Multi-Channel Systems in a Finite Volume

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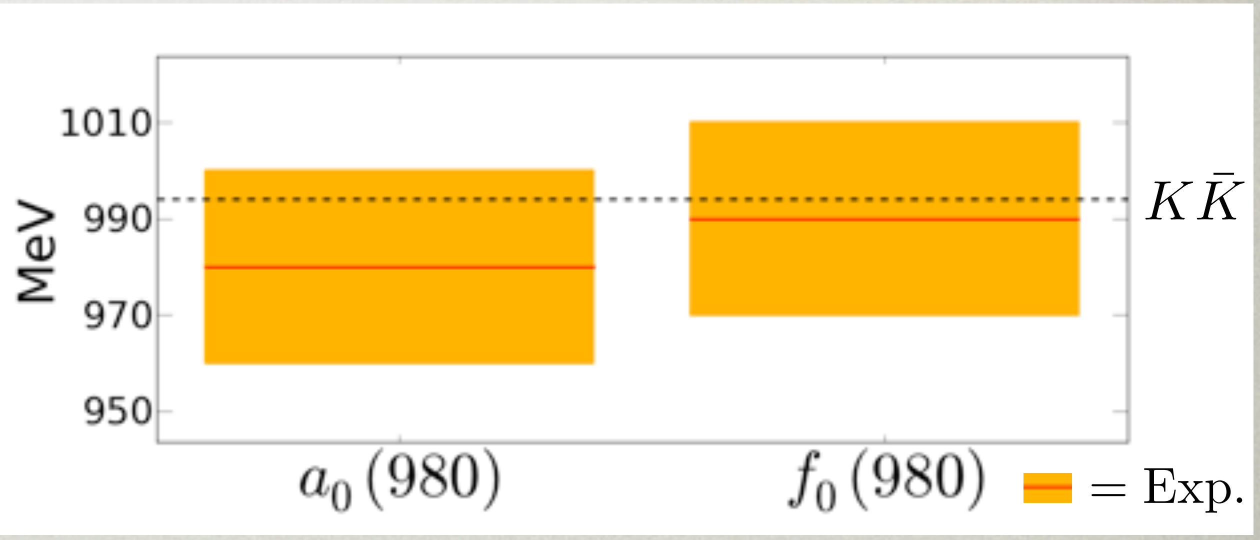
In collaboration with:
Zohreh Davoudi (arXiv:1204.1110)

Motivation: Scalar Sector

- The light scalar spectrum
- Its nature remains puzzling
- Pertinent examples:
  - $f_0(980),\ \mathrm{M_{f_0}} = 990 \pm 20\ \mathrm{MeV},\ \mathrm{I^G(J^{PC})} = 0^+(0^{++})$
  - $a_0(980),\ \mathrm{M_{a_0}} = 980 \pm 20\ \mathrm{MeV},\ \mathrm{I^G(J^{PC})} = 1^-(0^{++})$
- Possible interpretations:
  - Tetraquark states
  - $K\bar{K}$ molecular states
- Desired Ab Initio calculation: $\{\pi\pi, 4\pi, 6\pi, K\bar{K}, \eta\eta\}$

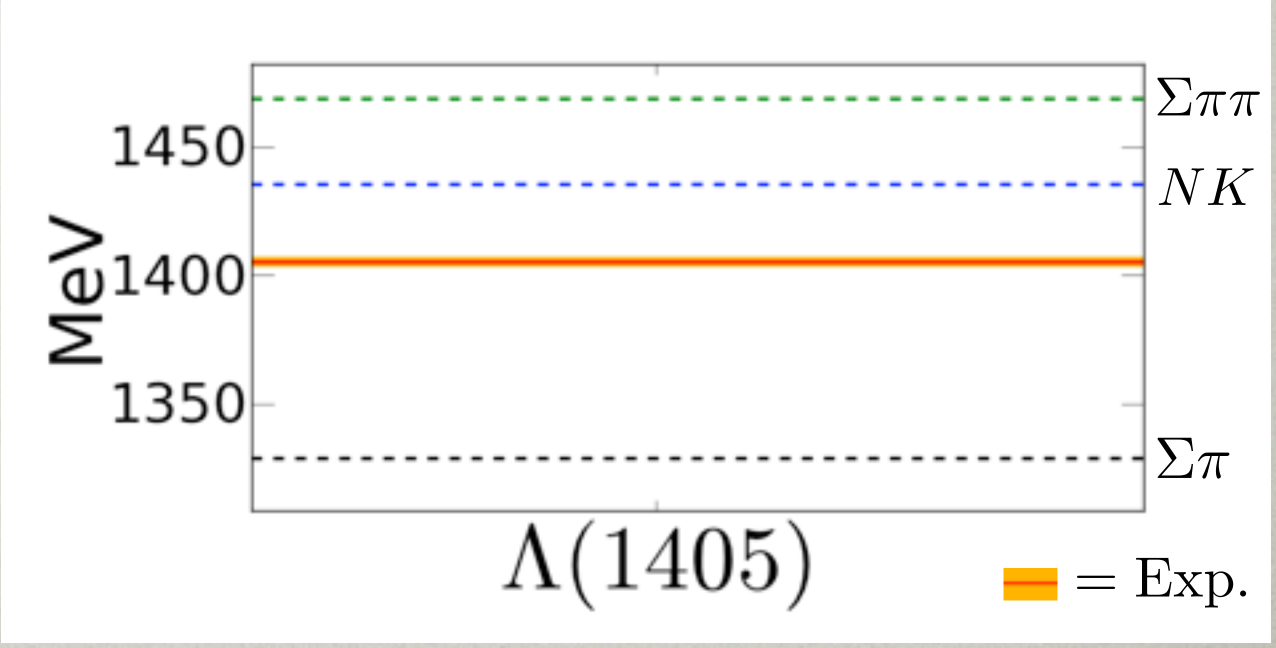


R. L. Jaffe (1977). Weinstein and Isgur (1982).

Motivation: Baryonic Sector

- Strongly-attractive iso-singlet NK ⟶ Kaon condensation
- $(a_{KN})_{I=0}$ suffers from large systematic errors
  - Multi-channels: $\{\Sigma\pi, KN, \Sigma\pi\pi, \Lambda\eta, \ldots\}$
  - Resonances: $\Lambda(1405), \Lambda(1520), \ldots$
  - Non-perturbative, model-dependent,...
- Desired Ab Initio calculation:

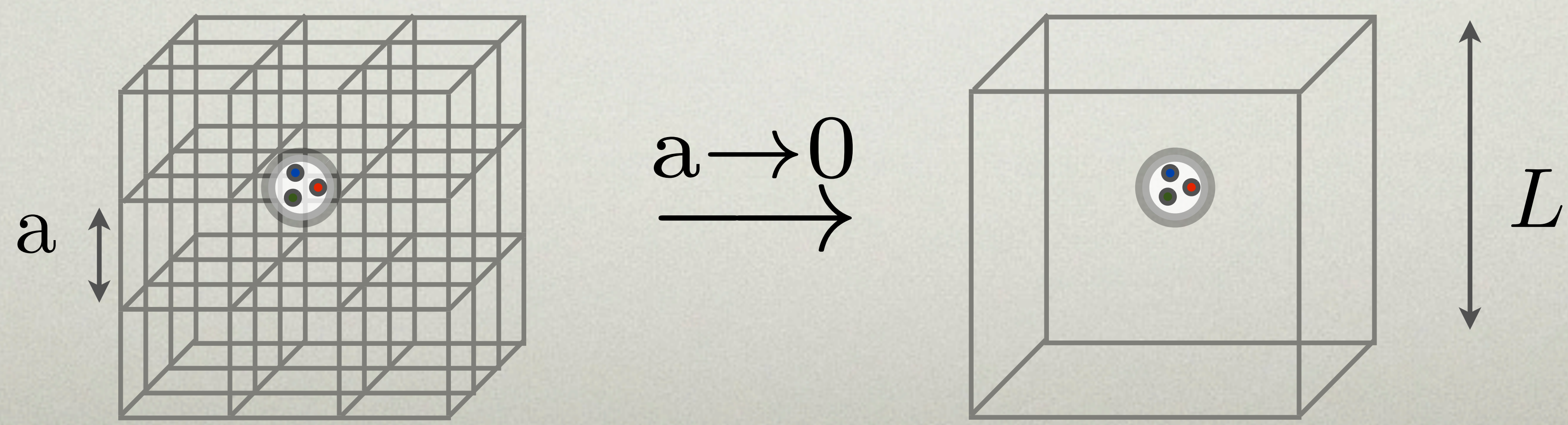
$$\{\Sigma\pi, KN, \Sigma\pi\pi, \Lambda\eta, \ldots\}$$



Kaplan and Nelson (1986).

LQCD & EFT in FV (I)

- LQCD: numerical evaluation of QCD
- LQCD artifacts: finite Euclidian spacetime, $\mathrm{a} \neq 0, \quad \mathrm{m}_\pi \gg 140 \ \mathrm{MeV}, \ldots$
- Maiani-Testa theorem
- Luscher's Method: S-matrix elements from FV effects
  - $E_{\pi\pi}(L) \longrightarrow \delta_{\pi\pi}$
- Periodic finite volume, $\mathrm{a} \to 0, \quad \mathrm{T} \to \infty$
  - Discretized momenta: $\mathbf{p} = \dfrac{2\pi\mathbf{n}}{L}$



Maiani and M. Testa (1990)
Luscher (1991)

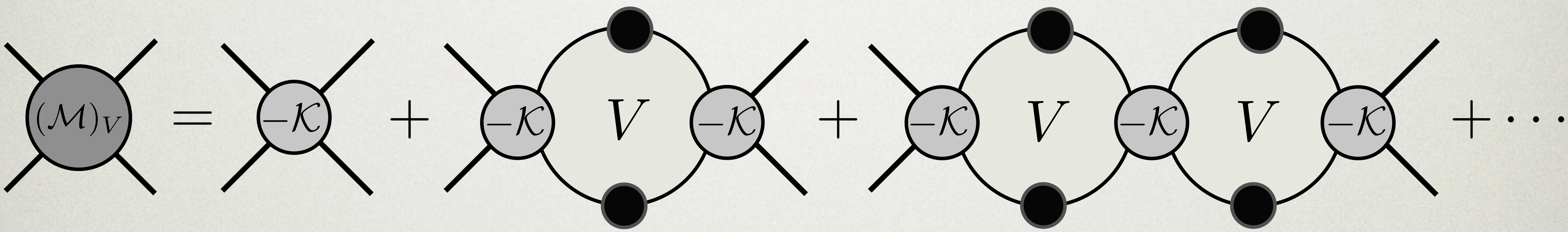
LQCD & EFT IN FV (II)

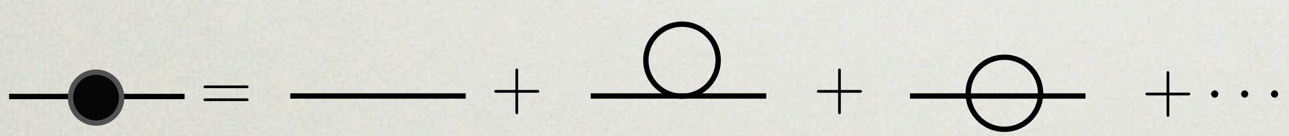
- Coupled channels (CC): [eg. $\pi\pi \to K\bar{K} \to \pi\pi$]
  - $E_n(L) \longrightarrow \{\delta_{\pi\pi}, \delta_{K\bar{K}}, \bar{\epsilon}_{\pi\pi-K\bar{K}}\}$
- Previous CC work: QM 2-body scattering, NR EFT, Relativistic EFT, UChPT
- CC more parameters : Twisted BC, asymmetric lattices,..., boosted systems,...
- Boosted systems:
  - More measurements/Reduce systematics
  - Reduce FV artifacts for bound states
- Work presented:
  - Model independent, relativistic EFT multi-coupled channels with non-zero momenta

He, Feng, and Liu (2005) Lage, Meissner, and Rusetsky (2009)....

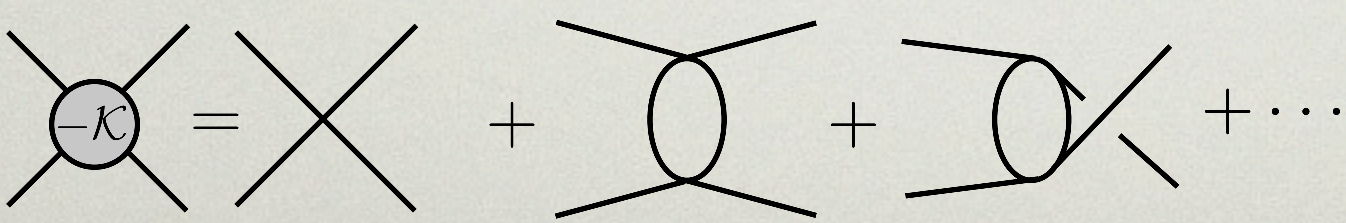
Review: Uncoupled System in a Moving Frame

- Obtain poles in the FV four-point correlation





(**1PI** diagrams)

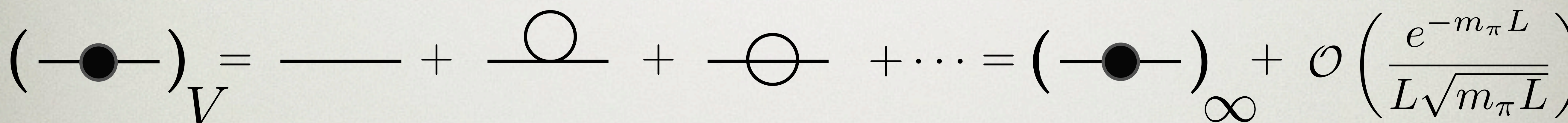


(s−**2PI** diagrams)

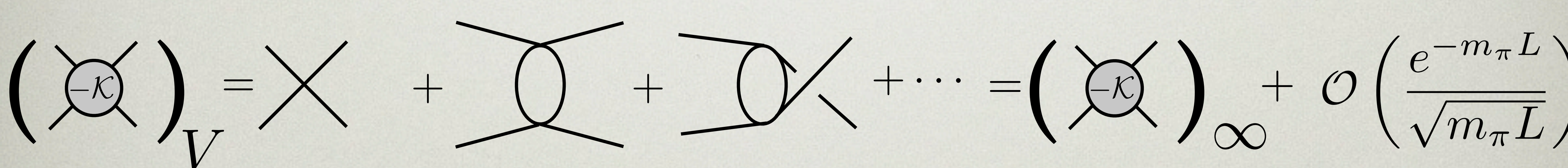
Kim, Sachrajda, and Sharpe (2005).

