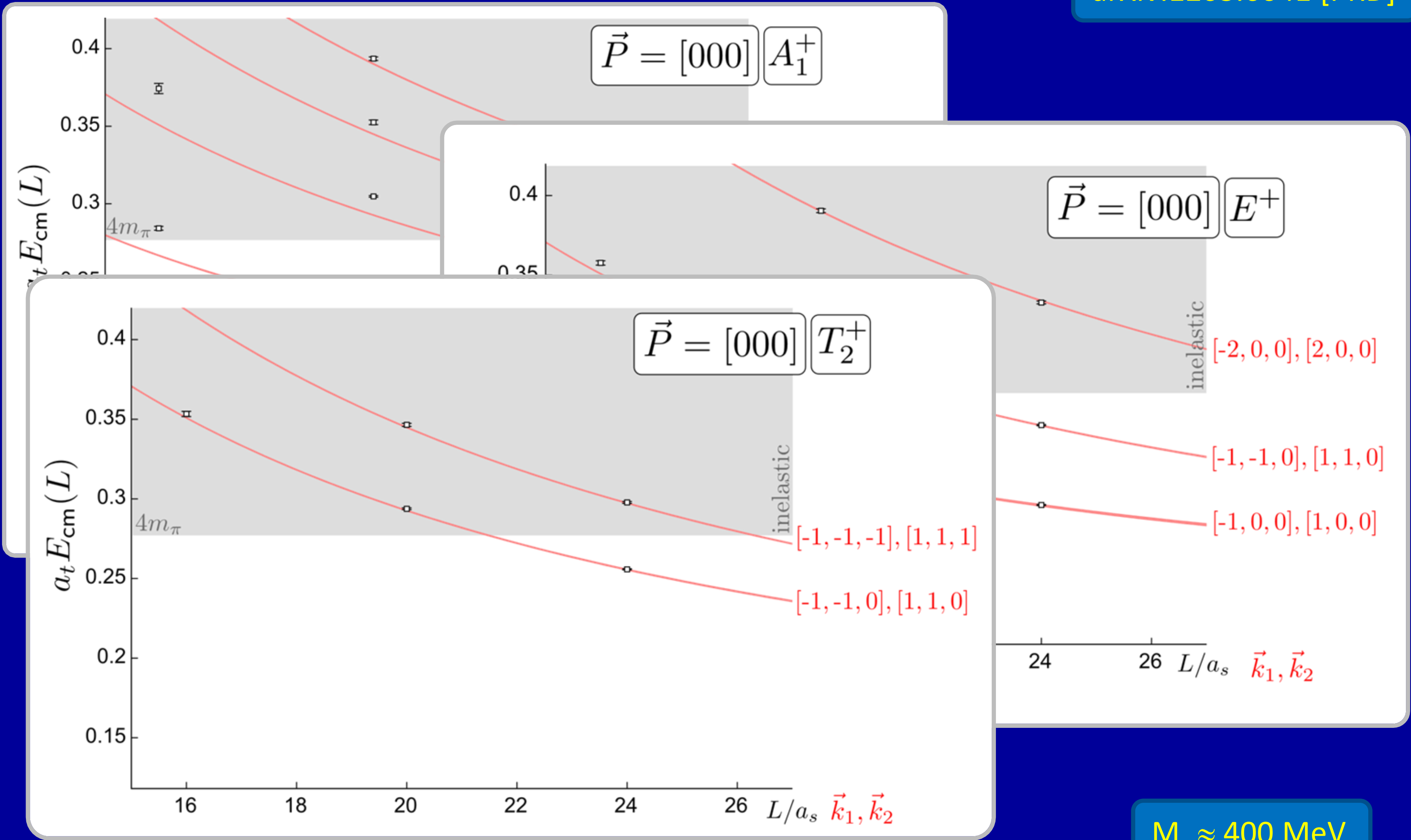
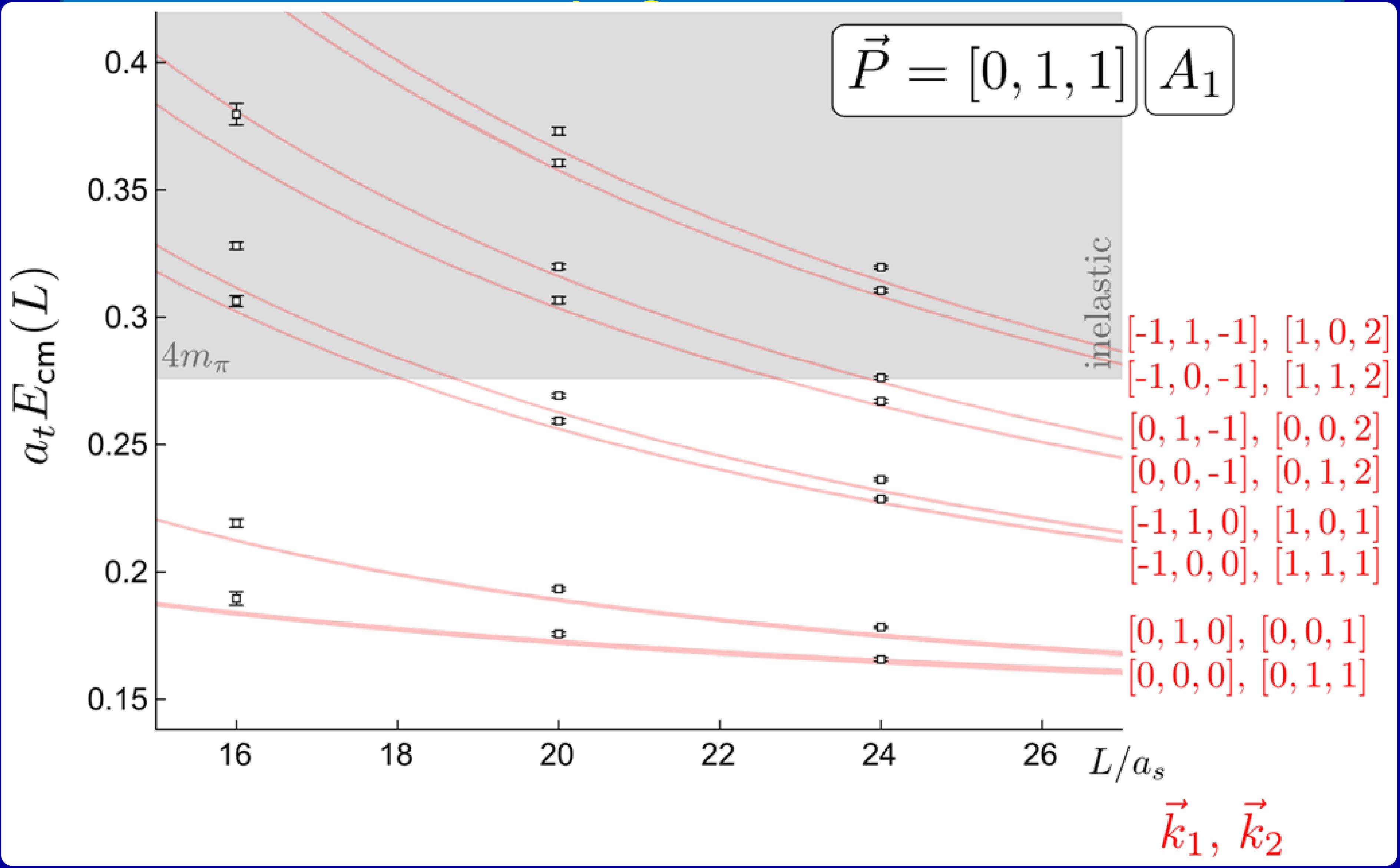
arXiv:1203.6041 [PRD]



$M_\pi \approx 400$ MeV



$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

Lüscher method

(Elastic) energy shifts in finite volume → phase shift

Lüscher, Nucl. Phys. B354, 531 (1991); Rummukainen & Gottlieb, Nucl. Phys. B450, 397 (1995)
Leskovec & Prelovsek, PR D85, 114507 (2012); many others...

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$$\det\left[e^{2i\delta(p_{\rm cm})} - \mathbf{U}^{(\vec{P},\Gamma)}\left(p_{\rm cm}\frac{L}{2\pi}\right)\right] = 0$$

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Matrices in space of l contributing to irrep Γ

e.g. $\vec{P} = \vec{0}$ $\Gamma = A_1^+$ $\delta = {\rm diag}(\delta_0, \delta_4, \ldots)$

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e.g. $\vec{P} = \vec{0}$ $\Gamma = A_1^+$ $\delta = \mathrm{diag}(\delta_0, \delta_4, \ldots)$

$$q^2 = \left(\frac{p_{\rm cm}L}{2\pi}\right)^2$$

$$M^{\vec{P}}_{\ell m,\ell' m'}(q^2) = \gamma^{-1}\frac{(-1)^\ell}{\pi^{3/2}}\sum_{j=|\ell-\ell'|}^{\ell+\ell'}\sum_{s=-j}^{j}\frac{i^j}{q^{j+1}}Z^{\vec{P}}_{js}(1;q^2)C_{\ell m,js,\ell' m'}$$

$$Z^{\vec{P}}_{jm}(s,q^2) = \sum_{\vec{r}\in P_{\rm d}} r^l Y_{lm}(\hat{r})(r^2-q^2)^{-s}$$

Lüscher method

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$$Z^{\vec{P}}_{jm}(s,q^2) = \sum_{\vec{r}\in P_{\rm d}} r^l Y_{lm}(\hat{r})(r^2-q^2)^{-s}$$

$$\mathbf{U} = (\mathbf{M}+i\mathbf{1})(\mathbf{M}-i\mathbf{1})^{-1}$$

Lüscher method symmetries

$$Z_{jm}^{\vec{P}}(s,q^2) = \sum_{\vec{r}\in P_{\mathrm{d}}} r^l Y_{lm}(\hat{r})(r^2-q^2)^{-s}$$

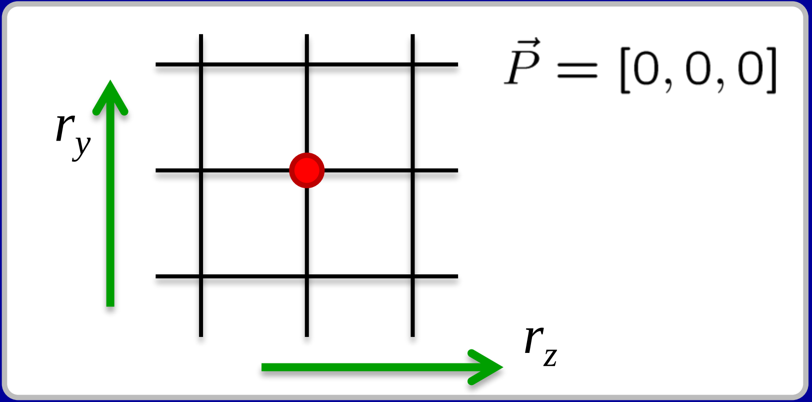
Lüscher method symmetries

$$Z^{\vec{P}}_{jm}(s,q^2) = \sum_{\vec{r}\in P_{\mathrm{d}}} r^l Y_{lm}(\hat{r})(r^2 - q^2)^{-s}$$

