# Non-relativistic field theories in a finite volume

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# Plan

- Introduction
- Resonances: Lüscher approach
- Non-relativistic EFT in a finite volume: essentials
- Lüscher-Lellouch formula
- Resonance matrix elements
- Three particles in a finite volume

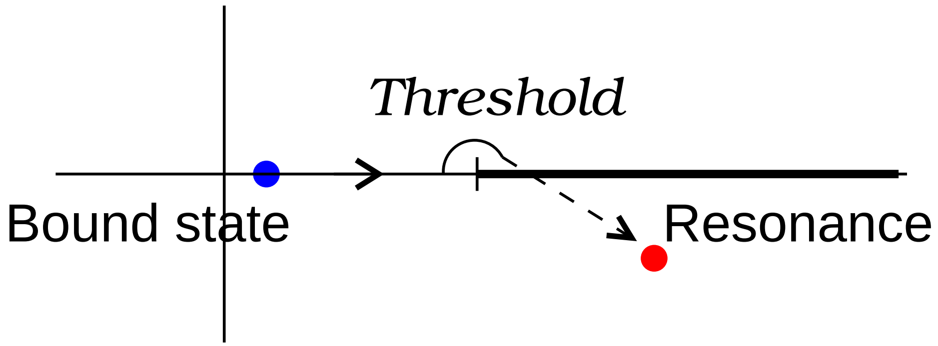
  *Separation of the infinite- and finite-volume contributions*

  *Disconnected diagrams*
- Conclusions, outlook

# Resonances in Quantum Field Theory

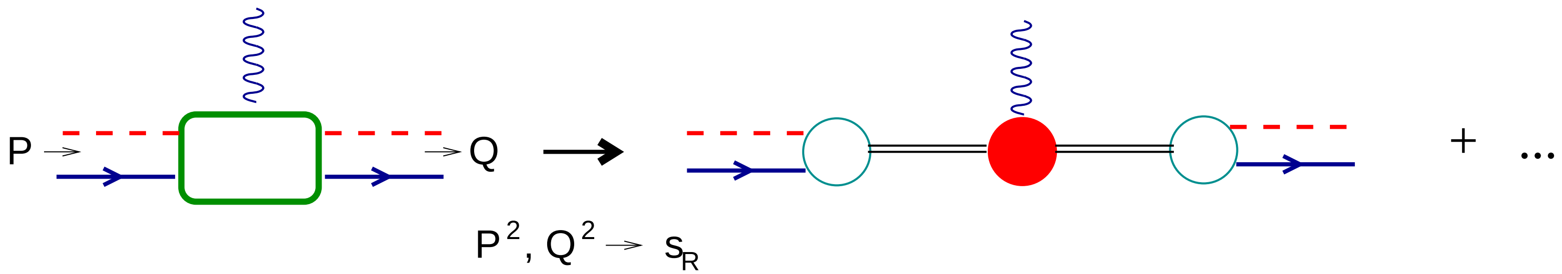
- Resonances are characterized by their mass, their lifetime, . . .
- These are the *intrinsic* properties of a resonance that should not depend neither on a particular *experiment* nor a particular *theoretical model* which is used to describe the data

$\hookrightarrow$ Resonances correspond to $S$-matrix poles on the unphysical Riemann sheets



# Matrix elements with the resonances

A consistent definition of a formfactor of an unstable particle in QFT




Example: electromagnetic formfactor of the $\Delta$-resonance:

- Gauge independent
- Invariant under field redefinitions

*Note: Definitions which do not imply analytic continuation, do not have the above properties*

How does one perform analytic continuation of the lattice data?

# Determining particle masses on the lattice

The two-point function in the Euclidean space

$$C(t)=\langle 0|\mathcal{O}(t)\mathcal{O}^\dagger(0)|0\rangle=\int dU\,d\psi\,d\bar\psi\;\mathrm{e}^{-S_{QCD}(U,\psi,\bar\psi)}\;\mathcal{O}(t)\mathcal{O}^\dagger(0)$$

yields the spectrum of stable particles at large $t$

$$C(t)=\sum_n|\langle 0|\mathcal{O}(0)|n\rangle|^2\mathrm{e}^{-E_nt}\quad\to\quad|\langle 0|\mathcal{O}(0)|1\rangle|^2\,\mathrm{e}^{-mt}+\cdots$$

The method does not apply to the case of unstable particles: $\rho$, $\Delta$, $\ldots$

- How does one study scattering processes in lattice QCD?
- How does one calculate the resonance properties?

