On the Art of Computing Resonances

– Masses

– Widths

– Form factors

G. Schierholz

Deutsches Elektronen-Synchrotron DESY

– QCDSF Collaboration –

With

M. Göckeler, R. Horsley, M. Lage, U.-G. Meißner, R. Millo,

Y. Nakamura, D. Pleiter, P.E.L. Rakow, A. Rusetsky and J. Zanotti

What makes Lattice Hadron Mass Calculations interesting?

- Can vary quark masses over a wide range of absolute and relative values

- Determination of low-energy constants

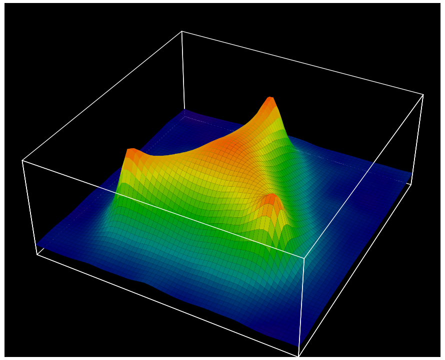
- Study of pattern of flavor symmetry breaking

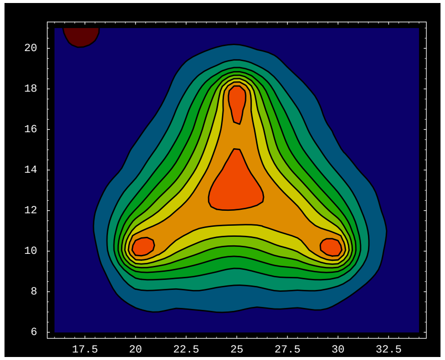
- Origin of mass (differences)

- $$\lambda \frac{\partial m_H}{\partial \lambda} = \langle H | \mathcal{O} | H \rangle$$

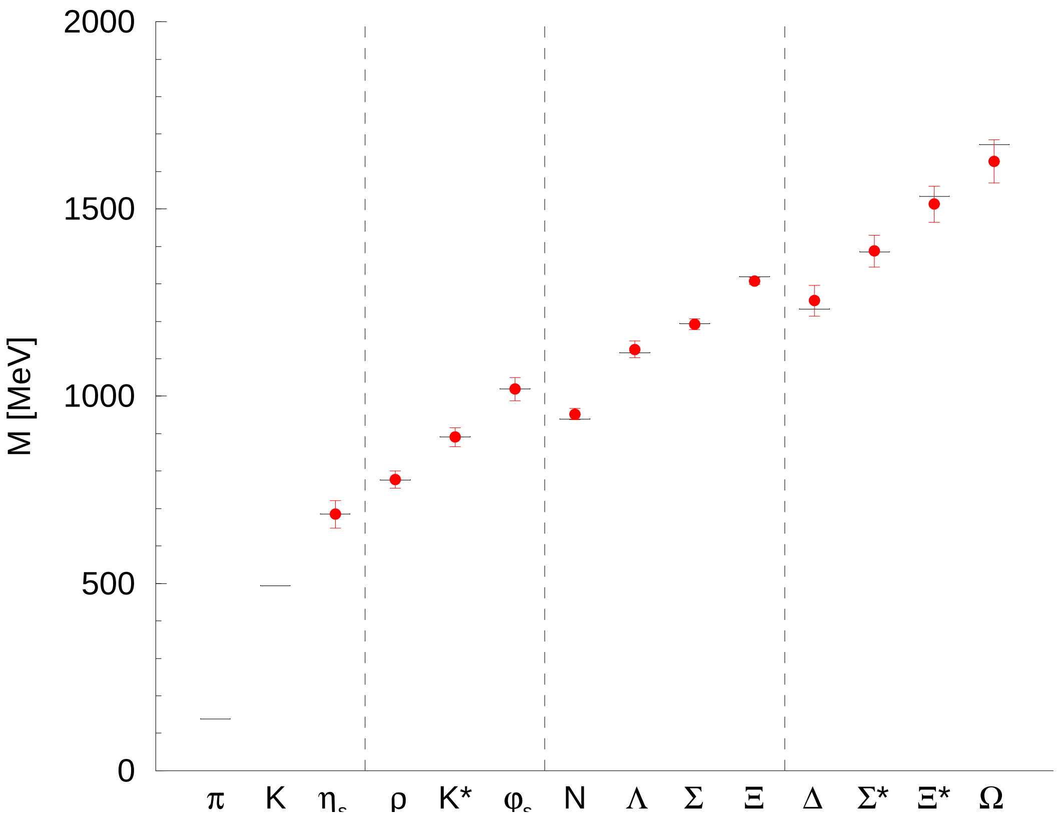
  $$S \to S + \lambda \mathcal{O}\,, \ \ \mathcal{O} = \bar{q}\gamma_\mu \gamma_5 q\,, \cdots$$

