Restoration of Rotational Symmetry From the Continuum Limit of Lattice Field Theories

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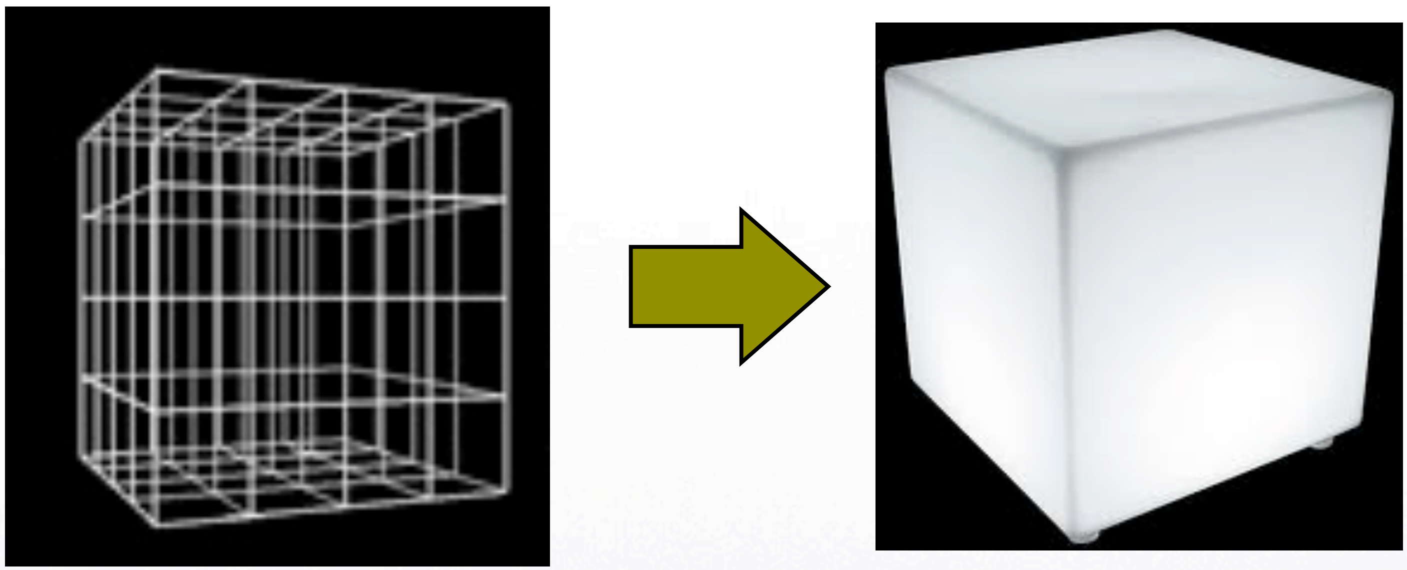
ZD, M. J. Savage, hep-lat: 1204.4146, To be published in Phys. Rev. D

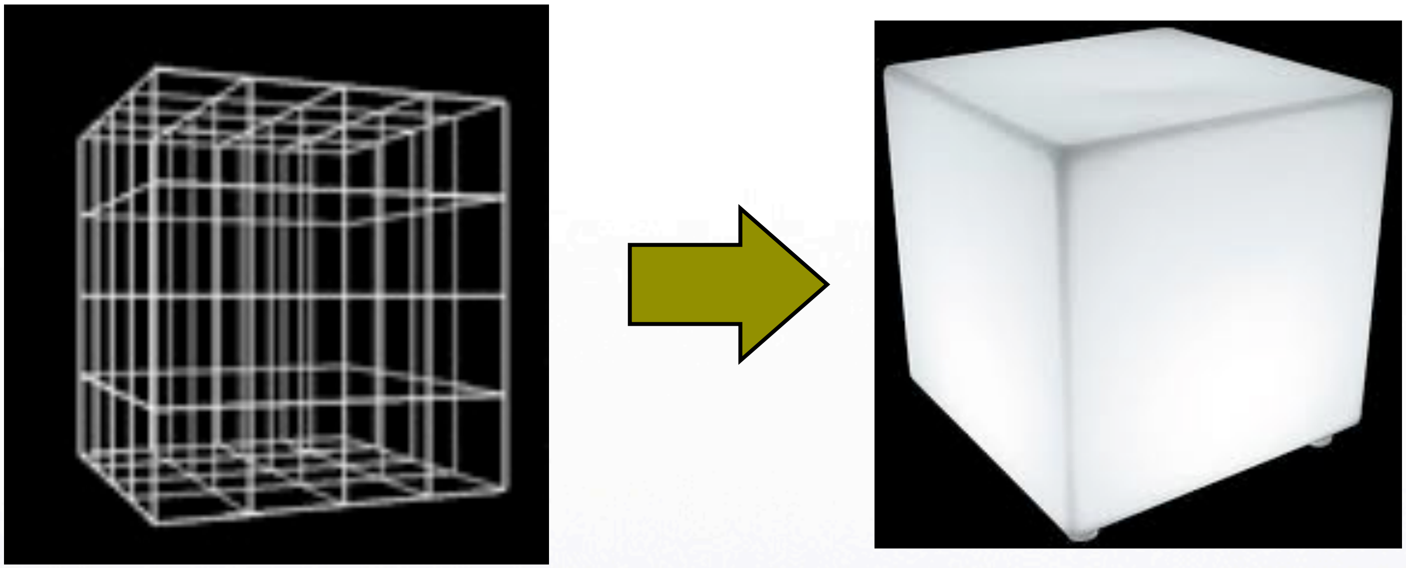
FROM LATTICE TO CONTINUUM?



UV rotational invariance recovery: $a \to 0$

IR rotational invariance recovery: $L \to \infty$

FROM LATTICE TO CONTINUUM?



UV rotational invariance recovery: $a \to 0$

IR rotational invariance recovery: $L \to \infty$

A PROBLEM

LQCD on HYPER-CUBIC lattices

Full rotational group → Infinite number of irreps

Cubic group → Only 10 irreps

L	
0	A_1^+
1	T_1^-
2	$E^+ \oplus T_2^+$
3	$A_2^- \oplus T_1^- \oplus T_2^-$
4	$A_1^+ \oplus T_1^+ \oplus T_2^+$
5	$E^- \oplus T_1^- \oplus T_1^- \oplus T_2^-$

Operators with different angular momentum can not mix in the continuum.

Not true for lattice operators → Less symmetries, less constraints.

POWER DIVERGENCE

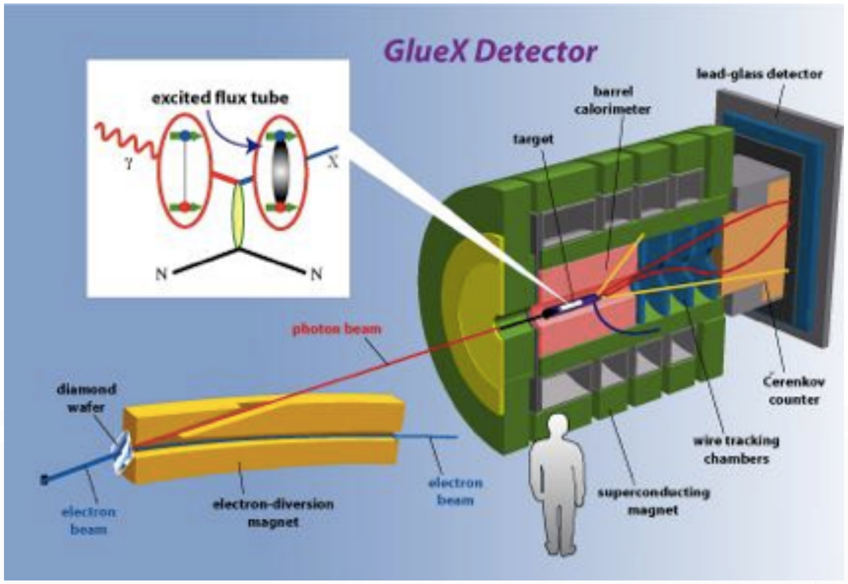
Two examples

1 Excited states spectroscopy

$$C(t) = \langle 0 | \mathcal{O}^{\dagger}(t)\, \mathcal{O}(0) | 0 \rangle$$

Continuum states up to $\mathcal{O}(a^n)$.

To be built on the lattice



Spin identification?

