Multiple-channel generalization of Lellouch-Luscher formula

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Based on M.T. Hansen and S.R. Sharpe, Phys.Rev. D86 (2012) 016007 [arXiv:1204.0826 [hep-lat]]

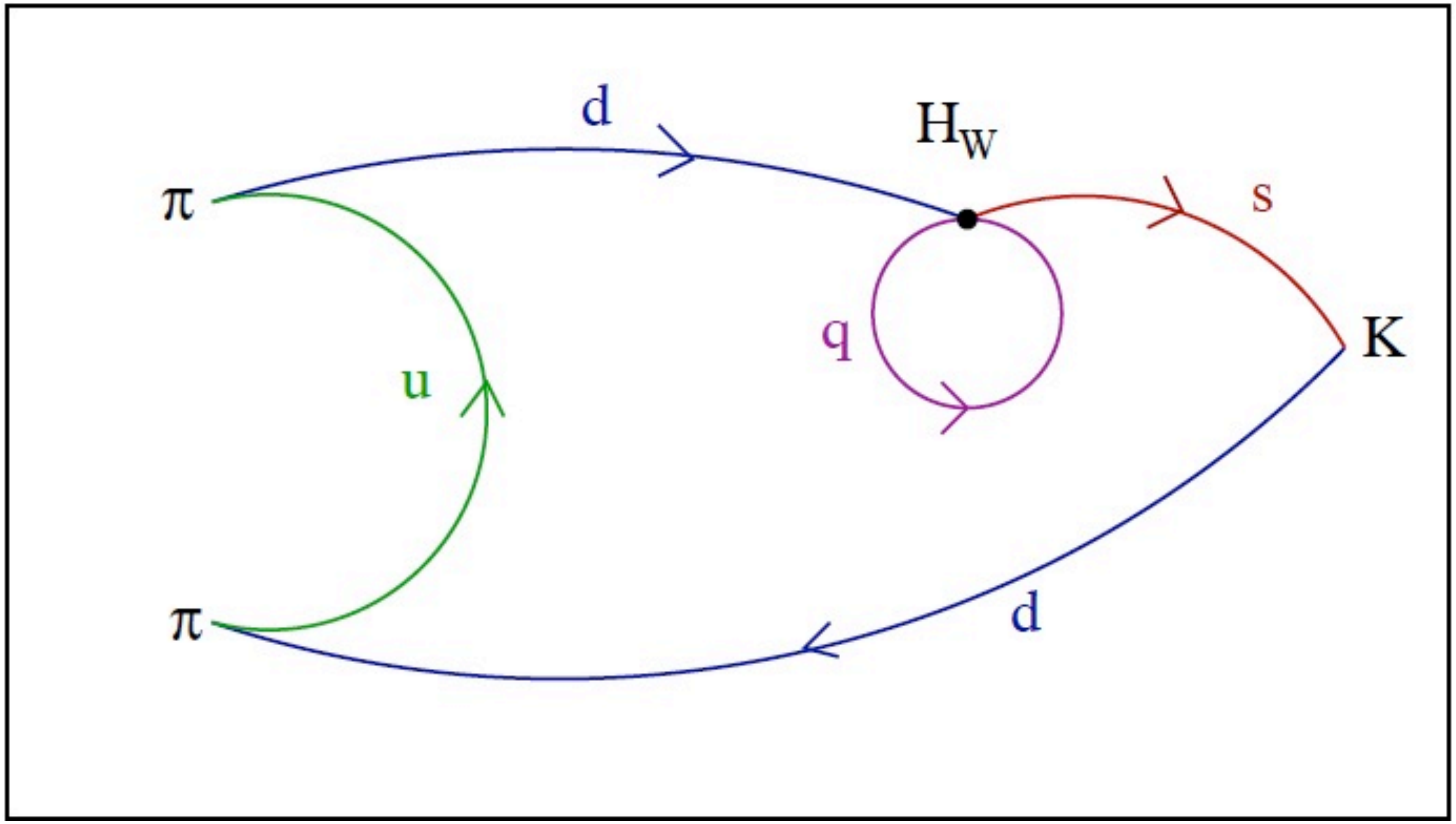
Outline

- Motivation
- Problem to be solved (multiple 2-particle channels)
- Generalized Luscher quantization condition
- Generalized Lellouch-Luscher formula
- Technical details
- Other applications & outlook

Motivation

- Lattice QCD has made significant progress in the calculation of $K \to \pi\pi$ weak decay amplitudes
  - ✶ Complete calculation for isospin 2 final state [RBC 2012]
  - ✶ Pilot calculation with unphysical masses for I=0 [RBC 2012]
  - ✶ We will know soon (few years?) whether QCD explains the $\Delta I=1/2$ rule in K decays, and whether CP-violation in K decays (ε'/ε) is consistent with the standard model (SM)
- LHCb recently presented evidence for CP-violation in $D \to \pi\pi$, KK decays
  - ✶ Larger rate than (naively) expected in SM, but large hadronic uncertainties in estimates
  - ✶ Is a LQCD calculation possible?

Ingredients for K→ππ



$$\Rightarrow \ {}_L\langle \pi\pi | \mathcal{H}_W | K \rangle_L$$

- Lattice details (discretization errors, chiral extrapolation, methods for multiple Wick contractions, ...) very important in practice but not discussed here

Ingredients for K→ππ

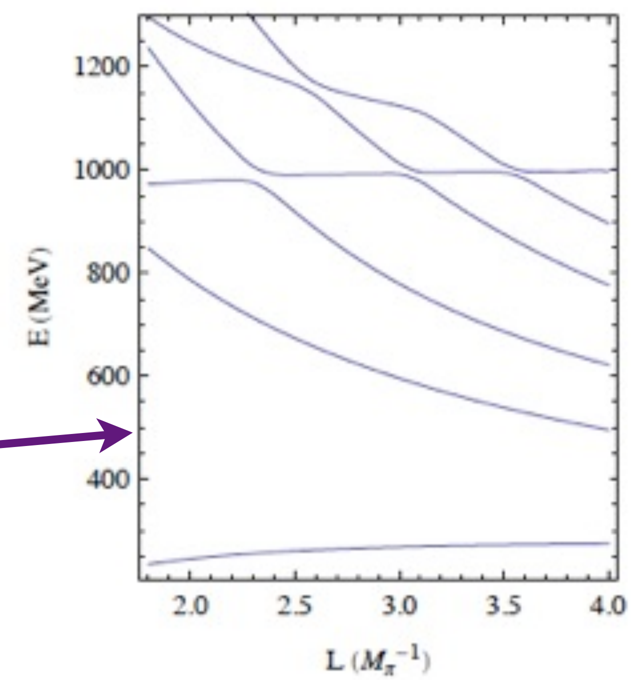
- Key theoretical issues arise from use of finite volume
  - ✴ Single-particle states (K) have exponentially suppressed FV effects--- these fall as exp(-m_πL) and often can be ignored
  - ✴ Two-particle states (ππ) have enhanced (power-law) FV effects--- distorted significantly compared to "out-states" of QFT

$$_L\langle \pi\pi| = c_0 \langle \pi\pi(\ell = 0), \mathrm{out}| + c_4 \langle \pi\pi(\ell = 4), \mathrm{out}| + \dots$$

  - ✴ Spectrum of two-particle states is discrete---energy does not match E_K for a general box size L

P=0, I=0 spectrum from UChPT with periodic BC (ignoring 4π etc.)
[Hansen, Lat2012]

M_K

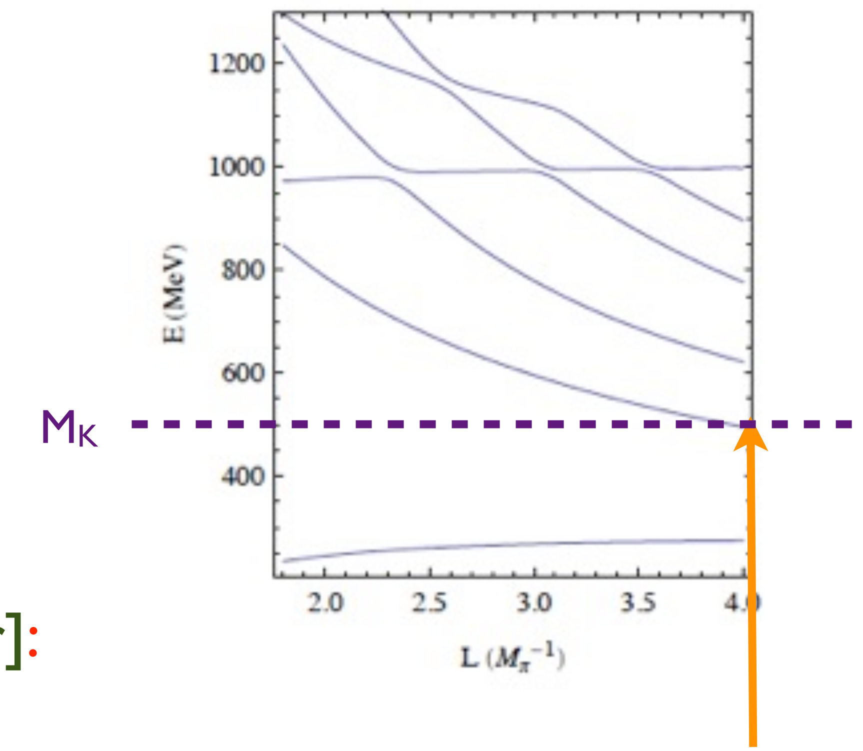


Ingredients for K→ππ

- Resolutions:

  ✶ Spectrum of two-particle states encodes information about scattering phase shifts---can extract $\delta(M_K)$ if assume only low partial waves are important (and L > range of interaction) [Luscher]

  ✶ Adjust L (and total 3-momentum) so $E_{\pi\pi} = E_K$



  ✶ Determine c_0 using method of [Lellouch & Luscher]:

    - Requires δ & dδ/dE at $E=E_K$, which can be determined from the slope of the spectral energy w.r.t. L

  ✶ Infinite volume amplitude given by (up to exponentially small FV effects)

$$\langle \pi\pi(\ell=0),\mathrm{out}|\mathcal{H}_W|K\rangle = \frac{e^{i\delta(M_K)}}{c_0}\,{}_L\langle\pi\pi|\mathcal{H}_W|K\rangle_L$$

Generalize to $D \to \pi\pi, KK$?