  Feynman-Hellmann



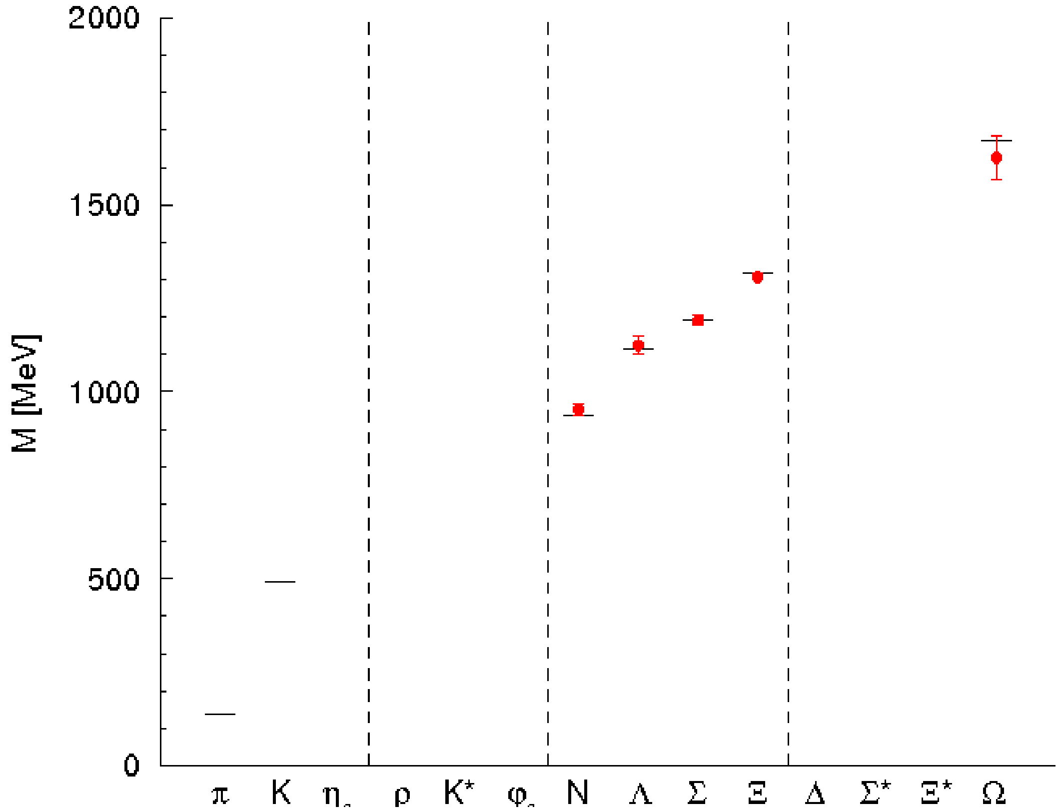


Meson & Baryon Spectrum

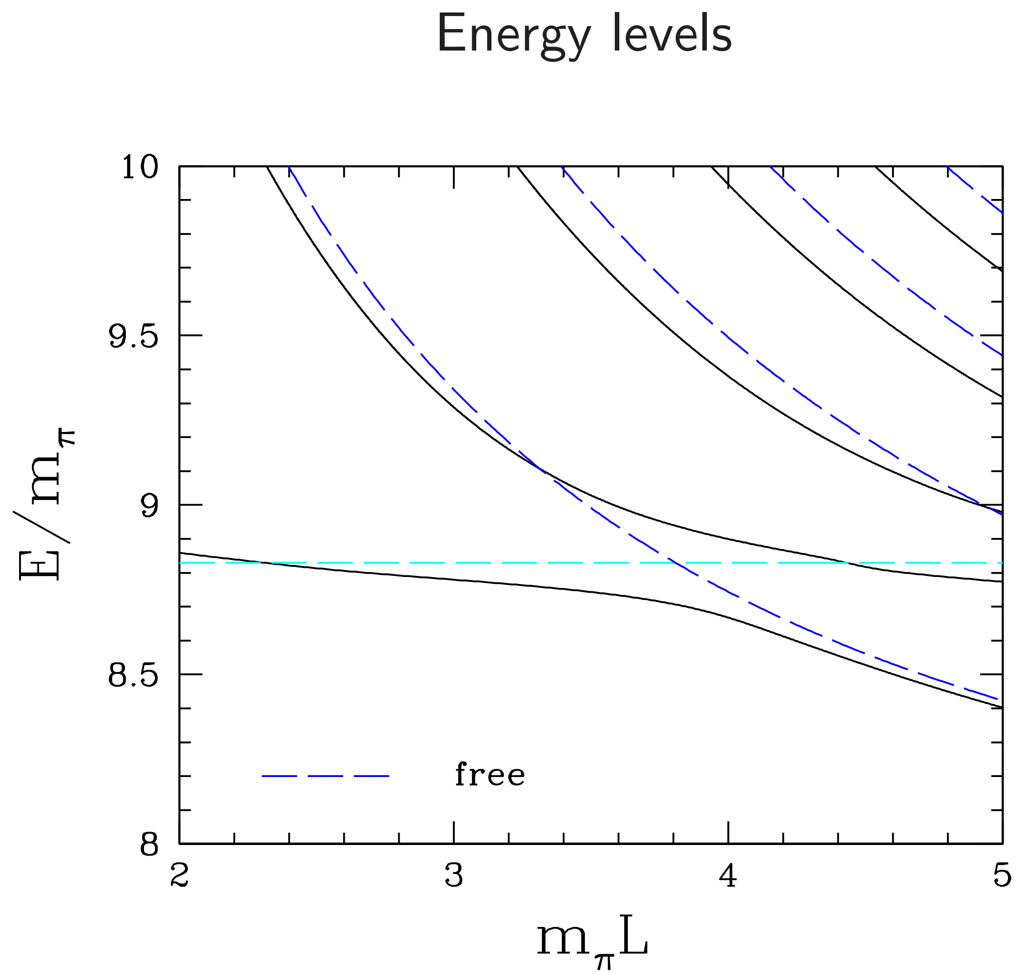


Most hadrons are resonances

Stable particles (under strong interactions)



Agreement pure coincidence $\cdots$



$$\Delta(1232) \to N\,\pi \qquad m_\pi = 140\,\text{MeV}$$

Outline

Resonances on the Lattice

Moving Frame vs. Rest Frame

Application to $\Delta(1232)$

Conclusions

Resonances on the Lattice

Quantization Conditions in finite Volume

Consider, for simplicity, two-pion resonance at rest $\boldsymbol{P} = 0$

Free case $E = 2\sqrt{k^2 + m_\pi^2}$ $\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{n}\,, \quad \boldsymbol{n} \in \mathbb{Z}^3$