# Lüscher's approach

M. Lüscher, lectures given at Les Houches (1988); NPB 364 (1991) 237, $\cdots$

- Lattice simulations are always done at a finite box size $L$

It is assumed: $R^{-1}L \simeq ML \gg 1\,.$

$R$: the range of interaction

- Momenta are small: $p \simeq 2\pi/L \ll$ the lightest mass
- Finite-volume corrections to the energy levels are only power-suppressed in $L$
- Studying the dependence of the energy levels on $L$ gives the scattering phase in the infinite volume $\Rightarrow$ Resonances

Non-relativistic effective field theories (NREFT) can be used to study the energy spectrum in a box

# Covariant NREFT in the infinite volume

G. Colangelo, J. Gasser, B. Kubis and AR, PLB 638 (2006) 187
J. Gasser, B. Kubis and AR, NPB 850 (2011) 96

The Lagrangian:

$$\begin{aligned}\mathcal{L} &= \sum_i \Phi_i^\dagger (2W_i)(i\partial_t - W_i)\Phi_i + C_0 \Phi_1^\dagger \Phi_2^\dagger \Phi_1 \Phi_2 \\ &+ C_1 \biggl( (\Phi_1^\dagger)^\mu (\Phi_2^\dagger)_\mu \Phi_1 \Phi_2 - M_1 M_2 \Phi_1^\dagger \Phi_2^\dagger \Phi_1 \Phi_2 + \text{h.c.} \biggr) + \cdots\end{aligned}$$

$$W_i = \sqrt{M_i^2 - \triangle}\,, \qquad (\Phi_i)^\mu = (W_i, i\nabla)\Phi_i$$

The propagator:

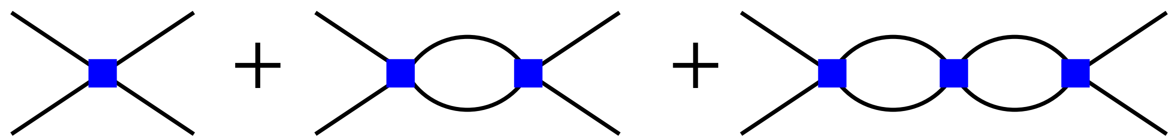
$$D_i(p) = \frac{1}{2W_i(\mathbf{p})}\,\frac{1}{W_i(\mathbf{p}) - p_0 - i0}$$

# Lippmann-Schwinger equation

- Expand $W_i(\mathbf{p}) = M_i + \mathbf{p}^2/(2M_i) + \cdots$ in all Feynman integrands, integrate in the dimensional regularization and sum up again

$$\hookrightarrow \text{loop} = \frac{ip}{8\pi\sqrt{s}}\,, \qquad p = \frac{\lambda^{1/2}(s, M_1^2, M_2^2)}{2\sqrt{s}}$$

T =



+ ...

$\hookrightarrow$ Scattering amplitude is <u>Lorentz-invariant:</u>

$$T_l = \frac{8\pi\sqrt{s}}{p\cot\delta_l(p) - ip}$$

- Important in nonrest frames (formfactors, 3-body scattering)

# Covariant NREFT in a finite volume

Loops modified, Lüscher's zeta-function emerges (nonrest frame):

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\mathbf{p}}, \qquad \text{loop} = \frac{ip}{8\pi\sqrt{s}} \to \frac{Z_{00}^{\mathbf{P}}}{4\pi^{3/2}L\gamma\sqrt{s}} \quad \text{(S-wave)}$$

Poles in the LS equation = spectrum of the Hamiltionan

(see S. R. Beane *et al.,* NPA 747 (2005) 55)

$\hookrightarrow$ Gottlieb-Rummukainen equation:

$$\det\left(\delta_{ll'}\delta_{mm'} - \tan\delta_l(s)\mathcal{M}_{lm,l'm'}\right) = 0$$

- $\mathcal{M}_{lm,l'm'}$ is a linear combination of $Z_{lm}^{\mathbf{P}}$
- Partial-wave mixing occurs in a finite volume