- Luscher's quantization and Lellouch-Luscher formula predicated on lying below inelastic thresholds
  - ✴ Works for $K \to \pi\pi$ because $M_K < 4M_\pi$
  - ✴ Fails spectacularly for D meson (M_D=1865 MeV), which can decay to 2π, KK, 4π, $\eta\eta$, 6π, ...
  - ✴ D also decays to many other channels (e.g. $K\pi$) but on lattice can treat each strongly interacting sector separately, and focus here on I=S=0
- Describe here first step on way to full calculation
  - ✴ Keep only two-particle channels ($\pi\pi$, KK, $\eta\eta$), ignoring 4π, 6π, etc.
  - ✴ Not expected to be realistic for D [$f_0(1500)$ has 50% BR to 4π], though might give semi-quantitative guide

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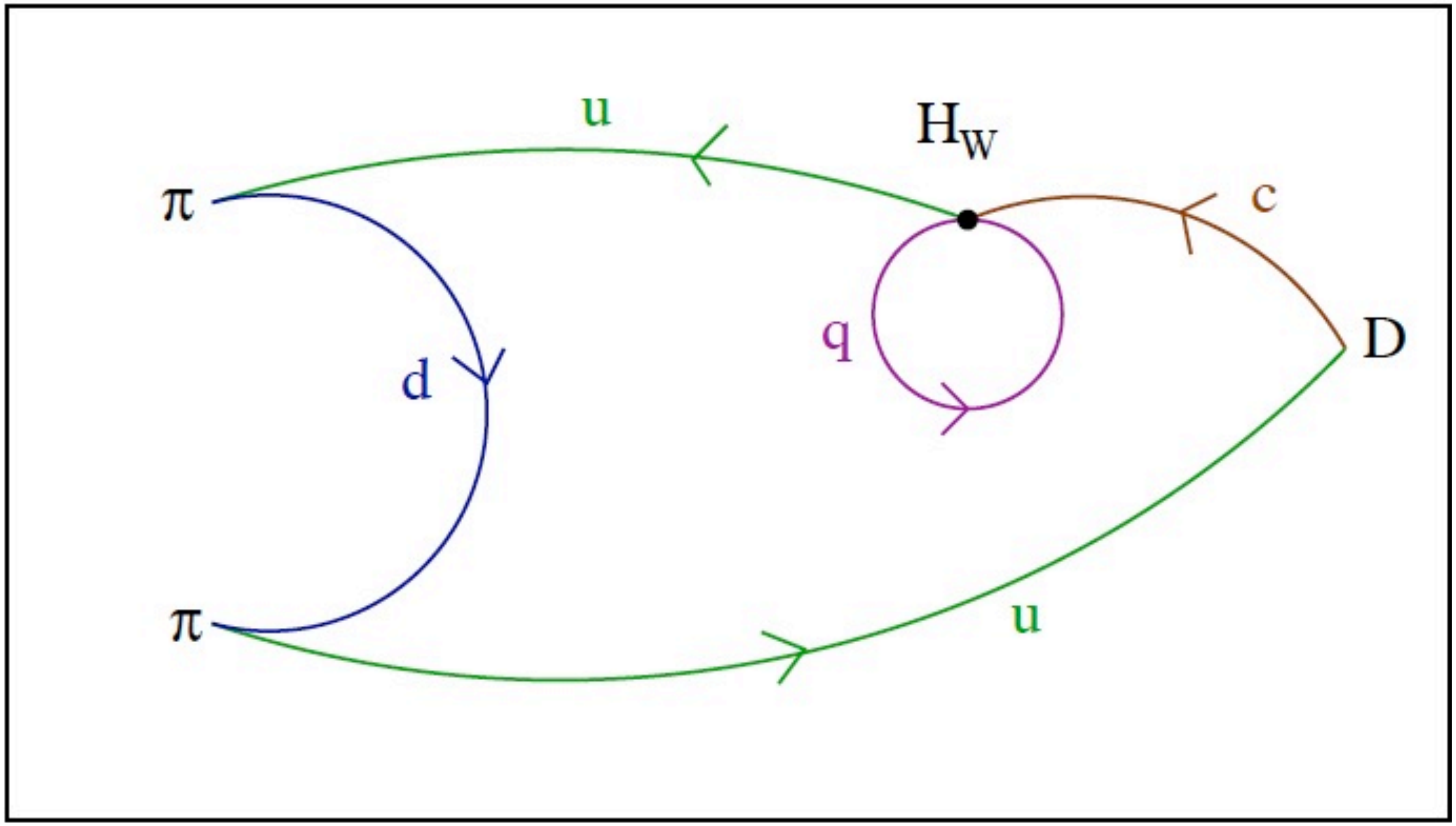
Problem to be solved

- "D" decays weakly to "ππ" & "$K\overline{K}$"
  - ✷ $2M_\pi < 2M_K < M_D < 4M_\pi$ so only two decay channels (though can add any number of two-particle channels and formalism generalizes)
  - ✷ Particles in each channel can be identical or not, degenerate or not
  - ✷ Total 3-momentum $\vec{P} = \frac{2\pi\vec{n}_P}{L}$ is arbitrary
  - ✷ Kinematics is relativistic (derivation is in QFT)
  - ✷ Assume particles are scalars (should be straightforward to generalize)
- Given finite-volume matrix elements (from LQCD) want to determine infinite-volume matrix elements

$$M_{D\to n} = {}_L\langle n|\mathcal{H}_W|D\rangle_L \qquad \Rightarrow$$

$$\mathcal{A}_{D\to\pi\pi} = \langle\pi\pi|\mathcal{H}_W|D\rangle \quad \text{and} \quad \mathcal{A}_{D\to KK} = \langle K\bar{K}|\mathcal{H}_W|D\rangle$$

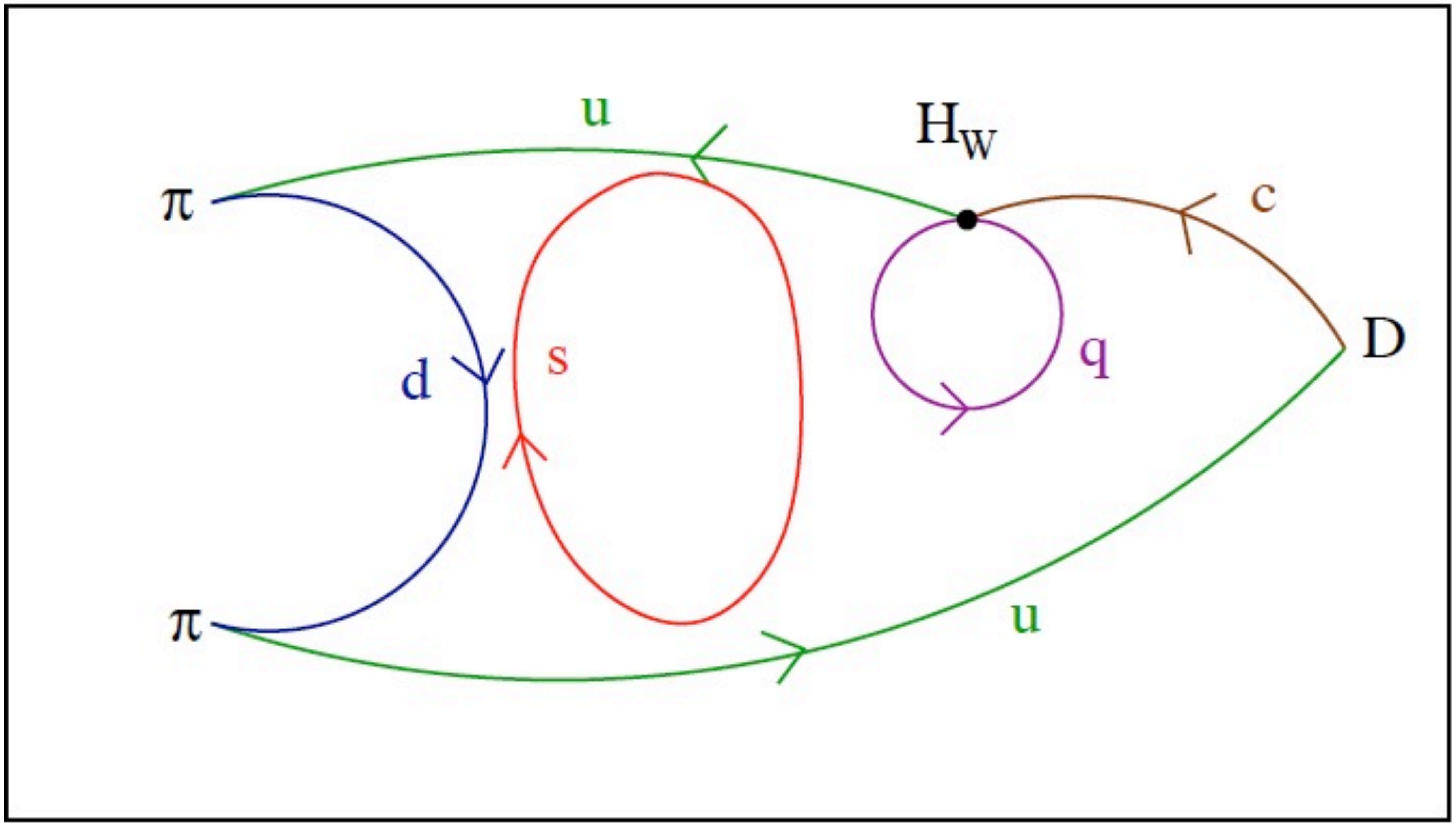
Ingredients for toy problem



$$\Rightarrow \quad {}_L\langle n|\mathcal{H}_W|D\rangle_L$$

- Choose $\vec{P}$ and L so that FV state has energy $E_n = E_D = \sqrt{M_D^2 + \vec{P}^2}$

Ingredients for toy problem



$$\Rightarrow \quad {}_L\langle n|\mathcal{H}_W|D\rangle_L$$

- Key point: even if create two-particle state with 2 pion operator, strong interactions will mix it with K-antiK, so FV states are mixtures!