Interacting case $E = 2\sqrt{k^2 + m_\pi^2}$ $n\pi - \delta(k) = \phi(q)\,, \quad q = \frac{kL}{2\pi}$ Lüscher

$$\cot\phi(q) = -\frac{1}{\pi^{3/2} q}\,\mathcal{Z}_{00}(1; q^2)$$

$$\underline{\delta(k) = \pi/2} \;\Rightarrow\; \cot\phi(q) = 0$$

$$\mathcal{Z}_{00}(1; q^2) \overset{s \to 1}{=} \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \frac{\mathcal{Y}_{00}(\boldsymbol{n})}{(\boldsymbol{n}^2 - q^2)^s}$$

Hypothetical Energy Levels

Effective range formula ρ Resonance

$$\frac{k^3}{E}\cot\delta_{11}(k) = \frac{24\pi}{g^2_{\rho\pi\pi}}\left(k^2_\rho - k^2\right) \qquad k = \frac{1}{2}\sqrt{E^2 - 4m_\pi^2}\,, \quad k_\rho = \frac{1}{2}\sqrt{m_\rho^2 - 4m_\pi^2}$$

$$\Gamma_\rho = \frac{g^2_{\rho\pi\pi}}{6\pi}\frac{k_\rho^3}{m_\rho^2} \qquad \frac{E}{m_\pi} = 2\sqrt{1 + \left(\frac{2\pi}{m_\pi L}\right)^2 q^2}$$

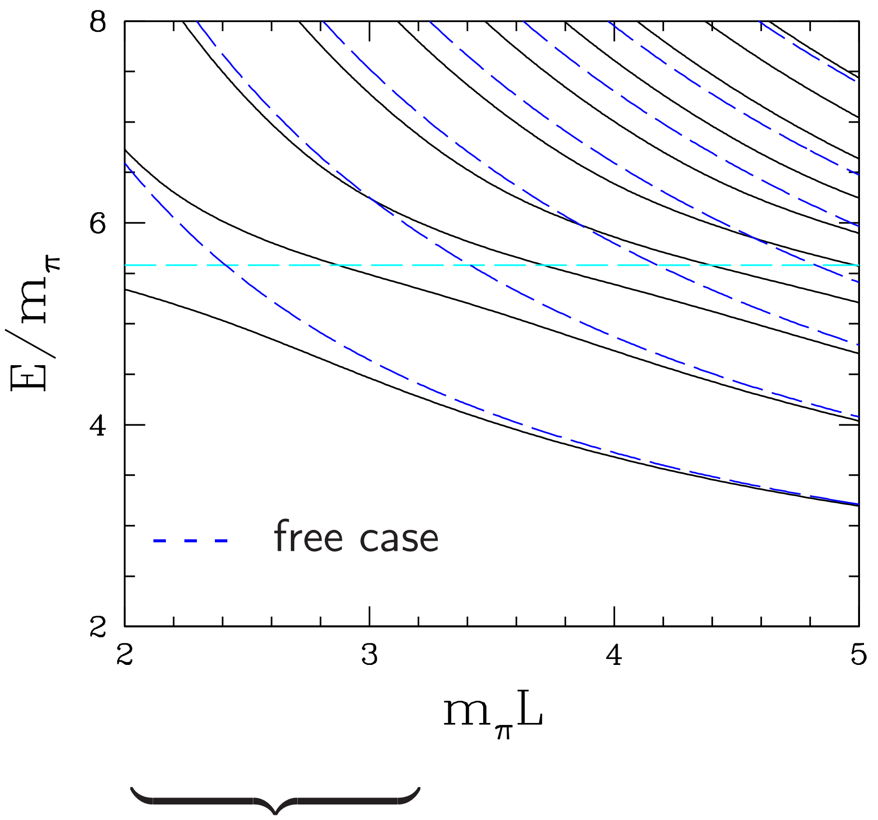
↑ solve

Free case

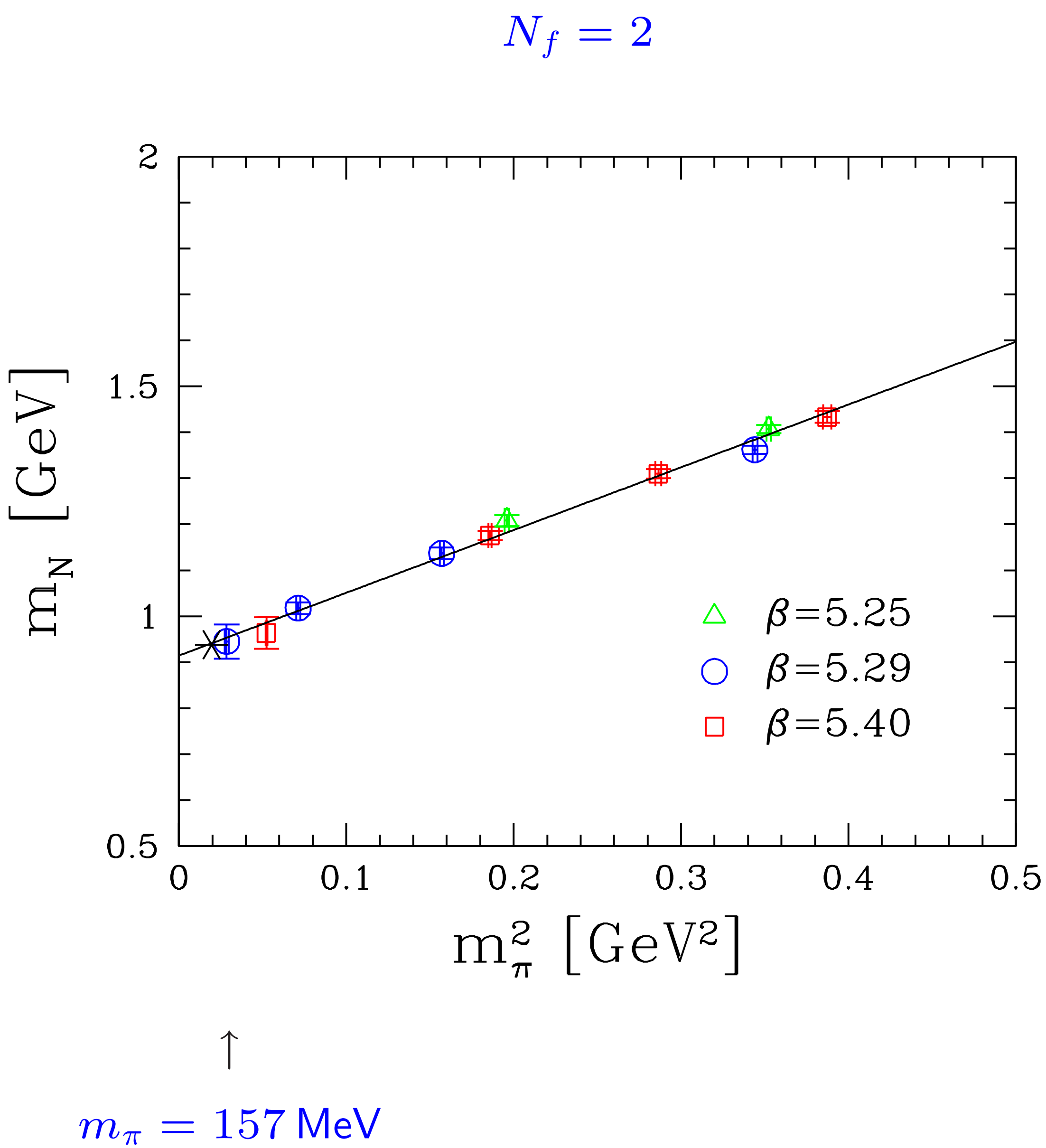
$$\frac{E}{m_\pi} = 2\sqrt{1 + \left(\frac{2\pi}{m_\pi L}\right)^2 \boldsymbol{n}^2}$$