2 Higher moments of hadron structure functions

$$\langle x^n \rangle_{q,\mu^2} = \int dx\, x^n q(x;\mu^2)$$

$$\langle p, s | \mathcal{O}_{\mu_1 \mu_2 \ldots \mu_n} | p, s \rangle |_{\mu^2} = 2 \langle x^n \rangle_{q,\mu^2} p^{\{\mu_1} p^{\mu_2} \ldots p^{\mu_n\}}$$

Two examples

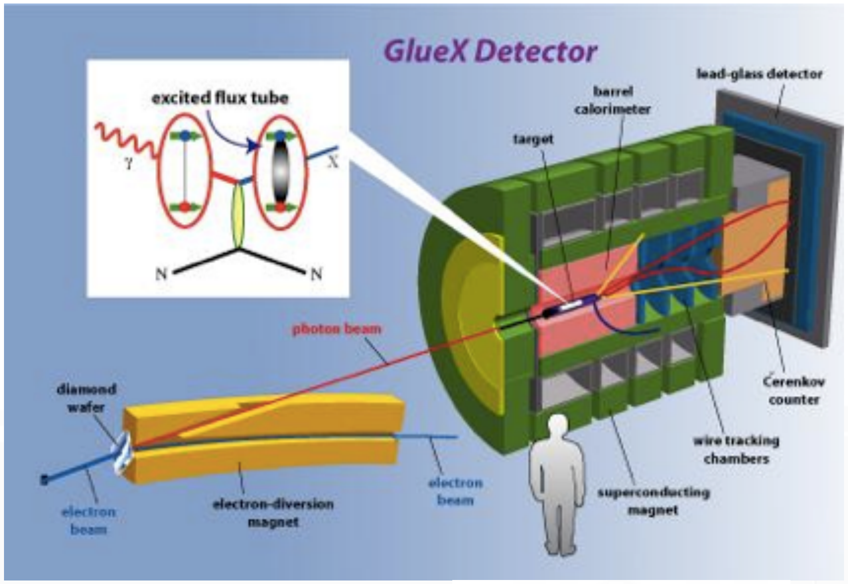
1 Excited states spectroscopy

$$C\left(t\right)=\left\langle 0\left|\mathcal{O}^{\dagger}\left(t\right)\mathcal{O}\left(0\right)\right|0\right\rangle$$

Continuum states up to $\mathcal{O}\left(a^{n}\right)$.

To be built on the lattice

Spin identification?



2 Higher moments of hadron structure functions

$$\left\langle x^{n}\right\rangle_{q,\mu^{2}}=\int dx\,x^{n}q\left(x;\mu^{2}\right)$$

$$\left\langle p,s\left|\mathcal{O}_{\mu_{1}\mu_{2}\ldots\mu_{n}}\right|p,s\right\rangle\Big|_{\mu^{2}}=2\left\langle x^{n}\right\rangle_{q,\mu^{2}}p^{\{\mu_{1}}p^{\mu_{2}}\ldots p^{\mu_{n}\}}$$

A successful empirical scenario*

$$C_{ij}(t) = \sum_n \frac{1}{2E_n} \langle 0 | \mathcal{O}_i^\dagger | n \rangle \underbrace{\langle n | \mathcal{O}_j | 0 \rangle}_{Z_i^n} e^{-E_n t}$$

Overlap function

How to build $\mathcal{O}_i$?

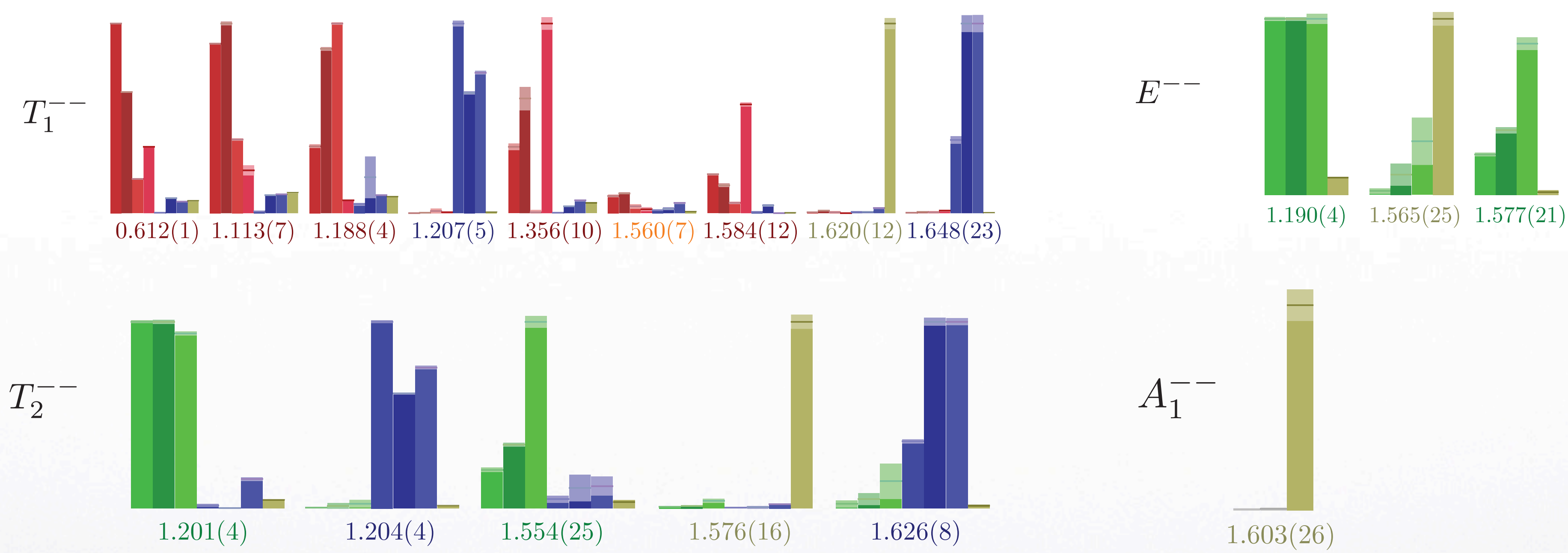
- Smear out the fields: $\begin{cases} \psi(x) \to \widetilde{\psi}(x) \\ U(x) \to \widetilde{U}(x) \end{cases}$
- Subduce it from a continuum angular momentum J :

$$\mathcal{O}^{[J]}_{\Lambda,\lambda} \equiv \sum_M \mathcal{S}^{J,M}_{\Lambda,\lambda} \mathcal{O}^{J,M}$$

$$\mathcal{S}^{J,M}_{\Lambda,\lambda} = \langle \Lambda, \lambda | J, M \rangle \qquad \mathcal{O}^{J,M} \equiv \left(\Gamma \times D^{n_D}\right)^{J,M}$$

* *J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards, and C. E. Thomas, Phys.Rev.Lett., 103, 262001 (2009); Phys. Rev. D82, 034508 (2010),*

The results for the overlap functions:*



$J=0$ $(a_1 \times D^{[1]}_{J=1})^{J=0}$ $(a_1 \times D^{[3]}_{J_{13}=2,J=1})^{J=0}$

$J=1$ $(\rho)^{J=1}$ $(a_1 \times D^{[1]}_{J=1})^{J=1}$ $(\rho \times D^{[2]}_{J=2})^{J=1}$ $(\pi \times D^{[2]}_{J=1})^{J=1}$

$J=2$ $(a_1 \times D^{[1]}_{J=1})^{J=2}$ $(\rho \times D^{[2]}_{J=2})^{J=2}$ $(a_1 \times D^{[3]}_{J_{13}=2,J=3})^{J=2}$

$J=3$ $(\rho \times D^{[2]}_{J=2})^{J=3}$ $(a_0 \times D^{[3]}_{J_{13}=2,J=3})^{J=3}$ $(a_1 \times D^{[3]}_{J_{13}=2,J=3})^{J=3}$

$J=4$ $(a_1 \times D^{[3]}_{J_{13}=2,J=3})^{J=4}$

* *J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards, and C. E. Thomas, Phys.Rev.Lett., 103, 262001 (2009).*

THE LESSONS

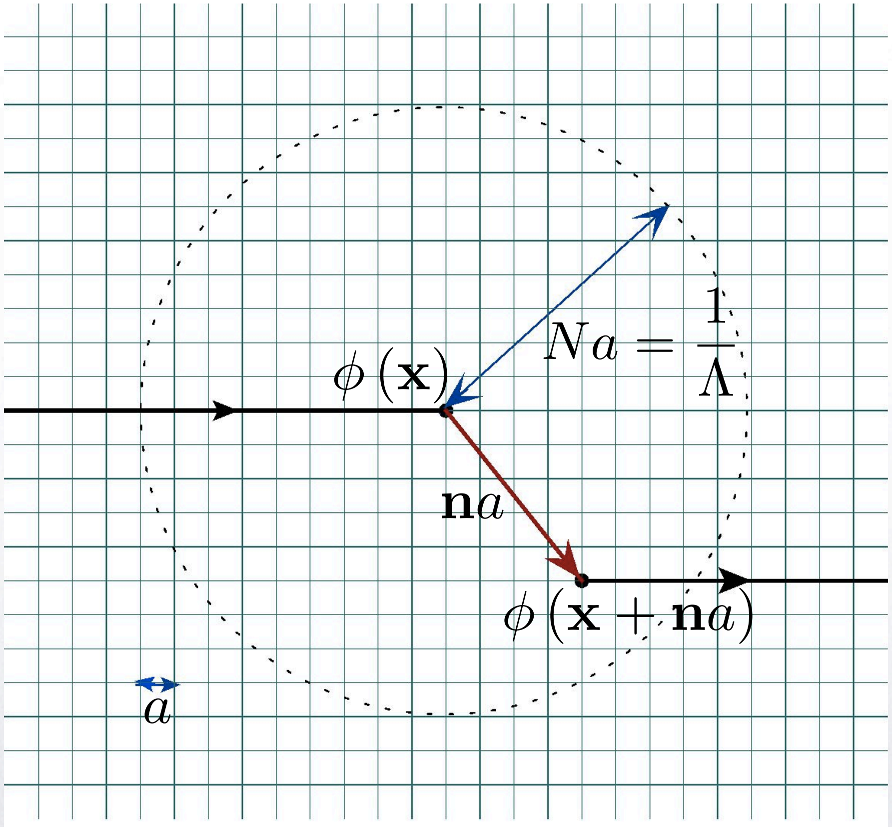
- Smearing the fields
- Angular momentum “memory” function

WHY?

