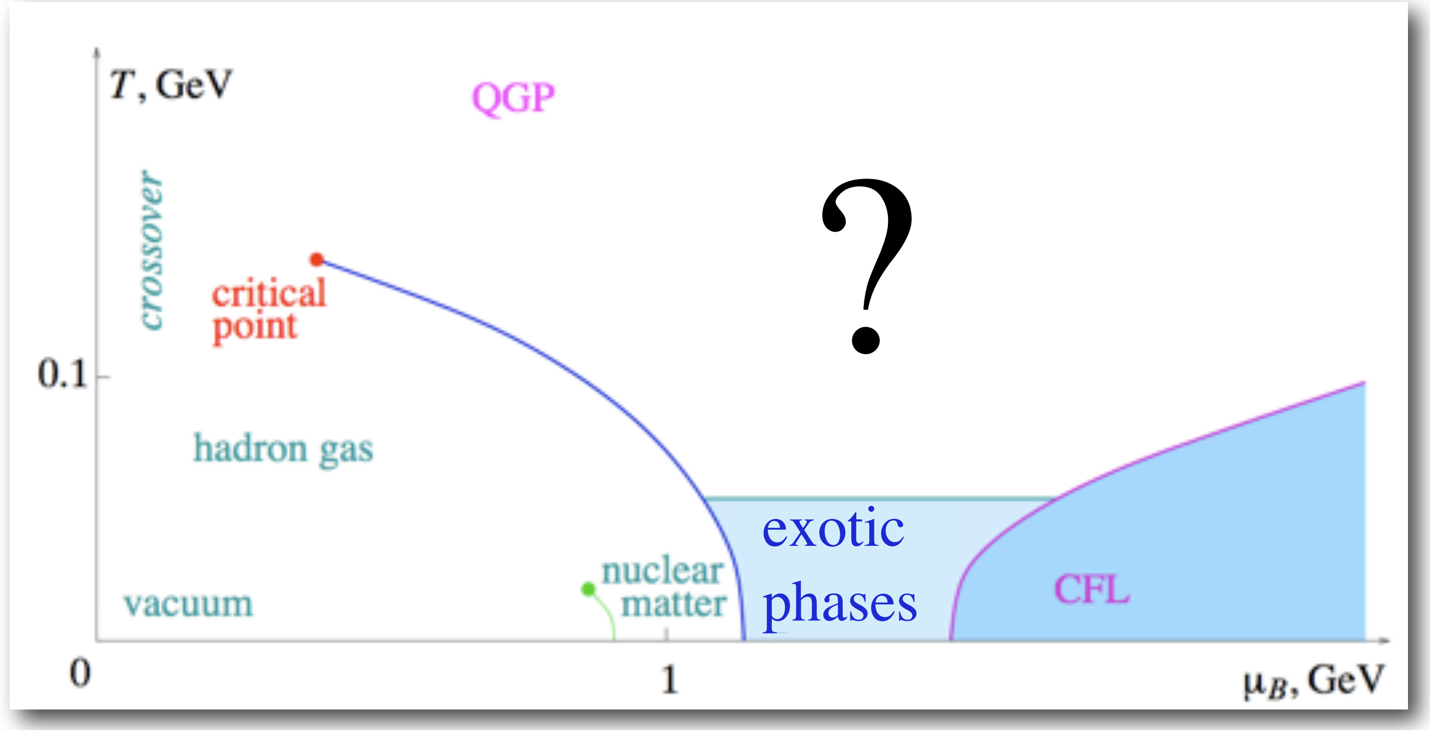
Noise, sign problems and chiral symmetry breaking

Noise, sign problems, and statistics
arXiv:1106.0073 [hep-lat]
Michael Endres, D.K., Jong-Wan Lee, Amy Nicholson

Work in progress (!)
Dorota Grabowska, D.K., Amy Nicholson

Physics motivation: can't we get beyond this cartoon??



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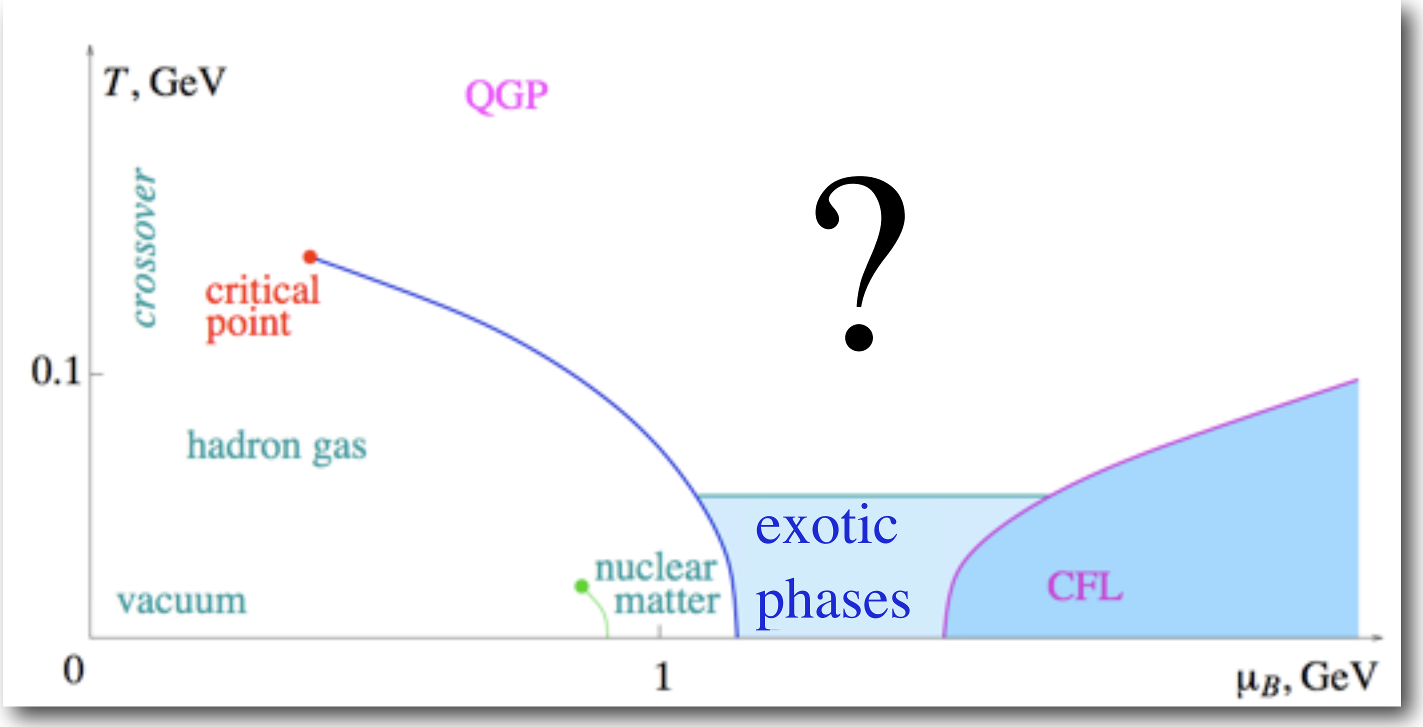


Sign problem!

This talk:

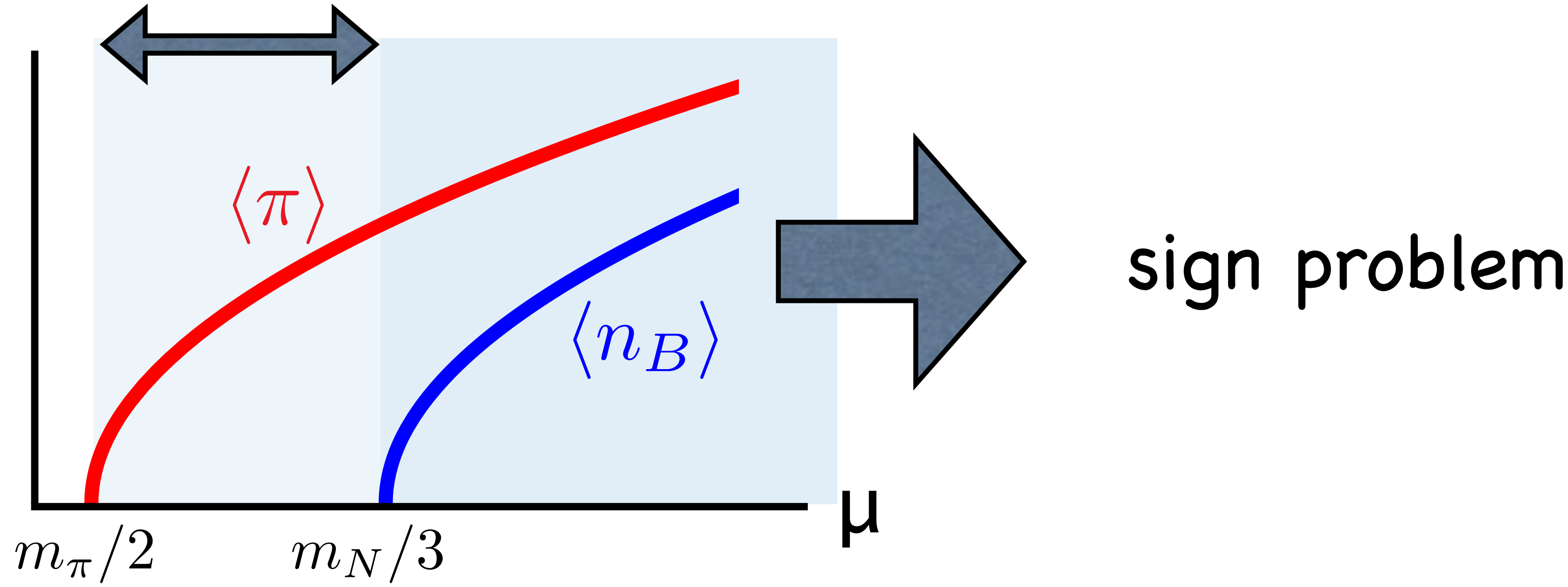
- From sign problem to noise in correlation functions
- The relation between noise and the pion
- Computing the distribution of noise in a model with chiral symmetry breaking
- Wild and unjustified speculations

The "sign" problem in the grand canonical approach:
Det($\not{D}+\mu\gamma^0$) complex

- physics happens for $\mu \geq m_N/3$...
- ...but sign problem starts at $\mu = m_\pi/2$!