Ingredients for toy problem

- Need generalization of Luscher's quantization condition for multiple 2-particle channels
  - ✶ Obtained for "rest frame" and assuming S-wave scattering by [Bernard, Lage, Meissner & Rusetsky] with related work by several groups [Liu et al., Lage et al., Doring et al., Aoki et al.]
  - ✶ We give the result for a "moving frame" and arbitrary scattering waves, generalizing methodology of [Kim, Sachrajda & SRS]
  - ✶ Identical result obtained independently by [Briceno & Davoudi]
- Need generalization of Lellouch-Luscher formula to account for mixing between channels
  - ✶ We give the result assuming S-wave dominance, any number of 2-particle channels, and arbitrary total 3-momentum

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Quantization condition

$$\det\left(F^{-1} + i\mathcal{M}\right) = 0$$

(neglecting exponentially suppressed FV effects)

Identical in form to single-channel moving-frame result of [Kim, Sachrajda & SRS]

Quantization condition

$$\det\left(F^{-1} + i\mathcal{M}\right) = 0$$

Known kinematic FV factor
(generalized Luscher zeta function)
Depends on E and 3-momentum P
Diagonal in channel space;
non-trivial in angular momentum

Scattering amplitude (infinite volume quantity)
Depends on CM energy E^*
Non-trivial matrix in channel space;
diagonal in angular momentum

$$\mathcal{M}_{ij;\ell_1,m_1;\ell_2,m_2} = \mathcal{M}_{ij}^{\ell_1,m_1}\delta_{\ell_1\ell_2}\delta_{m_1m_2}$$

channel indices

$1 = \pi\pi, \quad 2 = K\overline{K}$

For given $(L, \vec{P})$
find solutions only for discrete energies E_n

Quantization condition

$$\det\left(F^{-1} + i\mathcal{M}\right) = 0$$

$$F_{ij;\ell_1,m_1;\ell_2,m_2} \equiv \delta_{ij}\eta_i \left[\frac{\mathrm{Re}q_i^*}{8\pi E^*}\delta_{\ell_1\ell_2}\delta_{m_1m_2} + \frac{i}{2\pi EL}\sum_{\ell,m} x_i^{-\ell}\, \mathcal{Z}^P_{\ell m}[1;x_i^2] \int d\Omega Y^*_{\ell_1,m_1} Y^*_{\ell,m} Y_{\ell_2,m_2}\right]$$

$E^* = \sqrt{E^2 - \vec{P}^2}$ is CM energy

$q_j^* = \sqrt{E^{*\,2}/4 - m_j^2}$ is relative momentum in CM for j'th channel

$x_i \equiv q_i^* L/(2\pi)$ and $\mathcal{Z}^P_{\ell m}$ is a generalization of the zeta-function

$\eta = 1/2$ for identical particles and 1 for non-identical

Practical quantization condition

$$\det\left(F^{-1} + i\mathcal{M}\right) = 0$$

- If can ignore scattering above l_{max}, i.e. $\mathcal{M}_{ij}^{\ell > \ell_{max}, m} = 0$
  - ✶ then condition collapses to determinant of a finite matrix
- E.g. if only s-wave scattering is significant then

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P}) \equiv \det\left[\begin{pmatrix} (F_1^s)^{-1} & 0 \\ 0 & (F_2^s)^{-1} \end{pmatrix} + i \begin{pmatrix} \mathcal{M}^s_{1\to 1} & \mathcal{M}^s_{2\to 1} \\ \mathcal{M}^s_{1\to 2} & \mathcal{M}^s_{2\to 2} \end{pmatrix}\right] = 0$$

$$\left(F_j^s\right)^{-1} = \frac{4\pi E^*}{\eta_j q_j^*}\left(1 - e^{2i\phi^P(q_j^*)}\right)$$

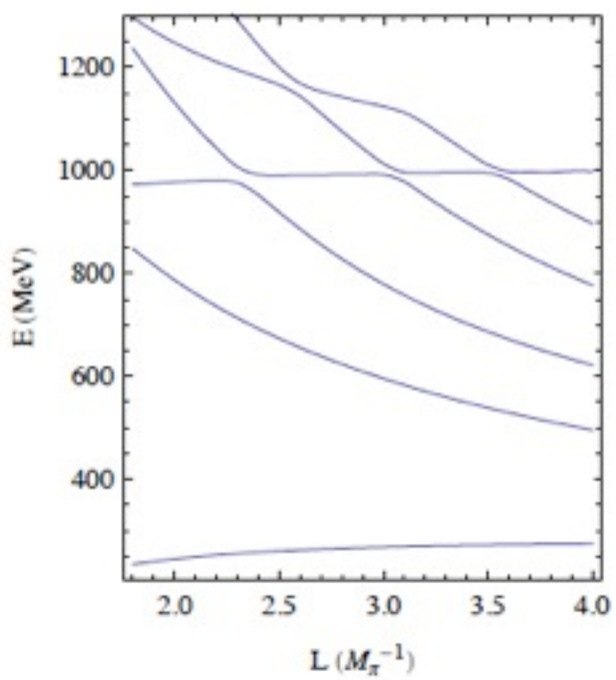
$\phi^P(q_j^*)$ is Lüscher's kinematic phase

- Consider "s-wave only" case for rest of talk

Practical quantization condition

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P}) \equiv \det\left[\begin{pmatrix} (F_1^s)^{-1} & 0 \\ 0 & (F_2^s)^{-1} \end{pmatrix} + i \begin{pmatrix} \mathcal{M}^s_{1\to1} & \mathcal{M}^s_{2\to1} \\ \mathcal{M}^s_{1\to2} & \mathcal{M}^s_{2\to2} \end{pmatrix}\right] = 0$$

Solving this equation with model for $\mathcal{M}^s$ leads to:



$$\Delta^{\mathcal{M}}(L, E^*, \vec{P} = 0) = 0$$

Extracting scatt. amp. from E_n

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P}) \equiv \det\left[\begin{pmatrix} (F_1^s)^{-1} & 0 \\ 0 & (F_2^s)^{-1} \end{pmatrix} + i \begin{pmatrix} \mathcal{M}_{1\to1}^s & \mathcal{M}_{2\to1}^s \\ \mathcal{M}_{1\to2}^s & \mathcal{M}_{2\to2}^s \end{pmatrix}\right] = 0$$

- Write scattering amplitude in terms of S-matrix

$$i\left(\mathcal{M}\right) = \left(Q\right)\left[\left(S\right) - 1\right]\left(Q\right)$$

$$\left(Q\right) = \sqrt{4\pi E^*}\begin{bmatrix} (q_1^*\eta_1)^{-1/2} & 0 \\ 0 & (q_2^*\eta_2)^{-1/2} \end{bmatrix}$$ is phase-space factor

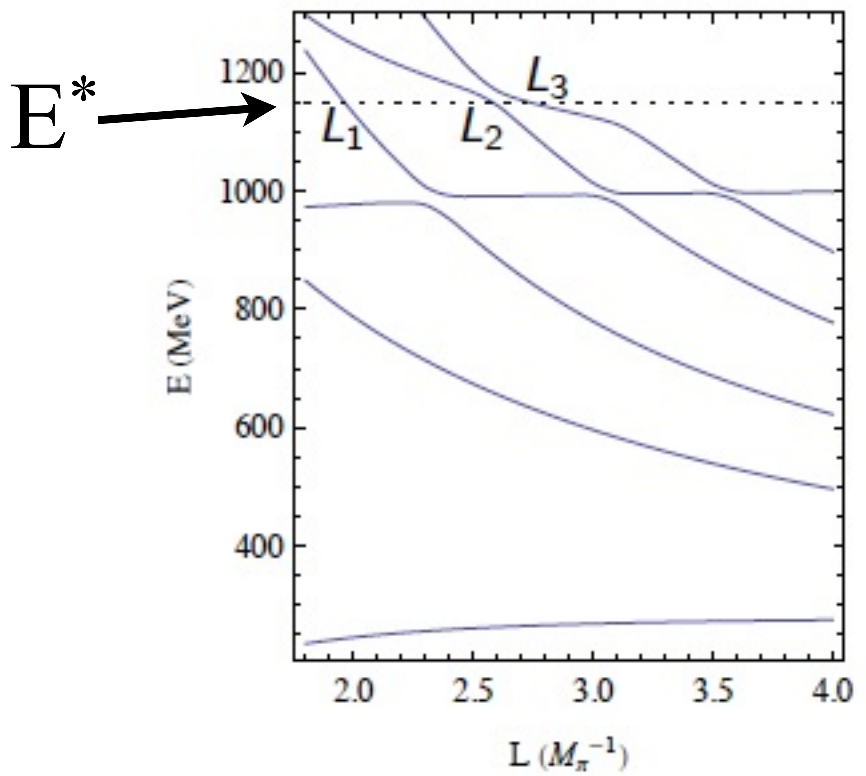
- S is unitary, and can be taken to be symmetric (T inv.)
  - ✴ Determined by 3 real physical parameters, e.g.

$$S^s = \begin{pmatrix} c_\epsilon & -s_\epsilon \\ s_\epsilon & c_\epsilon \end{pmatrix}\begin{pmatrix} e^{2i\delta_\alpha} & 0 \\ 0 & e^{2i\delta_\beta} \end{pmatrix}\begin{pmatrix} c_\epsilon & s_\epsilon \\ -s_\epsilon & c_\epsilon \end{pmatrix}$$