HOW?

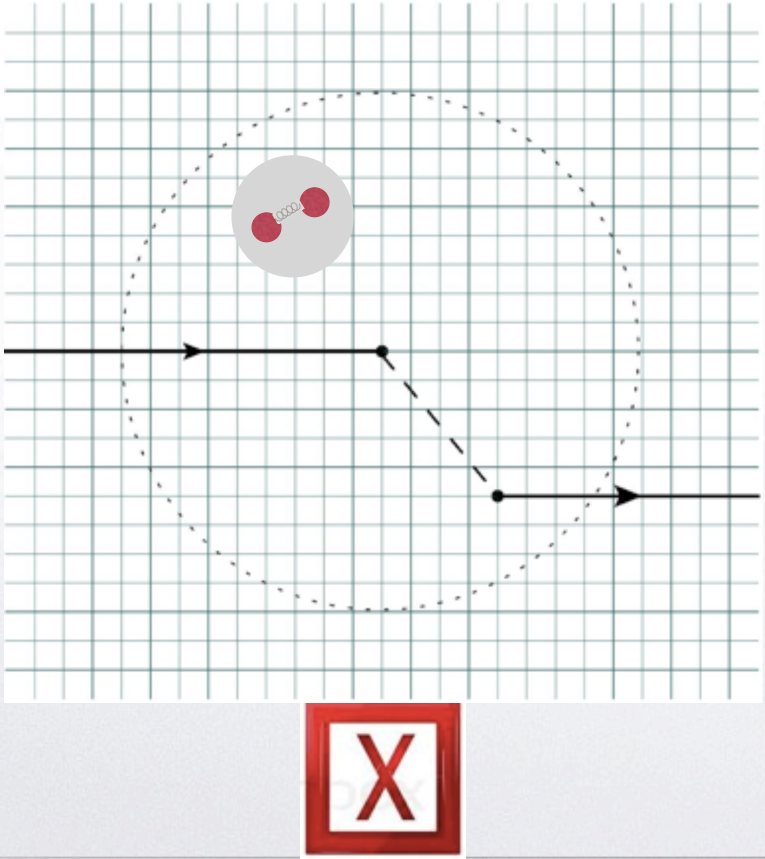
A TOY MODEL:

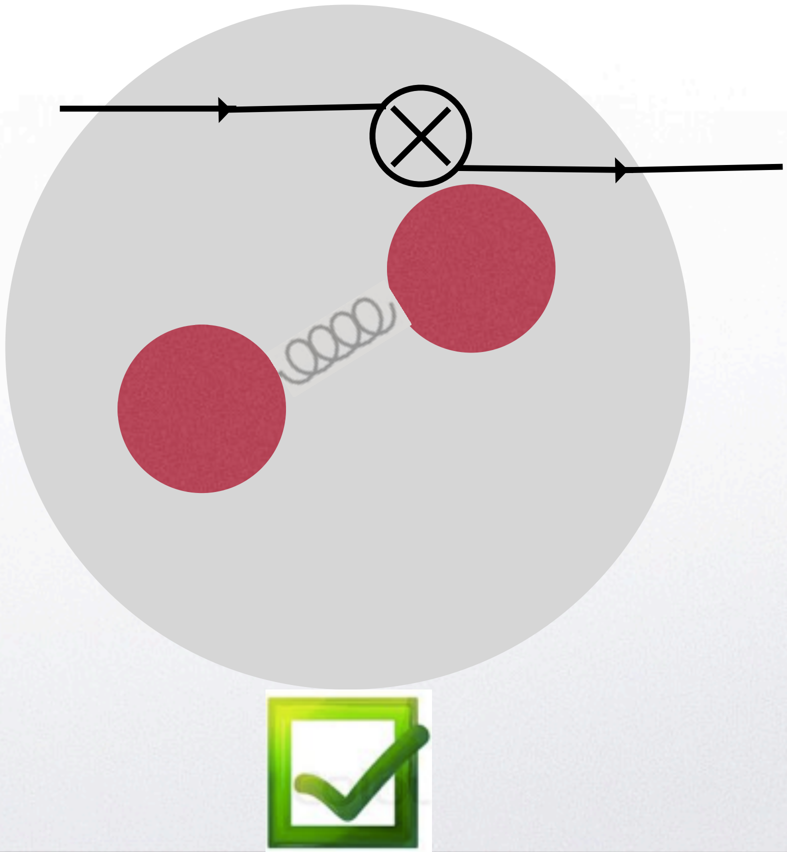
$$\hat{\theta}_{L,M}(\mathbf{x}; a, N) = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} \phi(\mathbf{x})\,\phi(\mathbf{x}+\mathbf{n}a)\;\; Y_{L,M}(\hat{\mathbf{n}})$$



Relevant scales:

$$a \ll \frac{1}{\Lambda} \ll \frac{1}{\Lambda_{Hadron}}\left(\frac{1}{p}\right)$$





A derivative expansion

$$\hat{\theta}_{L,M}(\mathbf{x}; a, N) = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} \phi(\mathbf{x})\, \phi(\mathbf{x} + \mathbf{n}a)\; Y_{L,M}(\hat{\mathbf{n}})$$

⬇

$$\hat{\theta}_{L,M}(\mathbf{x}; a, N) = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} \sum_{k} \frac{1}{k!}\, \phi(\mathbf{x})\, (a\mathbf{n}\cdot\nabla)^k\, \phi(\mathbf{x})\; Y_{L,M}(\hat{\mathbf{n}})$$

⬇

$$\hat{\theta}_{L,0}(\mathbf{x}; a, N) = \sum_{L',d} \frac{C^{(d)}_{L0;L'0}(N)}{\Lambda^d}\, \mathcal{O}^{(d)}_{z^{L'}}(\mathbf{x})$$

The number of derivatives

L' number of free z indices

What is the operator basis $\mathcal{O}^{(d)}_{z\,L'}(\mathbf{x})$?

Example

Operators with L=1, M=0

$$\mathcal{O}^{(1)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\,\nabla_z\phi(\mathbf{x})$$

$$\mathcal{O}^{(3)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\,\nabla^2\nabla_z\phi(\mathbf{x})$$

$$\mathcal{O}^{(5)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\left(\nabla^2\right)^2\nabla_z\phi(\mathbf{x})$$

Violates rotational invariance.

$$\mathcal{O}^{(5,RV)}_{z}(\mathbf{x}) = \phi(\mathbf{x})\sum_j \nabla_j^4\nabla_z\phi(\mathbf{x})$$

Mixing in the classical operator

Not a surprise! → Operator is on a grid!

Example

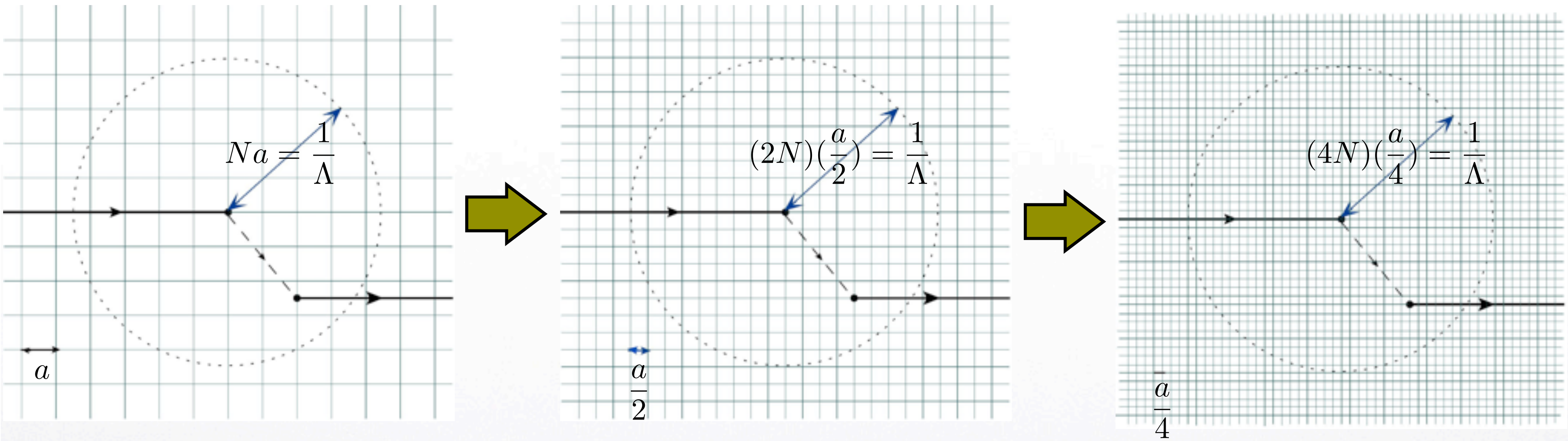
$$\hat{\theta}_{3,0}\left(\mathbf{x};a,N\right)=\frac{C^{(1)}_{30;10}\left(N\right)}{\Lambda}\mathcal{O}^{(1)}_{z}\left(\mathbf{x};a\right)+\frac{C^{(3)}_{30;10}\left(N\right)}{\Lambda^{3}}\mathcal{O}^{(3)}_{z}\left(\mathbf{x};a\right)+\frac{C^{(5)}_{30;10}\left(N\right)}{\Lambda^{5}}\mathcal{O}^{(5)}_{z}\left(\mathbf{x};a\right)+$$