$\boldsymbol{P} = 0$

Physical m_π, m_ρ and Γ_ρ



Useful region

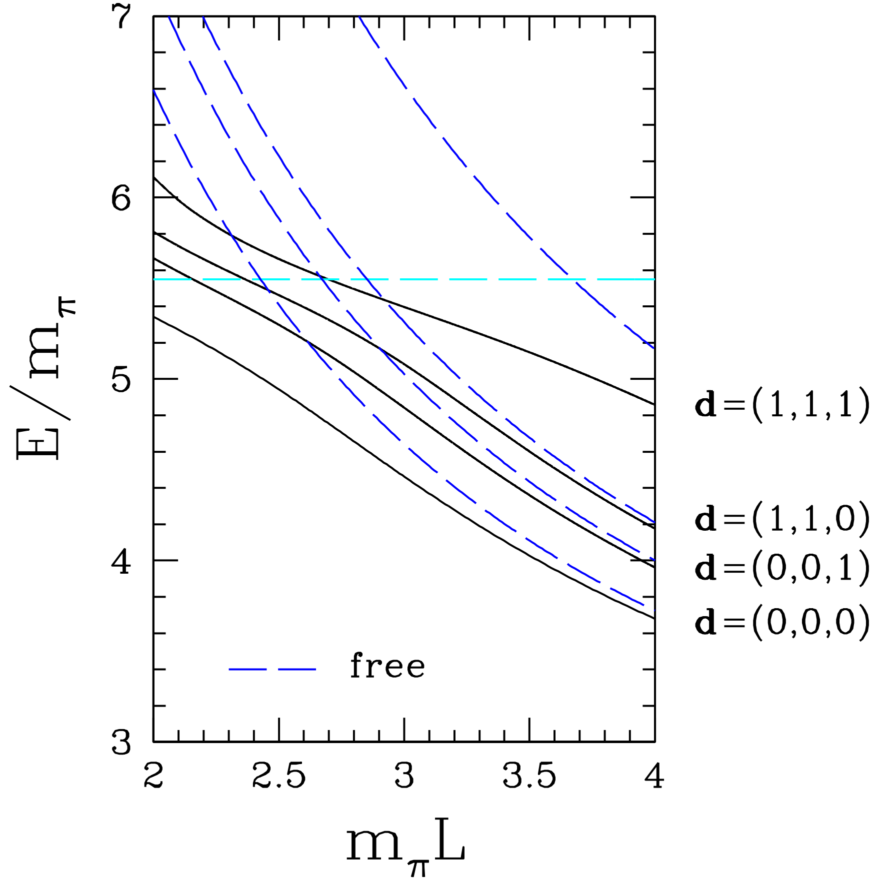


$N_f = 2$
2
1.5
1
0.5
0
0.1
0.2
0.3
0.4
0.5
m_N [GeV]
m_π^2 [GeV^2]
β=5.25
β=5.29
β=5.40
↑
$m_\pi = 157$ MeV