# Where are the resonance poles?

Suppose that there exists an isolated narrow resonance in the vicinity of the elastic threshold. Assume that effective range expansion for the quantity $p\cot\delta(p)$ is convergent in the resonance region.

$$p\cot\,\delta(p) = A_0 + A_1 p^2 + \cdots$$

$\Rightarrow$ $A_0, A_1, \cdots$ are measured on the lattice

$\Rightarrow$ Resonance pole in the complex momentum plane:

$$\cot\,\delta(p_R) = -i \quad \surd$$

# Example of using NREFT: Lüscher-Lellouch formula

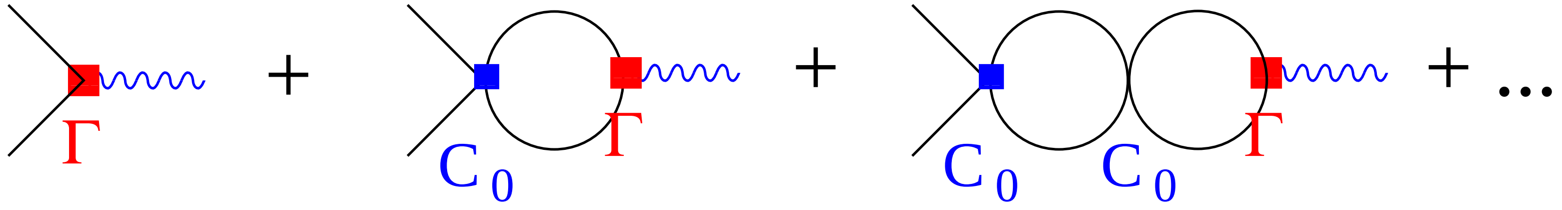
V. Bernard, D. Hoja, U.-G. Meißner and AR, arXiv:1205.4642

- Aim: extract the formfactor in a timelike region
- Method: Calculate the formfactor in NREFT, in the infinite and in a finite volume; single out the infinite-volume formfactor in the finite-volume expression

Most general NREFT Lagrangian with the external field:

$$\mathcal{L} = \sum_i \Phi_i^\dagger (2W_i)(i\partial_t - W_i)\Phi_i + C_0 \Phi_1^\dagger \Phi_2^\dagger \Phi_1 \Phi_2 + eA(\Phi_1^\dagger \Phi_2^\dagger + \text{h.c.}) + \cdots$$

Summing up bubble diagrams in the vertex function



# Derivation of the LL formula

Formfactor in a finite volume:

$$|\langle E_n(\mathbf{P})|j(0)|0\rangle| = L^{-3/2}\,|\Gamma(s_n)|\,\frac{p\cos\delta(s_n)}{2\pi\sqrt{s_n}E_n}\,\frac{1}{|\delta'(s_n)+\phi'(s_n)|^{1/2}}$$

Formfactor in the infinite volume, with $(k_1+k_2)^2=s_n$:

$$|F(s_n)| = |\langle k_1,k_2;out|j(0)|0\rangle| = |\Gamma(s_n)\cos\delta(s_n)|$$

$\hookrightarrow$ LL formula for the timelike formfactor in the nonrest frame

(see also H. Meyer, PRL 107 (2011) 072002)

$$|F(s_n)|^2 = |L^{3/2}\langle E_n(\mathbf{P})|j(0)|0\rangle|^2\,\frac{2\pi\sqrt{s_n}E_n}{p^2}\,|\delta'(s_n)+\phi'(s_n)| \qquad \surd$$

# Resonance matrix elements

D. Hoja, U.-G. Meißner and AR, JHEP 1004 (2010) 050

V. Bernard, D. Hoja, U.-G. Meißner and AR, arXiv:1205.4642

Field operators with resonance quantum numbers:

$$O_{\mathbf{P}}(t) = \sum_{\mathbf{x}} e^{-i\mathbf{P}\mathbf{x}} O(\mathbf{x}, t) \,,$$

Three- and two-point functions on the lattice:

$$\tilde{V}_\mu(\mathbf{P}, t'; \mathbf{Q}, t) = \langle 0 | T O_{\mathbf{P}}(t') J_\mu(0) O_{\mathbf{Q}}^\dagger(t) | 0 \rangle \,,$$