$\vec{P} = \vec{0}$

$P_{\mathrm{d}} = \left\{\vec{r} \in \mathbb{Z}^3\right\}$

O_h^{D}



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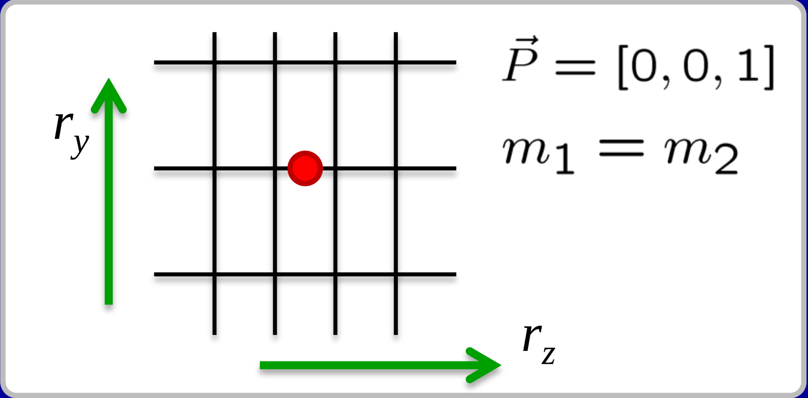

$\vec{P} = \vec{0}$

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O_h^{D}

$\vec{P} \neq \vec{0},\ m_1 = m_2$

$P_{\mathsf{d}} = \left\{\vec{r} \in \mathbb{R}^3 | \vec{r} = \vec{\gamma}^{-1}(\vec{n} - \vec{P}/2)\right\}$

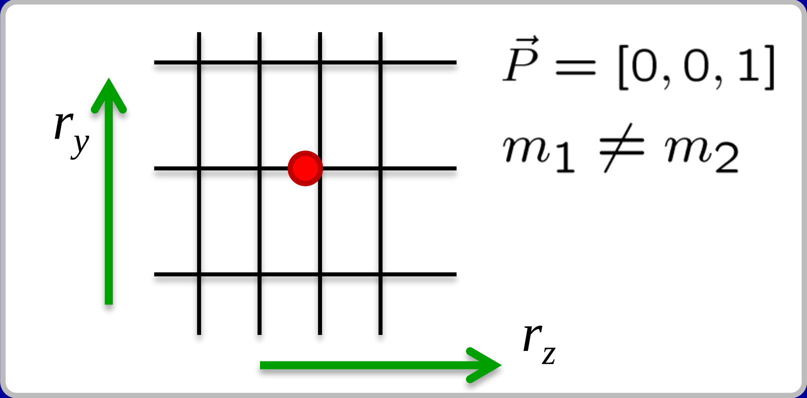
$n \in \mathbb{Z}^3$

$\left\{R \in O_h^{\mathsf{D}} | R\vec{P} = \pm\vec{P}\right\}$

LG($\vec{P}$) with extra ‘parity’

Lüscher method symmetries

$$Z_{jm}^{\vec{P}}(s,q^2) = \sum_{\vec{r}\in P_d} r^l Y_{lm}(\hat{r})(r^2-q^2)^{-s}$$



$\vec{P} = \vec{0}$ | $P_d = \left\{\vec{r} \in \mathbb{Z}^3\right\}$ | O_h^D

$\vec{P} \neq \vec{0},\ m_1 = m_2$ | $P_d = \left\{\vec{r} \in \mathbb{R}^3 | \vec{r} = \vec{\gamma}^{-1}(\vec{n} - \vec{P}/2)\right\}$ | $n \in \mathbb{Z}^3$

$\left\{R \in O_h^D | R\vec{P} = \pm\vec{P}\right\}$ | $\mathrm{LG}(\vec{P})$ with extra ‘parity’

$\vec{P} \neq \vec{0},\ m_1 \neq m_2$ | $P_d = \left\{\vec{r} \in \mathbb{R}^3 | \vec{r} = \vec{\gamma}^{-1}(\vec{n} - A\vec{P}/2)\right\}$

$\left\{R \in O_h^D | R\vec{P} = \vec{P}\right\}$ | $\mathrm{LG}(\vec{P})$

$$A = 1 + \frac{m_1^2 - m_2^2}{E_{cm}^2}$$

Lüscher Method

$$M^{\vec{P}}_{\ell m,\ell' m'}(q^2) = \gamma^{-1}\frac{(-1)^\ell}{\pi^{3/2}}\sum_{j=|\ell-\ell'|}^{\ell+\ell'}\sum_{s=-j}^{j}\frac{i^j}{q^{j+1}}Z^{\vec{P}}_{js}(1;q^2)C_{\ell m,js,\ell' m'}$$

$$\det\left[e^{2i\delta(p_{\rm cm})} - \mathbf{U}^{(\vec{P},\Gamma)}\left(p_{\rm cm}\frac{L}{2\pi}\right)\right] = 0$$

Matrices in space of l contributing to irrep Γ

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Block diagonalise

$$M^{(\vec{P},\Lambda,\mu)}_{\ell n;\ell' n'}(q^2)\;\delta_{\Lambda,\Lambda'}\delta_{\mu,\mu'} = \sum_{\substack{\hat{\lambda}=\pm|\lambda| \\ m=-\ell\ldots\ell}}\;\sum_{\substack{\hat{\lambda}'=\pm|\lambda'| \\ m'=-\ell'\ldots\ell'}} \mathcal{S}^{\tilde{\eta},\lambda *}_{\vec{P},\Lambda,\mu}\, D^{(\ell)*}_{m\lambda}(R)\cdot M^{(\vec{P})}_{\ell m;\ell' m'}(q^2)\cdot\mathcal{S}^{\tilde{\eta},\lambda'}_{\vec{P},\Lambda,\mu'}\, D^{(\ell')}_{m'\lambda'}(R)$$

$$\det\left[e^{2i\delta(p_{\mathrm{cm}})} - \mathbf{U}^{(\vec{P},\Gamma)}\left(p_{\mathrm{cm}}\frac{L}{2\pi}\right)\right] = 0$$

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$$M^{\vec{P}}_{\ell m,\ell' m'}(q^2) = \gamma^{-1}\frac{(-1)^\ell}{\pi^{3/2}} \sum_{j=|\ell-\ell'|}^{\ell+\ell'} \sum_{s=-j}^{j} \frac{i^j}{q^{j+1}} Z^{\vec{P}}_{js}(1;q^2) C_{\ell m,js,\ell' m'}$$

Block diagonalise

J_z = m → λ → Λ(μ)

or J → Λ(μ) at zero mom.

$$M^{(\vec{P},\Lambda,\mu)}_{\ell n;\ell' n'}(q^2)\,\delta_{\Lambda,\Lambda'}\delta_{\mu,\mu'} = \sum_{\substack{\hat{\lambda}=\pm|\lambda| \\ m=-\ell\ldots\ell}} \sum_{\substack{\hat{\lambda}'=\pm|\lambda'| \\ m'=-\ell'\ldots\ell'}} \mathcal{S}^{\tilde{\eta},\lambda *}_{\vec{P},\Lambda,\mu} D^{(\ell)*}_{m\lambda}(R)\, M^{(\vec{P})}_{\ell m;\ell' m'}(q^2)\, \mathcal{S}^{\tilde{\eta},\lambda'}_{\vec{P},\Lambda,\mu'} D^{(\ell')}_{m'\lambda'}(R)$$

$$\det\left[e^{2i\delta(p_{\rm cm})} - \mathrm{U}^{(\vec{P},\Gamma)}\left(p_{\rm cm}\frac{L}{2\pi}\right)\right] = 0$$

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$$M^{\vec{P}}_{\ell m,\ell' m'}(q^2) = \gamma^{-1}\frac{(-1)^\ell}{\pi^{3/2}}\sum_{j=|\ell-\ell'|}^{\ell+\ell'}\sum_{s=-j}^{j}\frac{i^j}{q^{j+1}}Z^{\vec{P}}_{js}(1;q^2)C_{\ell m,js,\ell' m'}$$

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$$\det\left[e^{2i\delta(p_{\mathrm{cm}})} - \mathrm{U}^{(\vec{P},\Gamma)}\left(p_{\mathrm{cm}}\frac{L}{2\pi}\right)\right] = 0$$

Matrices in space of
l contributing to irrep Γ

Lüscher Method

ππ isospin-2

$$M^{\vec{P}}_{\ell m,\ell' m'}(q^2) = \gamma^{-1}\frac{(-1)^{\ell}}{\pi^{3/2}} \dots$$

Block diagonalise

$J_z = \dots$

$$M^{(\vec{P},\Lambda,\mu)}_{\ell n;\ell' n'}(q^2)\,\delta_{\Lambda,\Lambda'}\delta_{\mu,\mu'} = \sum_{\substack{\hat{\lambda}=\pm|\lambda| \\ m=-\ell\dots\ell}} \dots (R)$$

$$\det\left[e^{2i\delta(p_{\text{cm}})} - \mathrm{U}^{(\vec{P},\Gamma)}\left(p_{\text{cm}} \dots\right.\right.$$

$\vec{P}$	$LG(\vec{P})$	$\Lambda^{(P)}$	$\pi\pi\ \ell^N$
$[0,0,0]$	$\mathrm{O}_h^{\mathrm{D}}$	A_1^+	$0^1,\ 4^1$
		T_1^+	4^1
		T_2^+	$2^1,\ 4^1$
		E^+	$2^1,\ 4^1$
$[n,0,0]$	Dic_4	A_1	$0^1,\ 2^1,\ 4^2$
		A_2	4^1
		E_2	$2^1,\ 4^2$
		B_1	$2^1,\ 4^1$
		B_2	$2^1,\ 4^1$
$[n,n,0]$	Dic_2	A_1	$0^1,\ 2^2,\ 4^3$
		A_2	$2^1,\ 4^2$
		B_1	$2^1,\ 4^2$
		B_2	$2^1,\ 4^2$
$[n,n,n]$	Dic_3	A_1	$0^1,\ 2^1,\ 4^2$
		A_2	4^1
		E_2	$2^2,\ 4^3$
$[n,m,0]$ $[n,n,m]$	C_4	A_1	$0^1,\ 2^3,\ 4^5$
		A_2	$2^2,\ 4^4$

Lüscher Method

ππ isospin-2

$$M^{\vec{P}}_{\ell m,\ell' m'}(q^2) = \gamma^{-1}\frac{(-1)^{\ell}}{\pi^{3/2}}$$

$\vec{P}$	$LG(\vec{P})$	$\Lambda^{(P)}$	$\pi\pi\ \ell^N$
$[0,0,0]$	O_h^D	A_1^+	$0^1,\ 4^1$
		T_1^+	4^1
		T_2^+	$2^1,\ 4^1$
		E^+	$2^1,\ 4^1$
$[n,0,0]$	Dic_4	A_1	$0^1,\ 2^1,\ 4^2$
		A_2	4^1
		E_2	$2^1,\ 4^2$
		B_1	$2^1,\ 4^1$
		B_2	$2^1,\ 4^1$
$[n,n,0]$	Dic_2	A_1	$0^1,\ 2^2,\ 4^3$
		A_2	$2^1,\ 4^2$
		B_1	$2^1,\ 4^2$
		B_2	$2^1,\ 4^2$
$[n,n,n]$	Dic_3	A_1	$0^1,\ 2^1,\ 4^2$
		A_2	4^1
		E_2	$2^2,\ 4^3$
$[n,m,0]$ $[n,n,m]$	C_4	A_1	$0^1,\ 2^3,\ 4^5$
		A_2	$2^2,\ 4^4$