Power-Law vs. Exponential FV Effects

- Kinematically allowed regime: $0 \leq E^2 - P^2 \leq (4m)^2$
- p-regime: $\frac{m_\pi L}{2\pi} \gg 1$
- Dressed propagator



- Bethe-Salpeter kernel:



- Power-law corrections

Gasser and Leutwyler (1987).
Bedaque, Sato and Walker-Loud (2006)

Power-Law vs. Exponential FV Effects

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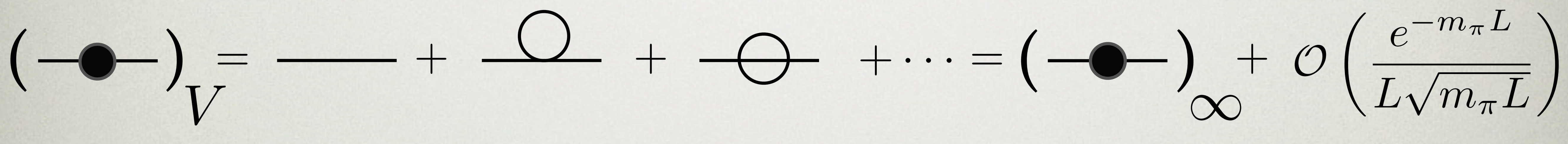

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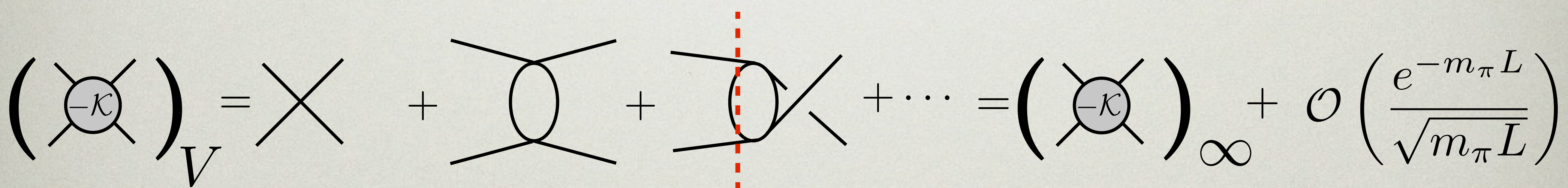


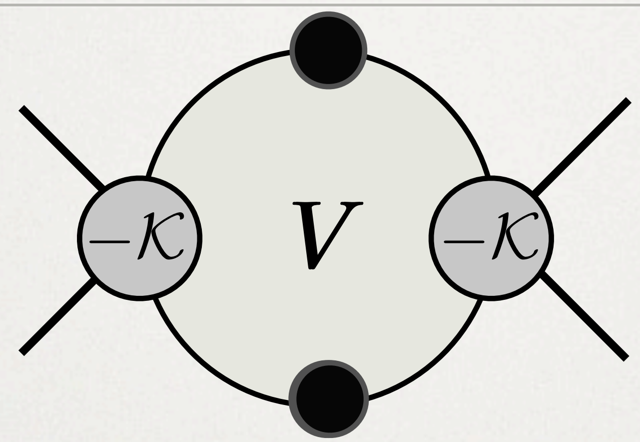
"particles on-shell explore the boundaries of the volume"

- Power-law corrections

Gasser and Leutwyler (1987).
Bedaque, Sato and Walker-Loud (2006)

Generic Loop

$P = (E, \mathbf{P})$



$n =$ symmetry factor

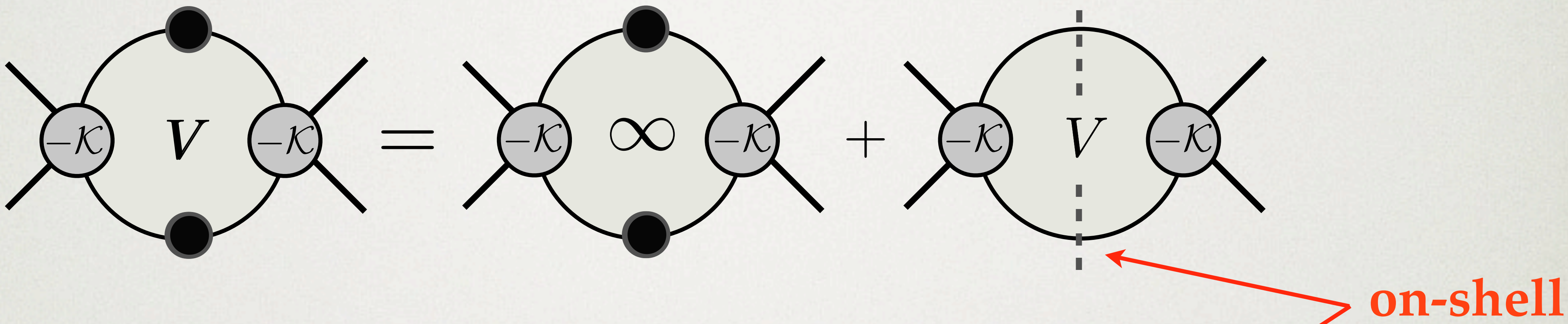
$$\delta G^V(\mathbf{k}_i^*, \mathbf{k}_f^*) \equiv G^V(\mathbf{k}_i^*, \mathbf{k}_f^*) - G^\infty(\mathbf{k}_i^*, \mathbf{k}_f^*)$$
$$= n\left(\frac{1}{L^3}\sum_{\mathbf{l}} - \int \frac{dl^3}{(2\pi)^3}\right)\frac{n\mathcal{K}(\mathbf{k}_i, \mathbf{l})\mathcal{K}(\mathbf{l}, \mathbf{k}_f)}{2\omega_{\mathbf{l}}[(E-\omega_{\mathbf{l}})^2 - \omega_{\mathbf{Pl}}^2 + i\epsilon]}$$

$k_i^*, k_f^* -$ C.M. coordinates

$$\omega_{\mathbf{l}}^2 = m^2 + \mathbf{l}^2$$
$$\omega_{\mathbf{Pl}}^2 = m^2 + (\mathbf{P}\text{-}\mathbf{l})^2$$

Kim, Sachrajda, and Sharpe (2005).