Extracting scatt. amp. from E_n

$$\Delta^{\mathcal{M}}(L, E^*, \vec{P}) \equiv \det\left[\begin{pmatrix} (F_1^s)^{-1} & 0 \\ 0 & (F_2^s)^{-1} \end{pmatrix} + i \begin{pmatrix} \mathcal{M}^s_{1\to1} & \mathcal{M}^s_{2\to1} \\ \mathcal{M}^s_{1\to2} & \mathcal{M}^s_{2\to2} \end{pmatrix}\right] = 0$$

- Need three FV states at given E^* to determine $\mathcal{M}^s$
- Example



$$\Delta^{\mathcal{M}}(L_1, E^*, \vec{P}_1) = 0$$
$$\Delta^{\mathcal{M}}(L_2, E^*, \vec{P}_2) = 0$$
$$\Delta^{\mathcal{M}}(L_3, E^*, \vec{P}_3) = 0$$

Here $\vec{P}_1 = \vec{P}_2 = \vec{P}_3 = 0$

- Or use three different values of $\vec{P}$

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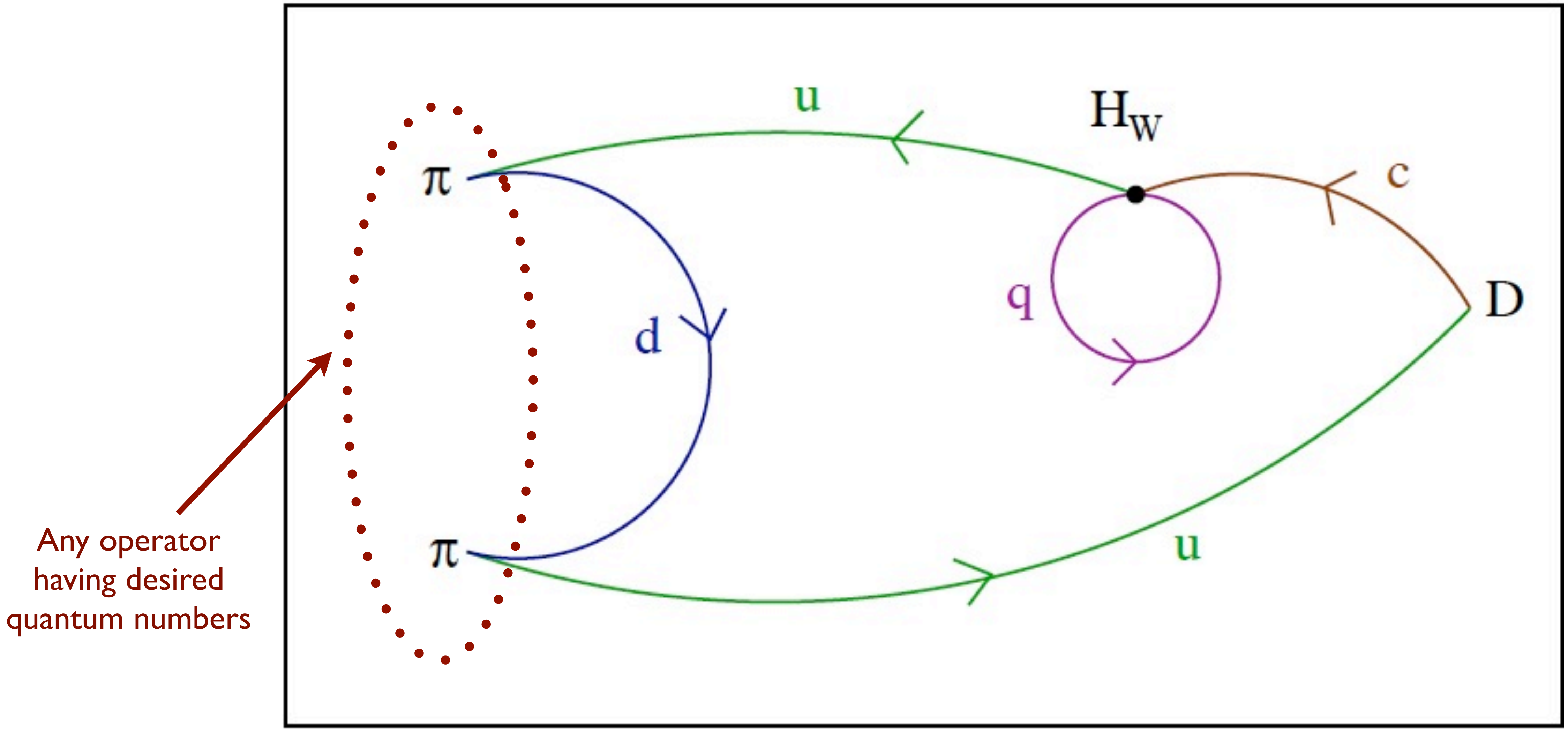
Recall need for LL formula



$\Rightarrow \quad M_{D\to n} = {}_L\langle n|\mathcal{H}_W|D\rangle_L$ for several choices of $\vec{P}$ and L s.t. $E_n = \sqrt{M_D^2 + \vec{P}^2}$

From these want to obtain $\mathcal{A}_{D\to\pi\pi} = \langle \pi\pi|\mathcal{H}_W|D\rangle$ & $\mathcal{A}_{D\to KK} = \langle K\bar{K}|\mathcal{H}_W|D\rangle$

Result

- Need to determine coefficients in decomposition:

$$_L\langle n| = C_\pi \langle \pi\pi, \mathrm{out}| + C_K \langle K\overline{K}, \mathrm{out}| + \cdots$$

- We find that each choice of $(L, \vec{P})$ determines

$$M_\infty = e^{i\phi_1}\sqrt{q_1^* \eta_1} A_{D\to\pi\pi} + z e^{i\phi_2}\sqrt{q_2^* \eta_2} A_{D\to KK}$$

$$z = \tan(\epsilon)\frac{\sin(\delta_\beta+\phi_1)}{\sin(\delta_\beta+\phi_2)}$$

$$S^g = \begin{pmatrix} c_\epsilon & -s_\epsilon \\ s_\epsilon & c_\epsilon \end{pmatrix} \begin{pmatrix} e^{2i\delta_\alpha} & 0 \\ 0 & e^{2i\delta_\beta} \end{pmatrix} \begin{pmatrix} c_\epsilon & s_\epsilon \\ -s_\epsilon & c_\epsilon \end{pmatrix}$$

$\phi_j \equiv \phi^P(q_j^*)$ are Lüscher's kinematic phases

- Can show using generalized quantization condition and Watson's theorem that M_∞ is real

- Need to use generalized quantization condition to determine δ_β & ε in order to extract amplitudes from M_∞

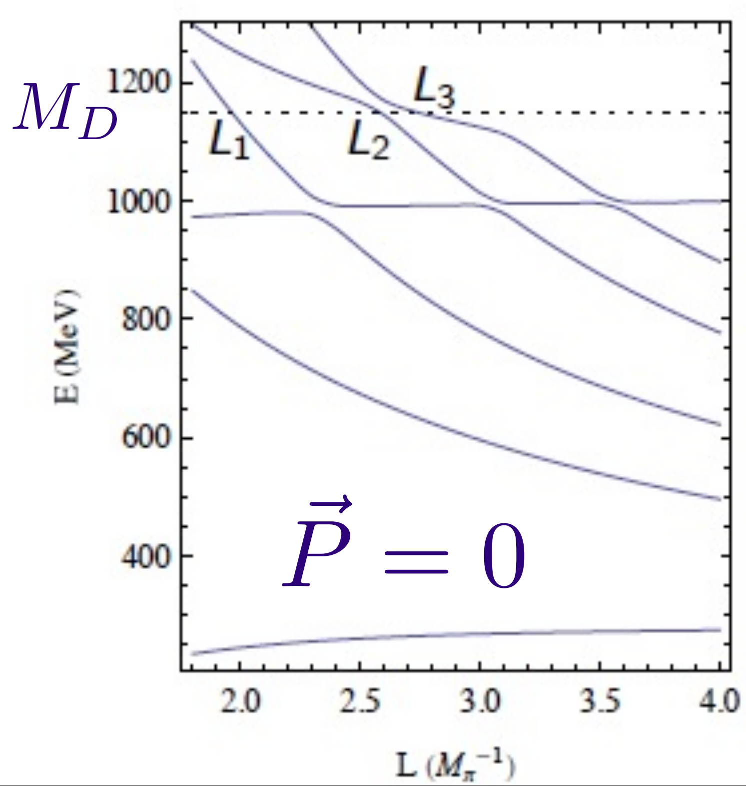
Result

- Explicit result for $M_\infty = e^{i\phi_1}\sqrt{q_1^*\eta_1}A_{D\to\pi\pi} + ze^{i\phi_2}\sqrt{q_2^*\eta_2}A_{D\to KK}$

$$M_\infty^2 = -|M_{D\to n}|^2(16\pi M_D E_n L^6)\frac{\frac{\partial\phi_1}{\partial L} + z^2\frac{\partial\phi_2}{\partial L}}{\frac{dE_n}{dL} + \frac{\vec{P}^2}{E_n L}}$$

✴ In addition to δ_β & ε, need to determine dE_n/dL to use this formula

- To separately extract the two (complex) infinite volume amplitudes, need 3 values of $(L,\vec{P})$ satisfying $E_n = E_D$
- Positivity of LHS implies $dE_n^*/dL < 0$ for all spectral lines above all thresholds
- Example



Need $M_{D\to n}$ and slope dE_n/dL at L_1, L_2, L_3

Clearly very challenging in practice!