Block diagonalise

$J_z =$

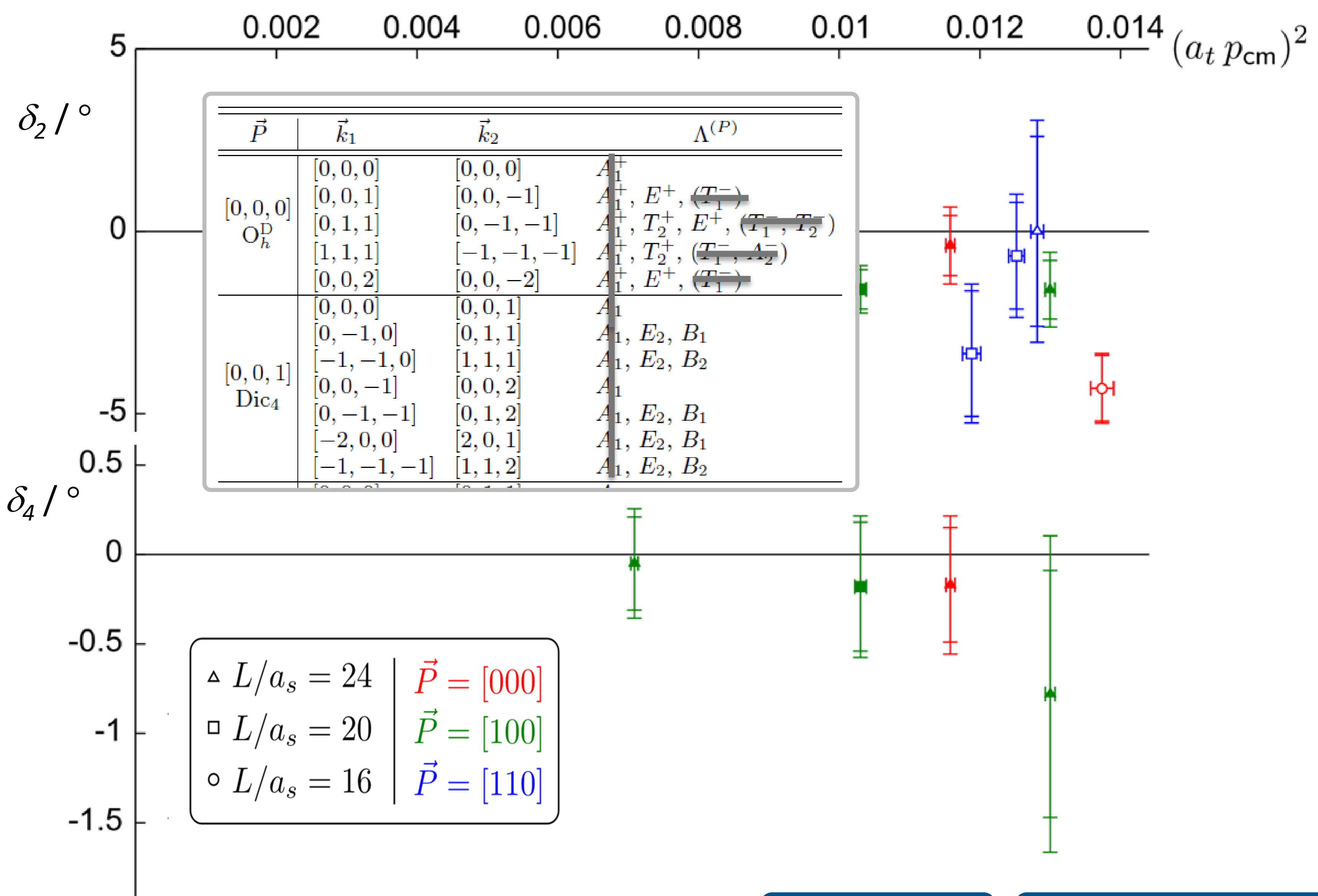
$$M^{(\vec{P},\Lambda,\mu)}_{\ell n;\ell' n'}(q^2)\,\delta_{\Lambda,\Lambda'}\delta_{\mu,\mu'} = \sum_{\substack{\hat{\lambda}=\pm|\lambda| \\ m=-\ell\ldots\ell}} \quad \sum_{\substack{\hat{\lambda}' \\ m'}} \ldots (R)$$

$$\det\left[e^{2i\delta(p_{\rm cm})} - \mathrm{U}^{(\vec{P},\Gamma)}\left(p_{\rm cm}\frac{L}{2\pi} \ldots\right.\right.$$

Assume $\delta_{l>4} \approx 0$ in this energy range

$$\delta_l(p) \xrightarrow[p\to 0]{} a_l\ p^{2l+1}$$

ππ I=2 phase shift: l = 2, 4

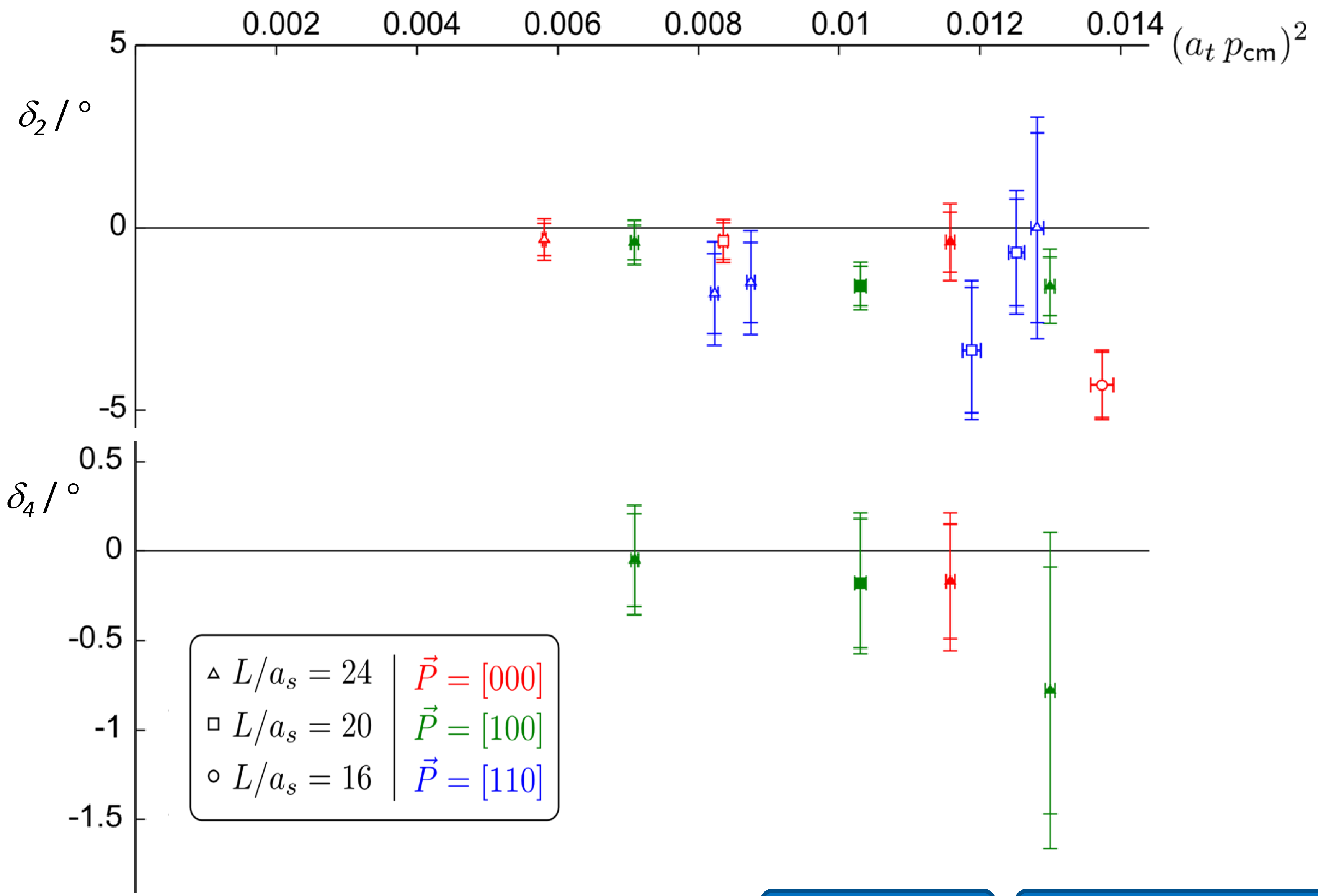


$\vec{P}$	$\vec{k}_1$	$\vec{k}_2$	$\Lambda^{(P)}$
[0, 0, 0] O_h^D	[0, 0, 0]	[0, 0, 0]	A_1^+
	[0, 0, 1]	[0, 0, −1]	A_1^+, E^+, (T_1^-)
	[0, 1, 1]	[0, −1, −1]	A_1^+, T_2^+, E^+, (T_1^-, T_2^-)
	[1, 1, 1]	[−1, −1, −1]	A_1^+, T_2^+, (T_1^-, A_2^-)
	[0, 0, 2]	[0, 0, −2]	A_1^+, E^+, (T_1^-)
[0, 0, 1] Dic_4	[0, 0, 0]	[0, 0, 1]	A_1
	[0, −1, 0]	[0, 1, 1]	A_1, E_2, B_1
	[−1, −1, 0]	[1, 1, 1]	A_1, E_2, B_2
	[0, 0, −1]	[0, 0, 2]	A_1
	[0, −1, −1]	[0, 1, 2]	A_1, E_2, B_1
	[−2, 0, 0]	[2, 0, 1]	A_1, E_2, B_1
	[−1, −1, −1]	[1, 1, 2]	A_1, E_2, B_2