P.E. Gibbs, 1986

$$\Delta\mu = \left(\frac{m_N}{3} - \frac{m_\pi}{2}\right)$$



Explanation (2-flavor QCD):

|Det($\not{D}+\mu\gamma^0$)| ≈ isospin chemical potential

K. Splittorff
J. Verbaarschot, 2006

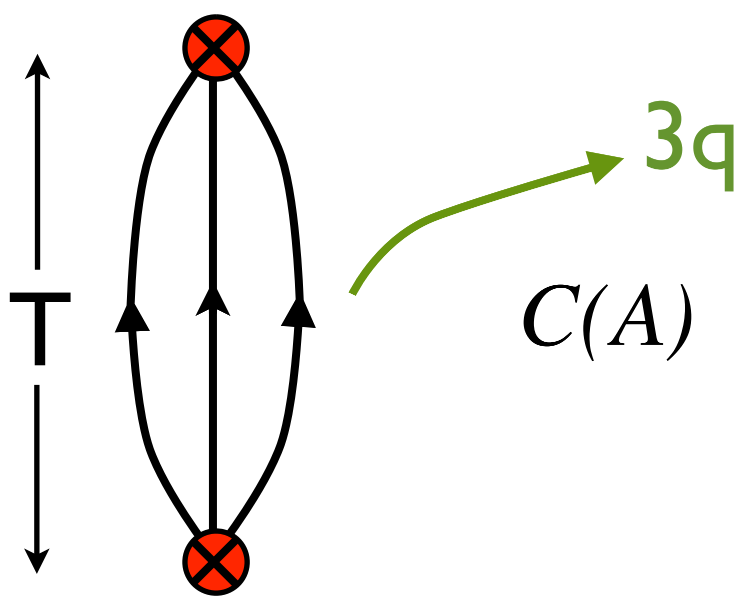
Role of phase: eliminate pion condensate for $\mu \geq m_\pi/2$!

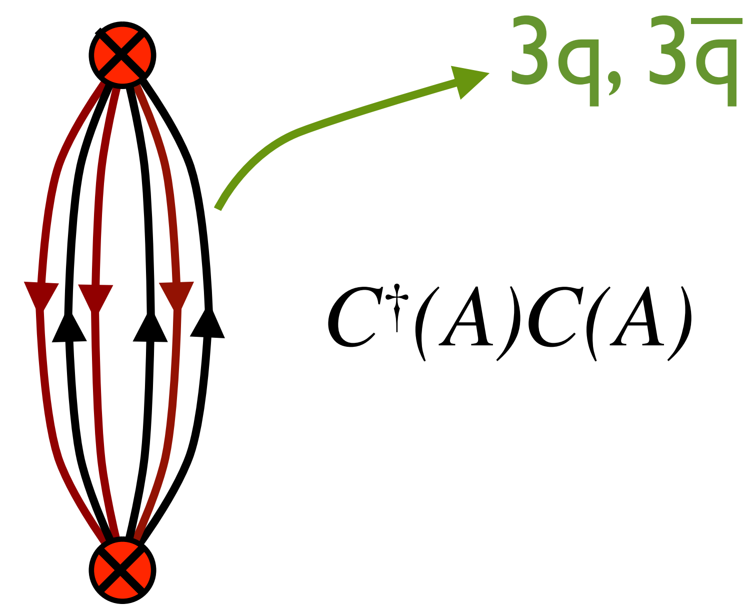
Canonical approach?

Compute correlator of N quarks with μ=0

No sign problem...but now a noise problem

Parisi, Lepage 1980's





nucleon correlator

signal: $\sim e^{-m_N T}$

noise: $\sim \frac{1}{\sqrt{N_{\text{conf.}}}} e^{-\frac{3}{2} m_\pi T}$

$$\frac{\text{signal}}{\text{noise}} \sim \sqrt{N_{\text{conf.}}}\, e^{-3T\left(\frac{m_N}{3} - \frac{m_\pi}{2}\right)}$$

Same factor as Δμ in grand canonical

Martin Savage's generalization of Lepage argument:

Consider an operator X(t) = 2 Re[baryon correlator for time t]

Then leading time behavior:

$$\langle X(t)^k \rangle \sim \begin{cases} e^{-\left(M_B + \frac{3(k-1)}{2} m_\pi\right)t} & k \text{ odd} \\ e^{-\left(\frac{3k}{2} m_\pi\right)t} & k \text{ even} \end{cases}$$

So one expects ($\kappa_n \equiv n^{th}$ cumulant):

$$\text{skewness} = \frac{\kappa_3}{(\kappa_2)^{\frac{3}{2}}} \sim e^{-(M_B - \frac{3}{2} m_\pi)t}$$

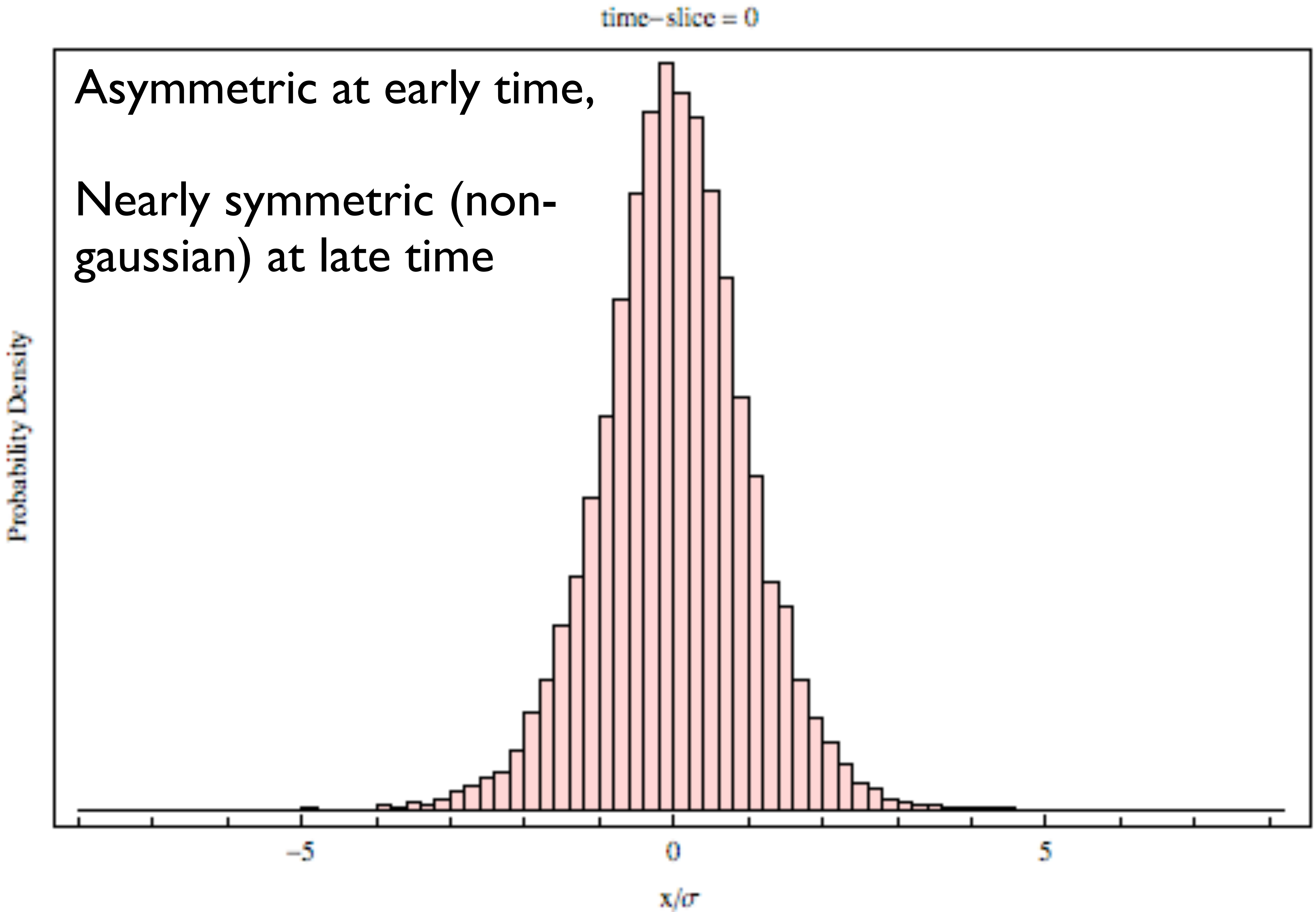
$$\text{kurtosis} = \frac{\kappa_4}{(\kappa_2)^2} \sim 1$$