$$\frac{C^{(5;RV)}_{30;10}\left(N\right)}{\Lambda^{5}}\mathcal{O}^{(5;RV)}_{z}\left(\mathbf{x};a\right)+\frac{C^{(3)}_{30;30}\left(N\right)}{\Lambda^{3}}\mathcal{O}^{(3)}_{zzz}\left(\mathbf{x};a\right)+\frac{C^{(5)}_{30;30}\left(N\right)}{\Lambda^{5}}\mathcal{O}^{(5)}_{zzz}\left(\mathbf{x};a\right)+$$

$$\frac{C^{(5)}_{30;50}\left(N\right)}{\Lambda^{5}}\mathcal{O}^{(5)}_{zzzzz}\left(\mathbf{x};a\right)+\mathcal{O}\left(\frac{\nabla^{7}_{z}}{\Lambda^{7}}\right)$$

Coefficients of $L=3$ operator

Reduce the pixelation of the lattice



$$\hat{\theta}_{3,0}\left(\mathbf{x}; a, N\right)?$$

A good operator if

$$C^{(d)}_{30;L'0}(N) \quad \text{is finite for} \quad L' = 3$$

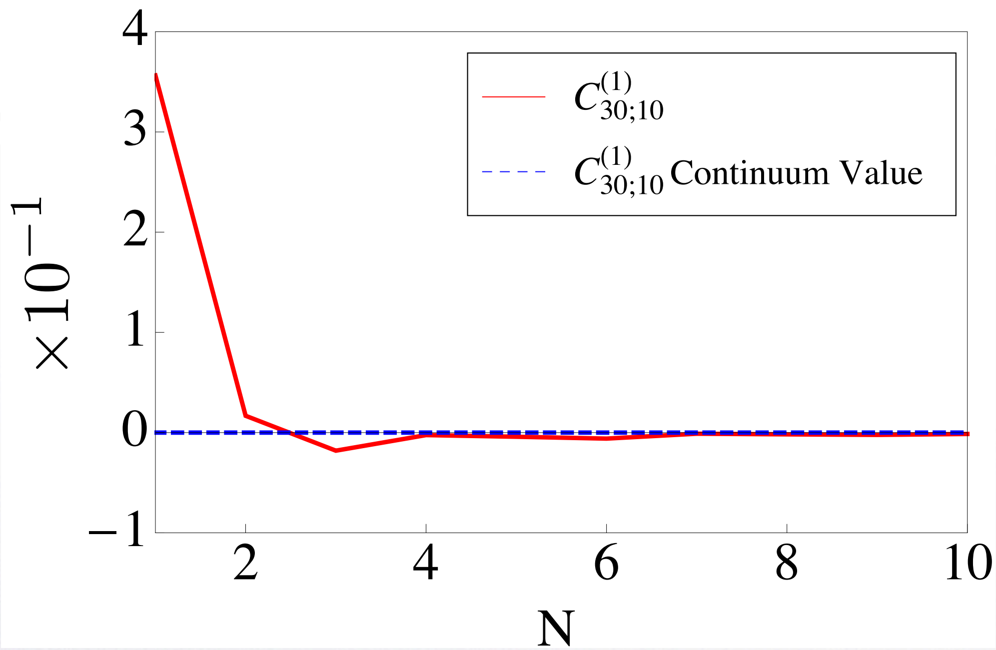
$$C^{(d)}_{30;L'0}(N) \to 0 \quad \text{for} \quad L' \neq 3$$

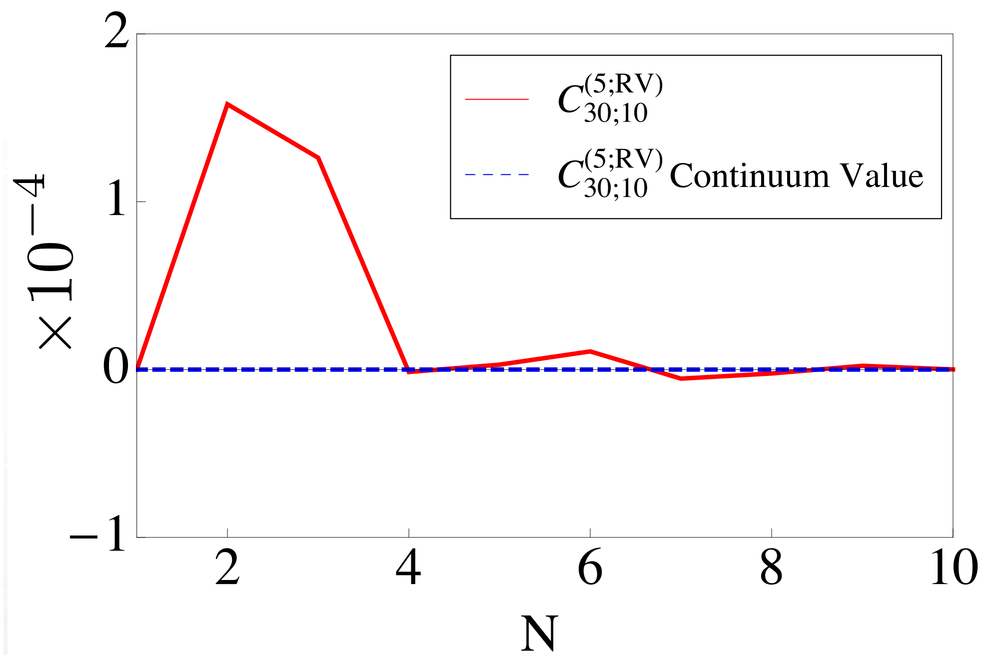
$$C^{(d;RV)}_{30;L'0}(N) \to 0$$

as $N \to \infty$.

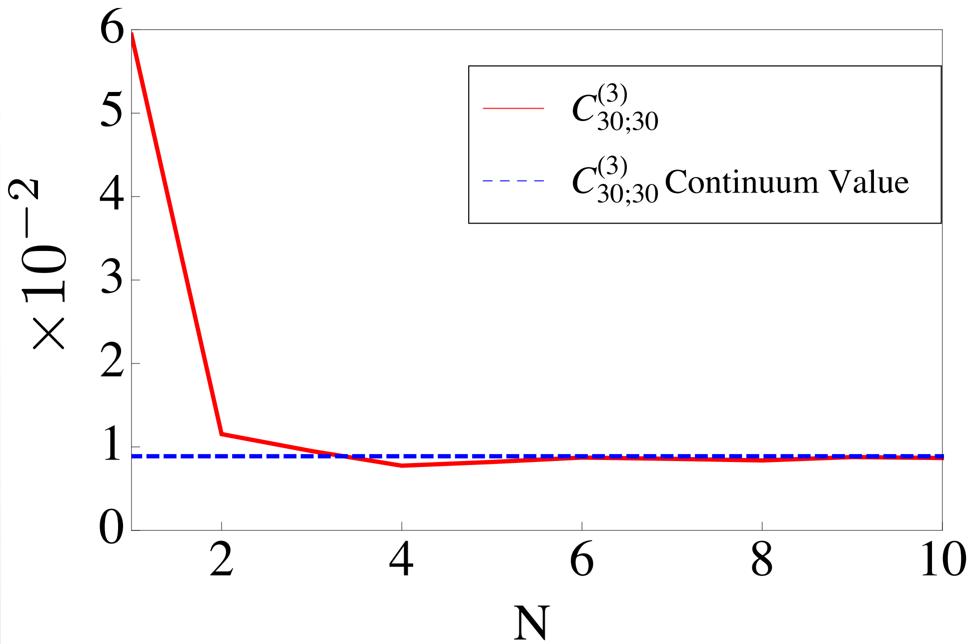
So it recovers a $L = 3$ operator!

