Finite Range Force models for EOS

INT 2012-2a Core-collapse supernova

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Nuclear Equation of State

What is it, why is it important ?

- Nuclear EOS is thermodynamic relations for given ρ, Y_e, T with wide range of varibles.
($\rho : 10^4 \sim 10^{14}$g/cm^3, $Y_e : 0.01 \sim 0.56$, $T : 0.1 \sim 50$MeV)
- Nuclear EOS is important to simulate core collapsing supernova explosion, proto-neturon stars, and compact binary mergers involving neutron stars.
- The nuclear equation of states are provided as a tabular form.

- There are only few EOS tables which are available.

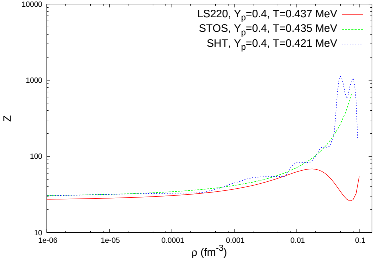
- LS EOS (Lattimer Swesty 1991)
  Use Skyrme type potential with Liquid droplet approach
  - Consider phase transition, several K
- STOS EOS (H Shen, Toki, Oyamastu, Sumiyoshi 1998), new version (2011)
  Use RMF with TF approximation and parameterized density profile (PDP)
  - Old : awkward grid spacing
  - New : finer grid spacing, adds Hyperon(Λ, $\Sigma^{+,-,0}$)
- SHT EOS (G Shen, Horowitz, Teige 2010)
  Use RMF with Hartree approximation

Table: Range of Tables

	LS 220	STOS	SHT
$\rho(\mathrm{fm}^{-3})$	$10^{-6} \sim 1$ (121)	$7.58 \times 10^{-11} \sim 6.022$ (110)	$10^{-8} \sim 1.496$ (328)
Y_p	$0.01 \sim 0.5$ (50)	$0 \sim 0.65$ (66)	$0 \sim 0.56$ (57)
T (MeV)	$0.3 \sim 30$ (50)	$0.1 \sim 398.1$ (90)	$0 \sim 75.0$ (109)

- Except LS model (liquid droplet approach), the thermodynamic quantities such as P, S can be obtained from numerical derivatives. $P = \rho^2 \frac{\Delta F}{\Delta \rho}, S = -\frac{\Delta F}{\Delta T}$

- STOS & SHT tables don't provide second derivative $\left(\frac{\partial(P,S)}{\partial(T,\rho,Y_p)}\right)$.



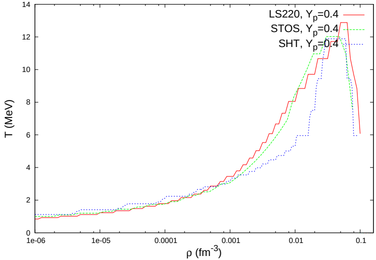


Figure: Left (right) figure shows the atomic number (phase boundaries).

- Hempel et al. 2010, 2012
  - Use Relativistic mean field model (TM1, TMA, FSUgold)
  - Nuclear statistical equilibrium (Alpha, Deutron, Triton)
- Blinnikov et al. 2011 (under construction)
  - Used Saha equation to find fraction of multi-component of nuclei
  - Nuclear mass formula
- Furusawa, 2011 (under construction)
  - Used Saha equation to find fraction of multi-component of nuclei
  - Used Relativistic mean field model
  - Consider phase transtion using geometric function

- The EOS table should be thermodynamically consistent up to second order,
  (numerical derivative may not be smooth !! $\frac{\partial p}{\partial \rho}$, $\frac{\partial p}{\partial T}$ $\frac{\partial p}{\partial Y_e}$...)
- EOS should also fit astrophysical observation (M-R relation)

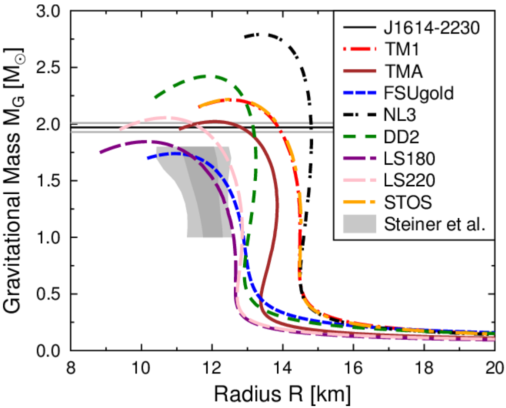


Figure: Mass and radius relation from various E.O.S. Only LS220 satisfies maximum mass of neutron star and allowed region for give neutron star's mass.