Generic Loop



$$G^V(\mathbf{k}_i^*, \mathbf{k}_f^*) = G^\infty(\mathbf{k}_i^*, \mathbf{k}_f^*) + \delta G^V(\mathbf{k}_i^*, \mathbf{k}_f^*)$$

- Angular momentum decomposition: $(\delta G^V)_{l,m;l',m'} = -i\left(\mathcal{K}\delta\mathcal{G}^V\mathcal{K}\right)_{l,m;l',m'}$

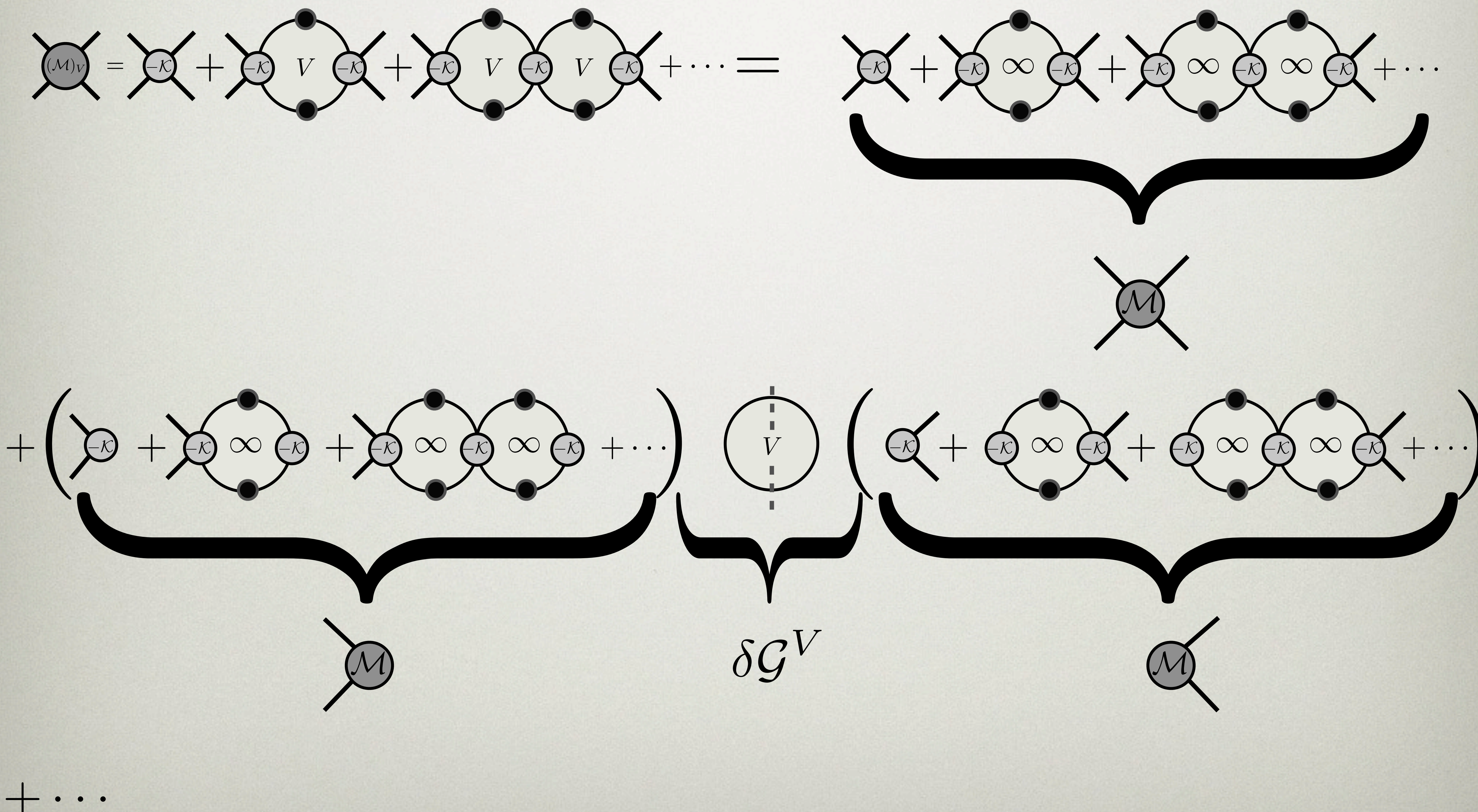
$$(\delta\mathcal{G}^V)_{l_1,m_1;l_2,m_2} = i\frac{q^*n}{8\pi E^*}\left(\delta_{l_1,l_2}\delta_{m_1,m_2} - i\frac{4\pi}{q^*}\sum_{l,m}\frac{\sqrt{4\pi}}{q^{*l}}c^P_{lm}(q^{*2})\int d\Omega\; Y^*_{l_1m_1}Y^*_{lm}Y_{l_2m_2}\right)$$

partial wave mixing!

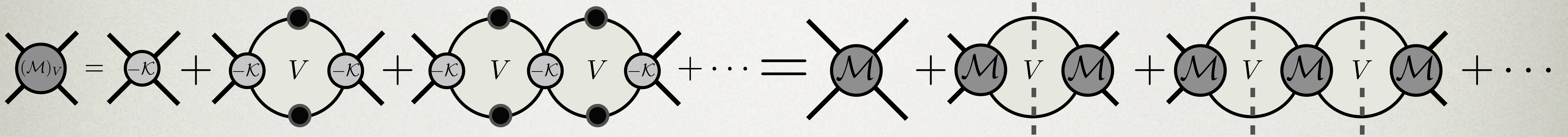
where $c^P_{lm}(q^{*2}) \sim \mathcal{Z}^d_{lm}[1;(q^*L/2\pi)^2]$

Kim, Sachrajda, and Sharpe (2005).

FV Scattering Amplitude and Quantization Condition



FV Scattering Amplitude and Quantization Condition



$$= i\mathcal{M}\frac{1}{1+\delta\mathcal{G}^V\mathcal{M}}$$

- Quantization condition (QC): $\boxed{\det\left(1+\delta\mathcal{G}^V\mathcal{M}\right)=0}$
- In practice: $\det\left(1+\delta\mathcal{G}^V\mathcal{M}\right)_{l_{max}}=0$
- A1-cubic irrep, $l_{\max}=0:\quad q^*\cot(\phi)\equiv 4\pi c^P(q^*)$
- QC can be written as: $\cot(\phi)=-\cot(\delta)\Rightarrow\delta+\phi=m\pi$

$$\underset{E^*}{\uparrow}\qquad\qquad\underset{\delta}{\downarrow}$$

Kim, Sachrajda, and Sharpe (2005).
Rummukainen and Gottlieb (1995)

Coupled Channels (I)