Moving Frame vs. Rest Frame

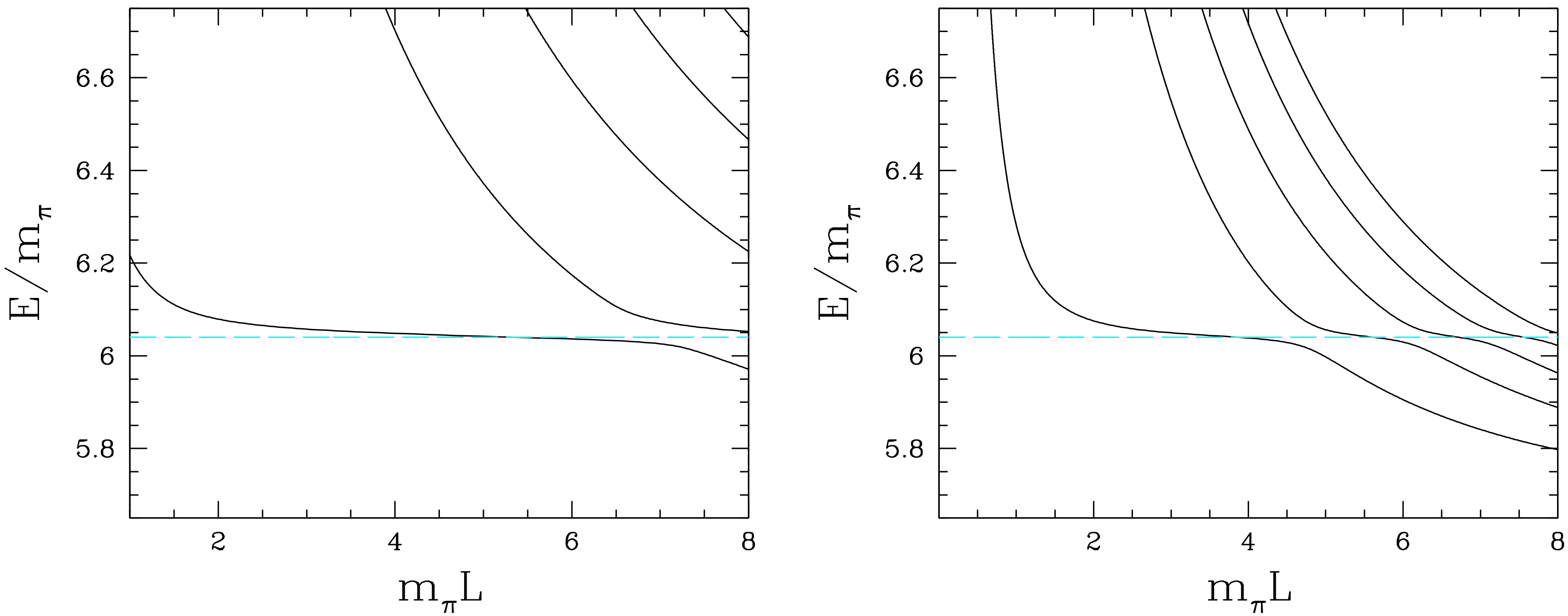
$$\boldsymbol{P} = \frac{2\pi}{L}\,\boldsymbol{d}$$

Physical m_π, m_ρ and Γ_ρ



Lowest level only

$\Sigma^*(1385) \to \Lambda\pi\,, \quad m_\pi = 230\,\text{MeV}$



$\boldsymbol{P} = 0$ $\qquad\qquad$ $\boldsymbol{P} = \dfrac{2\pi}{L}\,(0, 0, 1)$

Besides allowing for many more energy levels on a single lattice volume, $P > 0$ shifts avoided level crossing

- for ρ resonance, and generally for resonances of equal mass particles, to larger values of $m_\pi L$

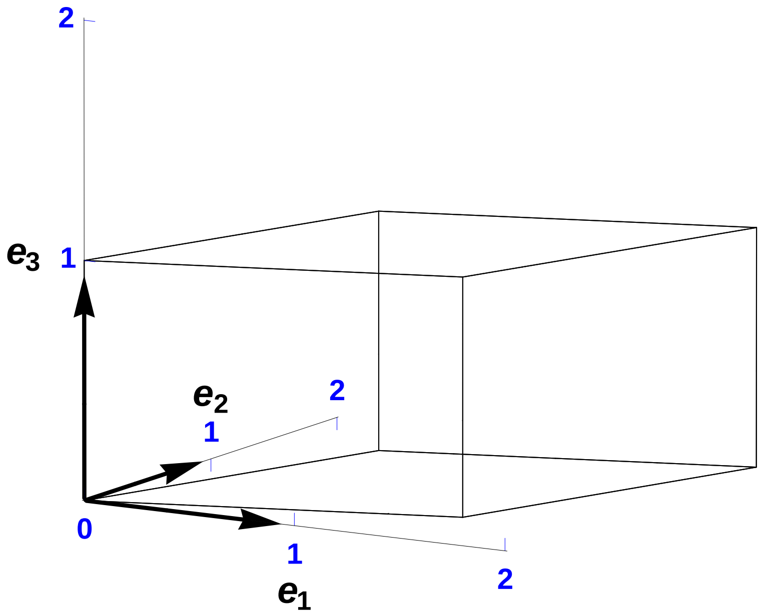
- for Δ, Σ^* resonances, and generally for P-wave resonances of unequal mass particles, to smaller values of $m_\pi L$

$P > 0$ greatly improves the computation of phase shifts on the lattice

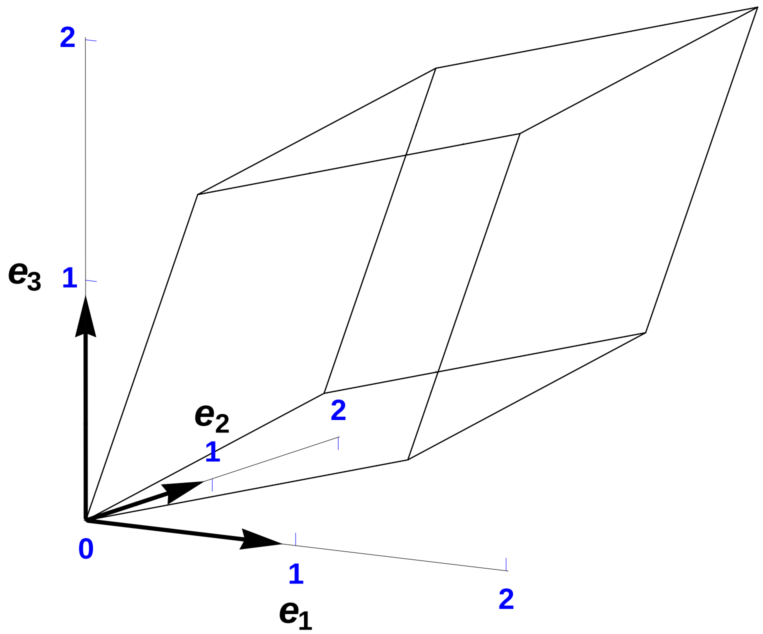
Preliminaries

CM

$$\boldsymbol{x} \to \boldsymbol{\gamma x}\,, \quad \boldsymbol{\gamma} = 1 + \frac{1-\sqrt{1-v^2}}{\sqrt{1-v^2}}\,\frac{\boldsymbol{v}}{v}$$