At late times distribution is non-gaussian but almost symmetric

QCD data (M. Savage, NPLQCD): *Un-averaged Λ baryon correlator data*

Asymmetric at early time,

Nearly symmetric (non-gaussian) at late time



Noise: consider distribution of correlators over ensemble of gauge fields

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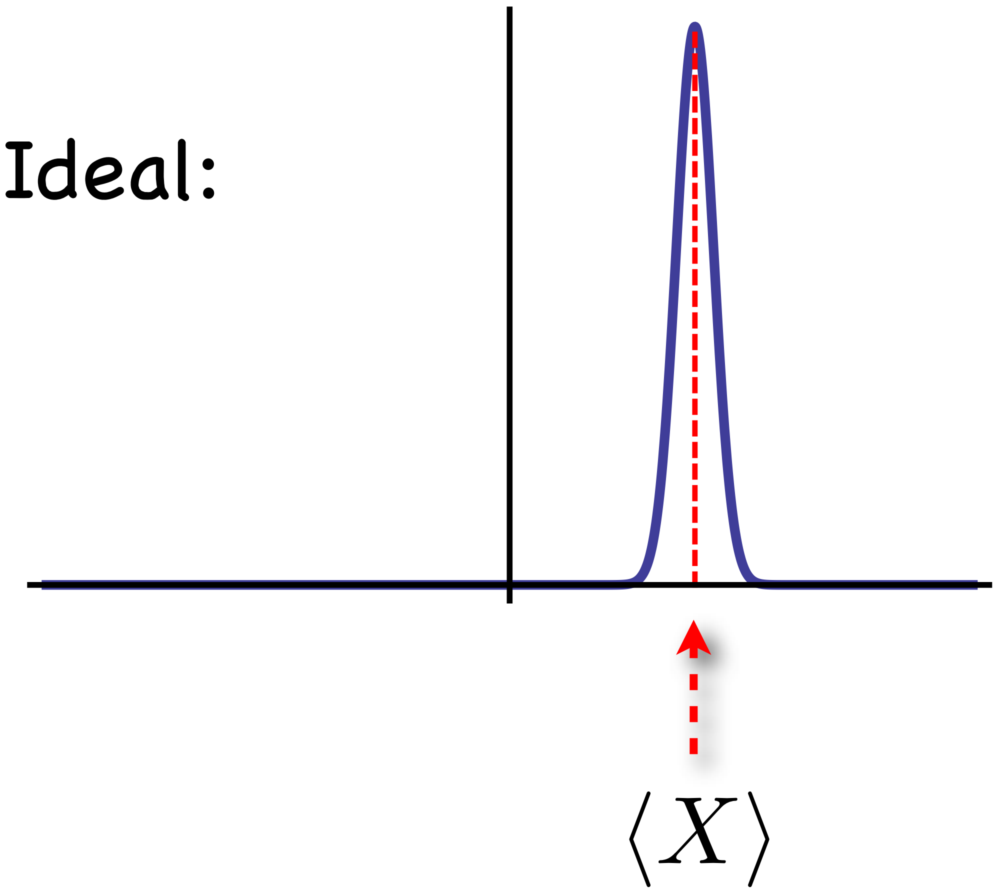
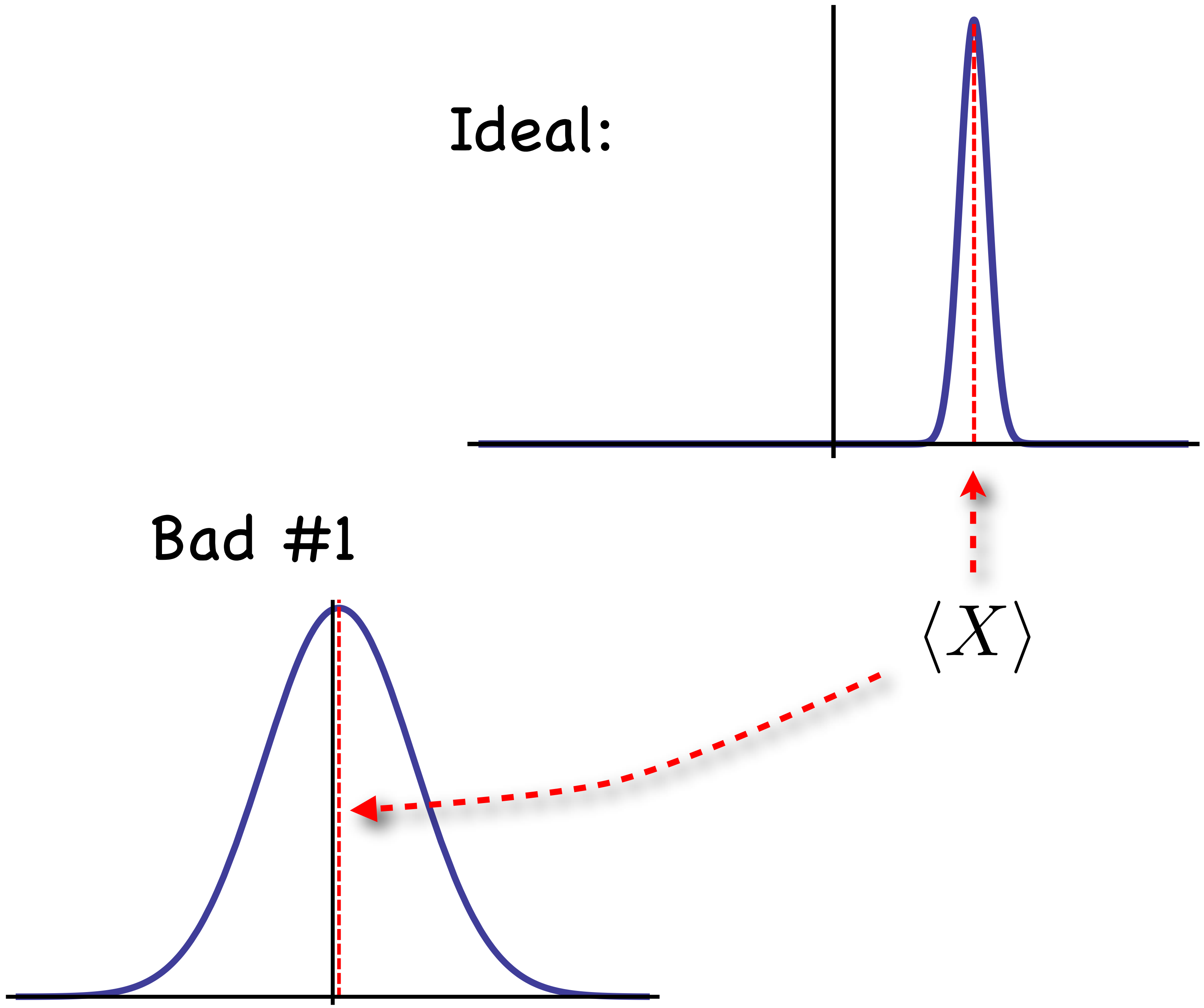
Ideal:

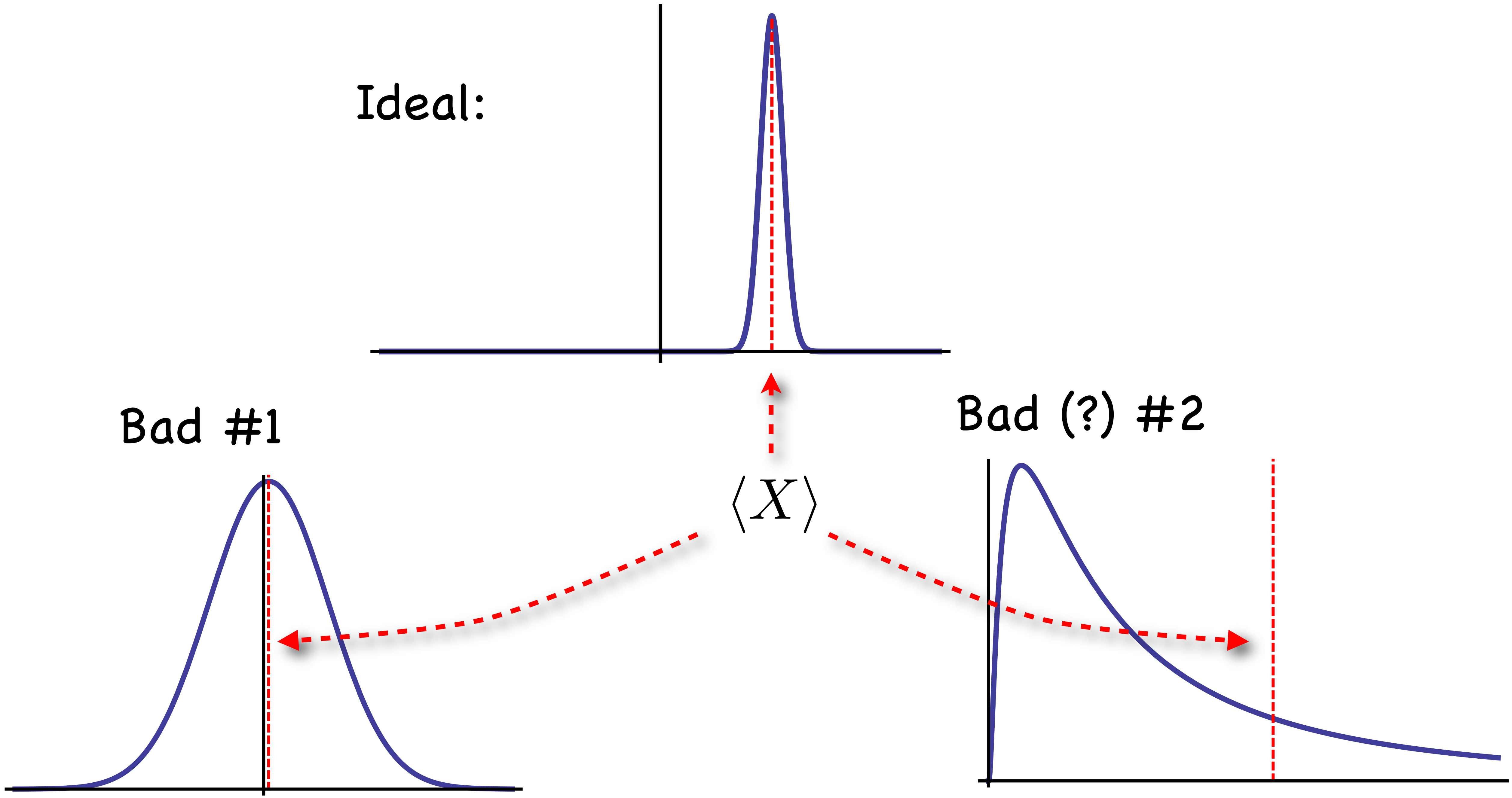


Noise: consider distribution of correlators over ensemble of gauge fields



- *Almost symmetric, small mean*
- *"sign problem" big cancellations*

Noise: consider distribution of correlators over ensemble of gauge fields



- *Almost symmetric, small mean*
- *"sign problem" big cancellations*

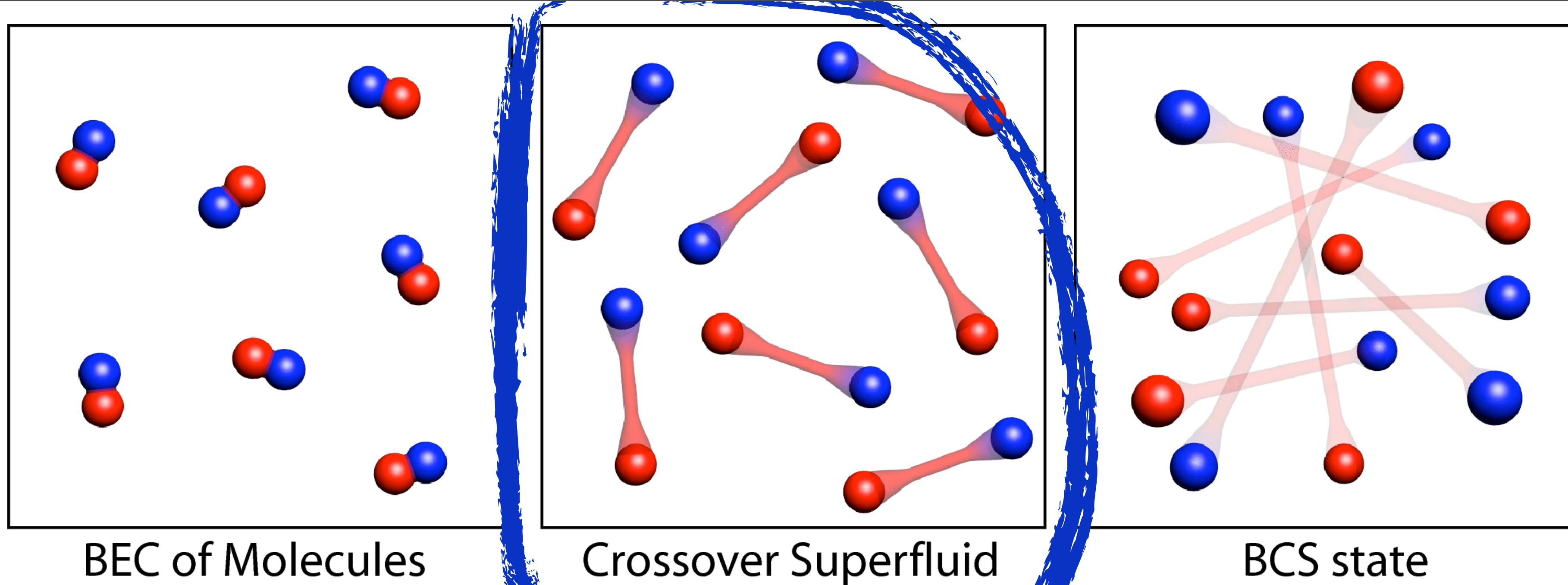
- *Long tail, small mean*
- *"overlap problem" poor sampling*

Challenge:

- Can we understand the correlator distribution quantitatively?
- Can we relate the distribution to the physical spectrum of the theory?
- Is it possible to use this knowledge to ameliorate the sign problem and study QCD at finite density or large baryon number?
  - either use knowledge of distribution + statistical methods to improve signal/noise, or
  - reformulate the theory in a way to improve the distribution

Some reason for optimism:

- experience simulating cold atoms at a Feshbach resonance
- analytical work (in progress) in a QFT with chiral symmetry breaking



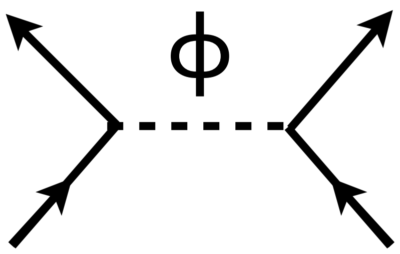
$$\frac{1}{a} \to +\infty \qquad \frac{1}{a} = 0 \qquad \frac{1}{a} \to -\infty$$

Unitary fermions: nonrelativistic fermions with zero-range interaction, tuned to infinite scattering length (conformal system)

M. Endres, D.K., J.W. Lee, A. Nicholson, arXiv:1106.5725 [hep-lat]

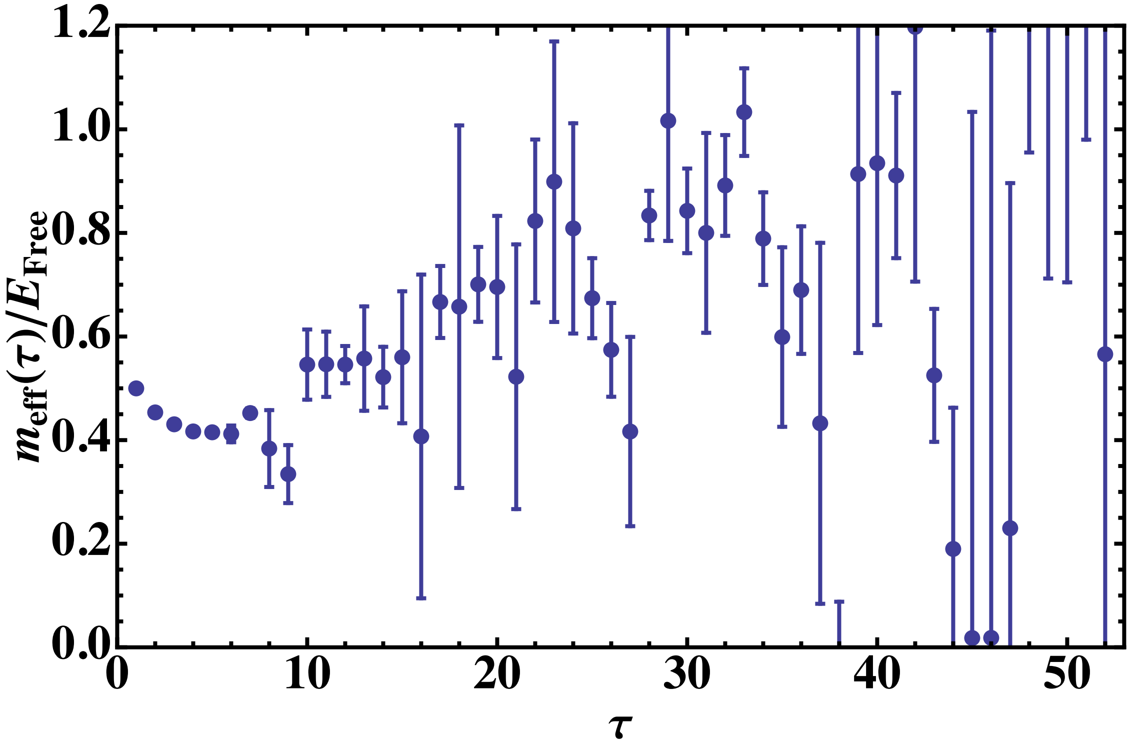
Lattice model:

- Short-range momentum-dependent 4-fermion interaction induced by auxiliary scalar field
- Interaction tuned to conformal fixed pt.

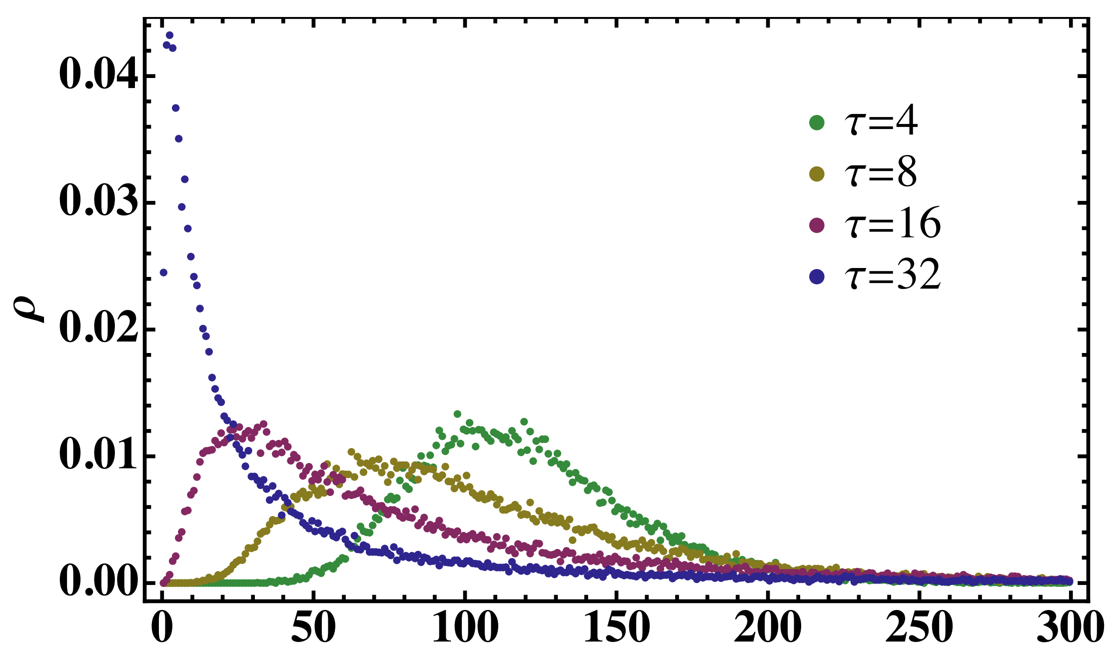


- Simulated up to ~N=70 fermions on 14^3 x 64 lattice
- ~1% accuracy in energies
- ~2 billion configurations for auxiliary field