How can we construct EOS table ?

How can we construct EOS table ?
We need nuclear force model and numerical method.

Nuclear force model	Numerical technique
Skyrme Force model (non-relativistic potential model)	Liquid Drop(let) approach (LDM)
Relativistic Mean Field model (RMF)	Thomas Fermi Approximatoin (TF)
Finite-Range Force model	Hartree-Fock Approximation (HF)

- LS EOS $\Rightarrow$ Skyrme force + LDM (without neutron skin)
- STOS $\Rightarrow$ RMF + Semi TF (parameterized density profile)
- SHT $\Rightarrow$ RMF + HARTREE

Nuclear force model should be picked up to represent both finite nuclei and neutron star observation.
Only LS220 can fit neutron star observations; the other EOS's have too large radii (L).
Our choice was Finite-Range Force model + LDM (with neutron skin)

Finite Range Force Model

- Finite Range Force
  - The finite-range force simplifies the calculation of unit cells
  - It more efficiently calculates the density profiles within the cell compared to traditional HF and TF approaches, because the Euler-Lagrange equations from energy minimization are integral, not differential, equations
  - it correctly includes Coulomb modifications to the surface energy which Danielewicz has criticized concerning the liquid droplet model

- Finite Range Force model (Seyler+Blanchard, Myers+Swiatecki)
  - Nuclear force is strong, but short ranged $\Rightarrow f(r)$.
  - The distance dependence is given by

$$f(r) = \begin{cases} \frac{e^{-r/a}}{r}, & \text{for Yukawa type ;} \\ e^{-r^2/r_0^2}, & \text{for Gaussian type.} \end{cases} \quad (1)$$

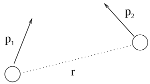
Cf. The intensity of a semi-infinite slab in the radiative transfer equation can be found from

$$I(\mu, 0) = \int_0^\infty S(t) e^{-t/\mu} \, \frac{dt}{\mu} \, , \quad (2)$$

where $S(t)$ is a source function.

  - The total interaction energy comes from phase space integration of

$$W \sim W(r, |\vec{p}_1 - \vec{p}_2|, \rho_1, \rho_2) \quad (3)$$



Truncated model - Yukawa type

$$W = -\frac{1}{h^3}\int d^3r_1 \int d^3r_2 f(r_{12}/a)\sum_t \left[\int C_L f_{t1} f_{t2} d^3p_{t1} d^3p_{t2} + \int C_U f_{t1} f_{t'2} d^3p_{t1} d^3p_{t'2}\right] \quad (4)$$

where f_t is Fermi-Dirac distribution function.

- Functional $C_{L,U}$

$$C_{L,U}(p_1, p_2, \bar{\rho}) = \frac{h^3}{4} T_o \rho_o \left(\frac{3}{4\pi P_o^3}\right)^2 \left[\alpha_{L,U} - \beta_{L,U}\left(\frac{p_{12}}{P_o}\right)^2 - \sigma_{L,U}\left(\frac{2\bar{\rho}}{\rho_o}\right)^{2/3}\right] \quad (5)$$

- α, β : Selyer & Blanchard
- The original paper (Myer & Swiatecki) has a term $\gamma_{L,U}\frac{1}{p_{12}}$ in $C_{L,U}$
  - $\gamma_{L,U}$ was added to improve optical potential, but didn't fix incompressibility problem.
  $K \sim 375$ MeV $\Rightarrow$ Stiff EOS, high surface tension
  $\Rightarrow$ Neglect $\gamma_{L,U}$ in $C_{L,U}$ (The Truncated model)

The Modified model

- Determination of $\alpha_{L,U}$, $\beta_{L,U}$, $\sigma_{L,U}$, 'a'
  - The parameters in $C_{L,U}$ can be determined from standard nuclear matter properties : E/A=-16 MeV, p=0 MeV/fm^3