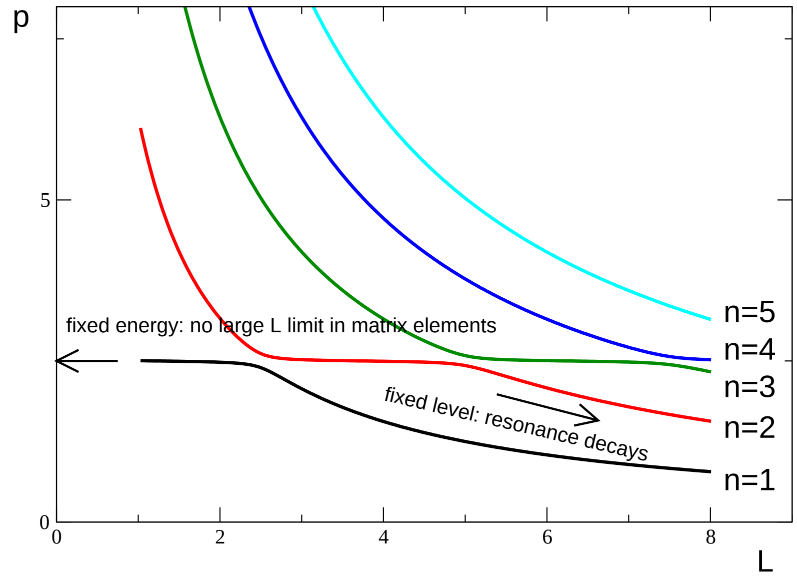
$$D(\mathbf{P}, t) = \langle 0 | T O_{\mathbf{P}}(t) O_{\mathbf{P}}^\dagger(0) | 0 \rangle \,.$$

Extraction of the formfactor (ground-state):

$$\langle \mathbf{P} | J_\mu(0) | \mathbf{Q} \rangle_0 = \lim_{\substack{t' \to \infty \\ t \to -\infty}} \tilde{V}_\mu(\mathbf{P}, t'; \mathbf{Q}, t) \sqrt{\frac{D(\mathbf{Q}, t') D(\mathbf{P}, t)}{D(\mathbf{Q}, t) D(\mathbf{Q}, t'-t) D(\mathbf{P}, t-t') D(\mathbf{P}, t')}}$$

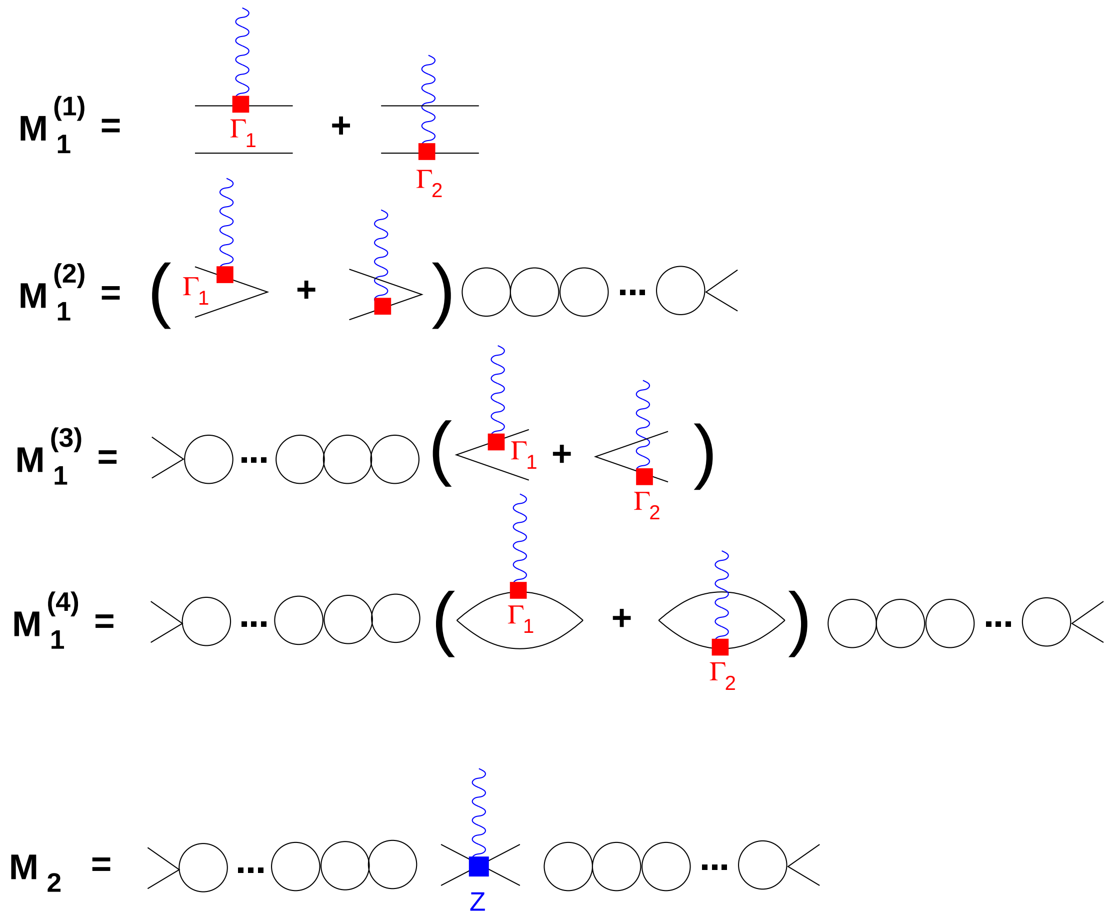
# Infinite-volume limit of the matrix elements