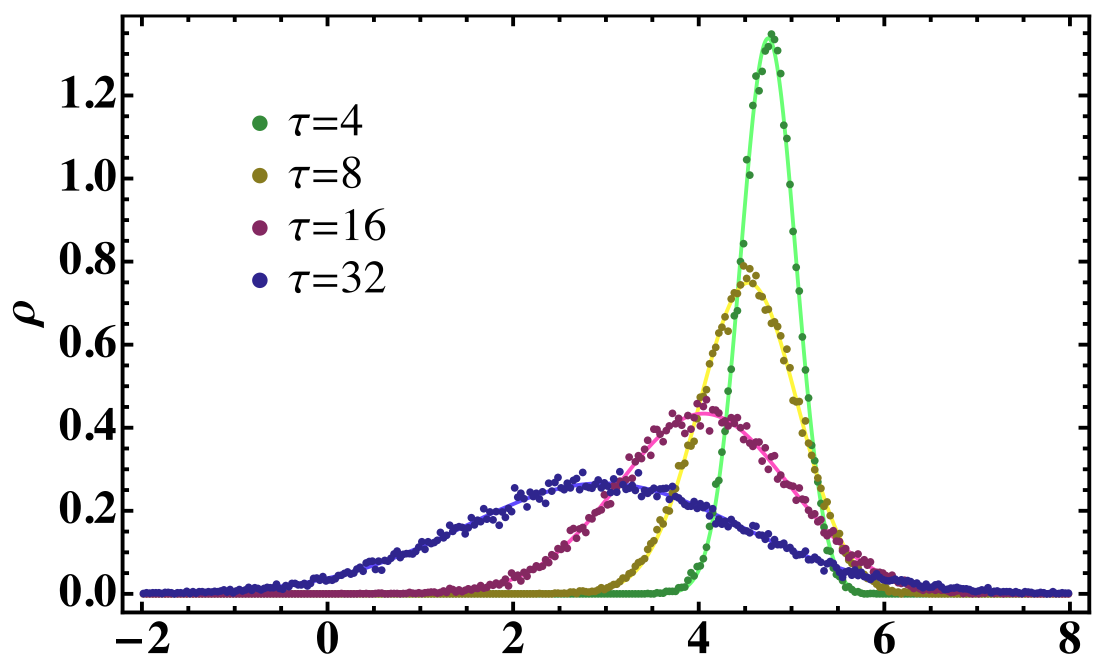
Effective mass plot: 46 fermions, $V=12^3$, 40M configurations



noise



C

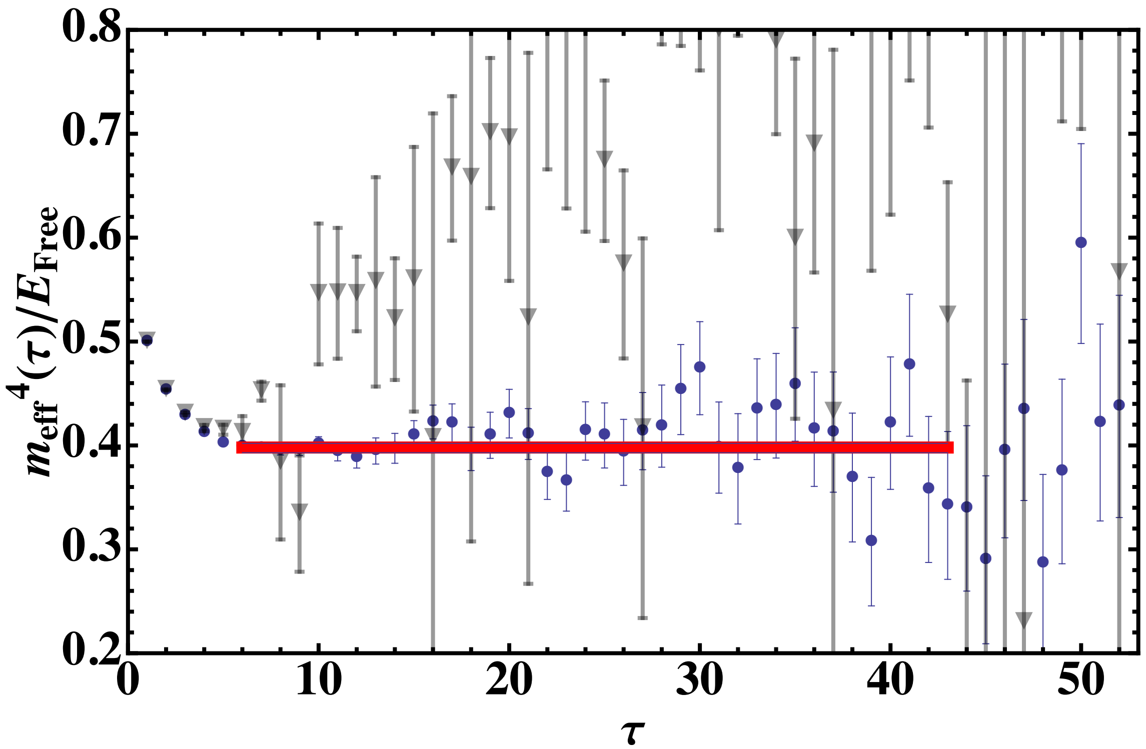


ln C

In unitary fermion case, found:

Noise, sign problems, and statistics.
M. G. Endres, D. B. Kaplan, J.-W. Lee, A. N. Nicholson
Phys.Rev.Lett. 107 (2011) 201601
arXiv:1106.0073

- Could compute probability distribution for correlator in mean field theory, and show why it was nearly log-normal
- Could exploit that it was nearly log-normal and use statistical methods to greatly improve signal/noise



Gray = unprocessed data
Blue = processed data
Red = extracted mass plateau

QCD will behave very differently:

Noise depends critically on the pion!

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WANTED:

- Tractable model with chiral symmetry breaking
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FOUND: Large-N NJL model in d=3

Fascinating model because it has two equivalent formulations: one with a QCD-like sign problem, one without!

Large-N NJL model in d=3

$$\mathcal{L} = N\left(\bar\psi_a(\partial\!\!\!/ - m)\psi_a - \frac{C}{2}\left[(\bar\psi_a\psi_a)^2 + (\bar\psi_a i\gamma_5\psi_a)^2\right]\right)$$

- a=1,...,N
- d=4 theory dimensionally reduced to d=3
- ψ = 4 component spinor (like d=4)
- γ-matrices = 4x4 (like d=4)
- Symmetry = $U(N)_V \times U(1)_A$ (approximate if m≠0)

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Two equivalent formulations with auxiliary fields:

$$\mathcal{L} = N\left(\frac{1}{2C}\left(\sigma^2 + \pi^2\right) + \bar\psi_a\left[\partial\!\!\!/ - m + \sigma + i\pi\gamma_5\right]\psi_a\right)$$ σ/π

$$\mathcal{L} = N\left(\frac{1}{C}\,\mathrm{Tr}\,\left(V_\mu V_\mu + A_\mu A_\mu\right) + \bar\psi_a\left[\partial\!\!\!/ + iV\!\!\!/ + A\!\!\!/\gamma_5\right]_{ab}\psi_b\right)$$ A/V

(obtained by Fierz rearrangement of 4-fermion interaction)

For even N, the σ/π formulation has no sign problem at μ≠0:

$$(\partial\!\!\!/ + \sigma + i\pi\gamma_5 + \mu\gamma_1)^* = C(\partial\!\!\!/ + \sigma + i\pi\gamma_5 + \mu\gamma_1)C$$

⇒ $$\det(\partial\!\!\!/ + \sigma + i\pi\gamma_5 + \mu\gamma_1)^N$$ = real, positive

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...but the equivalent A/V formulation has a QCD-like sign problem:

$$\det\left[\partial\!\!\!/ + iV\!\!\!/ + A\!\!\!/\gamma_5 + \mu\gamma_1\right]^N$$ = complex!

(Can give a Splittorff-Verbaarschot argument for why it is complex, relating phase fluctuations of determinant to pion condensation in 2N flavor theory, just as in QCD.)

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Same theory, different formulation!
As we will see: having an explicit pion field cures the sign problem

Program:

Look at these two formulations and compute the probability distribution for measurements of a single fermion correlator.