$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

ππ I=2 phase shift: l = 2, 4



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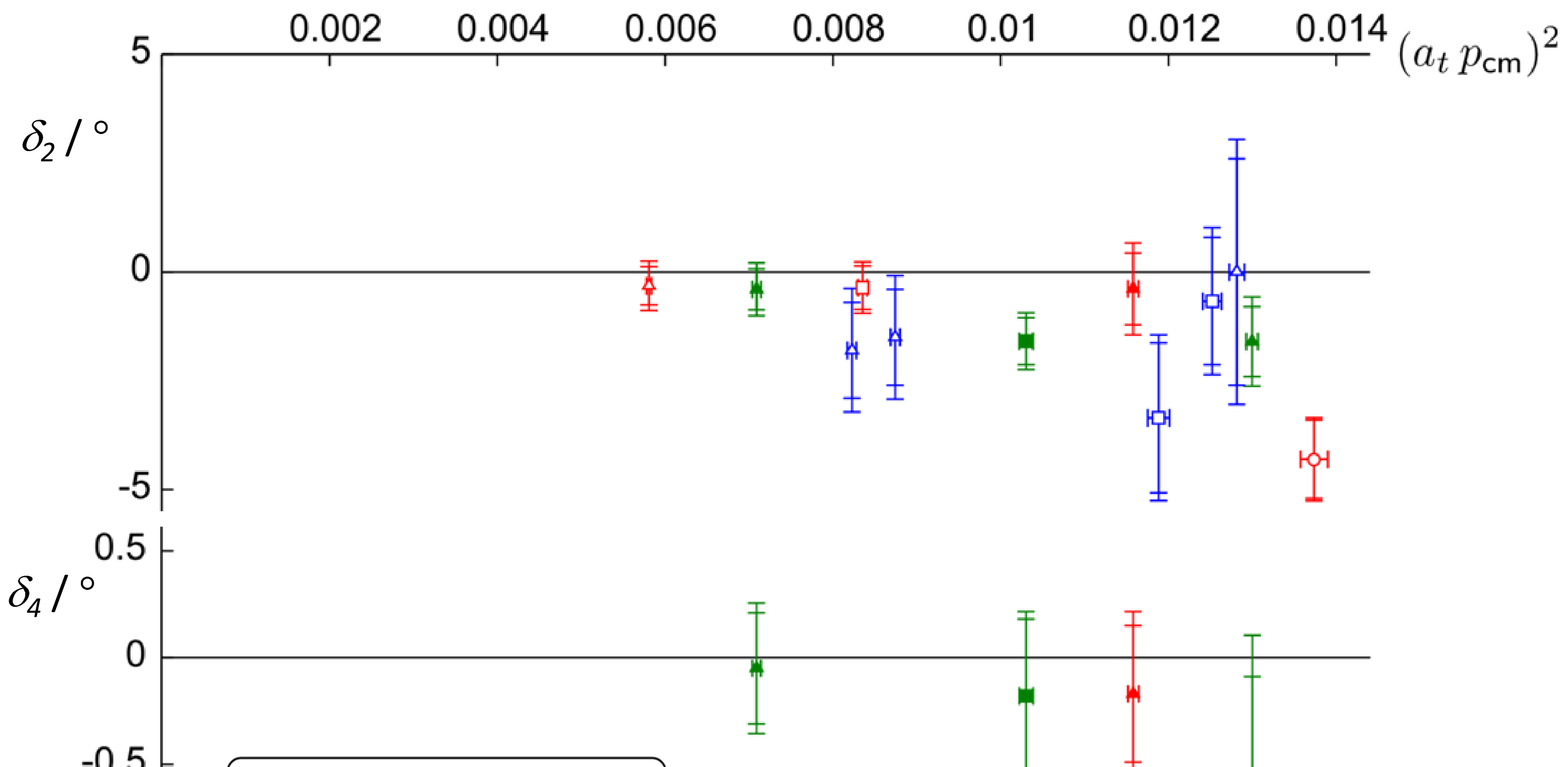
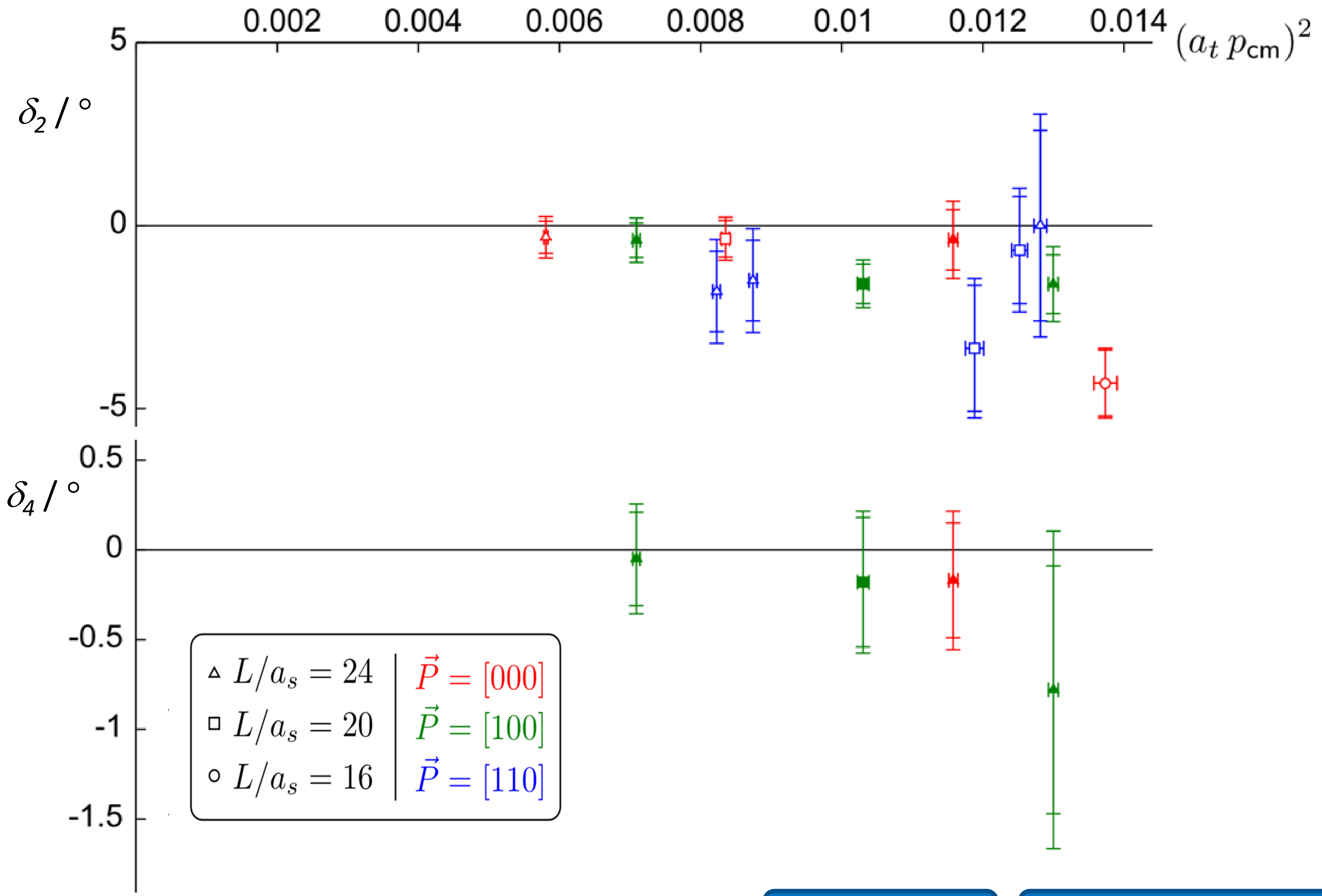


L/a_s	levels	$a_t p_{\mathrm{cm}}$	δ_2 / °	δ_4 / °
24	$[0,0,0]$, E^+, $\mathfrak{n}=1$ $[0,0,0]$, T_2^+, $\mathfrak{n}=0$	0.10766(23)(8) 0.10764(23)(8)	−0.39(82)(67)	−0.17(32)(22)
24	$[0,0,1]$, B_1, $\mathfrak{n}=0$ $[0,0,1]$, E_2, $\mathfrak{n}=0$	0.08427(25)(11) 0.08418(25)(11)	−0.40(47)(39)	−0.05(26)(16)
24	$[0,0,1]$, B_2, $\mathfrak{n}=0$ $[0,0,1]$, E_2, $\mathfrak{n}=1$	0.11412(29)(8) 0.11393(28)(8)	−1.60(80)(64)	−0.78(69)(55)
20	$[0,0,1]$, B_1, $\mathfrak{n}=0$ $[0,0,1]$, E_2, $\mathfrak{n}=0$	0.10174(35)(9) 0.10131(37)(9)	−1.59(54)(36)	−0.018(36)(17)