- For stable particles, the limit $L \to \infty$ exists
- Both methods give the matrix element sandwiched by the eigenvectors of the Hamiltonian. The resonances, however, do not correspond to a single energy level. How does one calculate the infinite-volume limit for these matrix elements?



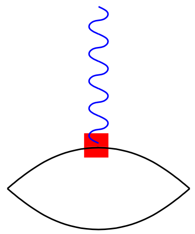

- Fixed energy levels decay in the limit $L \to \infty$
- The matrix elements at fixed energy oscillate in the limit $L \to \infty$

# Framework: non-relativistic EFT with the external fields




- Use NREFT in a finite volume to calculate the matrix element
- Extract the matrix element in the infinite volume

# Loop graph: analytic continuation (rest system)

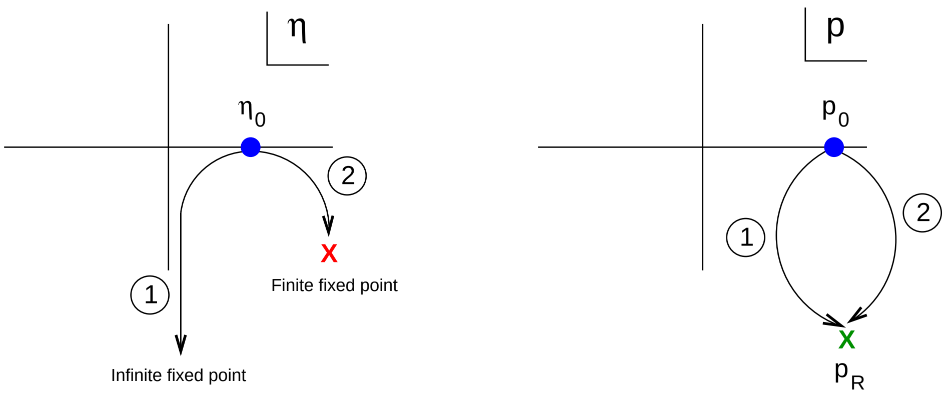


$$= \underbrace{\frac{m^2-p^2}{8\pi E^3 p^2}\, p\cot\delta(p)}_{(\text{polynomial in } p^2)/p^2} + \underbrace{\frac{1}{32\pi E p}\,(1+\cot^2\delta(p))\eta\phi'(\eta)}_{\text{culprit}}$$

$$p = p_n = \sqrt{\frac{E_n^2}{4} - m^2}\,, \qquad \tan\phi(\eta) = \frac{\pi^{3/2}\eta}{Z_{00}(1;\eta^2)}\,, \qquad \eta = \frac{pL}{2\pi}$$

- A polynomial in $p^2$, can be analytically continued $p^2 \to p_R^2$
- An analytic continuation of $\eta\phi'(\eta)$ is ambiguous