Comments

$$M_\infty^2 = -|M_{D\to n}|^2 (16\pi M_D E_n L^6) \frac{\frac{\partial \phi_1}{\partial L} + z^2 \frac{\partial \phi_2}{\partial L}}{\frac{dE_n}{dL} + \frac{\vec{P}^2}{E_n L}}$$

- Result in terms of spectral energies E_n is unfamiliar
- Single channel result (original LL result) can be expressed similarly

$$\Gamma_{D\to\pi\pi} = \frac{2E_n L^6 |M_{D\to n}|^2}{M_D} \left[\frac{-\frac{\partial \phi}{\partial L}}{\frac{dE_n}{dL} + \frac{\vec{P}^2}{E_n L}} \right]$$

✶ Need only determine E_n & dE_n/dL if want width

✶ Since $\partial\varphi/\partial L<0$ must have $dE_n^*/dL < 0$ if spectral line above threshold

Comments

- Generalized Watson's theorem shows that following combinations are real

$$v_1 = e^{-i\delta_\alpha}\left[\sqrt{q_1^*\eta_1}A_{D\to\pi\pi}\,\mathrm{c}_\epsilon + \sqrt{q_2^*\eta_2}A_{D\to KK}\,\mathrm{s}_\epsilon\right],$$
$$v_2 = e^{-i\delta_\beta}\left[-\sqrt{q_1^*\eta_1}A_{D\to\pi\pi}\,\mathrm{s}_\epsilon + \sqrt{q_2^*\eta_2}A_{D\to KK}\,\mathrm{c}_\epsilon\right]$$

- Can express M_∞ in terms of these real quantities

$$M_\infty = \sin(\phi_1-\phi_2)\left[-v_1\frac{c_\epsilon}{\sin(\delta_\alpha+\phi_2)} + v_2\frac{s_\epsilon}{\sin(\delta_\beta+\phi_2)}\right]$$

  - Relative sign of v_1 & v_2 is physical
  - Need three determinations of $|M_\infty|$ to determine v_j

Sketch of derivation

- Determining coefficients in decomposition of FV state knows nothing about D

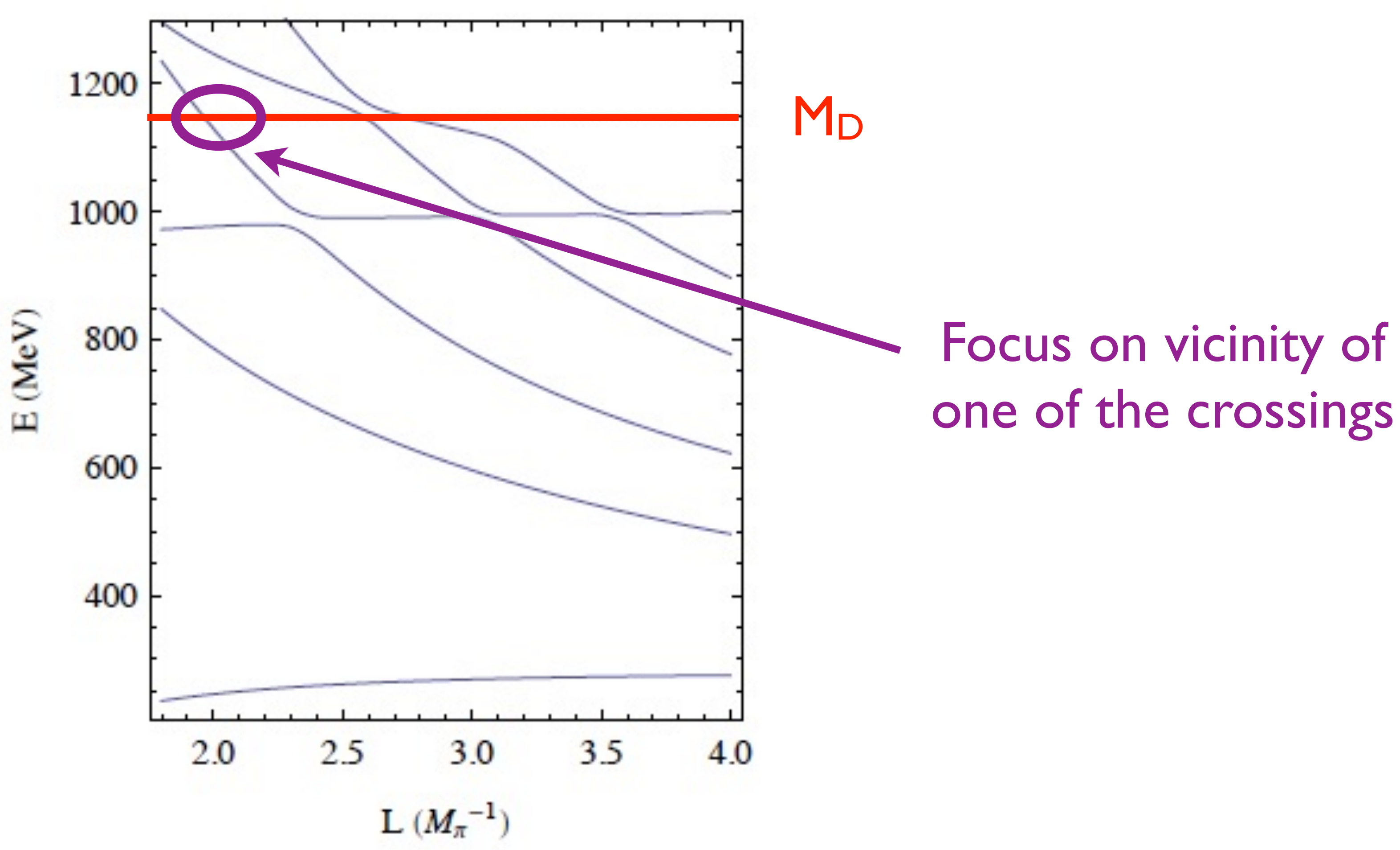
$$ {}_L\langle n| = C_\pi \langle \pi\pi, \mathrm{out}| + C_K \langle K\overline{K}, \mathrm{out}| + \cdots $$

- However, LL used a trick involving the D & the quantization condition to pick out the coefficients, and we generalize it to the multiple-channel case

Sketch of derivation

- Begin with strong-interaction spectrum for (say) I=0 and chosen $\vec{P}$
  - ✳ Two-particle spectrum E_n plus D meson (exactly stable)
- Example

$$\vec{P} = 0$$



Sketch of derivation

- Add an (arbitrarily) weak interaction into the full Hamiltonian, so that the D couples to ππ, etc. and thus becomes a resonance

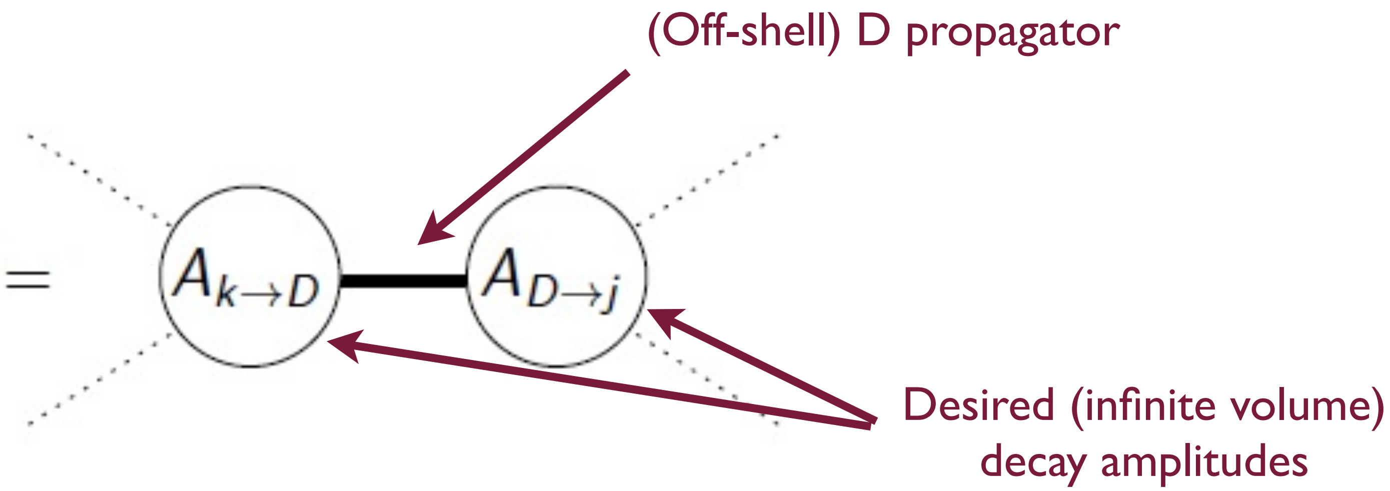
$$\mathcal{H}(x) \longrightarrow \mathcal{H}(x) + \lambda \mathcal{H}_W(x)$$

- Mixing implies level splitting: $E_n \rightarrow E_D + \Delta E$

$$\Delta E = \pm \lambda L^3 \left| {}_L\langle n | \mathcal{H}_W | D \rangle_L \right| \equiv \pm \lambda L^3 |M_{D\to n}|$$

- Scattering amplitude changes by

$$i\Delta\mathcal{M}_{jk} =$$



- The quantization condition changes to

$$\Delta^{\mathcal{M}+\Delta\mathcal{M}}(L, E^* = M_D + \lambda \Delta E^*, \vec{P}) = 0$$

- Expand in λ; linear term gives relation between FV and infinite volume amplitudes
- As expected, result is independent of form of H_W

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Deriving generalized quantization condition

- Method of KSS does not use auxiliary QM theory, but is directly based in QFT
- Simplifies generalization to arbitrary 3-momentum and multiple channels
- Work in EFT describing interactions between stable hadrons (π, K) with NO assumptions about form of interactions (other than range < L)
- Consider two-point correlator in finite volume with fixed total $\vec{P}$; poles in E give positions of spectral lines