$\boldsymbol{d} = (1, 1, 0)$



$\boldsymbol{d} = (1, 1, 1)$

Little groups

Group	$\boldsymbol{d}$	Little Group	$R_i\,\boldsymbol{d} = \boldsymbol{d}$	$R_i\,\boldsymbol{d} = -\boldsymbol{d}$
O_h	$(0, 0, 1)$	C_{4v}	$\{R_i \| i = 1, 14, 15, 24\}$	$\{R_i \| i = 18, 19, 22, 23\}$
	$(1, 1, 0)$	C_{2v}	$\{R_i \| i = 1, 18\}$	$\{R_i \| i = 19, 24\}$
	$(1, 1, 1)$	C_{3v}	$\{R_i \| i = 1, 2, 3\}$	$\{R_i \| i = 17, 19, 21\}$
${}^2O_h$	$(0, 0, 1)$	${}^2C_{4v}$	$\{R_i \| i = 1, 4, 7, 10, 13, 16, 19, 48\}$	$\{R_i \| i = 2, 3, 5, 6, 38, 39, 44, 45\}$
	$(1, 1, 0)$	${}^2C_{2v}$	$\{R_i \| i = 1, 38, 44, 48\}$	$\{R_i \| i = 4, 7, 39, 45\}$
	$(1, 1, 1)$	${}^2C_{3v}$	$\{R_i \| i = 1, 20, 24, 28, 32, 48\}$	$\{R_i \| i = 37, 39, 41, 43, 45, 47\}$

Representations Γ, conjugacy classes and characters χ

$^2C_{4v}$

$\{R_i, IR_i\}$	R_1	$\{R_4, R_7\}$	$\{R_{10}, R_{13}\}$	$\{R_{16}, R_{19}\}$	$\{IR_2, IR_3, IR_5, IR_6\}$	$\{IR_{38}, IR_{39}, IR_{44}, IR_{45}\}$	R_{48}
Class / Γ	I	$2C_4$	$2C_8'$	$2C_8$	$4IC_4$	$4IC_4'$	J
A_1	1	1	1	1	1	1	1
A_2	1	1	1	1	-1	-1	1
B_1	1	1	-1	-1	1	-1	1
B_2	1	1	-1	-1	-1	1	1
E	2	-2	0	0	0	0	2
G_1	2	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	-2
G_2	2	0	$-\sqrt{2}$	$\sqrt{2}$	0	0	-2

Quantization Condition

$l \leq 1$

$$\det\left(M_{Jl,J'l'} - \delta_{JJ'}\delta_{ll'}\cot\,\delta_{Jl}\right) = 0$$

$$\left(M^{\Gamma}_{Jl,J'l'}\right) = \begin{pmatrix} M_{\frac{1}{2}0,\frac{1}{2}0} & M_{\frac{1}{2}0,\frac{1}{2}1} & M_{\frac{1}{2}0,\frac{3}{2}1} \\ M_{\frac{1}{2}1,\frac{1}{2}0} & M_{\frac{1}{2}1,\frac{1}{2}1} & M_{\frac{1}{2}1,\frac{3}{2}1} \\ M_{\frac{3}{2}1,\frac{1}{2}0} & M_{\frac{3}{2}1,\frac{1}{2}1} & M_{\frac{3}{2}1,\frac{3}{2}1} \end{pmatrix}$$

G_1

$$\left(M_{Jl,J'l'}\right) = \begin{pmatrix} w_{00} & iw_{10} & i\sqrt{2}w_{10} \\ -iw_{10} & w_{00} & \sqrt{2}w_{20} \\ -i\sqrt{2}w_{10} & \sqrt{2}w_{20} & w_{00}+w_{20} \end{pmatrix}$$

G_2

$$\left(M_{Jl,J'l'}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w_{00}-w_{20} \end{pmatrix}$$

Basis vectors $|\frac{3}{2},+\frac{3}{2}\rangle$ $|\frac{3}{2},-\frac{3}{2}\rangle$

$$\mathcal{Z}^{\boldsymbol{\Delta}}_{lm}(1;q^2) \overset{s\to 1}{=} \sum_{\boldsymbol{z}\in P_{\boldsymbol{\Delta}}} \frac{\mathcal{Y}_{lm}(\boldsymbol{z})}{(\boldsymbol{z}^2-q^2)^s}$$

$$w_{lm} = \frac{1}{\pi^{3/2}\sqrt{2l+1}}\gamma^{-1}q^{-l-1}\,\mathcal{Z}^{\boldsymbol{\Delta}}_{lm}(1;q^2)$$

$$P_{\boldsymbol{\Delta}} = \left\{\boldsymbol{z}\,\middle|\,\boldsymbol{z} = \boldsymbol{\gamma}^{-1}\left(\boldsymbol{n}-\frac{1}{2}\boldsymbol{\Delta}\right),\ \boldsymbol{n}\in\mathbb{Z}^3\right\}$$