# $p$- and $\eta$- planes




- Problem: $\cot\phi(\eta)+i\propto(\eta-\eta_R)$ and $\phi(\eta)\propto\ln(\eta-\eta_R)$

$$(\cot\phi(\eta)+i)\phi'(\eta)\to \text{const}$$

- Remedy: $\eta\phi'(\eta)$ depends on the energy level $n$, since $\eta=\eta_n(p)$. The culprit can be eliminated by measuring <u>two</u> energy levels:

$$\bar{V}(p)=\frac{b_mV_{nn}(p)-b_nV_{mm}(p)}{b_n-b_m}\qquad \text{where}\quad b_n=\eta_n\phi'(\eta_n)$$

# How does one extract resonance formfactors?

i) Measure the quantities $\langle \mathbf{P}|J_\mu(0)|-\mathbf{P}\rangle_n$ on the lattice, Breit frame

ii) $$V_{nn}(p) = \underbrace{\frac{\delta'(p)+\phi'(\eta)}{4\sin^2\delta(p)}\,\frac{L^3E_n}{2\pi\sqrt{E_n^2-\mathbf{P}^2}}}_{\text{Lüscher-Lellouch factor}}\,\langle \mathbf{P}|J_\mu(0)|-\mathbf{P}\rangle_n$$

iii) Form the linear combination:

$$\bar V(p) = \frac{b_m(p,\mathbf{P})V_{nn}(p) - b_n(p,\mathbf{P})V_{mm}(p)}{b_n(p,\mathbf{P}) - b_m(p,\mathbf{P})}$$

iv) Effective-range expansion for $\bar V(p)$ holds

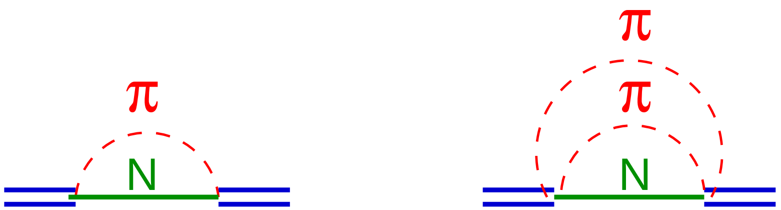
$$\bar V(p) = \frac{V_{-1}}{p^2} + V_0 + V_1 p^2 + \cdots \to \frac{V_{-1}}{p_R^2} + V_0 + V_1 p_R^2 + \cdots$$

v) Resonance formfactor: $\langle \mathbf{P}|J_\mu(0)|-\mathbf{P}\rangle = \underbrace{B_R}_{\text{w.f. norm.}}\;\bar V(p_R)\qquad\surd$

# Three-body intermediate states

K. Polejaeva and AR, EPJA 48 (2012) 67

The problem:
finite-volume effects in the spectrum of the Roper resonance

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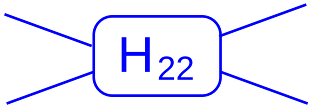
Approximations:

- No Lorentz-invariance
- No 4- and more particle states
- No 2- and 3-particle bound states

$$H=\sum_{i=1}^{3} H_0^{(i)} +$$

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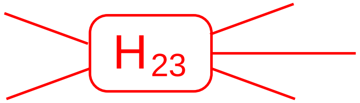



+h.c.)

# Two-body case: Splitting

Two-body propagator in a finite volume, with $q_0^2 = 2\mu z$:

$$G_0^{\mathsf{L}}(\mathbf{p};z) = \frac{2\mu}{L^3}\sum_{\mathbf{k}} \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})}{\mathbf{k}^2 - q_0^2} = G^{\mathsf{K}}(\mathbf{p};z) + G^{\mathsf{F}}(\mathbf{p};z)$$

$$G^{\mathsf{K}}(\mathbf{p};z) = \text{P.V.}\frac{2\mu}{\mathbf{p}^2 - q_0^2}$$

$$G^{\mathsf{F}}(\mathbf{p};z) = \sum_{lm} \frac{2}{\eta^{l+1}} Y_{lm}^*(\hat{p}) Z_{lm}(1;\eta^2)\delta(\mathbf{p}^2 - q_0^2)\,, \qquad \eta = \frac{q_0 L}{2\pi}$$

Derivation of Lüscher equation:

$$T^{\mathsf{L}} = V + VG_0^{\mathsf{L}}T^{\mathsf{L}} \quad \Rightarrow \quad K = V + VG^{\mathsf{K}}K\,, \quad \underbrace{T^{\mathsf{L}} = K + KG^{\mathsf{F}}T^{\mathsf{L}}}_{\text{Lüscher equation}}$$

- Due to the presence of $\delta(\mathbf{p}^2 - q_0^2)$, only on-shell $K$-matrix elements determine the finite-volume spectrum!