$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

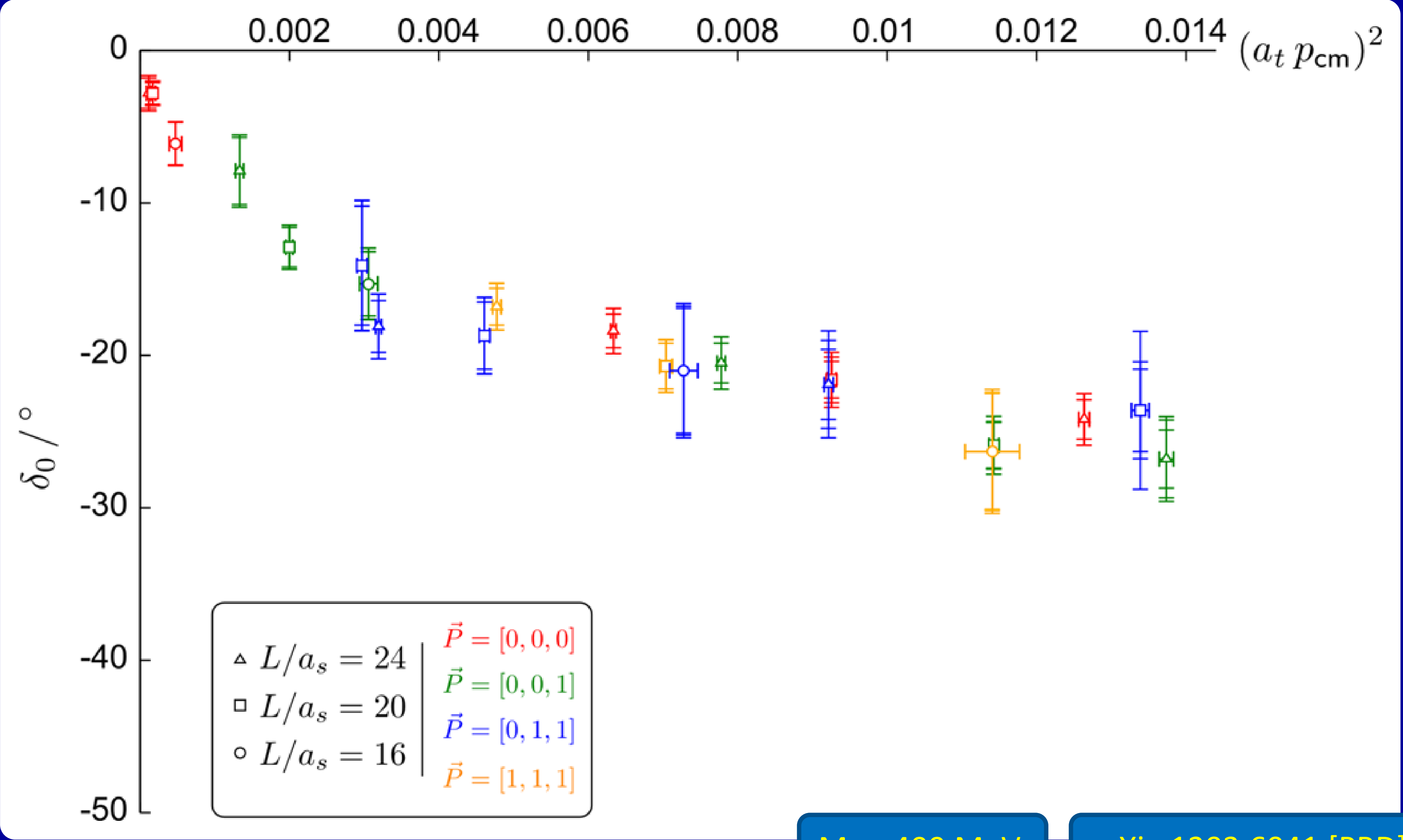
ππ I=2 phase shift: l = 2, 4



$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

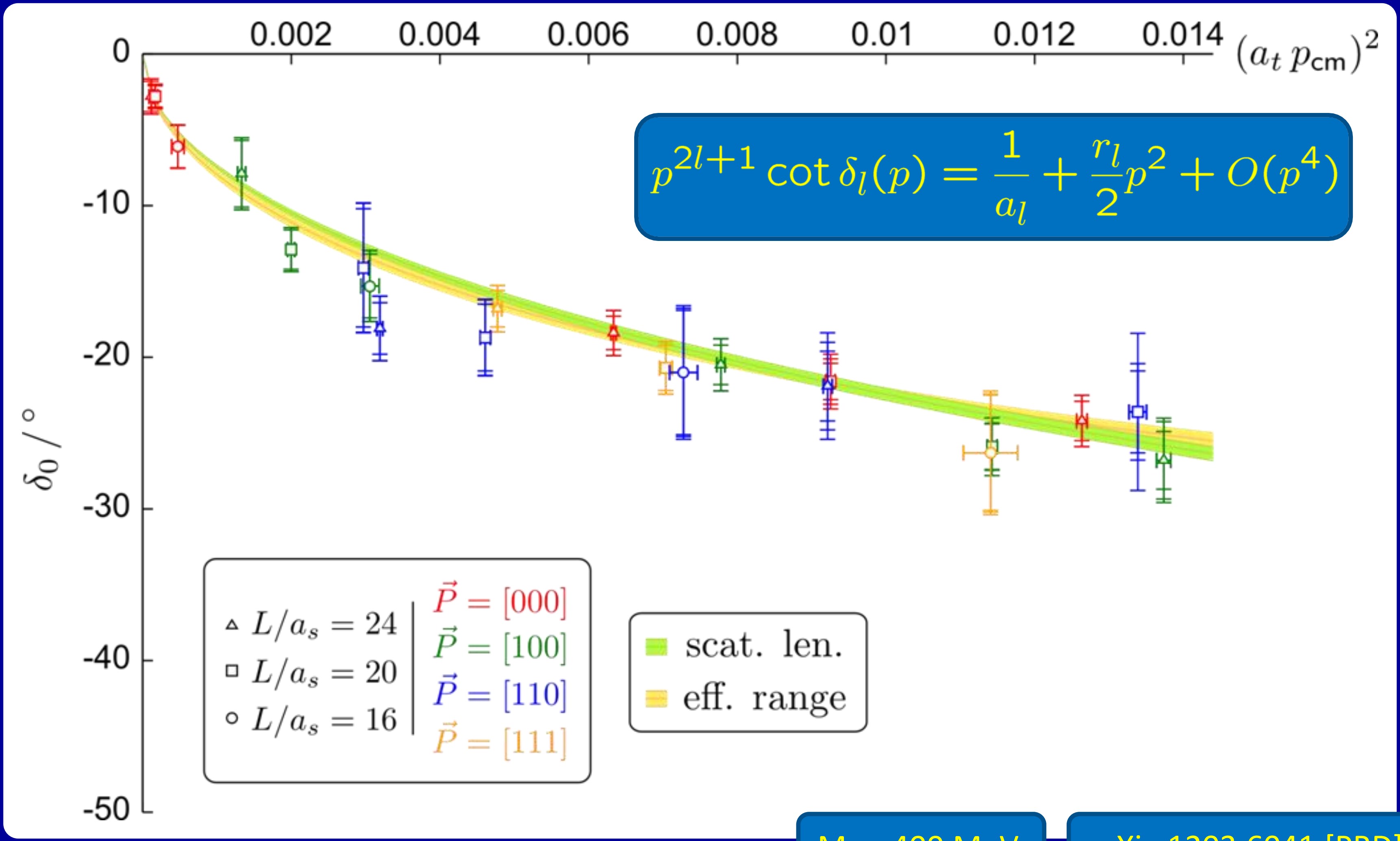
$\pi\pi$ I=2 phase shift: l = 0



$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

$\pi\pi$ I=2 phase shift: l = 0

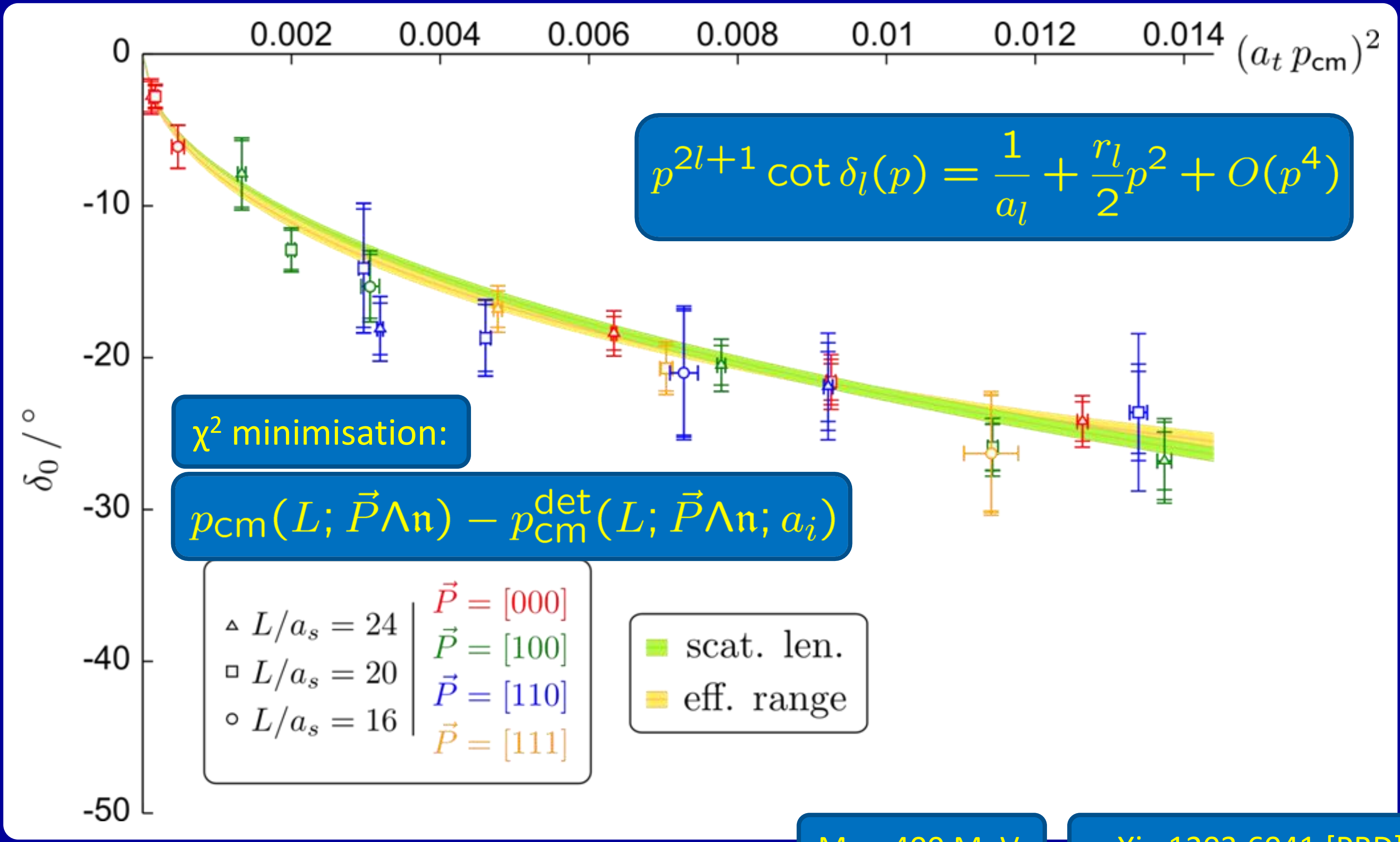


$$p^{2l+1} \cot \delta_l(p) = \frac{1}{a_l} + \frac{r_l}{2} p^2 + O(p^4)$$

$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

$\pi\pi$ I=2 phase shift: $l = 0$



$$p^{2l+1} \cot \delta_l(p) = \frac{1}{a_l} + \frac{r_l}{2} p^2 + O(p^4)$$

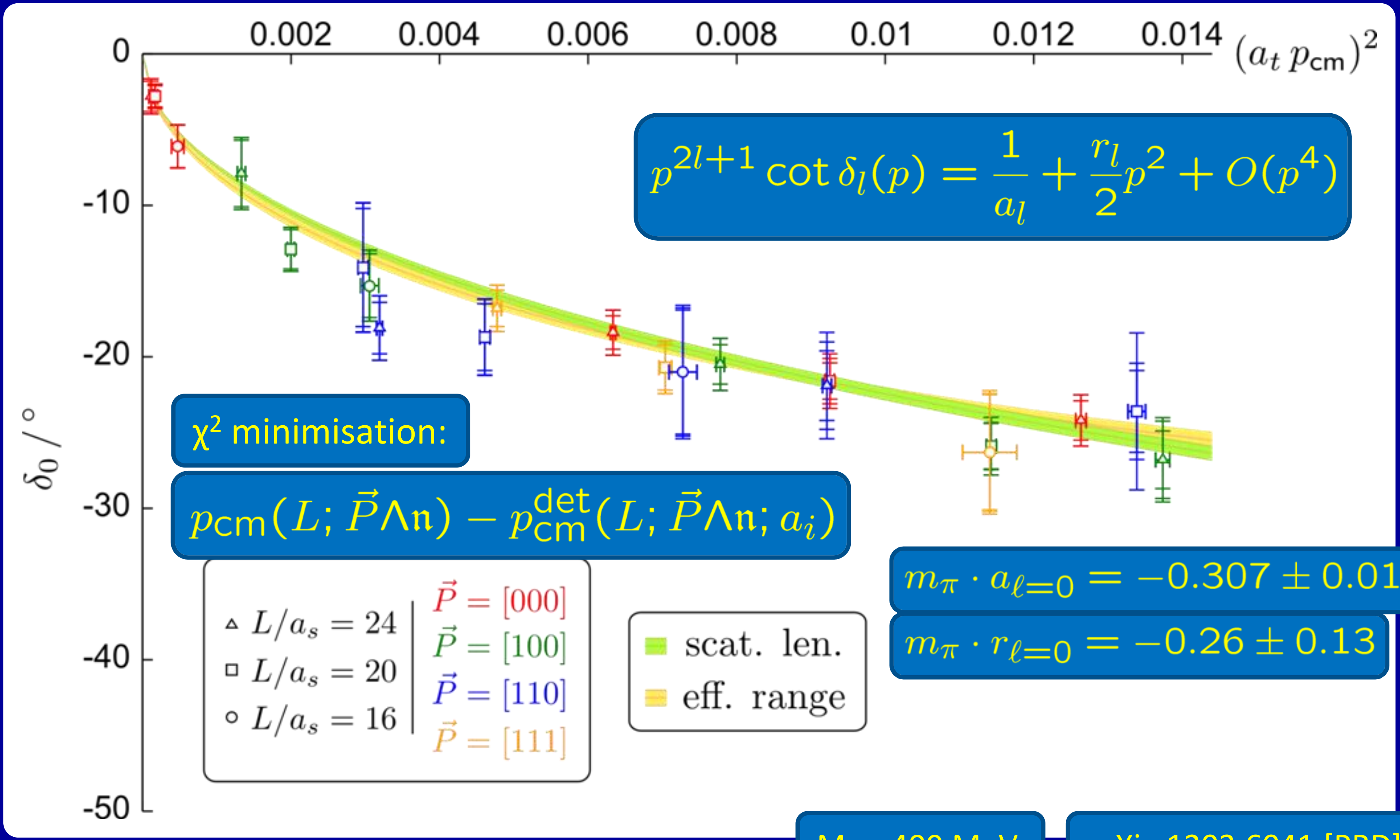
χ^2 minimisation:

$$p_{cm}(L; \vec{P}\Lambda\mathfrak{n}) - p_{cm}^{det}(L; \vec{P}\Lambda\mathfrak{n}; a_i)$$