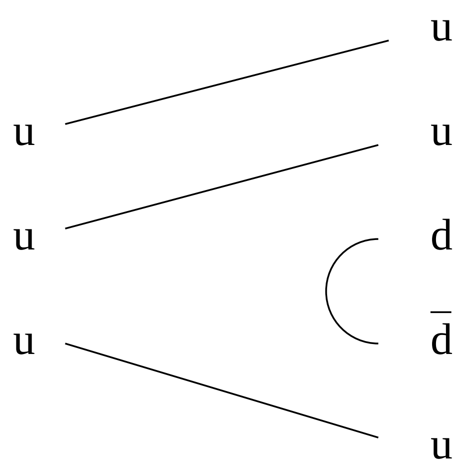
$$\boldsymbol{\Delta} = \boldsymbol{d}\left(1+\frac{m_1^2-m_2^2}{E}\right)$$

Operators

$$O^{\Gamma}(\boldsymbol{p},\boldsymbol{q},t) = \sum_{i=1}^{8} \chi_{\Gamma}^{*}(S_i) \sum_{\boldsymbol{x},\boldsymbol{y}} e^{i(\boldsymbol{p}\boldsymbol{x}+(S_i\boldsymbol{q})(\boldsymbol{x}-\boldsymbol{y}))} \phi_1(\boldsymbol{x},t)\,\phi_2(\boldsymbol{y},t)$$

i	$S_i\boldsymbol{q}$
1	(q_1, q_2, q_3)
2	$(q_2, -q_1, q_3)$
3	$(-q_2, q_1, q_3)$
4	$(-q_1, -q_2, q_3)$
5	$(-q_2, -q_1, q_3)$
6	(q_2, q_1, q_3)
7	$(-q_1, q_2, q_3)$
8	$(q_1, -q_2, q_3)$

Application to $\Delta(1232)$



$$
\begin{aligned}
C_{\Delta^{++}\,p\pi^+}(t',t) = \Phi_p^{(i,j,k)}(t') \\
\Big[& - \tau_u^{(i,\bar{i})}(t',t)\tau_u^{(j,\bar{j})}(t',t)\tau_d^{(k,l)}(t',t')\Phi_{\pi^+}^{(l,m)}(t')\tau_u^{(m,\bar{k})}(t',t) \\
& + \tau_u^{(i,\bar{i})}(t',t)\tau_u^{(j,\bar{k})}(t',t)\tau_d^{(k,l)}(t',t')\Phi_{\pi^+}^{(l,m)}(t')\tau_u^{(m,\bar{j})}(t',t) \\
& + \tau_u^{(i,\bar{j})}(t',t)\tau_u^{(j,\bar{i})}(t',t)\tau_d^{(k,l)}(t',t')\Phi_{\pi^+}^{(l,m)}(t')\tau_u^{(m,\bar{k})}(t',t) \\
& - \tau_u^{(i,\bar{j})}(t',t)\tau_u^{(j,\bar{k})}(t',t)\tau_d^{(k,l)}(t',t')\Phi_{\pi^+}^{(l,m)}(t')\tau_u^{(m,\bar{i})}(t',t) \\
& - \tau_u^{(i,\bar{k})}(t',t)\tau_u^{(j,\bar{i})}(t',t)\tau_d^{(k,l)}(t',t')\Phi_{\pi^+}^{(l,m)}(t')\tau_u^{(m,\bar{j})}(t',t) \\
& + \tau_u^{(i,\bar{k})}(t',t)\tau_u^{(j,\bar{j})}(t',t)\tau_d^{(k,l)}(t',t')\Phi_{\pi^+}^{(l,m)}(t')\tau_u^{(m,\bar{i})}(t',t) \Big] \, \Phi_{\Delta^{++}}^{(\bar{i},\bar{j},\bar{k})*}(t')
\end{aligned}
$$

$$\Box_{xy}(t) = \sum_{m=1}^{M} e_x^m(t)\, e_y^{m\,\dagger}(t) \equiv E(t)\, E^\dagger(t)$$

$$\tau_{\alpha\beta}(t',t) = E^\dagger(t')\, \mathcal{M}^{-1}_{\alpha\beta}(t',t)\, E(t)$$

$$\Phi^{(i,j,k)}_{B,\,\alpha_1\alpha_2\alpha_3}(t) = \epsilon^{abc} e^{(i)a} e^{(j)b} e^{(k)c}(t)\, S^B_{\alpha_1\alpha_2\alpha_3}$$

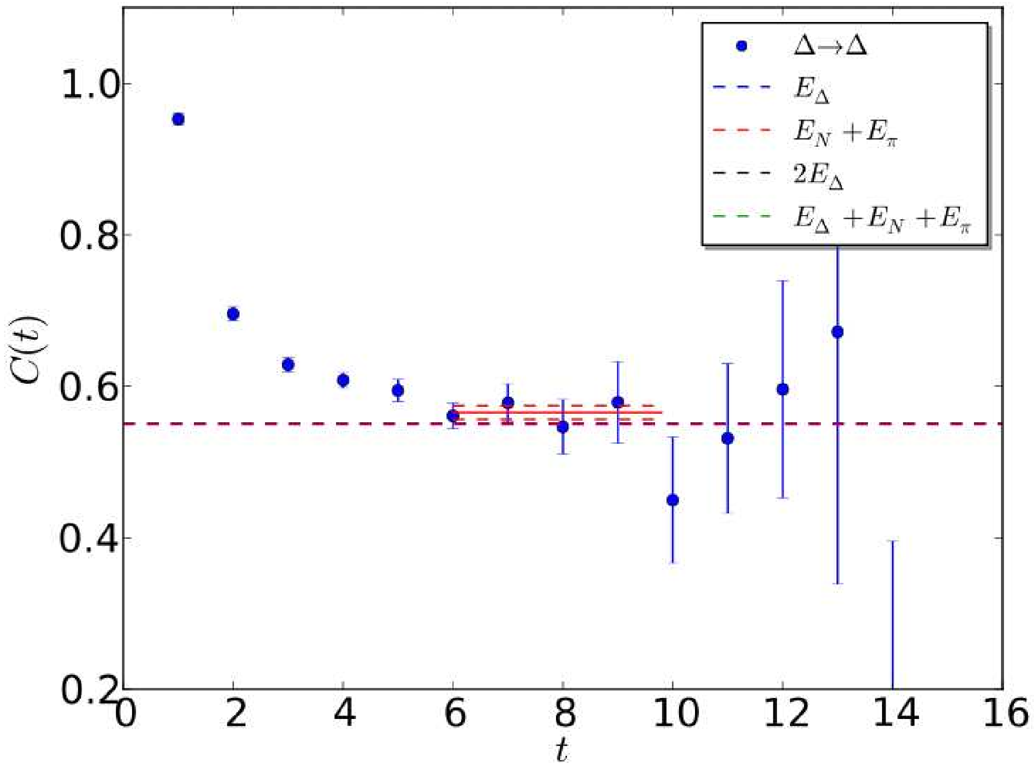
$$\Phi^{(i,j)}_{\pi,\,\alpha\beta}(t) = e^{(i)a}(t)^*\, [\Gamma^\pi(t)]_{\alpha\beta}\, e^{(j)a}(t)$$