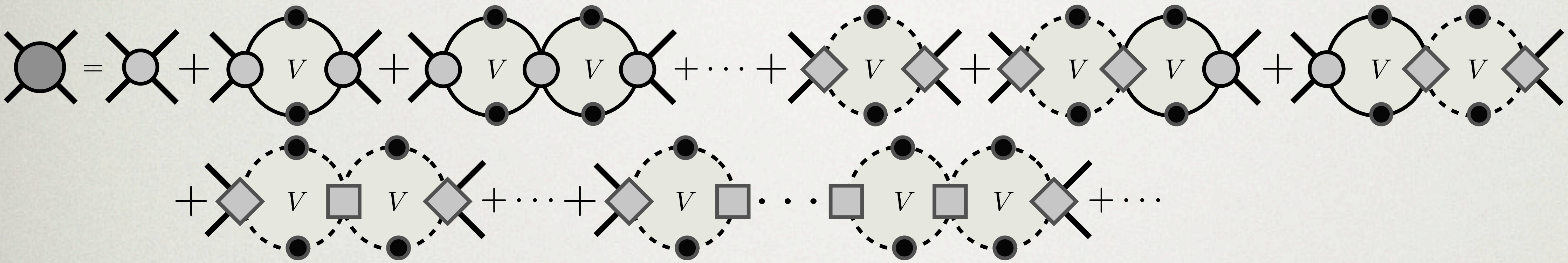
- Above the inelastic threshold: $\pi\pi \to K\bar{K} \to \pi\pi$
- S-matrix for N=2 channels :

$$S_2 = \begin{pmatrix} e^{i2\delta_I}\cos 2\bar{\epsilon} & ie^{i(\delta_I+\delta_{II})}\sin 2\bar{\epsilon} \\ ie^{i(\delta_I+\delta_{II})}\sin 2\bar{\epsilon} & e^{i2\delta_{II}}\cos 2\bar{\epsilon} \end{pmatrix},$$

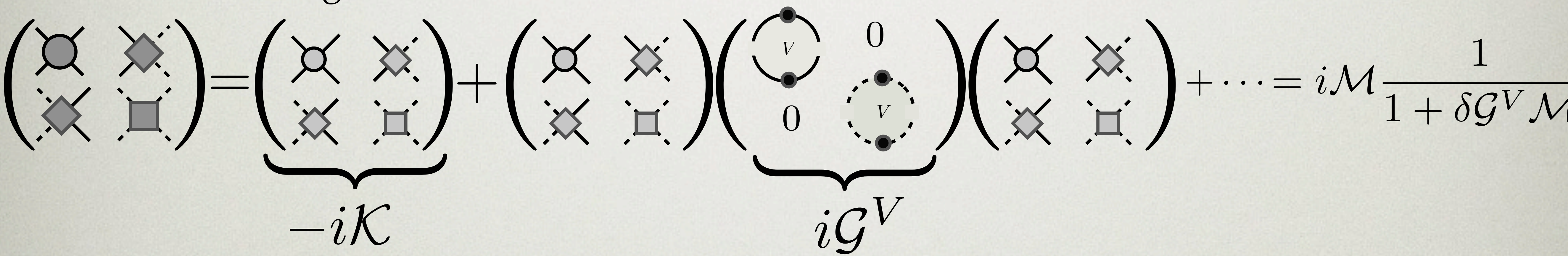
- Channels: I,II
- Scattering amplitude: $\mathcal{M} \to \begin{pmatrix} \mathcal{M}_{I,I} & \mathcal{M}_{I,II} \\ \mathcal{M}_{II,I} & \mathcal{M}_{II,II} \end{pmatrix}$
- S-wave-three parameters : $\delta_I^{(0)},\ \delta_{II}^{(0)},\ \bar{\epsilon}^{(0)}$

Coupled Channels (II)

- Finite volume scattering amplitude for I to I:



- Full scattering matrix



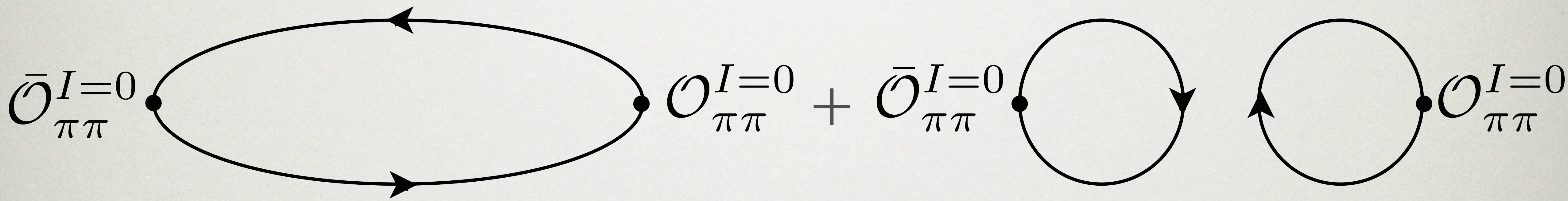
- Quantization condition:

$$\det\left(1+\delta\mathcal{G}^V\mathcal{M}\right) = \det\begin{pmatrix} 1+\delta\mathcal{G}^V_I\mathcal{M}_{I,I} & \delta\mathcal{G}^V_I\mathcal{M}_{I,II} \\ \delta\mathcal{G}^V_{II}\mathcal{M}_{II,I} & 1+\delta\mathcal{G}^V_{II}\mathcal{M}_{II,II} \end{pmatrix} = 0$$