Probability distribution for an observable X(φ) -- e.g. a correlation function -- where φ is a stochastic field:

$$\mathcal{P}(x) = \mathcal{N}\int [\mathrm{d}\phi]\, e^{-S[\phi]}\, \delta(X[\phi] - x)$$

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$$W(s) = \text{ sum of connected diagrams} = -\sum_n \frac{(is)^n}{n!}\kappa_n$$

n^{th} cumulant of X

I. The σ/π theory (chiral limit)

- Integrate out fermions
- Minimize effective action for mesons at leading order in $1/N$ (dim reg)
- find chiral symmetry breaking $<\sigma>=f$ at coupling $C=-\pi/f$
- Compute π,σ propagators

$$G_\sigma(k) = -\left[-\frac{f}{\pi}+\frac{\left(4f^2+k^2\right)\cot^{-1}\left(2f/k\right)+2fk}{2\pi k}\right]^{-1} ,$$

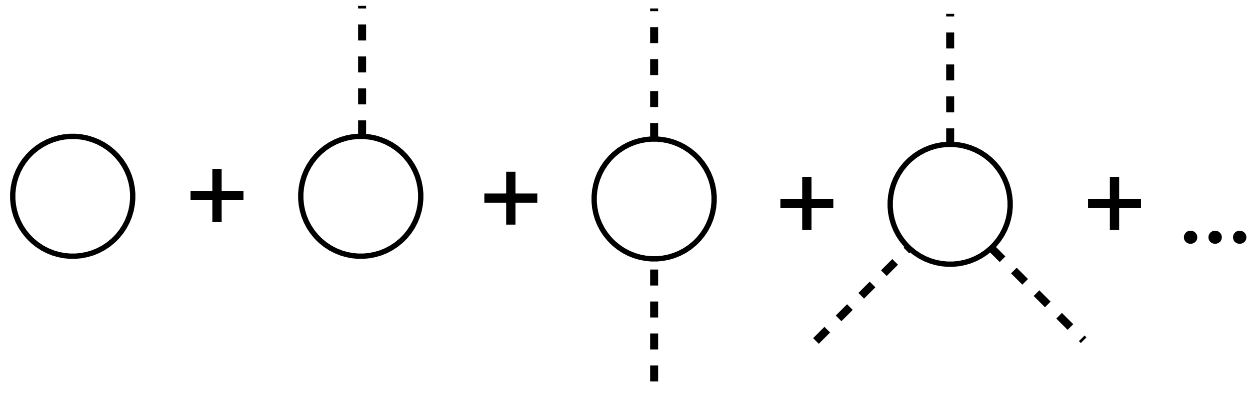
$$G_\pi(k) = -\left[\frac{k\cot^{-1}\left(2f/k\right)}{2\pi}\right]^{-1} .$$

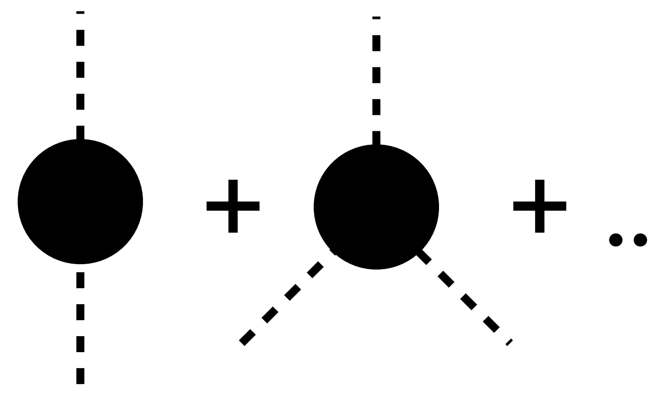
- Compute cumulants for the single fermion correlator:

$$X_\Gamma = \langle \mathbf{p}=0, t=T \,|\mathrm{Tr}\,\Gamma\left(\partial\!\!\!/ + f + \sigma + i\pi\gamma_5\right)^{-1}|\mathbf{p}=0, t=0\rangle$$

E.g. $\Gamma = (1+\gamma_1)/2$

Graphs:

$X_\Gamma[\phi] =$ 

$-\mathrm{Tr}\ \ln(\partial\!\!\!/ + f + \sigma + i\pi\gamma_5)$ = 

Can show:

- κ_n = sum of graphs with n white vertices
- these graphs proportional to $N^{-(n-1+L)}$, L= # loops

Leading in 1/N:

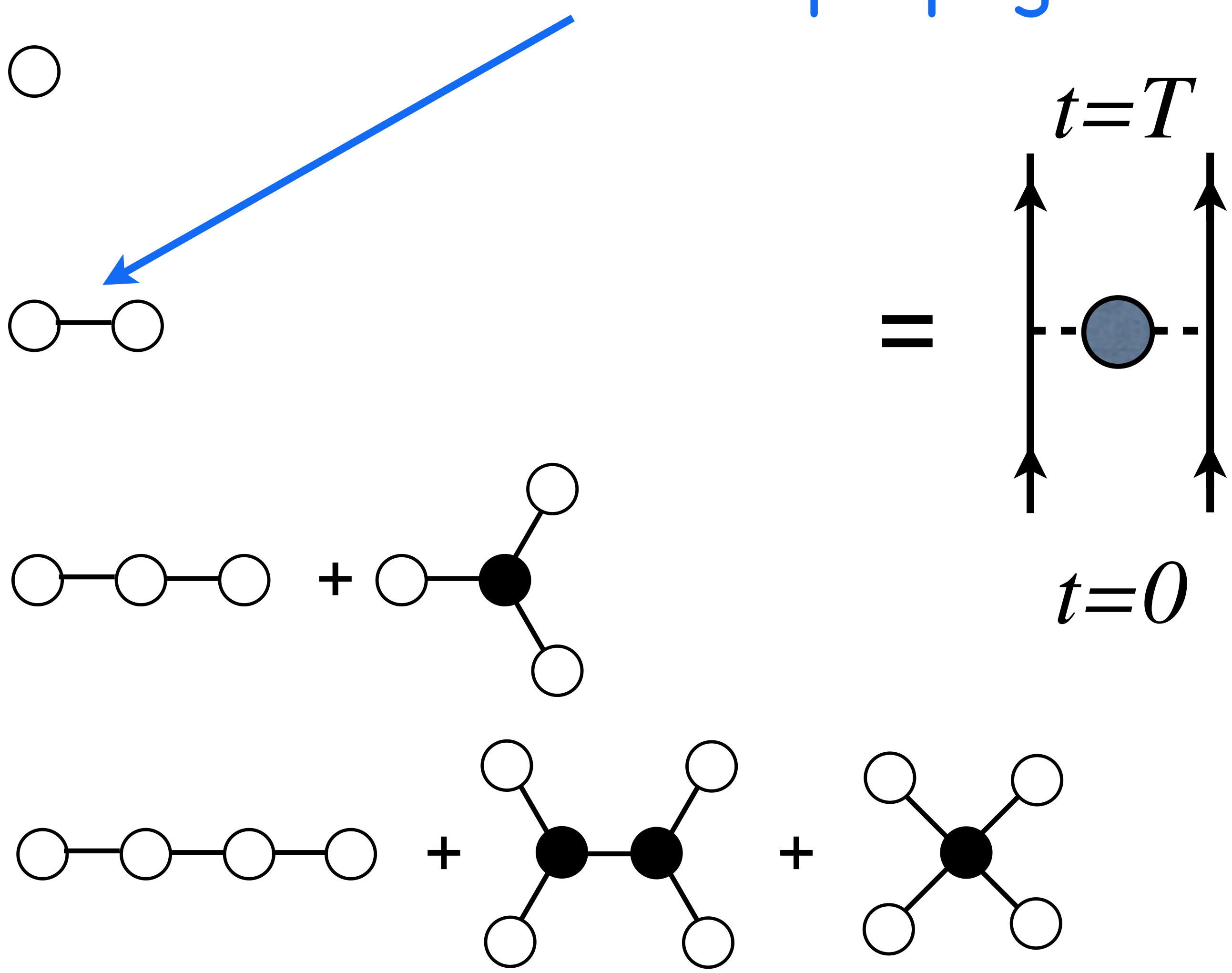
Full σ/π propagators

K_1

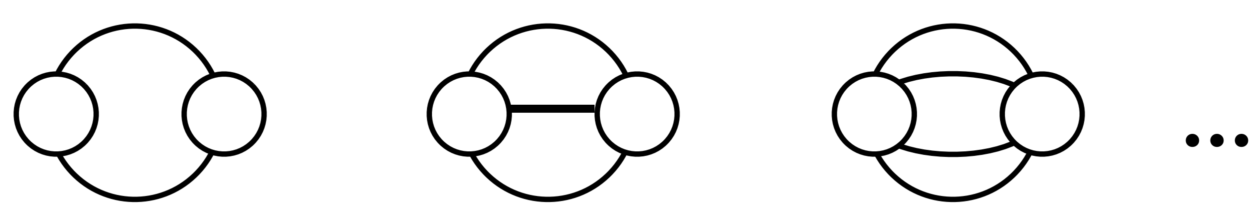
K_2

K_3

K_4



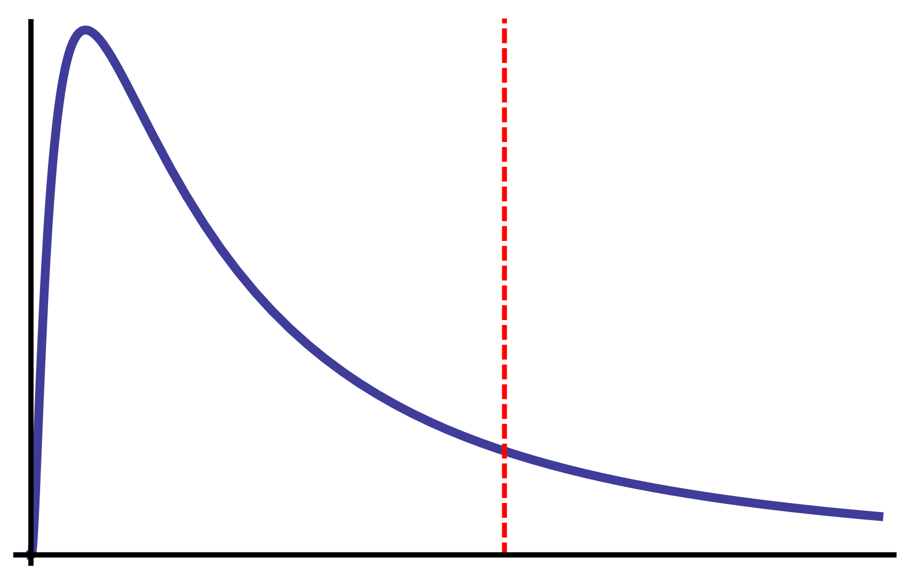
Subleading K_2 :