$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

$\pi\pi$ I=2 phase shift: $l = 0$



$$p^{2l+1} \cot \delta_l(p) = \frac{1}{a_l} + \frac{r_l}{2} p^2 + O(p^4)$$

χ^2 minimisation:

$$p_{cm}(L; \vec{P}\Lambda\mathfrak{n}) - p_{cm}^{det}(L; \vec{P}\Lambda\mathfrak{n}; a_i)$$

$$m_\pi \cdot a_{\ell=0} = -0.307 \pm 0.013$$

$$m_\pi \cdot r_{\ell=0} = -0.26 \pm 0.13$$

$M_\pi \approx 400$ MeV

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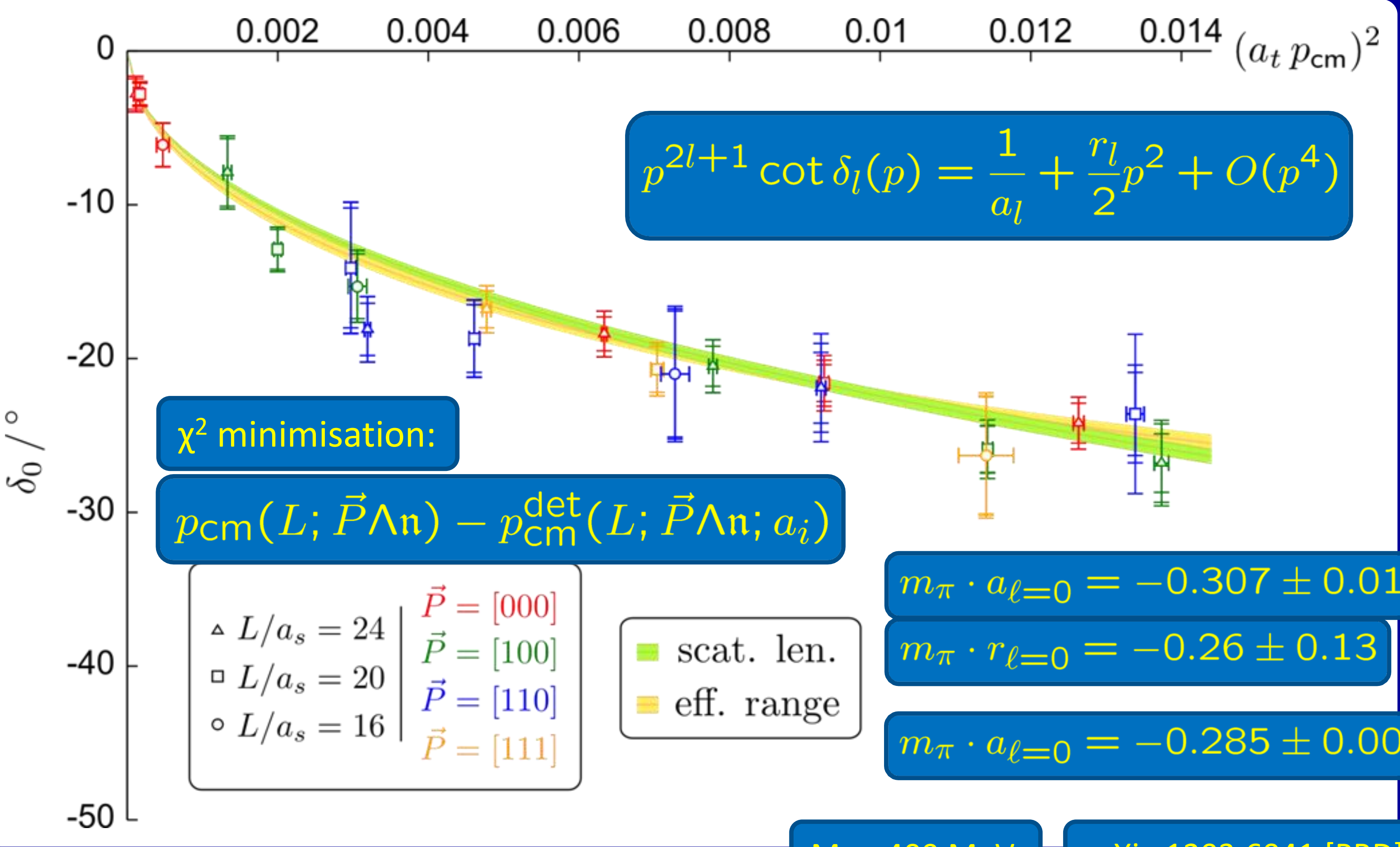

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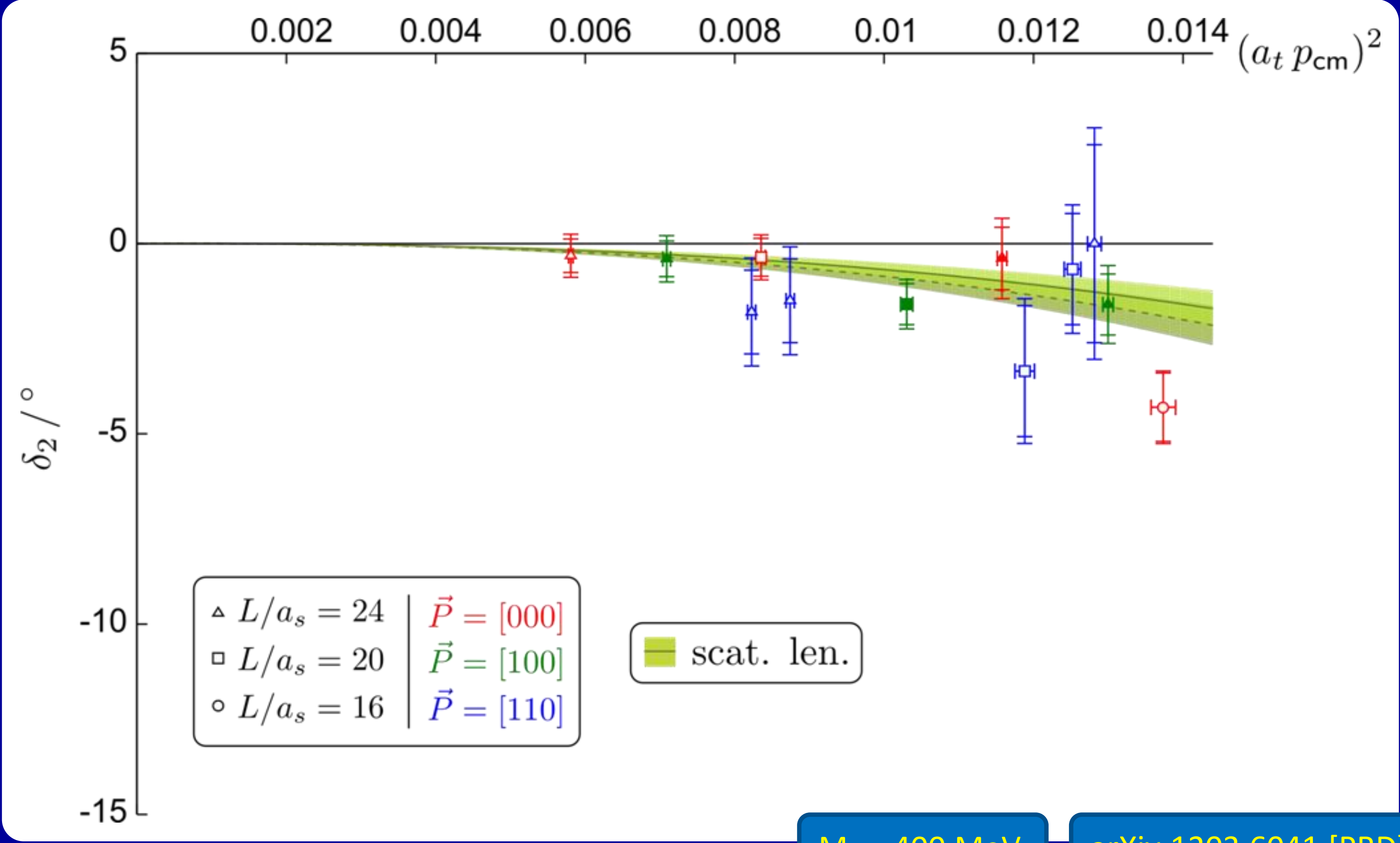
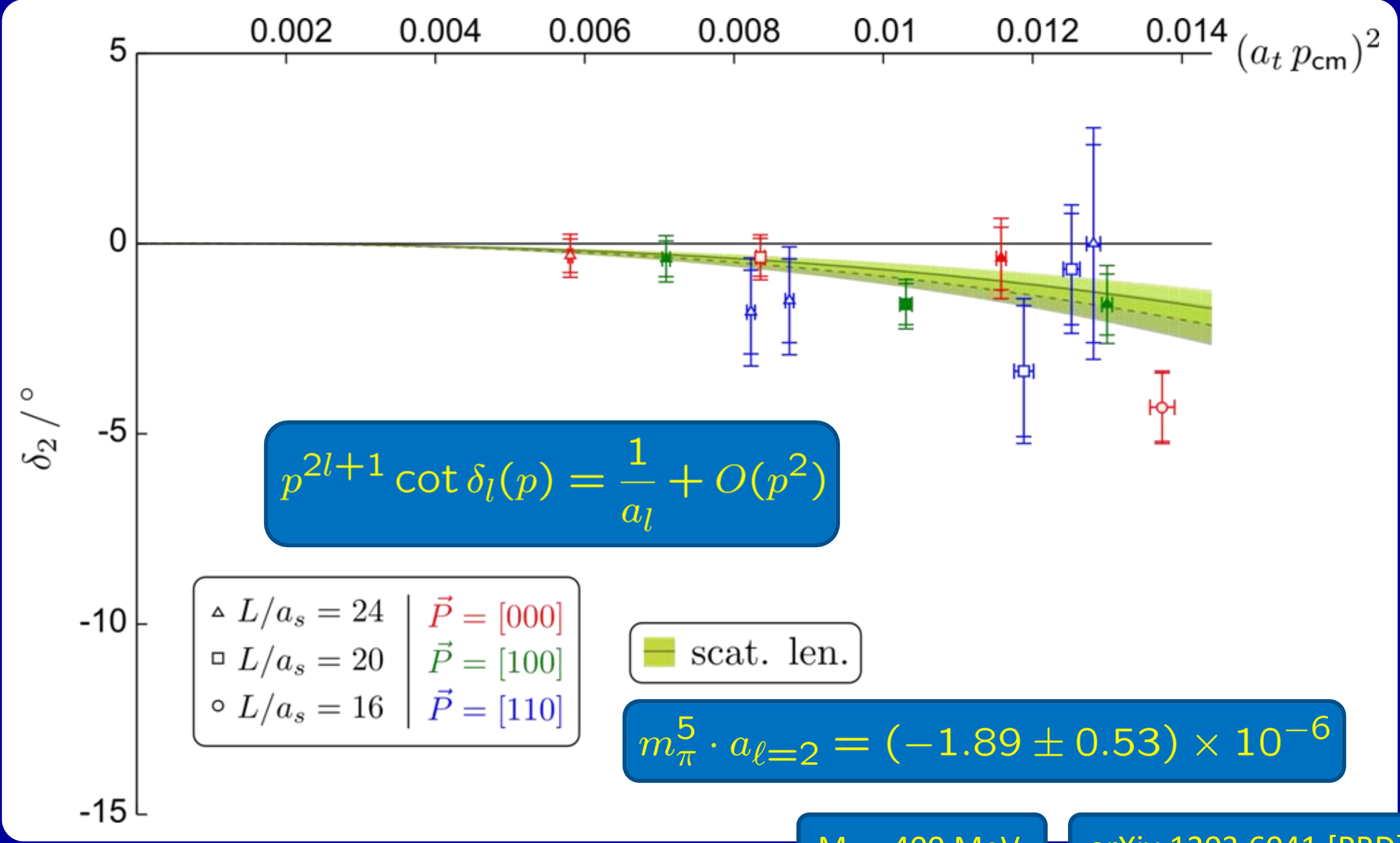
$\pi\pi$ I=2 phase shift: l = 2