# Splitting in the 3-particle case

$$G^{\mathsf{L}}_{0\alpha} = \frac{1}{L^6}\sum_{\mathbf{pq}} \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})(2\pi)^3\delta^3(\mathbf{q}-\mathbf{l})}{M+\frac{\mathbf{p}^2}{2M_\alpha}+\frac{\mathbf{q}^2}{2\mu_\alpha}-z} = G^{\mathsf{K}}_\alpha + G^{\mathsf{F}}_\alpha$$

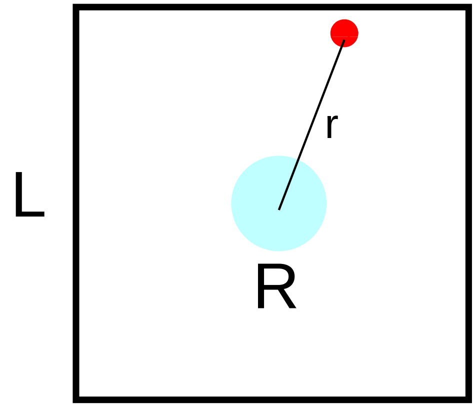
Can be the splitting used in the 3-body LS equations as well, in order to prove that the finite-volume energy spectrum is determined by the on-shell $K$-matrix elements only?

- Cusp singularity at $q^2_{0\alpha}=0$, breakdown of the regular summation theorem

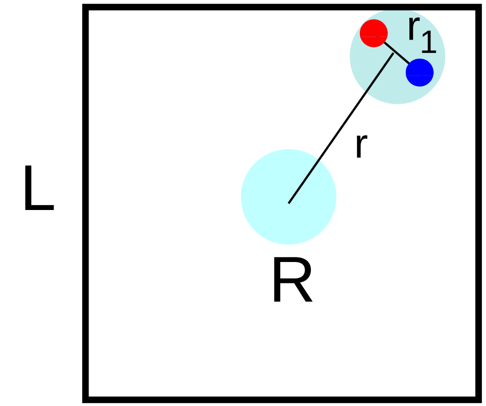
$$G^{\mathsf{F}}_\alpha \quad\sim\quad \delta(\mathbf{q}^2-q^2_{0\alpha})\,, \qquad q^2_{0\alpha}=2\mu_\alpha\left(z-M-\frac{\mathbf{p}^2}{2M_\alpha}\right)$$

- The splitting holds, if applied to the regular test functions. Disconnected diagrams in the 3-body scattering are not regular (contain the $\delta$-function).

# Physical interpretation




2 particles




3 particles

- In case of 2 particles: $r \gg R$, when particles are near the walls
- In case of 3 particles: it may happen that $r \gg R, \quad r_1 \simeq R$, when the particles are near the walls

The problem with the disconnected contributions: is the finite-volume spectrum in the 3-particle case determined solely through the on-shell scattering matrix?

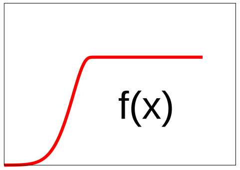
# The cusp singularity

The cusp singularity leads to the breakdown of the regular summation theorem:

$$\frac{1}{L^3}\sum_{\mathbf{p}}^{\Lambda}|\mathbf{p}| = \int^{\Lambda}\frac{d^3\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}| + \underbrace{\sum_{\mathbf{n}\in\mathbb{Z}\backslash\mathbf{0}}\int^{\Lambda}\frac{d^3\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}|e^{i\mathbf{n}\mathbf{p}L}}_{O(L^{-2}),\ \text{not exponent}}$$

remedy: $\delta(\mathbf{p}^2-q_0^2) \;\rightarrow\; \Delta(\mathbf{p}^2,q_0^2)$

where: $\displaystyle\int d\mathbf{p}^2\Delta(\mathbf{p}^2,q_0^2)\phi(\mathbf{p}^2) = f(q_0^2/\mu^2)\phi(q_0^2)$ :




- Smearing recovers the regular summation theorem
- Price to pay: information enters from the *subthreshold* region

(power-suppressed in $L$)