Preliminary results for σ/π formulation

- κ_n fall off as $e^{-nfT}/N^{(n-1)}$, f=fermion mass... pion mass doesn't appear
- No Lepage problem because no pion propagating in the "s-channel"
- 1/N corrections go as T/N: 1/N expansion breaks down at large T
- Cumulants κ'_n for log[X] fall off as $N^{-(n-1)}$
- κ'_2 grows proportional to T
- We think corrections to κ'_n go as 1/N, not T/N

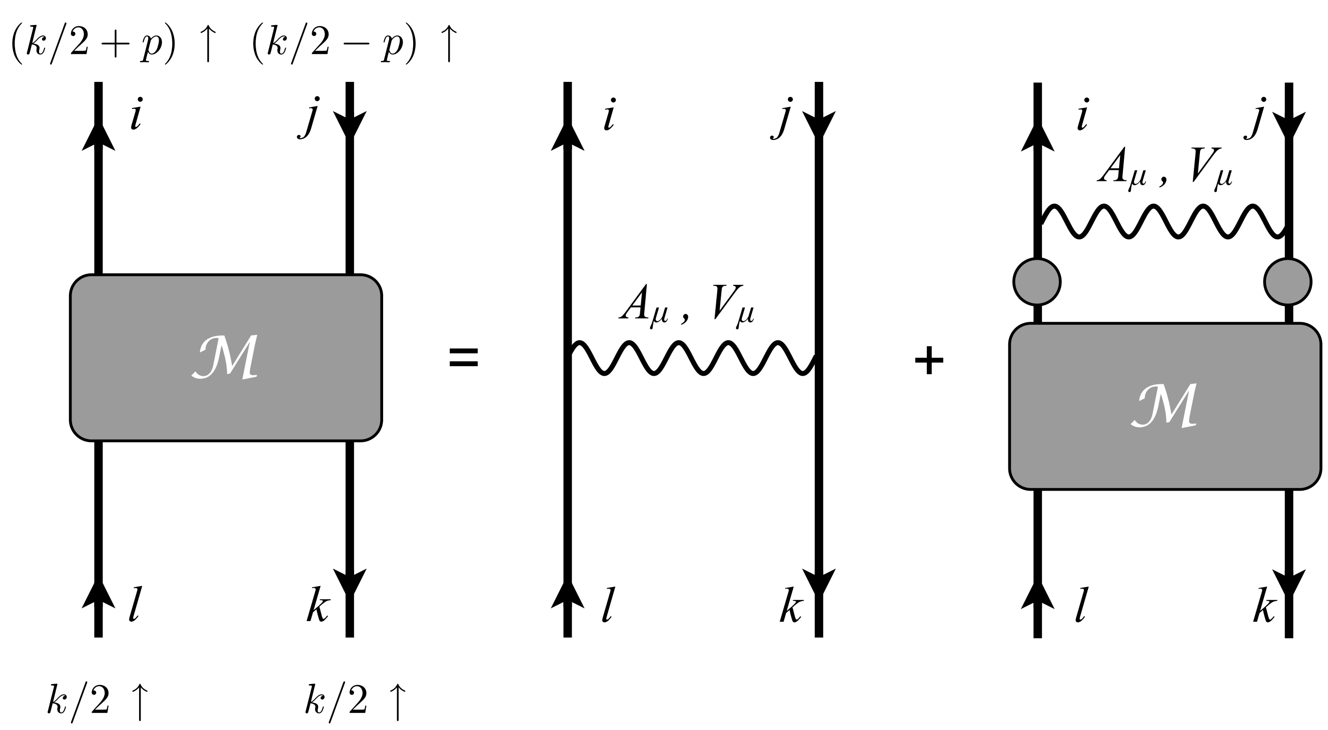
Would imply that the fermion correlator X has a long-tailed distribution, close to log-normal, similar to what was seen for unitary fermions



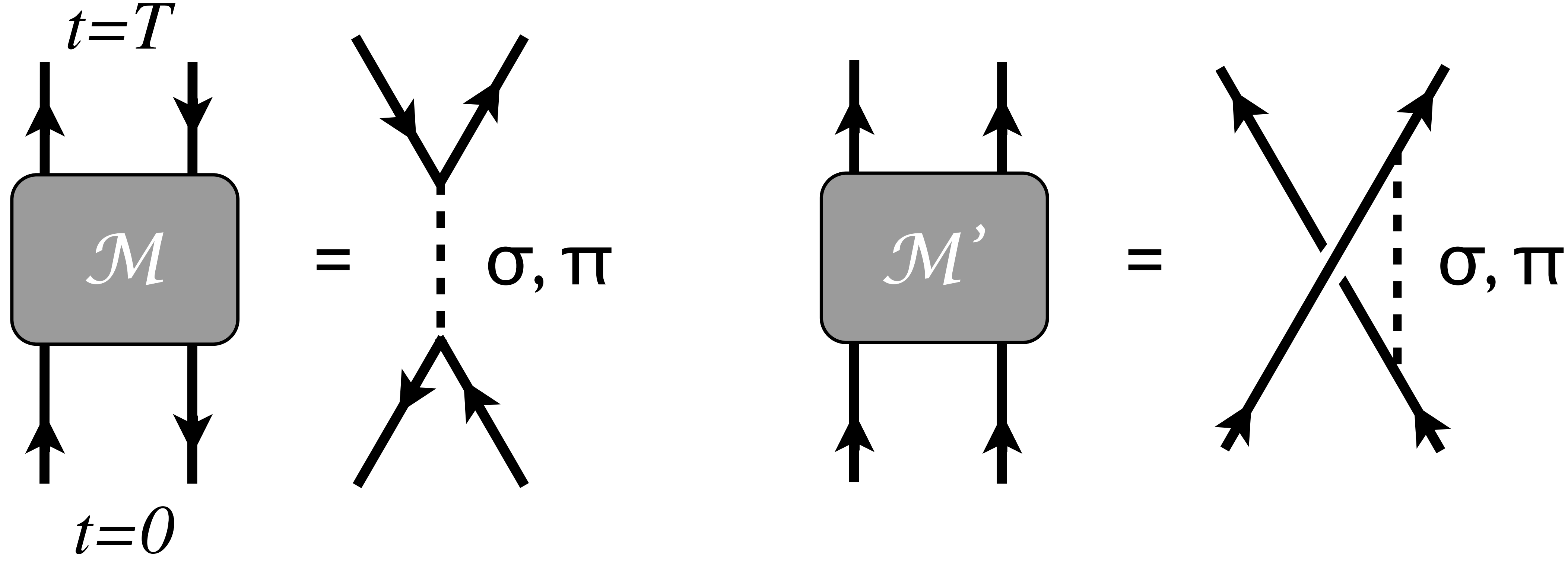
II. The A/V theory (chiral limit)

No mean field for chiral symmetry breaking: solve Schwinger-Dyson eq. instead (exact in large N). Power counting looks just like large-N QCD, but no nonlinear meson couplings before fermions integrated out.

To see π,σ mesons, must solve 4-pt function



and find...



--------- $= G_\sigma(k), G_\pi(k)$ found in σ/π formulation

Can now compute cumulants for the complex fermion correlator $X_\Gamma(T)$ in the A/V theory:

$$\Phi(s) = e^{-W(s)} = \mathcal{N} \int [\mathrm{d}\phi]\, e^{-S[\phi]+i(sX[\phi]+\bar{s}\bar{X}[\phi])}$$

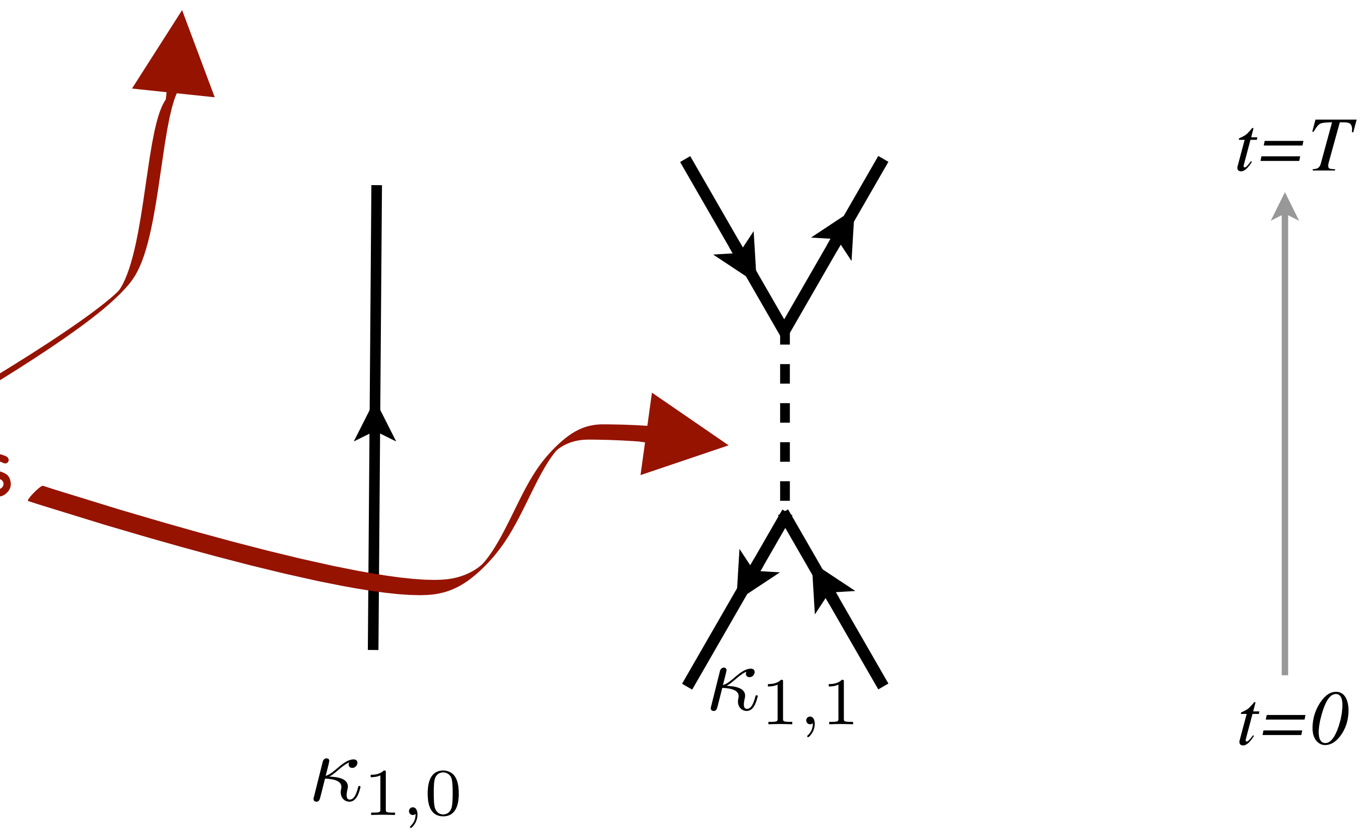
$$W(s) = -\sum_{m,n=1}^{\infty} \frac{(is)^m (i\bar{s})^n}{m!\, n!}\, \kappa_{m,n}$$