$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

$\pi\pi$ I=2 phase shift: l = 2



$$p^{2l+1} \cot \delta_l(p) = \frac{1}{a_l} + O(p^2)$$

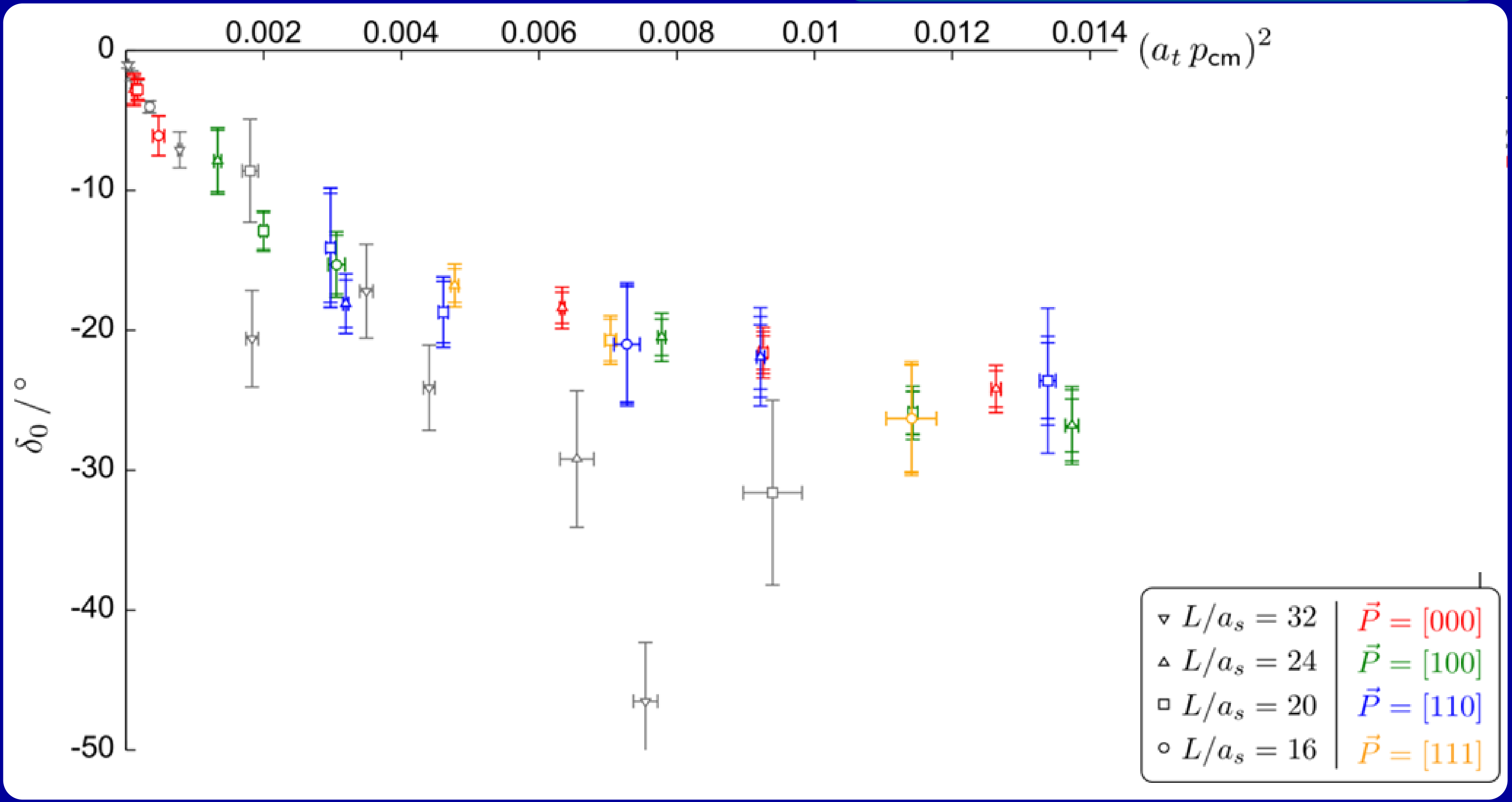
$$m_\pi^5 \cdot a_{\ell=2} = (-1.89 \pm 0.53) \times 10^{-6}$$

$M_\pi \approx 400$ MeV

arXiv:1203.6041 [PRD]

$\pi\pi$ I=2 phase shift: l = 0

c.f. NPLQCD, PR D85, 034505 (2012)



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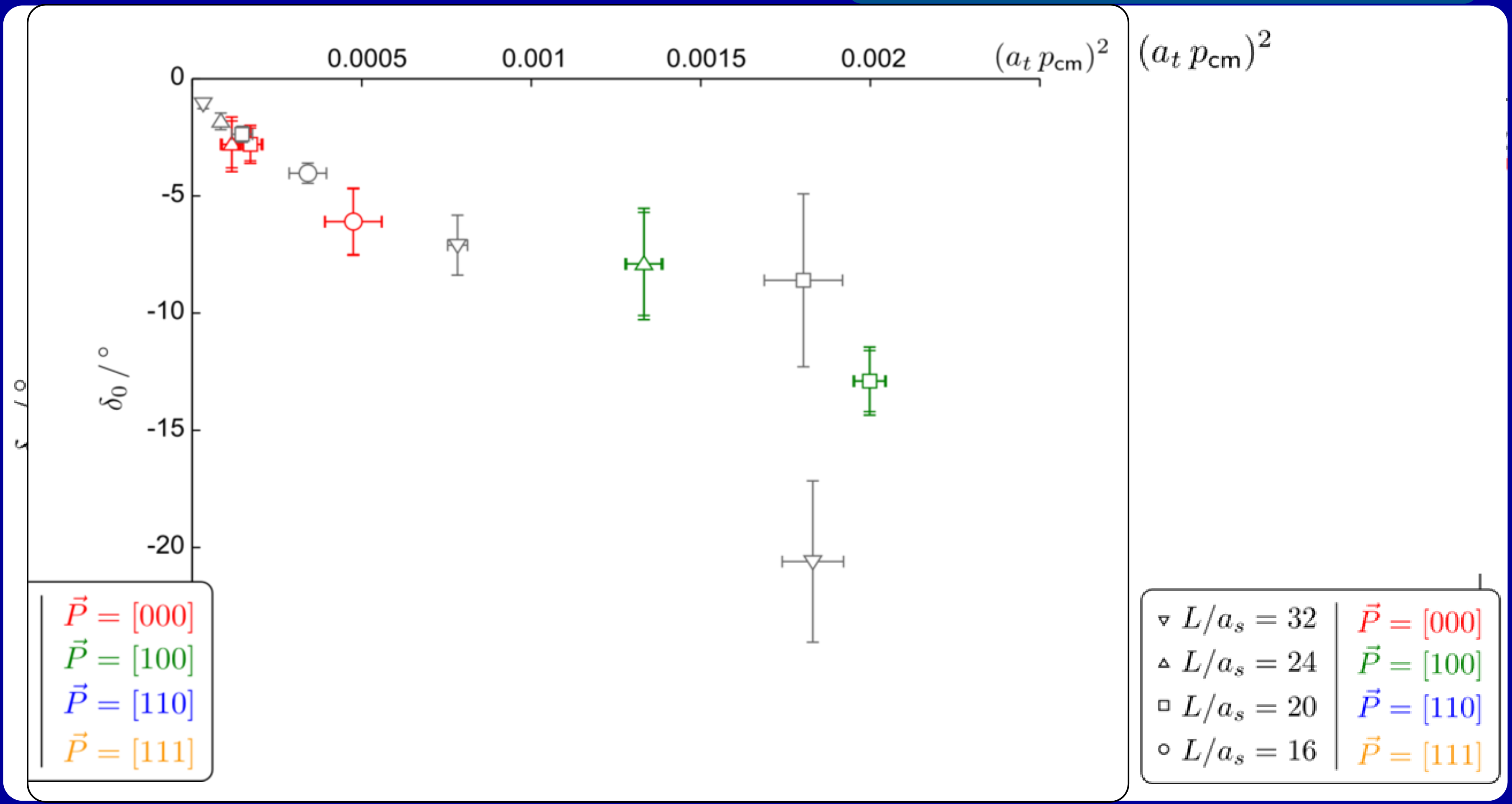