Bulava, Lin, Morningstar, Peardon *et al.*

$$C_{ij}(t)\, v_j = \lambda(t,t_0)\, C(t_0)_{ij}\, v_j$$

$$\lambda_k(t,t_0) = \mathrm{e}^{-(t-t_0)E_k} \left(1 + O(\mathrm{e}^{-(t-t_0)\Delta E_k})\right)$$

$32^3 \times 64$

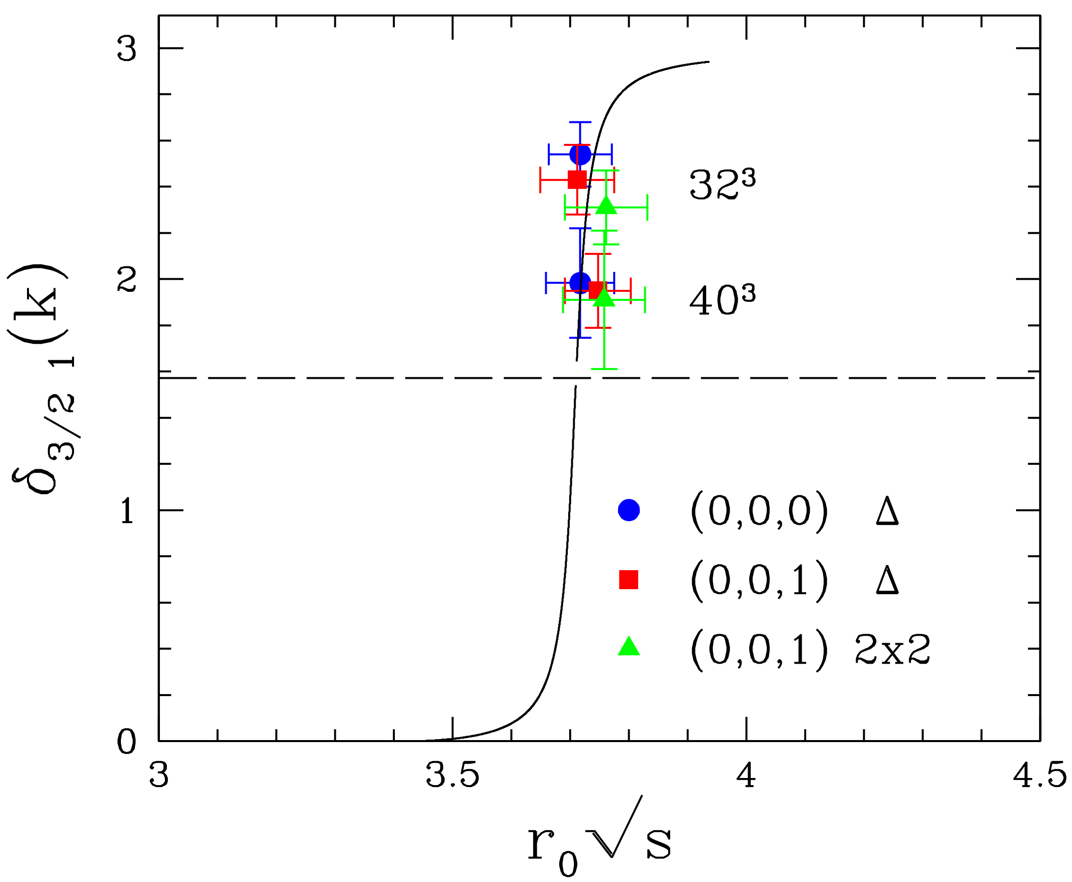


$N_f = 2$

$\beta = 5.29$ $\quad$ $m_\pi = 230$ MeV

$r_0/a = 7.0$ $\quad$ $O(50)$ eigenvectors

Preliminary



Effective range formula

$$\frac{k^3}{E} \cot \delta_{3/2\,1}(k) = \frac{24\pi}{g^2_{\Delta N\pi}} \left(m^2_{\Delta} - E^2\right)$$

$$\Gamma_{\Delta} = \frac{g^2_{\Delta N\pi}}{24\pi} \frac{k^3_{\Delta}}{m^2_{\Delta}} \qquad\qquad \frac{g^2_{\Delta N\pi}}{4\pi} = 14.6$$

Conclusions

- Simulations of $O(a)$ improved Wilson fermions at physical pion masses and relatively small volumes $(m_\pi L = 2 - 3)$ are feasible

- There are several advantages to employing nonvanishing total momenta: This includes making the avoided level crossing in P-wave decays occur at smaller volume, in case the scattering particles have different mass, and making a wider set of energy levels available on a single lattice volume

- The drawback is that the the individual partial waves will mix in general

- The success of the method depends on our ability to construct operators that will transform according to the desired representation of the little group

  arXiv:1206.4141

- Of particular interest are baryon resonances. A first attempt of computing the mass and width of the Δ resonance looks promising