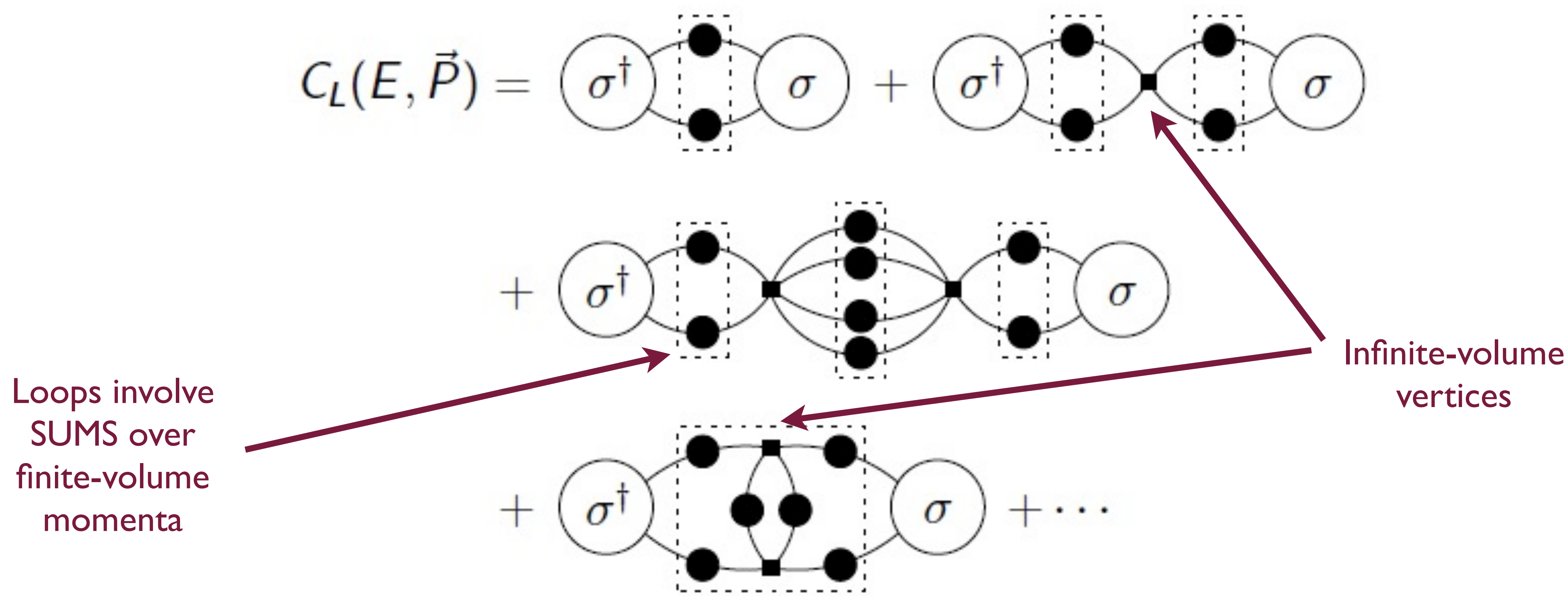
$$C_L(E,\vec{P}) \equiv \int_L d^4x\ e^{-i\vec{P}\cdot\vec{x}+iEt}\langle\Omega|T\sigma(x)\sigma^\dagger(0)|\Omega\rangle_L$$

Any operator coupling to desired two particle states

Deriving generalized quantization condition

$$C_L(E,\vec{P}) \equiv \int_L d^4x\; e^{-i\vec{P}\cdot\vec{x}+iEt}\langle\Omega|T\sigma(x)\sigma^\dagger(0)|\Omega\rangle_L$$

Consider first a single two-particle channel [KSS]



$$= i\frac{z(q)}{(q^0)^2 - \vec{q}^{\,2} - M_\pi^2 + i\epsilon}$$

Full propagators

Deriving generalized quantization condition

- Key point: finite-volume loop sums differ from infinite-volume loop integrals ONLY if ALL particles can go on shell for given E (aside from exponentially suppressed corrections)

- If all particles can go on shell then difference between FV and infinite volume loops is not exponentially suppressed---use summation formula of [KSS]

$$\left(\int \frac{dk_0}{2\pi}\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4}\right) f(k)\frac{1}{k^2-m_j^2+i\epsilon}\frac{1}{(P-k)^2-m_j^2+i\epsilon}g(k)$$

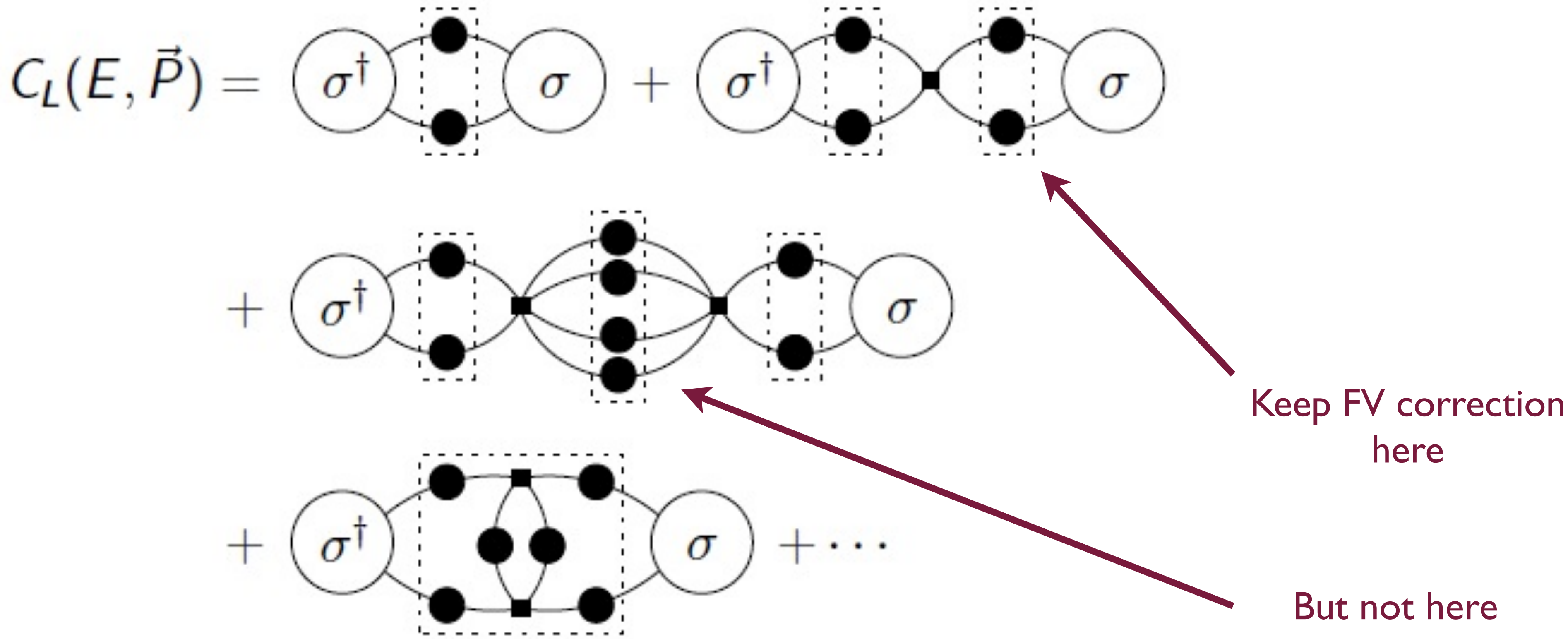
$$= \int d\Omega_{q^*} d\Omega_{q^{*\prime}} f_j^*(\hat{q}^*)\mathcal{F}_{jj}(q^*,q^{*'})g_j^*(\hat{q}^{*'})$$

When decompose in spherical harmonics get F given earlier

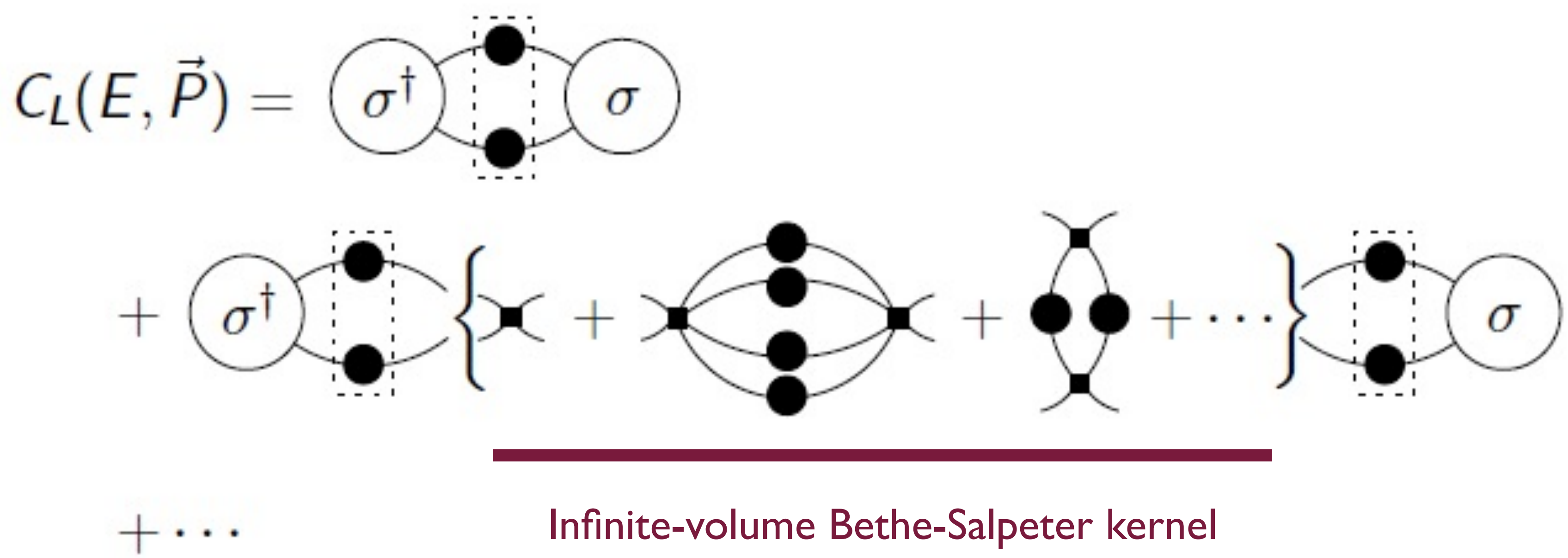
Evaluated at on-shell kinematics

- Only get FV corrections in toy model from ππ and KK loops (and not 4π, etc.)