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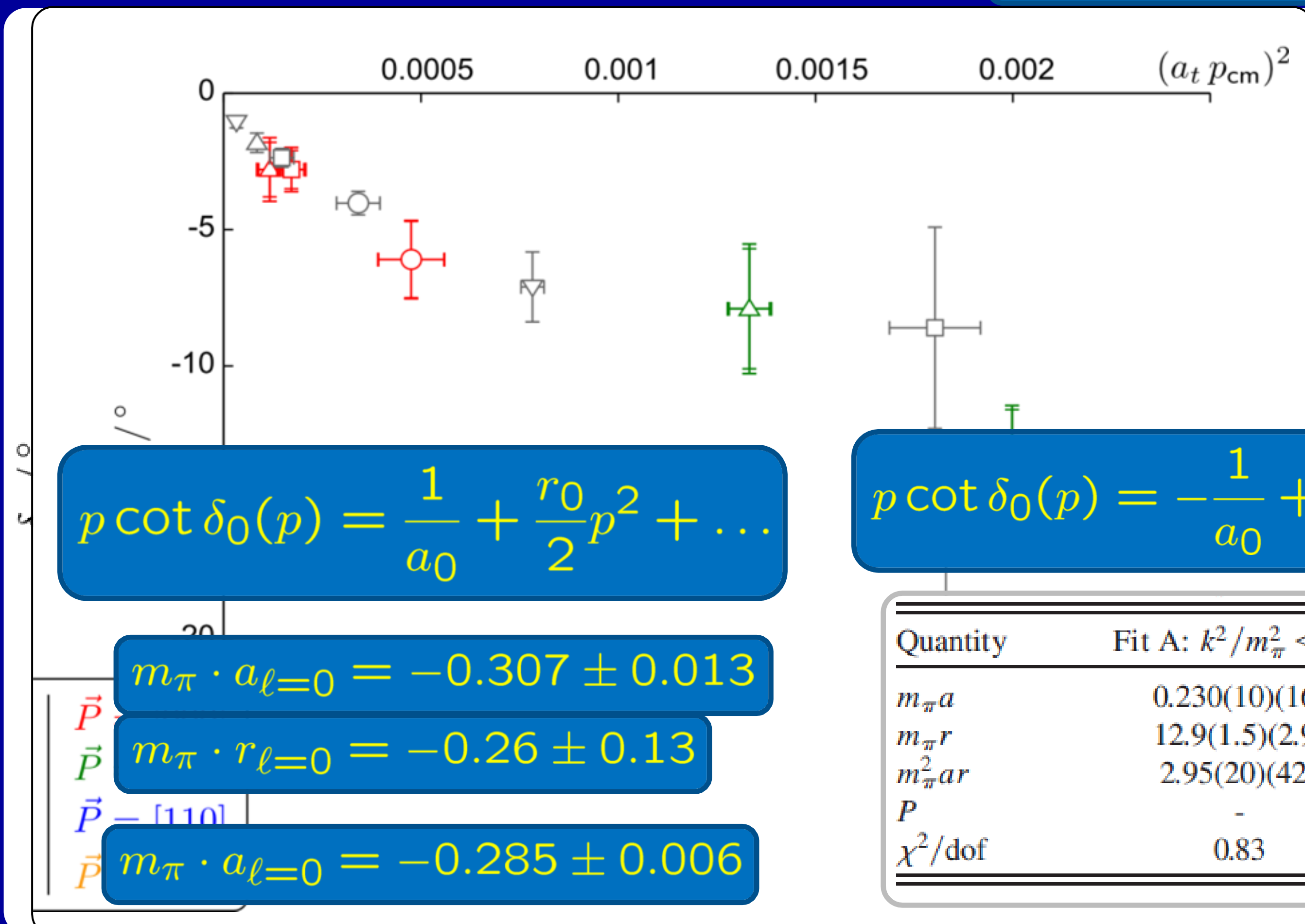

$(a_t\, p_{\mathrm{cm}})^2$

$$p \cot \delta_0(p) = \frac{1}{a_0} + \frac{r_0}{2} p^2 + \ldots$$

$$p \cot \delta_0(p) = -\frac{1}{a_0} + \frac{r_0}{2} p^2 + P r_0^3 p^4 + \ldots$$

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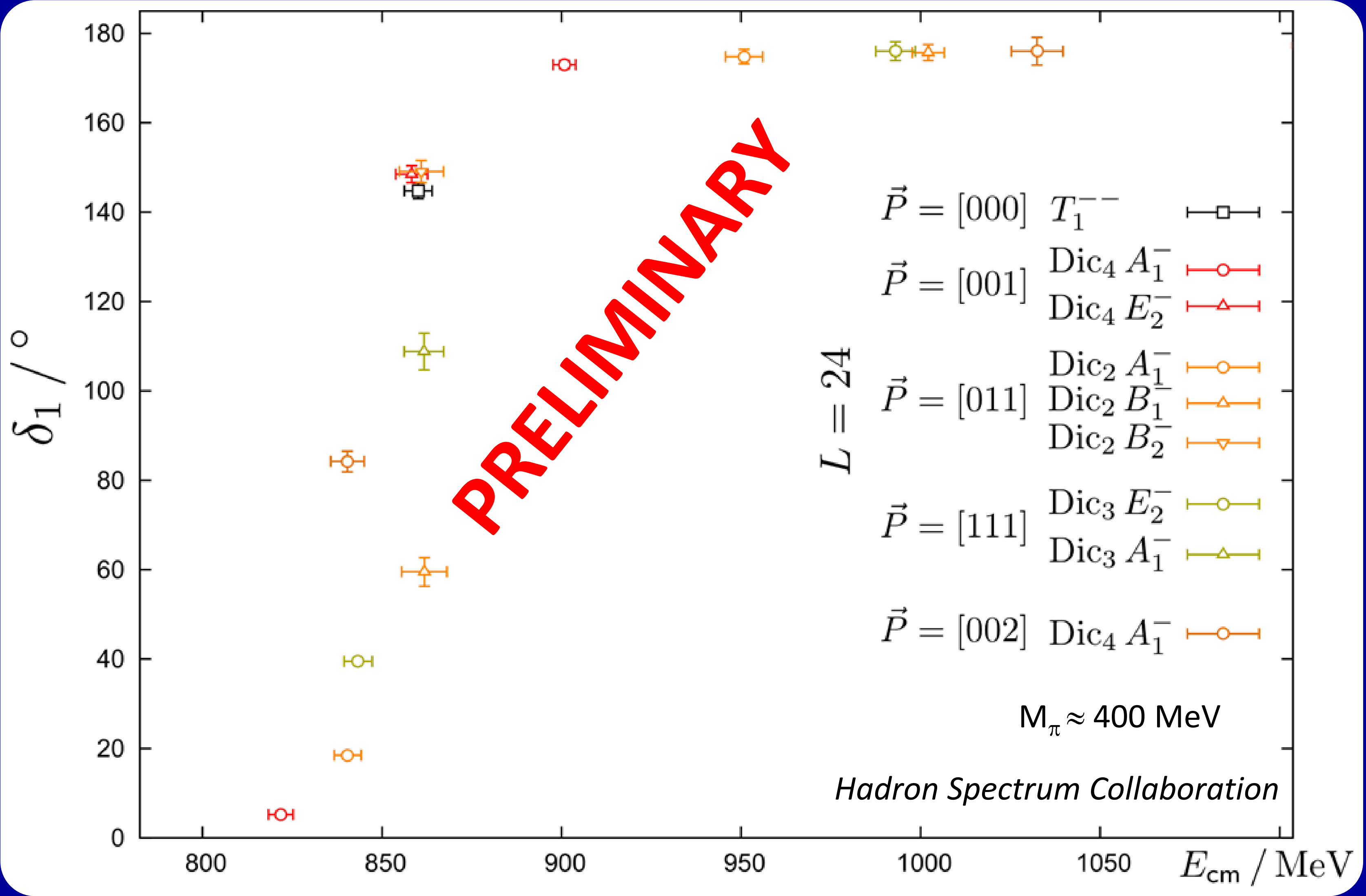
$$m_\pi \cdot a_{\ell=0} = -0.285 \pm 0.006$$

Quantity	Fit A: $k^2/m_\pi^2 < 0.5$	Fit B: $k^2/m_\pi^2 < 1$
$m_\pi a$	0.230(10)(16)	0.226(10)(16)
$m_\pi r$	12.9(1.5)(2.9)	18.1(2.4)(4.7)
$m_\pi^2 a r$	2.95(20)(42)	4.06(30)(57)
P	-	−0.001 23(30)(55)
χ^2/dof	0.83	0.79

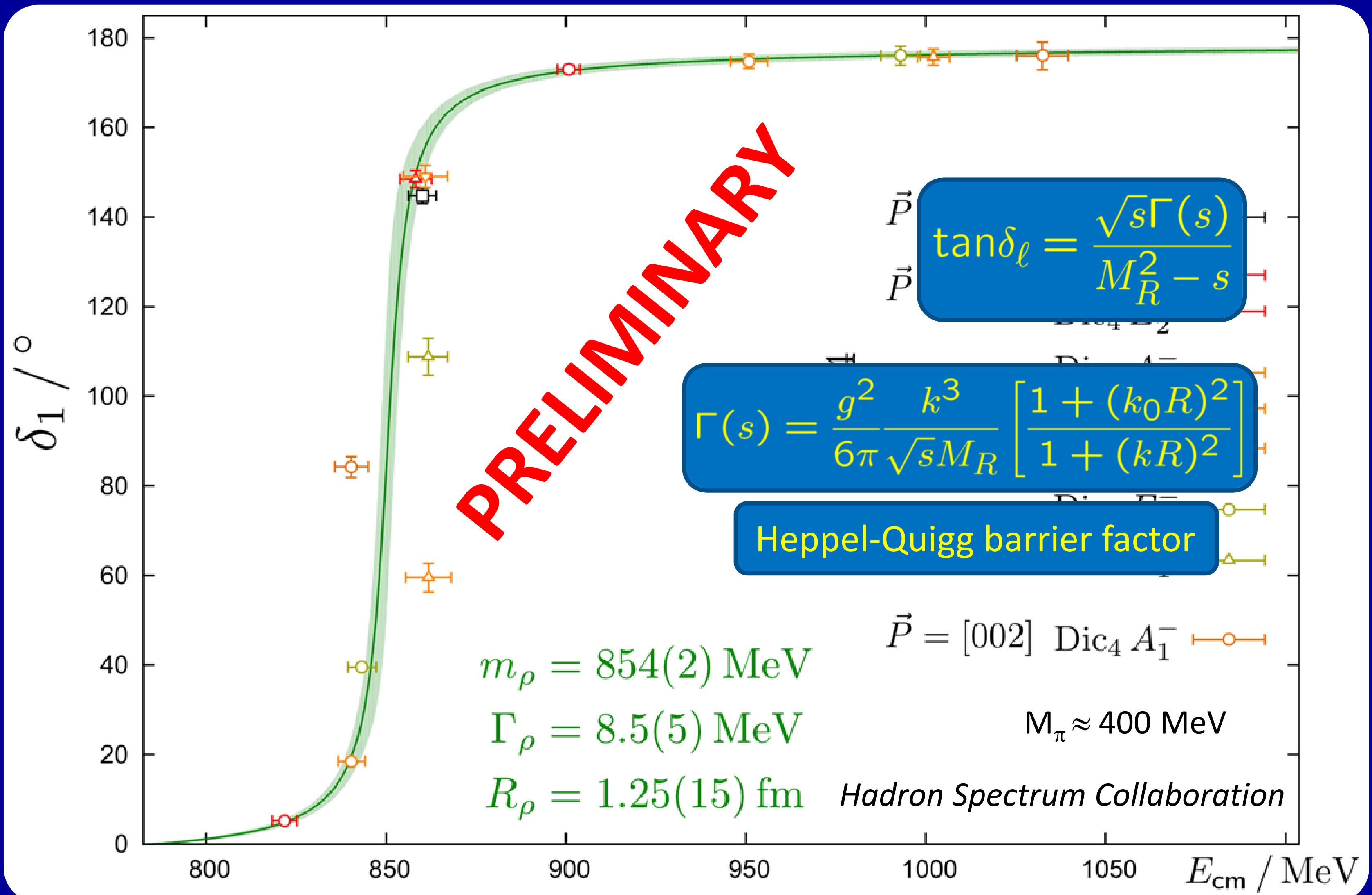
HadSpec, arXiv:1203.6041 [PRD]

NPLQCD, PR D85, 034505 (2012)

ρ meson – preliminary



ρ meson – preliminary



Summary and Outlook

Summary

- **Large basis** of **carefully constructed multi-hadron ops**
- **Distillation**, anisotropic lattices, variational method
- Variationally optimised π ops; finite-T effects
- → **Many** multi-meson energies in **many irreps reliably** and with **high precision**
- → ***l* = 0 and 2** phase shifts **mapped out**

MARIE CURIE ACTIONS

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Outlook

- Other **scattering** channels – **resonances**, decays, ...
- Lighter pion masses, larger volumes, ...
- Above inelastic thresholds?