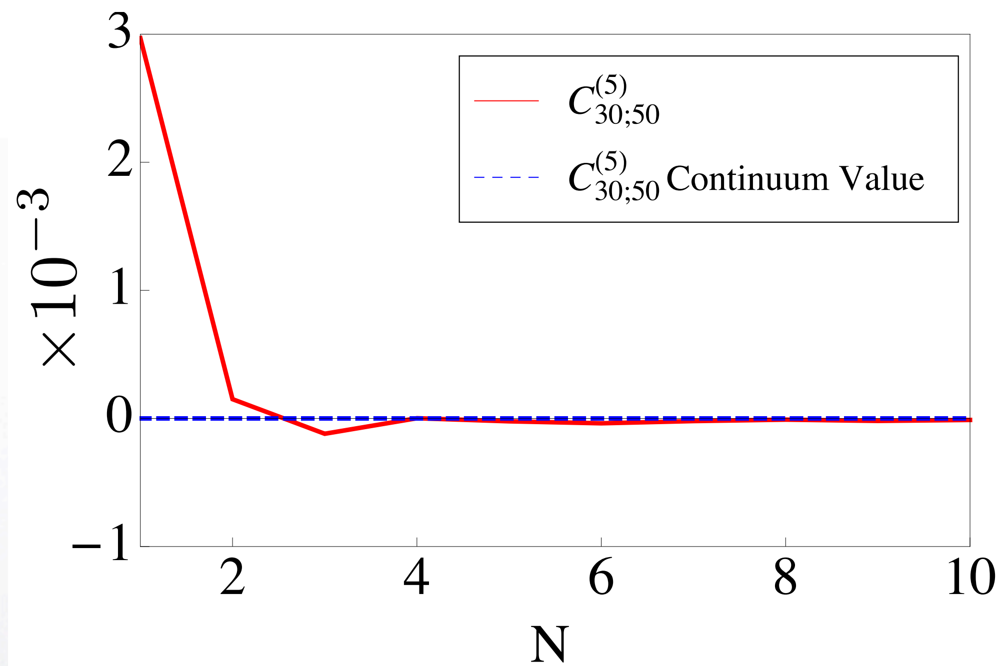
Coefficients as a function of N?



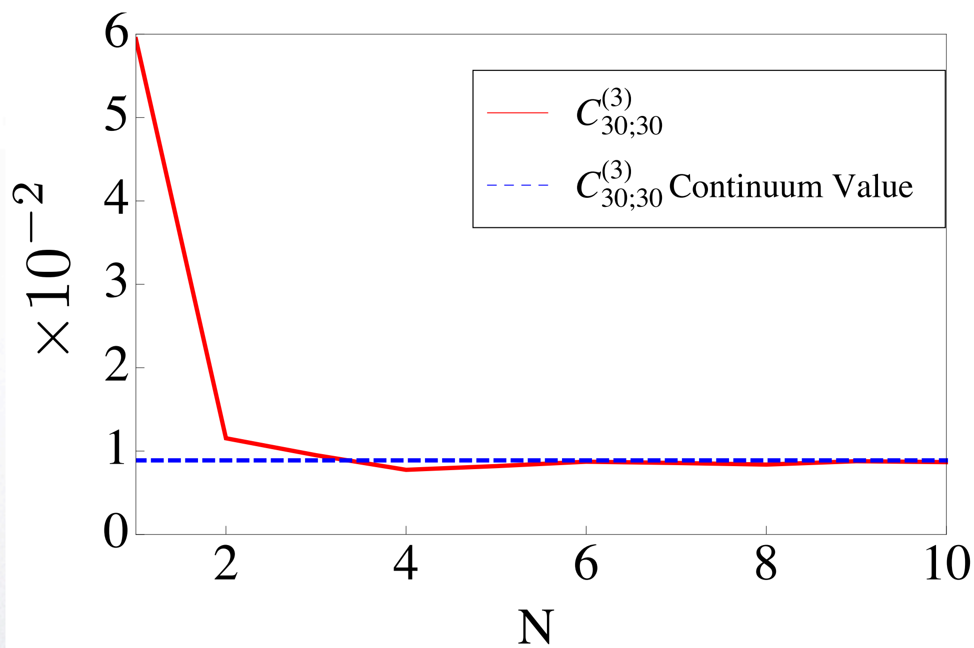


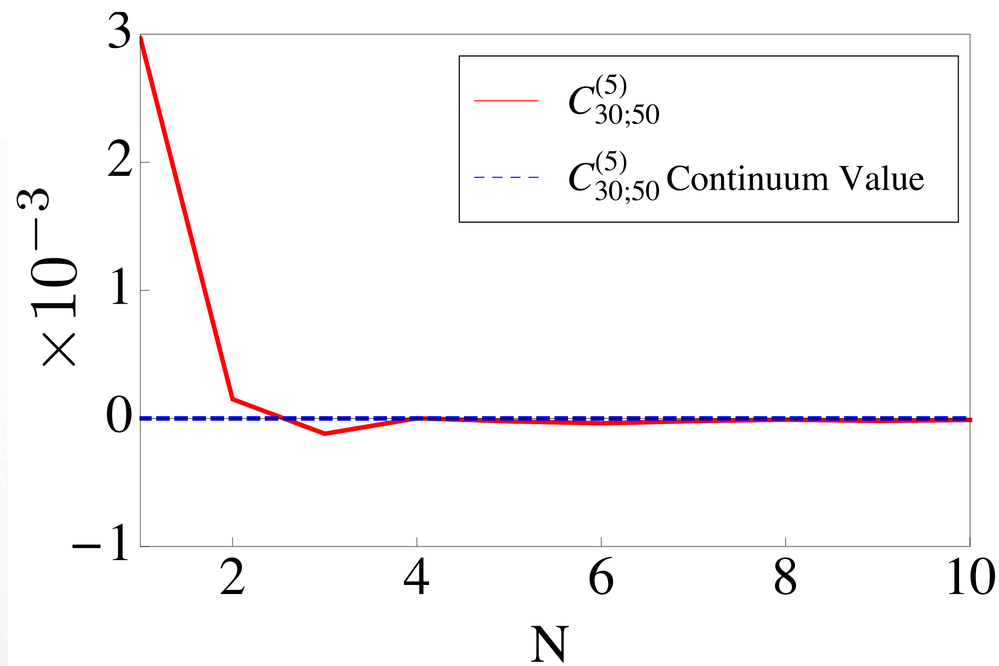
Coefficients as a function of N?





Coefficients as a function of N?





Analytically

$$C^{(d)}_{30;30} = \frac{15}{4}\sqrt{\frac{7}{\pi}}\,\frac{d^2-1}{(d+4)!} + \mathcal{O}\left(\frac{1}{N^2}\right) \qquad \text{with} \qquad d = 3, 5, \ldots$$

$$C^{(d)}_{30;L0} = \mathcal{O}\left(\frac{1}{N^2}\right) \qquad \text{with} \qquad L \neq 3 \qquad \text{and} \qquad d = L, L+1, \ldots$$

$$C^{(d;RV)}_{30;L0} = \mathcal{O}\left(\frac{1}{N^2}\right) \qquad \text{with} \qquad d = L, L+1, \ldots$$

UNIVERSAL $\frac{1}{N^2} = a^2\Lambda^2$ CORRECTIONS!

For large N

$$\Lambda^3 \hat{\theta}_{3,0}(\mathbf{x}; a, N) = \alpha_1 \frac{\Lambda^2}{N^2} \mathcal{O}_z^{(1)}(\mathbf{x}) + \alpha_2 \frac{1}{N^2} \mathcal{O}_z^{(3)}(\mathbf{x}) + \alpha_3 \frac{1}{\Lambda^2 N^2} \mathcal{O}_z^{(5)}(\mathbf{x}) +$$

$$\alpha_4 \frac{1}{\Lambda^2 N^2} \mathcal{O}_z^{(5;RV)}(\mathbf{x}) + \alpha_5 \mathcal{O}_{zzz}^{(3)}(\mathbf{x}) + \alpha_6 \frac{1}{\Lambda^2} \mathcal{O}_{zzz}^{(5)}(\mathbf{x}) +$$

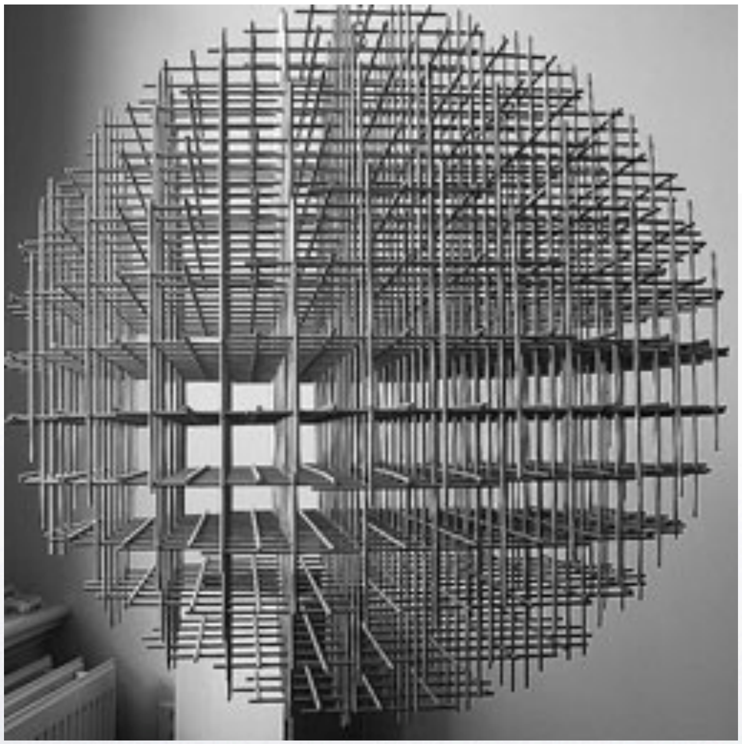
$$\alpha_7 \frac{1}{\Lambda^2 N^2} \mathcal{O}_{zzzzz}^{(5)}(\mathbf{x}) + \mathcal{O}\left(\frac{\nabla_z^7}{\Lambda^4}\right)$$

$N = 1$ ⇒

$$\alpha_1 \frac{1}{a^2} \mathcal{O}_z^{(1)} + \alpha_2 \mathcal{O}_z^{(3)} + \alpha_3 a^2 \mathcal{O}_z^{(5)} + \alpha_4 a^2 \mathcal{O}_z^{(5;RV)} +$$

$$\alpha_5 \mathcal{O}_{zzz}^{(3)} + \alpha_6 a^2 \mathcal{O}_{zzz}^{(5)} + \alpha_7 a^2 \mathcal{O}_{zzzzz}^{(5)} + \mathcal{O}\left(a^4 \nabla_z^7\right)$$

NO LARGE CONTAMINATION!