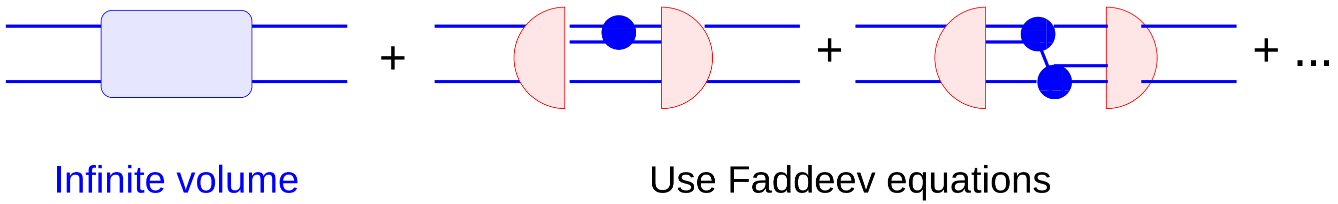
# Lüscher's equation with 3-particle intermediate states

Energy levels are determined from the Lüscher equation:

$$\tan\delta^L(p) = -\tan\phi(\eta)\,, \qquad \eta = \frac{pL}{2\pi}$$

The pseudophase is given by:

$p\cot\delta^L(p) =$



# Splitting in the 3-body equations

Naive analog of Faddeev equations in a finite volume:

$$
\begin{aligned}
\mathbf{R}_{4\beta} &= \boldsymbol{\theta}_4 \mathbf{G}_{\mathsf{F}} \Big( \boldsymbol{\theta}_\beta + \sum_{\gamma=1}^{3} \mathbf{R}_{\gamma\beta} \Big) \\
\mathbf{R}_{\alpha\beta} &= \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \boldsymbol{\theta}_\beta + \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \Big( \sum_{\gamma=1}^{3} (1-\delta_{\alpha\gamma}) \mathbf{R}_{\gamma\beta} + \mathbf{R}_{4\beta} \Big) \\
\mathbf{R}_{\alpha 4} &= \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \sum_{\gamma=1}^{3} (1-\delta_{\alpha\gamma}) \mathbf{R}_{\gamma 4} + \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \mathbf{R}_{44} \\
\mathbf{R}_{44} &= \boldsymbol{\theta}_4 + \boldsymbol{\theta}_4 \mathbf{G}_{\mathsf{F}} \sum_{\gamma=1}^{3} \mathbf{R}_{\gamma 4}
\end{aligned}
$$

$$
\boldsymbol{\theta}_\alpha = \mathbf{K}_\alpha + \mathbf{K}_\alpha \mathbf{G}_{\mathsf{F}} \boldsymbol{\theta}_\alpha \,, \qquad \boldsymbol{\theta}_4 = \mathbf{K}_4 + \mathbf{K}_4 \mathbf{G}_{\mathsf{F}} \boldsymbol{\theta}_4
$$

# Disconnected contributions

Naive Faddeev equations in a finite volume incorrect due to the presence of the disconnected contributions:


- One iteration of $\theta_\alpha$ and $\theta_\beta$ gives a tree diagram: no finite-volume effects
- The term $\theta_\alpha G^{\mathrm{F}} \theta_\beta$ in the naive Faddeev equations superfluous
- Dropping this term, the Born series of the Faddeev equations in a finite volume are shown to coinside order by order with that of the original Lippmann-Schwinger equation

# 3-body problem in a finite volume: summary

- Despite the presence of the disconnected contributions, the energy spectrum of the 3-particle system in a finite box is still determined by the on-shell scattering matrix elements in the infinite volume
- The information from the subthreshold region is needed. This is the price for recovering the regular summation theorem
- A full-fledged field- theoretical treatment of the problem (Lorentz-invariance, particle creation/annihilation) is planned

# Conclusions

- Use of the effective field theory methods in a finite volume enables one to carry out a detailed study of resonances on the lattice
- With the use of these methods, resonance matrix elements (e.g., magnetic moments of $\Delta, \rho, \cdots$) can be extracted from lattice data. The study of transition formfactors (e.g., $\Delta N \gamma$ vertex) is planned
- In the non-relativistic potential model, it was demonstrated that the finite-volume spectrum in the presence of the 3-body decay channels is still completely determined by the on-shell input in the infinite volume
- This result opens way to the investigation of the finite-volume effects in the spectrum of the Roper resonance