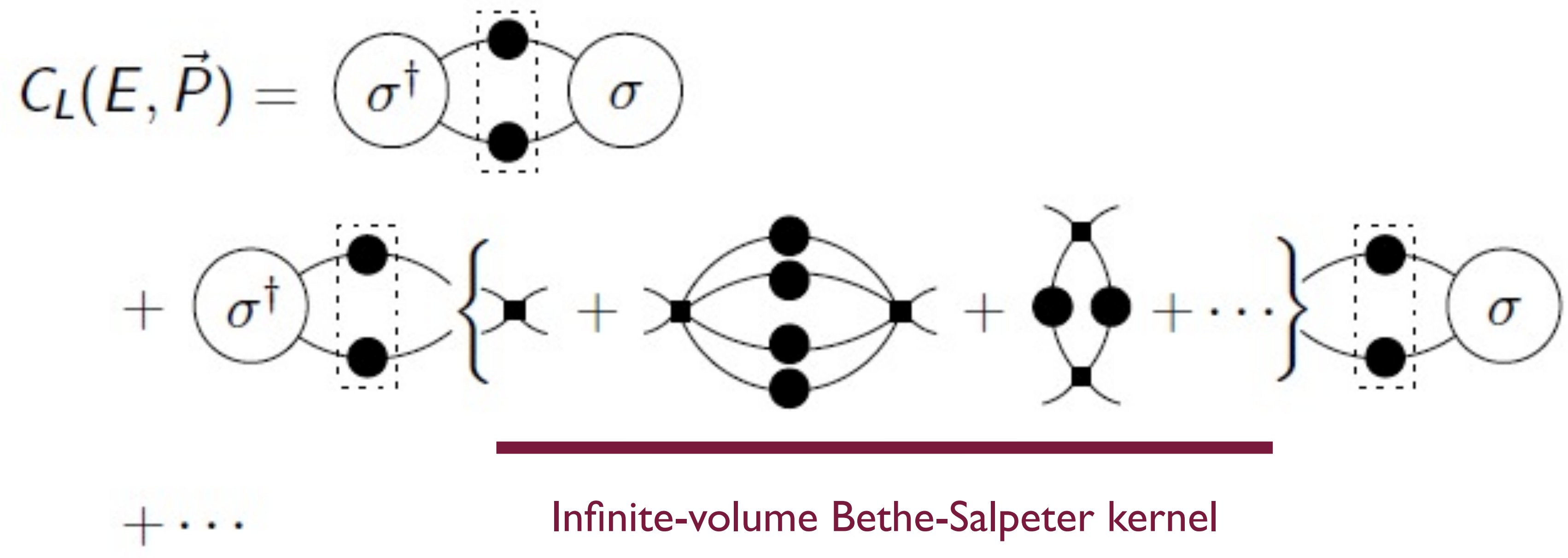
Deriving generalized quantization condition



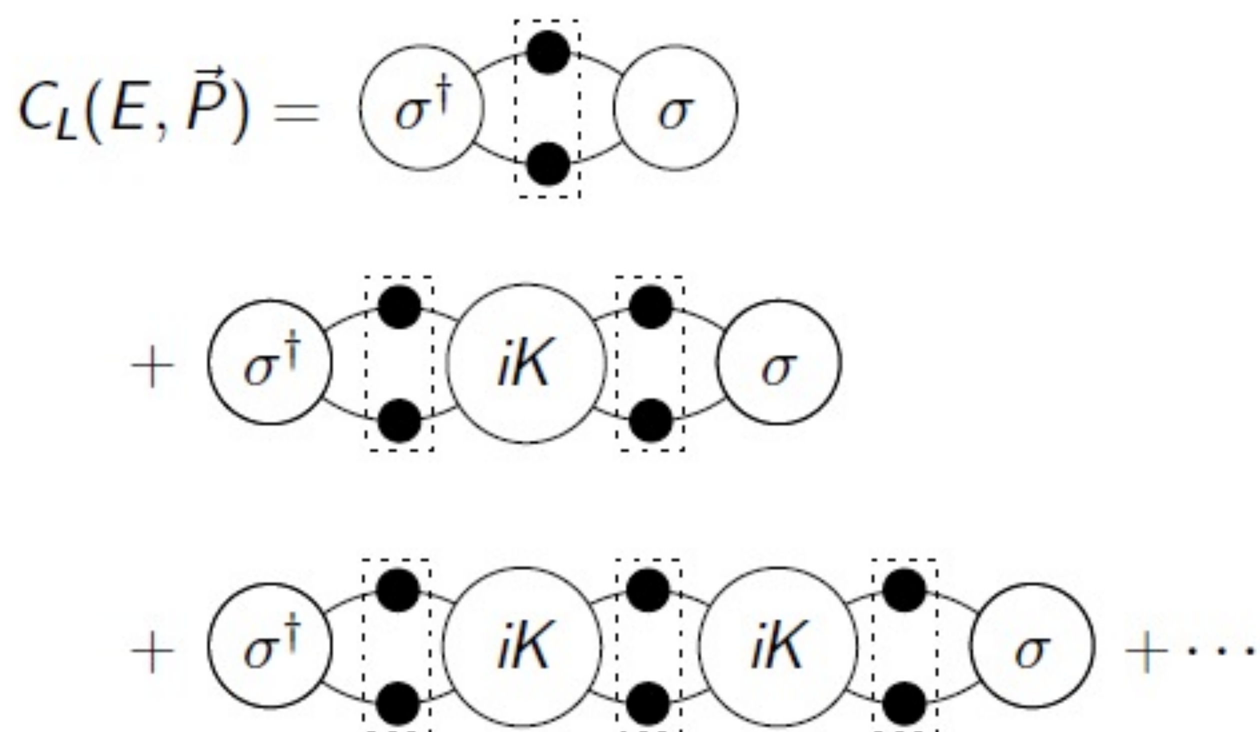
So can rewrite as:



Deriving generalized quantization condition

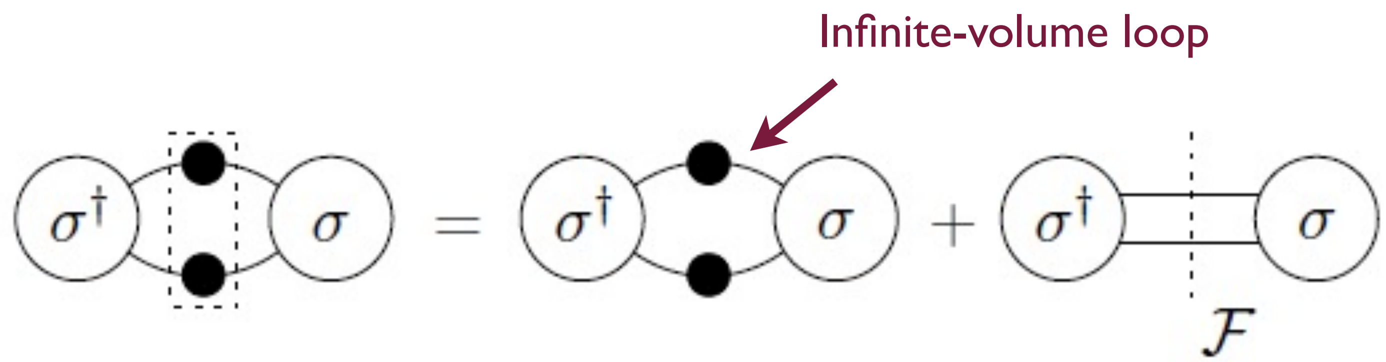


Working to higher order get:



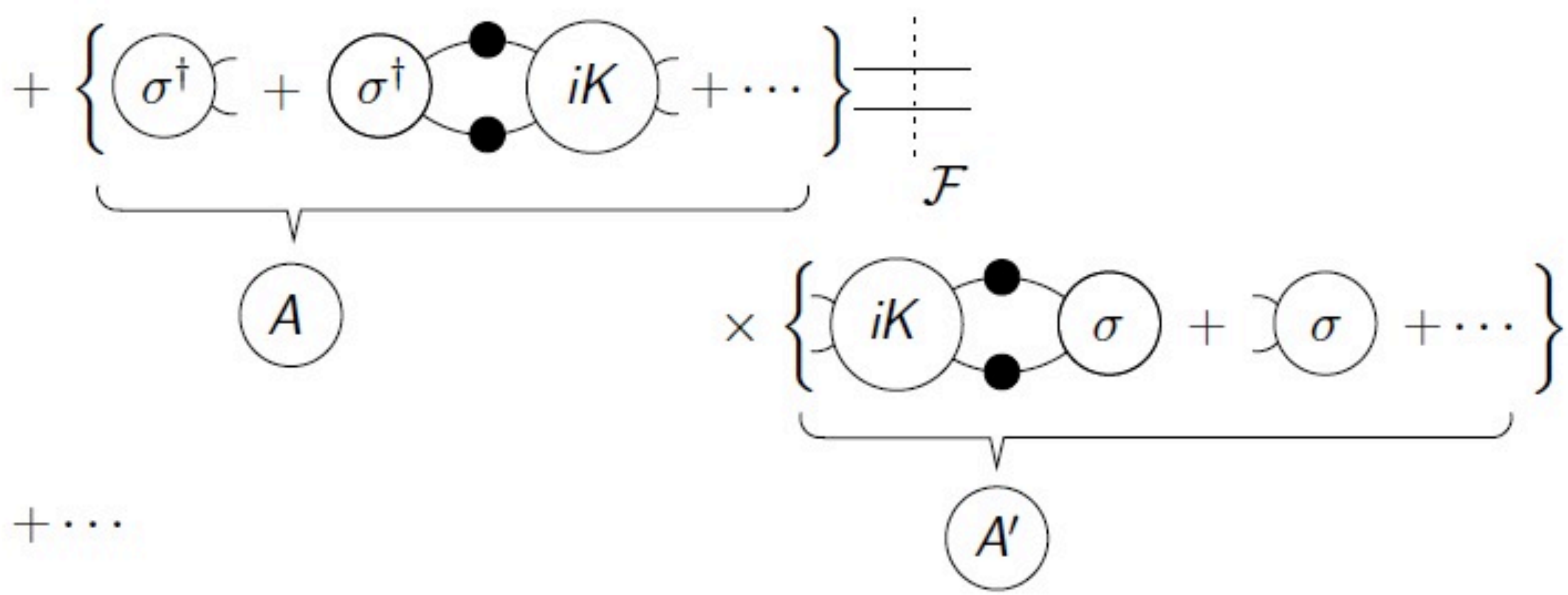
Deriving generalized quantization condition