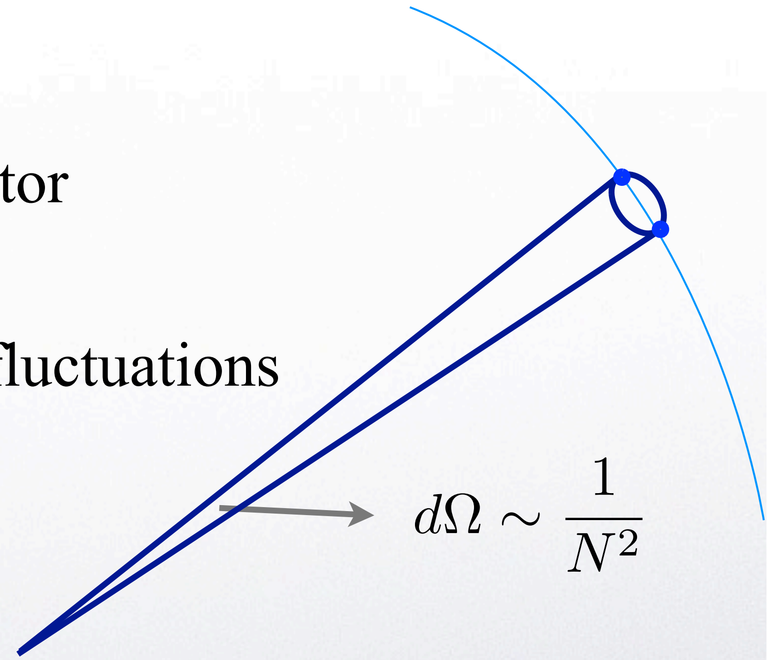


Why $\dfrac{1}{N^2}$?

Classical operator

⬇

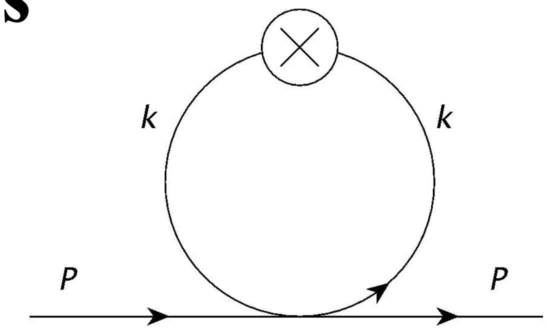
No short distance fluctuations



What about **QUANTUM FLUCTUATIONS**?

A perturbative analysis:

$\lambda\phi^4$



$$\frac{3\lambda}{4\pi N^3}\sum_{\mathbf{n}}^{|\mathbf{n}|\leq N}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^4k}{(2\pi)^4}\frac{e^{i\mathbf{k}\cdot\mathbf{n}a}}{\left(\frac{4}{a^2}\sum_{\mu}\sin^2\left(\frac{k_\mu a}{2}\right)^2+m^2\right)^2}Y_{LM}\left(\Omega_{\mathbf{n}}\right)$$

- Leading order → $L=0$ operator
- Sub-leading → $\mathcal{O}\left(\frac{\lambda}{N^2}\right)$
- Sub-leading to all orders → $\mathcal{O}\left(\frac{\lambda^n}{N^2}\right)$

⇒ $\hat{\theta}_{L,M}\left(\mathbf{x};a,N\right)$ ✓

The operator in QCD

Differences:
- Link
- Spin/Flavor

$$\hat{\theta}_{L,M}(\mathbf{x};a,N) = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\le N} \overline{\psi}(\mathbf{x}) \underbrace{U(\mathbf{x},\mathbf{x}+\mathbf{n}a)}\psi(\mathbf{x}+\mathbf{n}a)\; Y_{L,M}(\hat{\mathbf{n}})$$

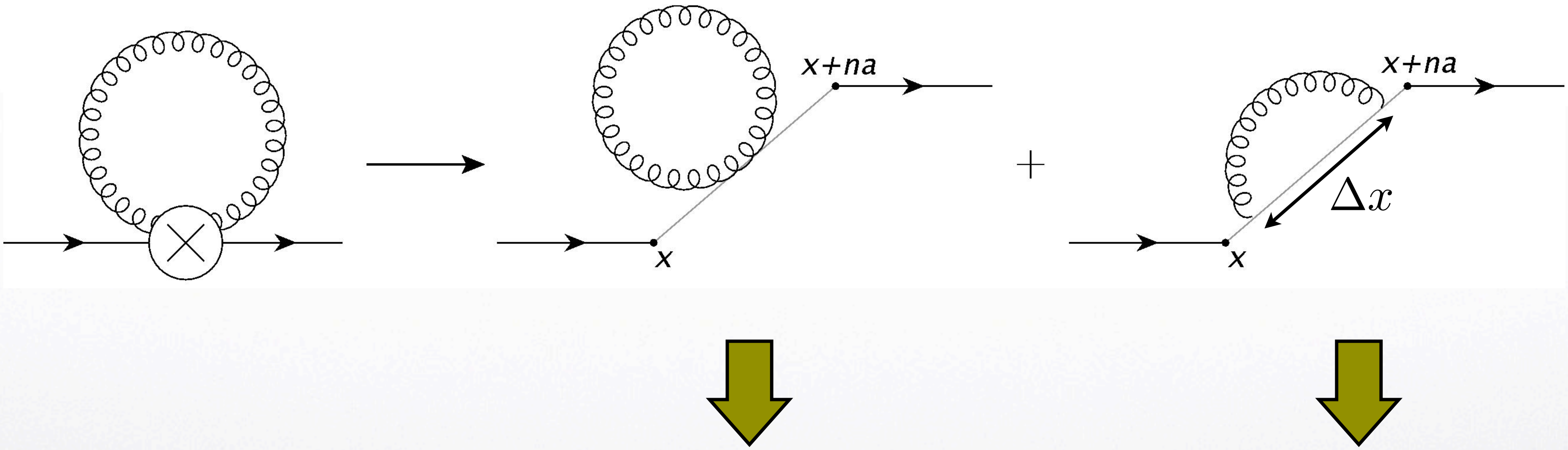
$$U(\mathbf{x},\mathbf{x}+\mathbf{n}a) = 1 + ig\int_{\mathbf{x}}^{\mathbf{x}+\mathbf{n}a} \mathbf{A}(z)\cdot d\mathbf{z} + \mathcal{O}\left(g^2\right)$$

Tree-level operator ➡ A $J = L$ operator with $1/N^2$ corrections

Quantum operator ➡ Two complications:
- Tadpoles
- Extended links

Tadpoles

- Tadpoles of the *continuum* operator



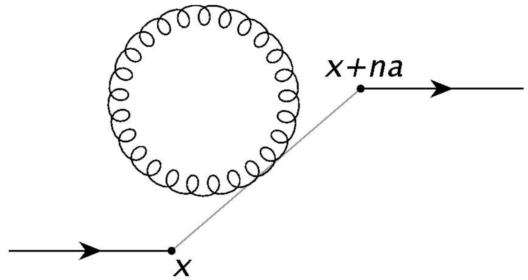
Vanishes!

$\sim \alpha_s / |\mathbf{\Delta x}|^2$

They are harmless in the continuum!

Tadpoles

- Tadpoles of the *lattice* operator



$$\sim \alpha_s a^2 \left(\frac{\pi}{a}\right)^2$$

Non-vanishing!

Perturbative LQCD is poorly convergent!