M. T. Hansen and S. R. Sharpe(2012), arXiv:1204.0826 PRD[hep-lat]

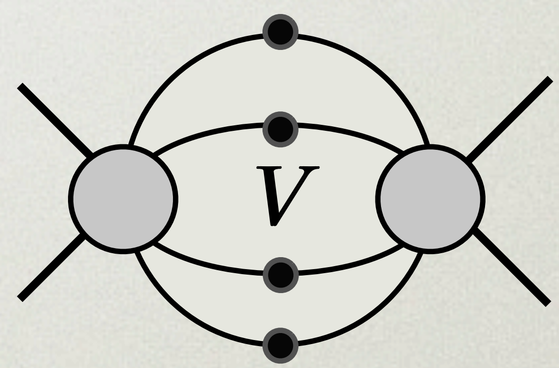
$\pi\pi - K\bar{K}$

- Numerical complications:
  - Disconnected diagrams



  - Physical m_π

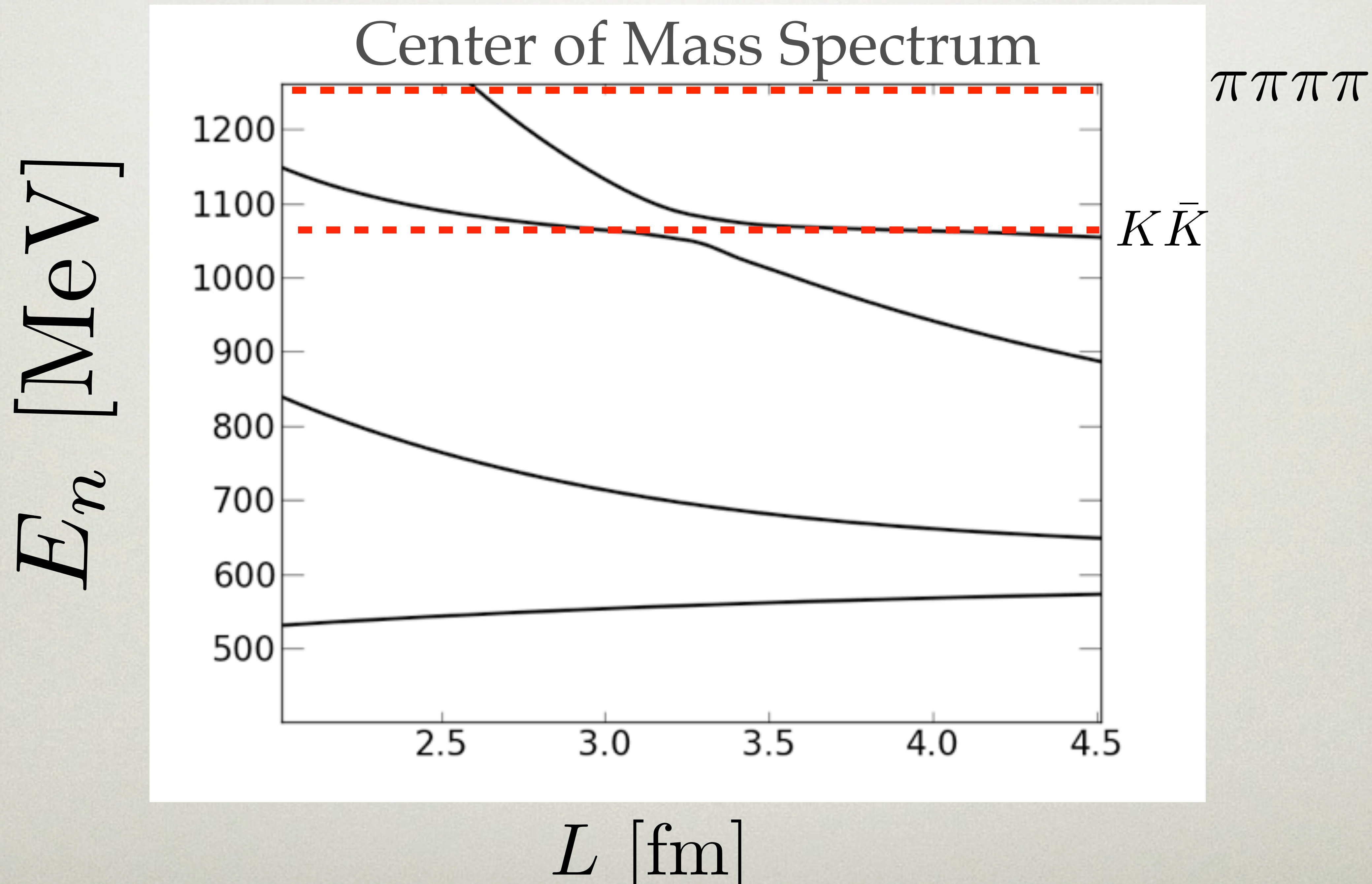
- Theoretical complications:
  - $\pi\pi\pi\pi$ threshold 560 MeV $<< 2\mathrm{M_K}$
  - Not a problem for $m_\pi \gtrsim 300$ MeV



$\pi\pi - K\bar{K}$

$[m_\pi \sim \ 310\mathrm{MeV}, \mathrm{m_K} \sim 530\mathrm{MeV}]$

$$\det\left(1 + \delta\mathcal{G}^V \mathcal{M}\right)_{l_{max}=0} = 0$$



Thanks to M. T. Hansen.

Coupled Channels (III)

- A1-cubic irrep and $l_{\max} = 0$:

$$\cos 2\bar{\epsilon} \cos\left(\phi_1 + \delta_1 - \phi_2 - \delta_2\right) = \cos\left(\phi_1 + \delta_1 + \phi_2 + \delta_2\right)$$

- $\bar{\epsilon} \to 0$ limit, channels decouple: $\delta_I + \phi_I = m\pi, \quad \delta_{II} + \phi_{II} = m'\pi$

- Generalization to arbitrary N:

$$\mathcal{M} \to \begin{pmatrix} \mathcal{M}_{I,I} & \mathcal{M}_{I,II} & \dots & \mathcal{M}_{I,N} \\ \mathcal{M}_{I,II} & \mathcal{M}_{II,II} & & \\ \vdots & & \ddots & \\ \mathcal{M}_{N,I} & & & \mathcal{M}_{N,N} \end{pmatrix} \qquad \delta\mathcal{G}^V \to \begin{pmatrix} \delta\mathcal{G}^V_I & 0 & \dots & 0 \\ 0 & \delta\mathcal{G}^V_{II} & & \\ \vdots & & \ddots & \\ 0 & & & \delta\mathcal{G}^V_N \end{pmatrix}$$

- Quantization condition: $\det\left(1 + \delta\mathcal{G}^V \mathcal{M}\right) = 0$

Weak-Matrix Elements

- Multi-hadron matrix elements $\longrightarrow$ power-law volume dependence
- (2001) Lellouch & Luscher: $K \to \pi\pi$ (LL-factor)
- (2012) Hansen & Sharpe: $D \to \{\pi\pi, KK\}$
- Extension to $2 \to 2$ (e.g. $K\pi \to \pi\pi$)
    - In the absence of 1-Body
- ${}^1S_0 - {}^3S_1$ mixing in the NN sector:

$$p + p \to d + e^+ + \nu_e$$
$$\nu + d \to \nu + p + n$$
$$\nu_e + d \to n + n + e^+$$

- $l_{\max} = 0$

Weak-Matrix Elements: Relativistic Case

- Starting point: $|\mathcal{M}_{I,II}|^2 e^{i2(\delta_I+\delta_{II})} - \left(\mathcal{M}_{I,I} + \frac{1}{\delta\mathcal{G}_I^V}\right)\left(\mathcal{M}_{II,II} + \frac{1}{\delta\mathcal{G}_{II}^V}\right) = 0$

- LL- "trick"

- Weak interactions lift degeneracy: $\Delta E^* = V|\mathcal{M}_{I,II}^V|$

- CM momenta and pseudo phase:

$$\Delta q_i^* = \Delta\tilde{q}_i^*\ V|\mathcal{M}_{I,II}^V|$$

$$\Delta\delta_i(q_i^*) = \delta_i'(q_i^*)\Delta\tilde{q}_i^*\ V|\mathcal{M}_{I,II}^V|$$

- Generalized LL factor:

requires high precision phase shift determination

$$|\mathcal{M}_{I,II}^\infty|^2 = V^2\left\{\Delta\tilde{q}_I^*\Delta\tilde{q}_{II}^*\left(\frac{8\pi E_0^*}{n_I q_I^*}\right)\left(\frac{8\pi E_0^*}{n_{II} q_{II}^*}\right)\left(\phi_I'(q_I^*) + \delta_I'(q_I^*)\right)\left(\phi_{II}'(q_{II}^*) + \delta_{II}'(q_{II}^*)\right)\right\}|\mathcal{M}_{I,II}^V|^2$$