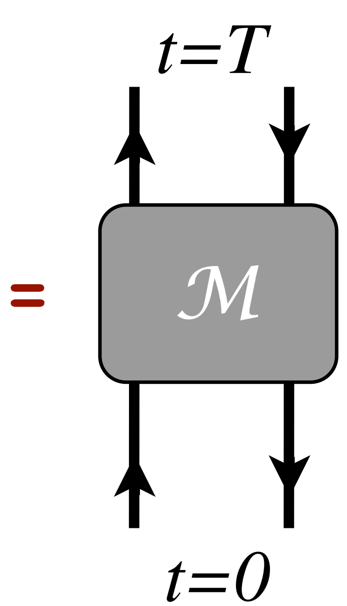
Find Lepage-Savage scaling (fermion mass f plays role of baryon mass)

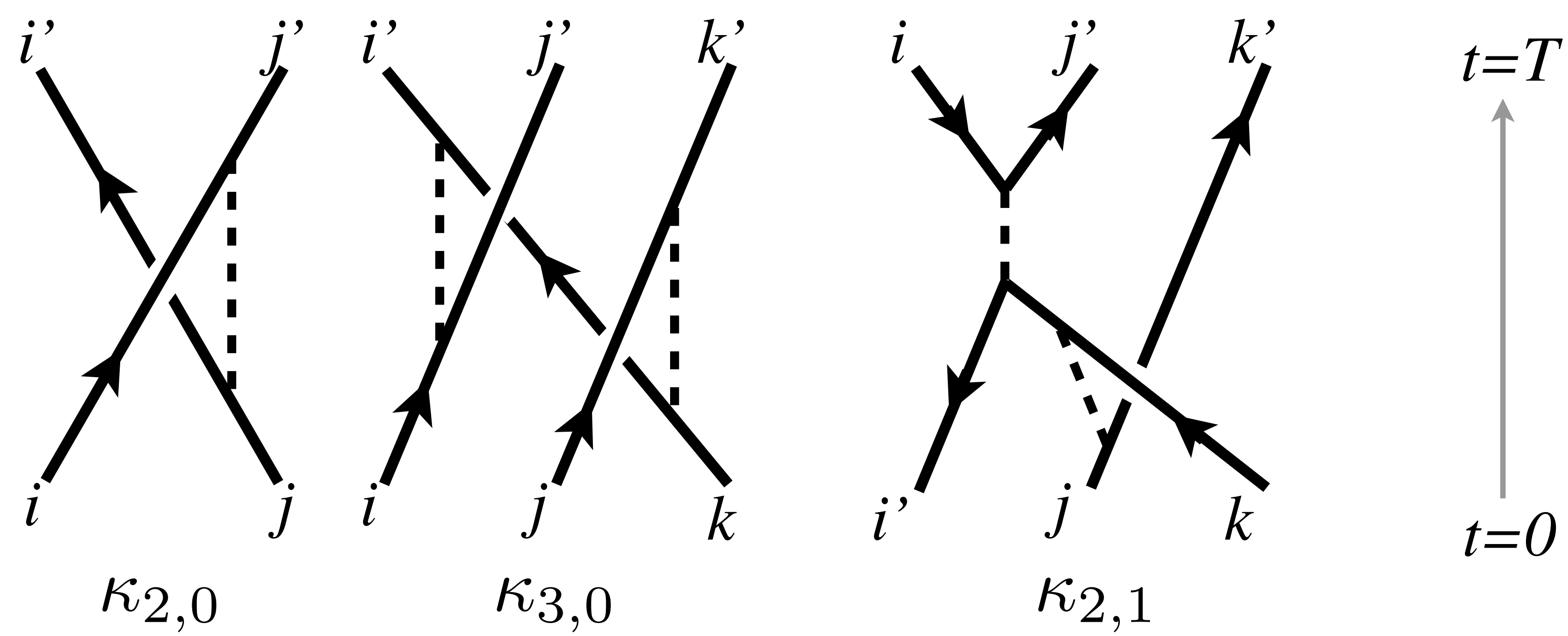
$$\kappa_{m,n} \propto e^{-\left(|m-n|f+\frac{(m+n)}{2}m_\pi\right)T}\ .$$

Same as in QCD

The culprit: s-channel pions

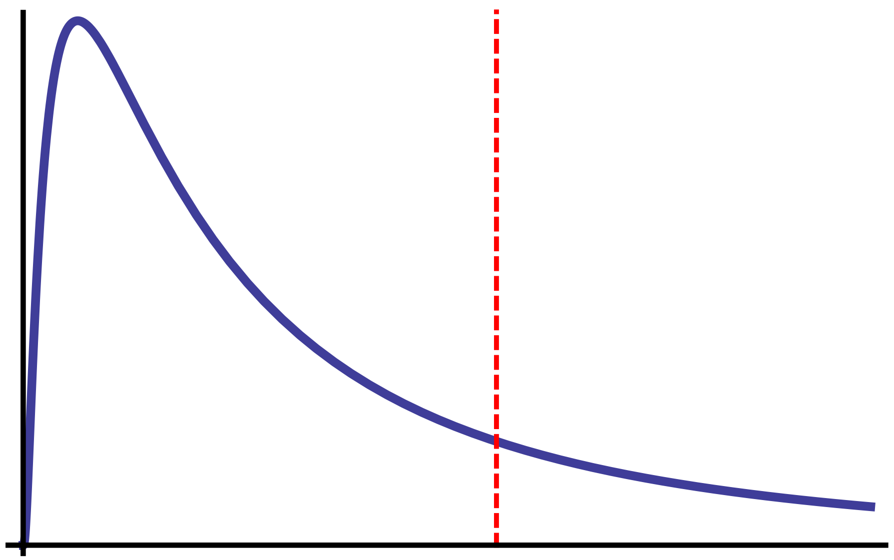






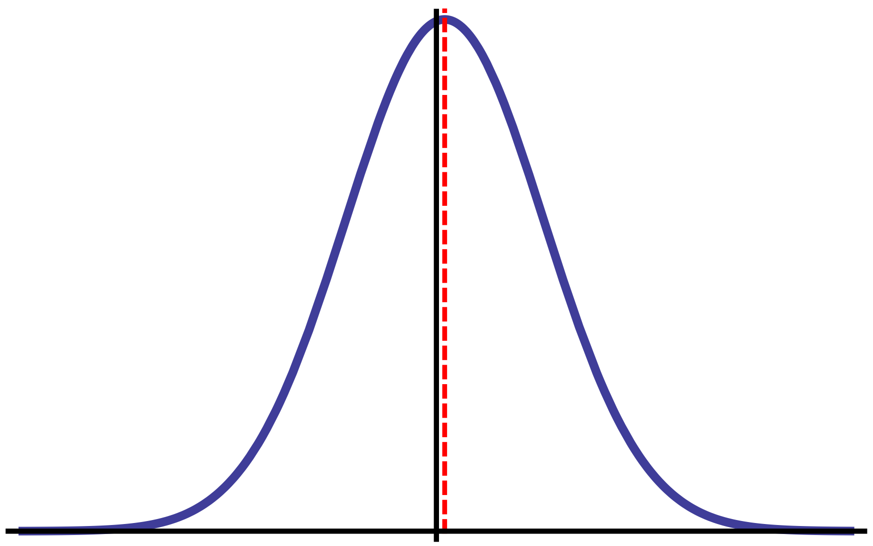
Distribution of fermion correlator at late time T

σ/π formulation,
no sign problem



might be tractable
with cumulant expansion

A/V formulation,
QCD-like sign problem



very hard

Same theory; σ/π formulation has explicit pion which does not behave as a fermion bound state

Speculations about QCD?

- Sign problem is not a "fermion sign problem" - it is directly attributable to chiral symmetry breaking and the existence of a light pion which is a q q* bound state
- If there is a way to include a mean field for the pion in QCD without changing the theory*, sign problem might be ameliorated or solved.

* Adding fundamental pion must also involve adding meson in t-channel to make gluon attraction between qq* weaker, so pion does not appear as bound state. Looks like quark model. Can this be done exactly without causing more problems??

Main conclusion to take away:

Since there exists a theory with a QCD-like sign problem at finite density which has another formulation without a sign problem

...and we understand why...

... maybe there is hope for QCD