Now make substitutions such as:



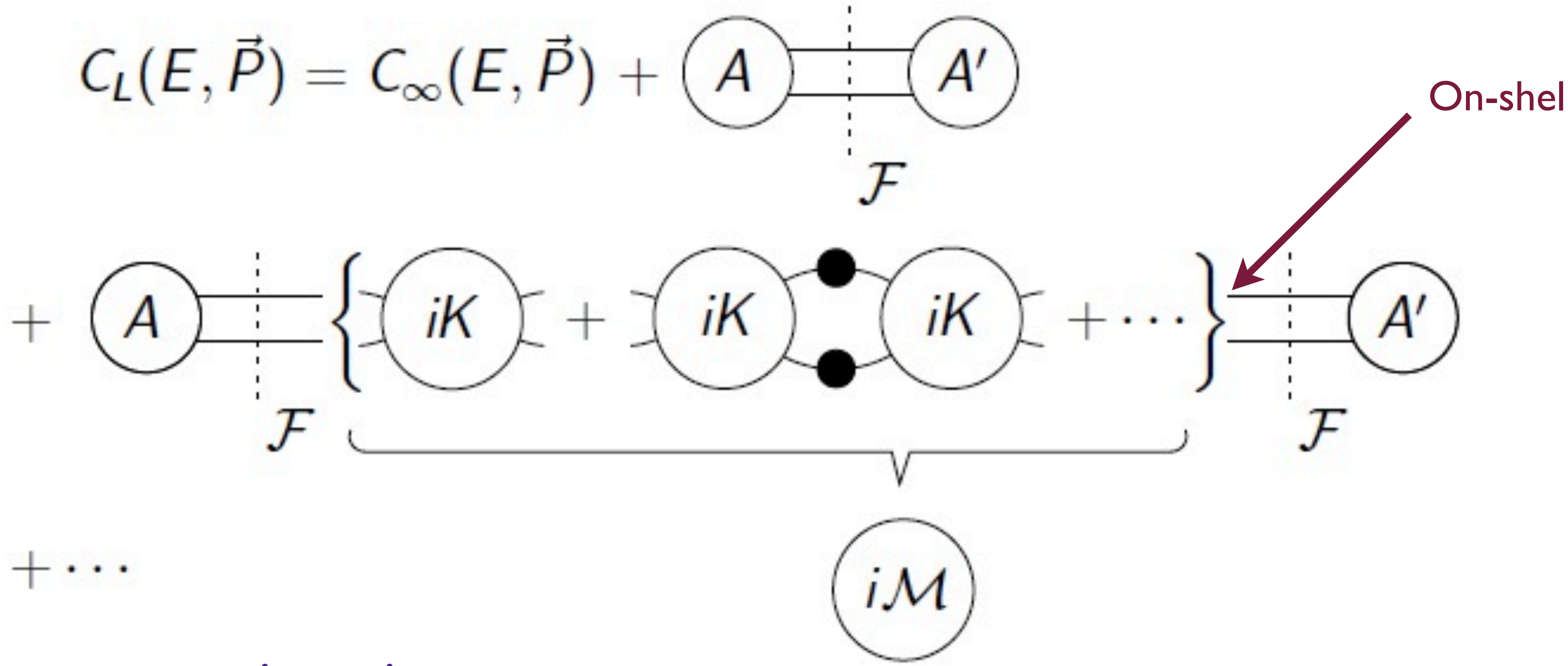
And obtain:

$$C_L(E,\vec{P}) = C_\infty(E,\vec{P})$$

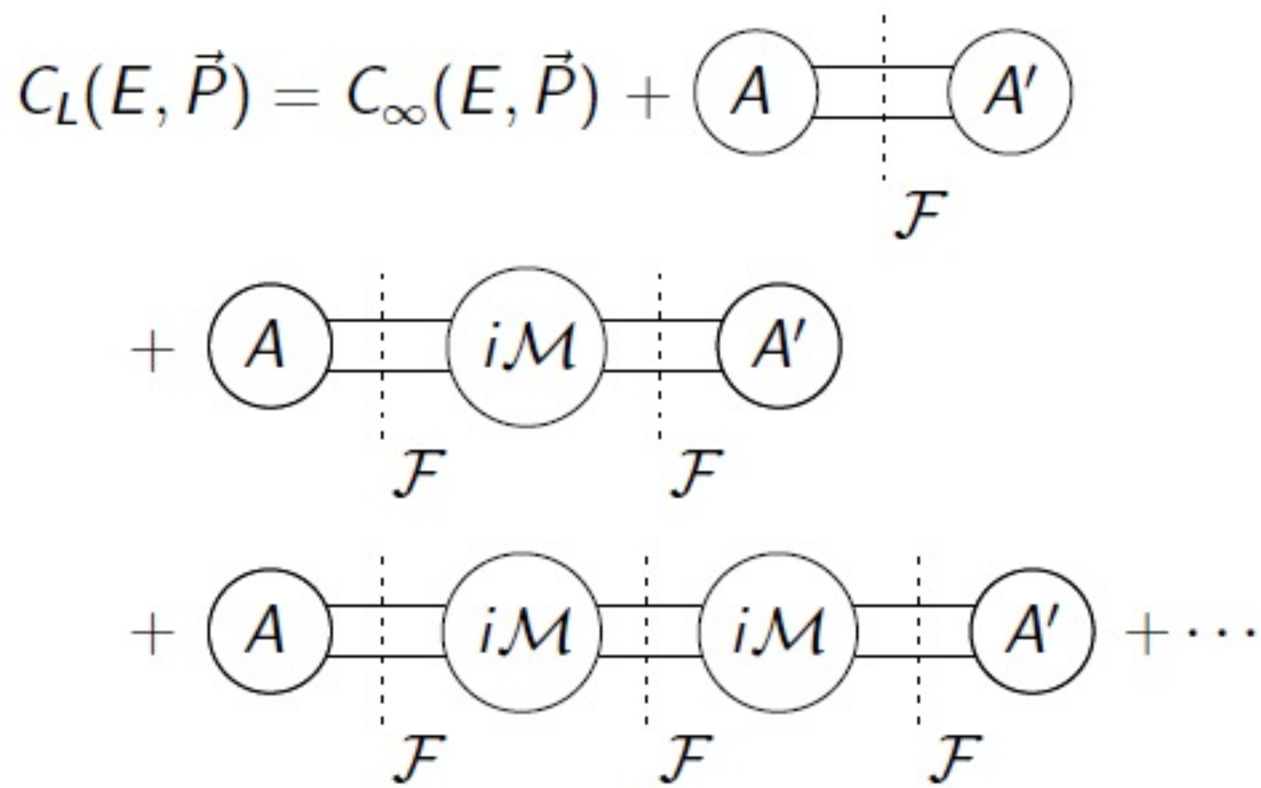


Deriving generalized quantization condition

At second-order in F full on-shell scattering amplitude appears:



And so get geometric series:



Deriving generalized quantization condition

We conclude

$$C_L(E,\vec{P}) - C_\infty(E,\vec{P}) = -\sum_{n=0}^{\infty} A' F[-i\mathcal{M}F]^n A = -A' \frac{1}{F^{-1} + i\mathcal{M}} A$$

So for given values of $\{L, \vec{n}_P\}$, the energies in the spectrum are all E^* for which

$$\det(F^{-1} + i\mathcal{M}) = 0\,.$$

Generalization to multiple 2-particle channels requires addition of channel index for different "cuts", and Bethe-Salpeter kernel becomes irreducible in s-channel w.r.t. both two pions and two kaons

Outline

- Motivation
- Problem to be solved (multiple 2-particle channels)
- Generalized Luscher quantization condition
- Generalized Lellouch-Luscher formula
- Technical details
- Other applications & outlook

Other applications of generalized LL formula

- K→ππ including isospin breaking
  - $\pi^+\pi^-$ and $\pi^0\pi^0$ not degenerate---so have two 2-particle channels and no further inelasticity
- K→ππ using staggered fermions including taste breaking
- $\Omega^- \to \Lambda K^-,\ \Xi^0\pi^-,\ \Xi^-\pi^0$
  - 3 body $\Xi\pi\pi$ channel highly suppressed
- Many applications of generalized Luscher quantization formula already investigated

Outlook for $D \to \pi\pi, KK$

- Including multiple 2-particle channels is only first step
- Even that step will be challenging numerically as need multiple spectral lines
- Some simplification may be possible by changing the boundary conditions (G-parity-like?)
- Key problem is including 4π channel
    - ✳ Quantization condition for 3 NR particles obtained by [Polejaeva & Rusetsky] with related work by [Kreuzer & Griesshammer]
- Should be possible in principle