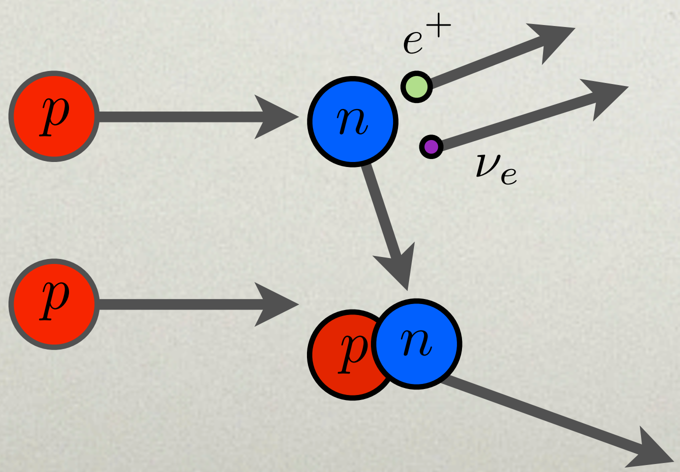
$\mathbf{2} \to \mathbf{2}$ boosted L.L. factor

caution!

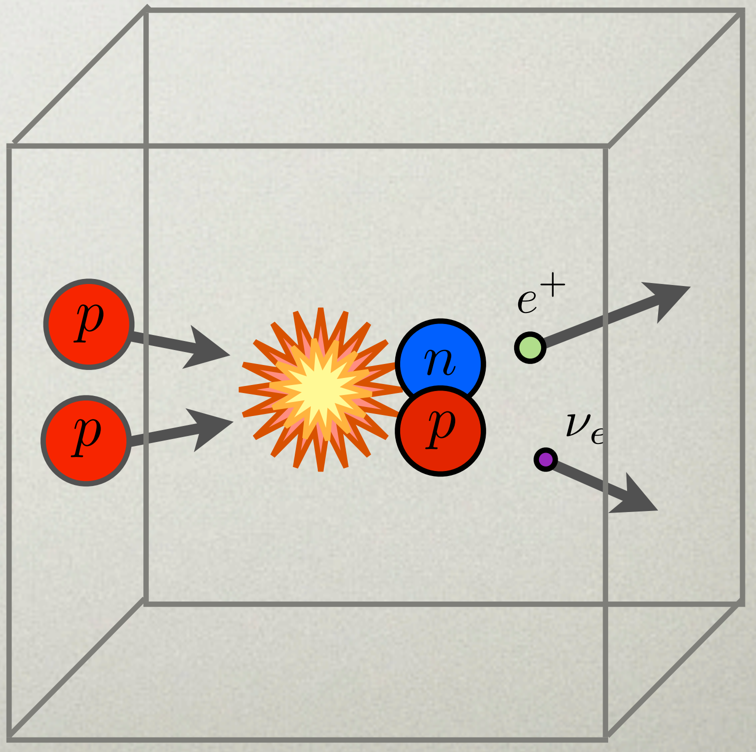
NN Weak Matrix Elements

- Axial vector current: $A^{\mu=3} = \frac{1}{2}\left(\bar{u}\gamma^3\gamma^5 u - \bar{d}\gamma^3\gamma^5 d\right)$
  - 1-Body + 2-Body NN current
  - 2-Body ~ dominant uncertainty in deuteron break-up
- (2004) Detmold & Savage: background field
  - $^1S_0 - {}^3S_1$ coupled channels
- 5-point correlation functions

1-body current:

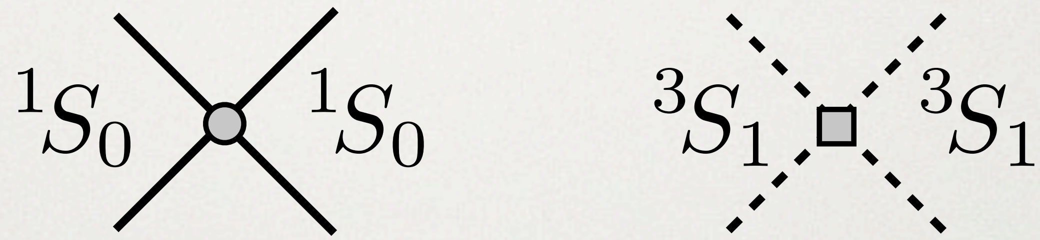


reaction in FV

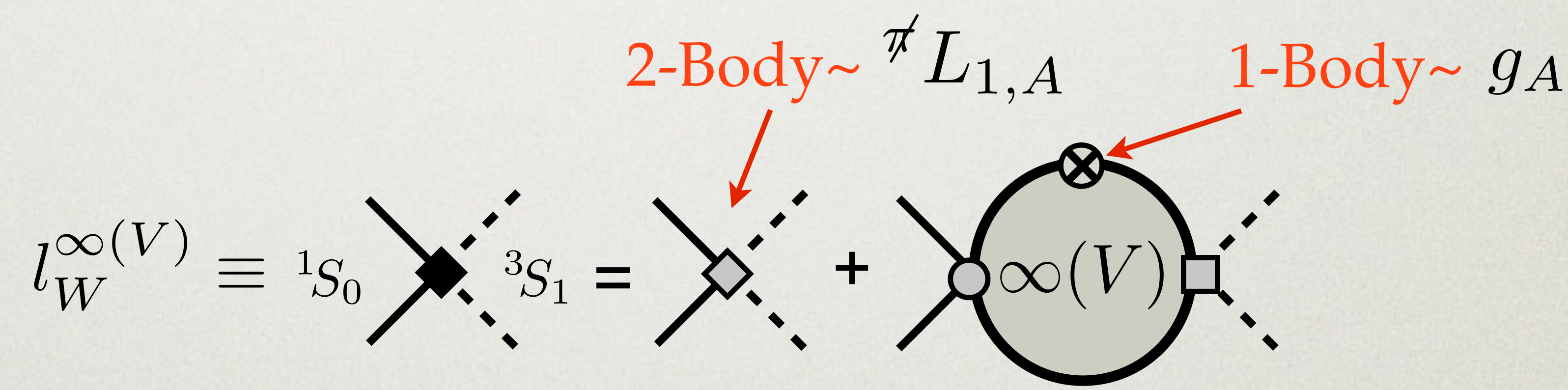


NN Weak Matrix Elements

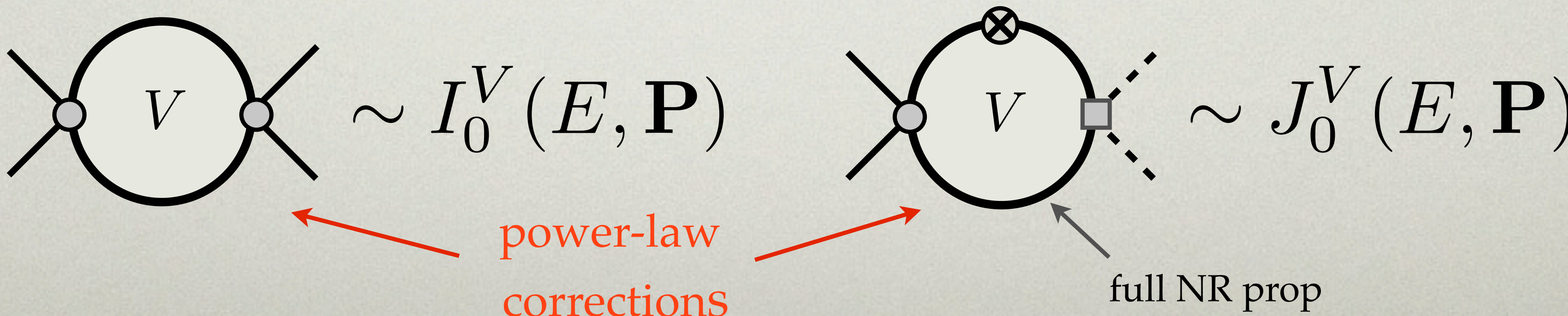
- Below pion production: $EFT(\not{\pi})$
- Strong interactions:



- Weak interaction:

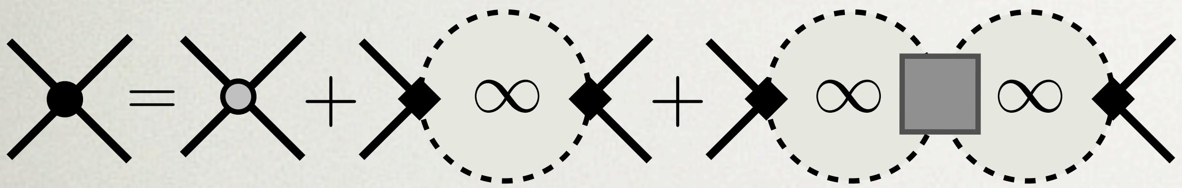


- Loops:



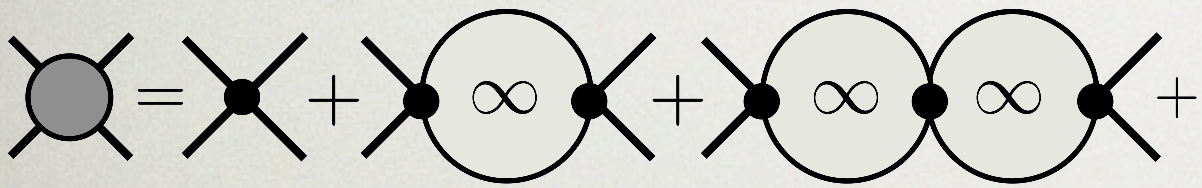
Renormalization Conditions

- Effective ${}^1S_0 - {}^1S_0$ vertex



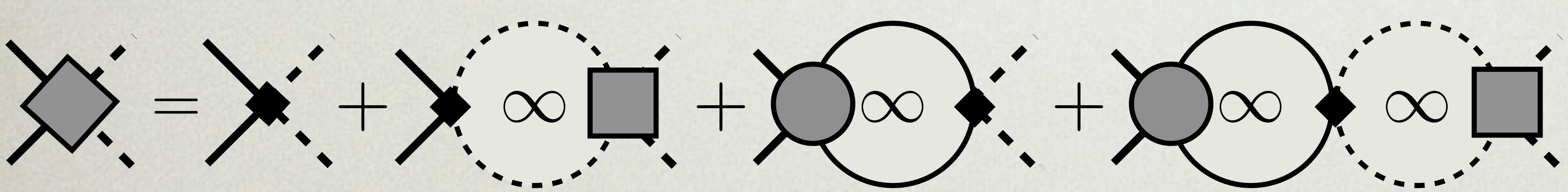
$$= D_1^{\infty}$$

- Infinite volume ${}^1S_0 - {}^1S_0$ scattering:



$$+\cdots = -\frac{iD_1^{\infty}}{1 - D_1^{\infty} I_0^{\infty}} \equiv i\mathcal{M}_{{}^1S_0}$$

- Infinite volume ${}^1S_0 - {}^3S_1$ transition amplitude:



$$\equiv i\mathcal{M}_{{}^1S_0 - {}^3S_1}$$

- Finite Volume : $D_1^{\infty} \to D_1^{V} \equiv D_1^{\infty} + \delta D_1^{V}$
  - Evaluate poles : $i\left(\mathcal{M}_{{}^1S_0}\right)_V = -\dfrac{iD_1^{V}}{1 - D_1^{V} I_0^{V}}$

Quantization Condition and FV Matrix Element

- Poles for ${}^1S_0 - {}^1S_0$ FV scattering amplitude:

$$\left(\mathcal{M}_{{}^1S_0-{}^3S_1} - \mathcal{M}_{{}^1S_0}\mathcal{M}_{{}^3S_1} g_A W_3 \delta J_0^V\right)^2 - \left(\mathcal{M}_{{}^1S_0} + \frac{1}{\delta I_0^V}\right)\left(\mathcal{M}_{{}^3S_1} + \frac{1}{\delta I_0^V}\right) = 0$$

where $\delta J_0^V = J_0^V - J_0^\infty, \quad \delta I_0^V = I_0^V - I_0^\infty$

- LL- "trick" $\delta E = V|\mathcal{M}^V_{{}^1S_0-{}^3S_1}| = V Z_A \left\langle NN;{}^1S_0\right| A^{\mu=3} \left|NN;{}^3S_1\right\rangle_V$

$$\left(|\mathcal{M}^\infty_{{}^1S_0-{}^3S_1}| - g_A W_3 \frac{\delta J_0^V e^{i2\phi}}{(\delta I_0^V)^2}\right)^2 = \left(\frac{2\pi V}{q_0^{*2}}\right)^2 (\phi' + \delta'_{{}^3S_1})(\phi' + \delta'_{{}^1S_0})|\mathcal{M}^V_{{}^1S_0-{}^3S_1}|^2$$

$$g_A = Z_A \langle N| A^{\mu=3} |N\rangle_V = Z_A \langle N| A^{\mu=3} |N\rangle_\infty + \mathcal{O}\left(e^{-Lm_\pi}\right)$$

Summary & Extension

- Strongly multi-coupled system
  - More parameters to extract ➔ Boosted systems
  - Possible to study $\{\pi\pi, KK\}, \{\pi\pi, KK, \eta\eta\}, \ldots$
- Weakly multi-coupled system
  - $\mathbf{2} \rightarrow \mathbf{2}$ boosted L.L. factor (no 1-Body oper.)
  - ${}^1S_0 - {}^3S_1$ mixing in the NN sector
- Extension:
  - Exponential corrections in the NN sector due to pions
  - Isoscalar NN electroweak matrix elements

Thank You!

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