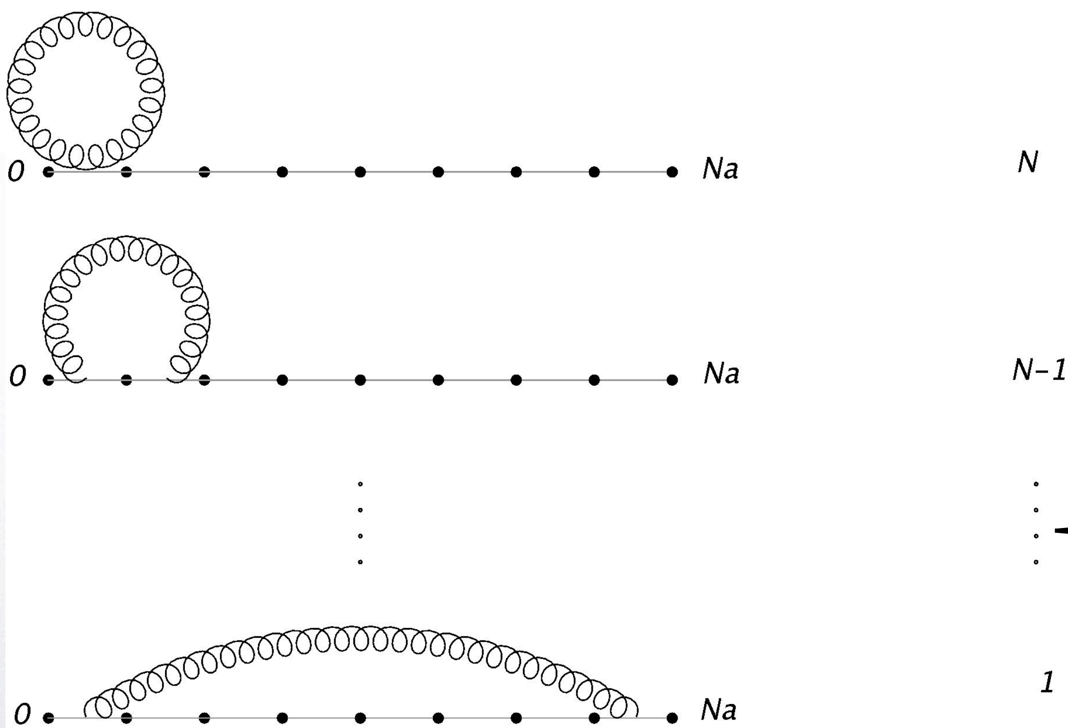
What to do?

Tadpole improvement*

$$U(x, x+a\hat{\mu}) \rightarrow \frac{1}{u_0} U(x, x+a\hat{\mu}) \qquad \text{with} \qquad u_0 \equiv \left\langle \frac{1}{3} \mathrm{Tr}\left(U_{plaq}\right) \right\rangle^{1/4}$$

*G. P. Lepage and P. B. Mackenzie, Phys. Rev., D48, 2250 (1993)

Even worse for $\hat{\theta}_{L,M}\,(\mathbf{x}; a, N)$!



$\mathcal{O}\,(N\alpha_s)$

$$\sum_{m=1}^{N-1}(N-m)\frac{\alpha_s}{m^2} = \mathcal{O}\,(N\alpha_s)$$

A CLOSER LOOK

Break-down of rotational invariance at $\mathcal{O}(N\alpha_s)$!

Example

$$\mathbf{n}^2 = 9 : \begin{cases} (2,2,1) & \Rightarrow \text{ 5 tadpoles of the first kind} \\ (3,0,0) & \Rightarrow \text{ 3 tadpoles of the first kind} \end{cases}$$

Different $A_1{}'s$

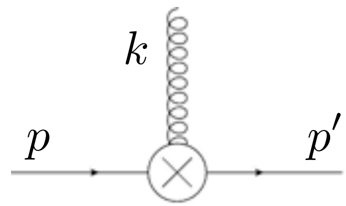
LESSON

Tadpole improvement is crucial.

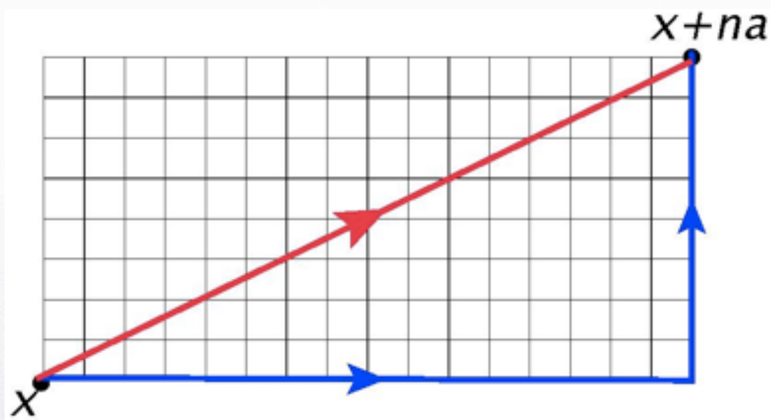
$$U_{A_1^i}(x, x + a\mathbf{n}) \rightarrow \frac{1}{u_{A_1^i}} U_{A_1^i}(x, x + a\mathbf{n})$$

Extended links on the grid

THE PATH?



☐ Continuum operator: Radial path vs. other paths?

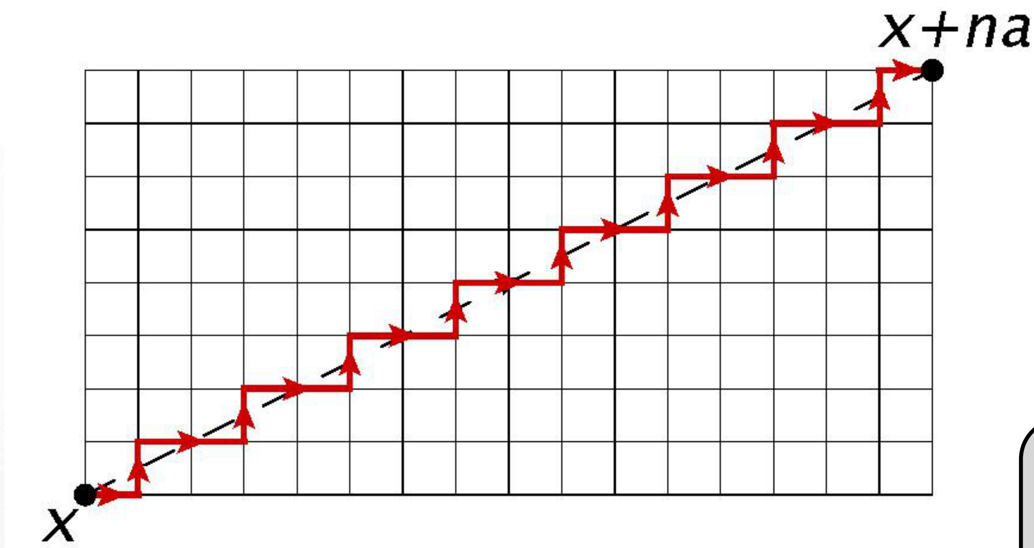


Explicitly rotational invariant gauge link

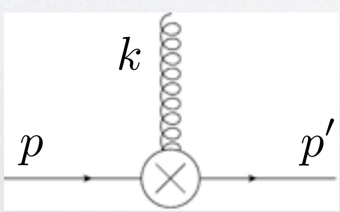
$$V_g^{\lambda} = \frac{3}{4\pi N^3} \sum_{\mathbf{n}}^{|\mathbf{n}|\leq N} g a n^{\lambda} \frac{1}{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{n}a} \left(e^{i(\mathbf{k}+\mathbf{p}')\cdot\mathbf{n}a} - e^{i\mathbf{p}'\cdot\mathbf{n}a} \right) \delta^4(p-p'-k)\ Y_{L,M}(\hat{\mathbf{n}})$$

- Lattice operator:

Closest to the radial path



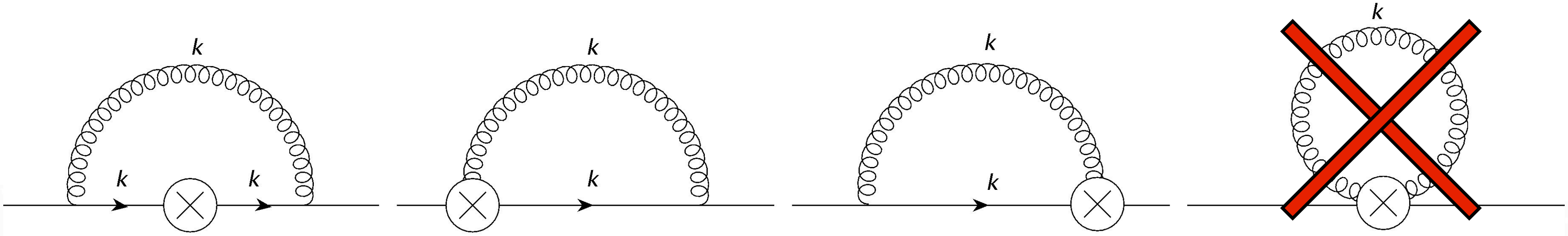
Both RI and RI violating terms



$$= V_g^{\lambda}(k) + \mathcal{O}(g\ k^2 a^2)$$

Operator renormalization at one-loop order:

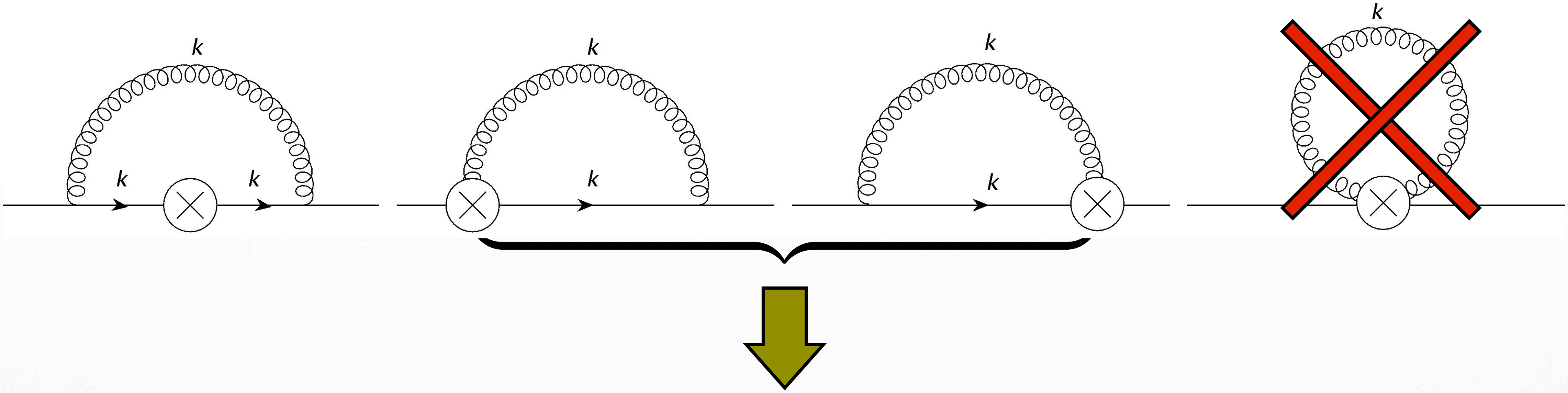
zero external momentum



$\left\{\begin{array}{l} \text{Continuum operator } (L=0,1): \quad \sim \alpha_s \\ \text{RI corrections for Wilson fermions:} \quad \sim \alpha_s/N \\ \text{RV corrections:} \quad \sim \alpha_s/N^2 \end{array}\right.$

Operator renormalization at one-loop order:

zero external momentum



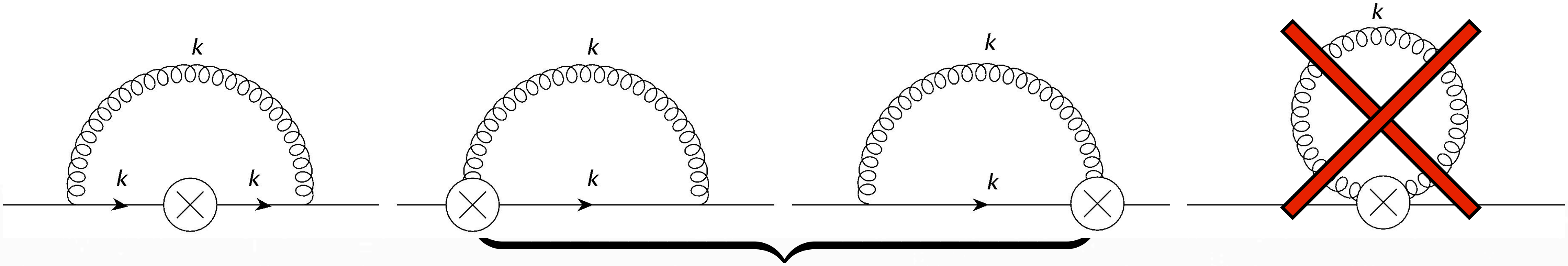
Continuum operator $(L = 0, 1)$: $\begin{cases} L = 0 \sim \alpha_s, \alpha_s \log N \\ L = 1 \sim \alpha_s m_q \end{cases}$

RI corrections for Wilson fermions: $\sim \alpha_s / N$

RV corrections: $\sim \alpha_s$

Operator renormalization at one-loop order:

zero external momentum



- Continuum operator $(L = 0, 1)$: $\begin{cases} L = 0 \sim \alpha_s, \alpha_s \log N \\ L = 1 \sim \alpha_s m_q \end{cases}$
- RI corrections for Wilson fermions: $\sim \alpha_s / N$
- RV corrections: $\sim \alpha_s$ CAUTION

$\mathcal{O}(\alpha_s)$ RI violation

WHY? → UV modes of the gauge fields!

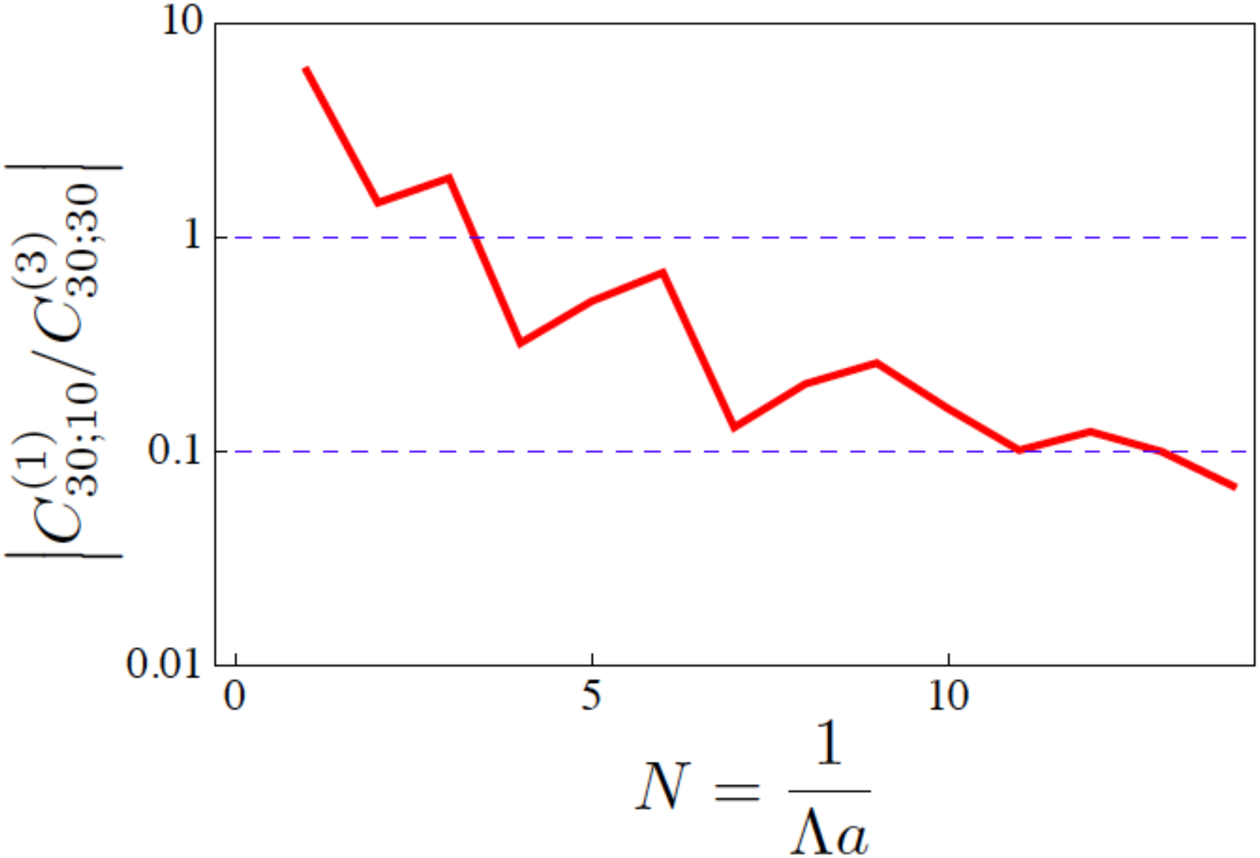
SOLUTION → Smear them over $aN_g = \dfrac{1}{\Lambda_g}$

⇒ RI violating corrections: $\sim \alpha_s a^2 \Lambda_g^2 \sim \dfrac{\alpha_s}{N_g^2}$

All RI violating corrections $\to 0$ as $a \to 0$ ✓

Practical implication?

Matrix elements of an $L = 3$ operator:



Set $\Lambda \sim 2$ GeV

$N = 1 \longrightarrow a \sim 0.1$ fm

$N = 2 \longrightarrow a \sim 0.05$ fm

$N = 10 \longrightarrow a \sim 0.01$ fm

IR rotational symmetry restoration

$\mathbf{P} = \frac{2\pi \mathbf{n}}{L}$ $L \uparrow$ ➡ The number of point-shells increases ➡ $A_1{}'s \uparrow$

Implication for RI restoration?*

Example

Two-particle scattering in the FV in A_1^+

$E \to \delta_0, \delta_4, \delta_6, \ldots$

$$\mathbf{n}^2 = 9 \to A_1^{+(1)}, A_1^{+(2)}$$

$$L \to \infty \Rightarrow \begin{cases} E^{(1)} = \frac{1}{2\mu}\left[\frac{9(2\pi)^2}{L^2} - c_1 \frac{\tan(\delta_0)}{L^2} + \ldots\right] \\ E^{(2)} = \frac{1}{2\mu}\left[\frac{9(2\pi)^2}{L^2} - c_2 \frac{\tan(\delta_4)}{L^2} + \ldots\right] \end{cases}$$

* *T. Luu and M. J. Savage, Phys.R ev., D83, 114508 (2011)*

CONCLUSION

- The smeared operator on the lattice approaches the continuum operator in a smooth way with the corrections that scale at most by a^2. Tadpole improvement and gauge field smearing are essential for this RI recovery in the lattice gauge theories.

- No power divergences! The spectrum of excited states/higher moments of Hadron structure functions are calculable from LQCD.

EXTENSION

- To investigate this universal scaling of the operator non-perturbatively.

- Other smearing profiles?

- Operator improvement.

- Restoration of SO(4) from hyper-cubic symmetry?

Thank you!