  $S_v \left(= \frac{1}{8} \frac{\partial^2(\mathcal{E}/\rho)}{dx^2}\Big|_{\rho=\rho_0, x=1/2} \right) \sim 32$ MeV,

  $L \left(= \frac{3}{8} \frac{\partial^3(\mathcal{E}/\rho)}{d\rho dx^2}\Big|_{\rho=\rho_0, x=1/2} \right) \sim$ 60MeV.
  - The range of force 'a' can be calculated from the surface thickness of semi-infinite nuclear matter.
- To improve K ($\sim$ 235MeV) we need to add another term

Adding new interaction form

$$C_{\eta L,U} = -\frac{h^3}{4} T_0 \rho_0 \left(\frac{3}{4\pi P_0^3} \right)^2 \eta_{L,U} \left(\frac{1}{\rho_0} \right)^{\epsilon} \left(\frac{\rho_1^{\epsilon} + \rho_2^{\epsilon}}{2} \right)$$

- This term now allows a good fit to K, neutron matter and optical potential.
- ϵ is a free parameter ($\epsilon \sim 1/3$).

Thomas Fermi Approximation

In the TF, we use local Density Approximation instead of finding the full wave function.

$$\rho_t = \frac{1}{4\pi^2\hbar^3}\int f_t d^3p = \frac{1}{2\pi^2}\left(\frac{2m_t^* T}{\hbar^2}\right)^{3/2} F_{1/2}(\Psi_t), \tag{6a}$$

$$\tau_t = \frac{1}{4\pi^2\hbar^5}\int f_t p^2 d^3p = \frac{1}{2\pi^2}\left(\frac{2m_t^* T}{\hbar^2}\right)^{5/2} F_{3/2}(\Psi_t) \tag{6b}$$

At T=0 MeV,

$$\rho = \frac{k_f^3}{3}, \quad \tau = \frac{3}{5}(3\rho)^{5/3} \tag{7}$$

- If T is finite
  Although the Fermi integral is not analytic, it can be approximated well by the analytic JEL polynomial fit

We combined Finite Range force with Thomas Fermi Approximation
$\Rightarrow$ Finite Range Thomas Fermi model (FRTF)

Numerical technique in FRTF

- Lagrange multiplier method to find nucleon density profiles.
  The total energy can be minimized for given N and Z

$$F = \int \mathcal{E} d^3r - \lambda_1 (\int \rho_n d^3r - N) - \lambda_2 (\int \rho_p d^3r - Z)$$
$$\frac{\partial \mathcal{E}}{\partial \rho_{n,i}} = \lambda_1 = \mu_n, \quad \frac{\partial \mathcal{E}}{\partial \rho_{p,i}} = \lambda_2 = \mu_p, \quad N = \int \rho_n d^3r, \quad Z = \int \rho_p d^3r \tag{8}$$

  so the μ_n and μ_p are constant in the cell.
- It is an integral equation
  - Because of finite range interaction in the total interaction energy
  - More stable than differential equation in the numerical sense
    (The differential equations normally obtained with Skyrme-like or RMF)
  - Developed a new integration scheme that is much more accurate
    (with relatively few zones) than other quadrature with fixed grid points
- Newton-Raphson
  - We assume spherically symmetric nuclei.
  - Guess density at each grid point (initial guess from Fermi density profile) .
  - Iterate until we get the enough accuracy.

Convergence is rapid.

Finite Nuclei in the Finite Range model

- Isolated Nuclei

Table: Binding energy per baryon from experiment and modified FRTF

Nucleus	Property	Experiment	modified FRTF
^{208}Pb	r_{ch} (fm)	5.50	5.53
	BE/A(MeV)	7.87	7.77
	δR(fm)	0.12 ± 0.05	0.13
		0.20 ± 0.04	
^{90}Zr	r_{ch} (fm)	4.27	4.21
	BE/A(MeV)	8.71	8.66
	δR(fm)	0.09 ± 0.07	0.055
^{40}Ca	r_{ch} (fm)	3.48	3.30
	BE/A(MeV)	8.45	8.44
	δR(fm)	-0.06 ± 0.05	-0.037
		-0.05 ± 0.04	

Optimized parameter set

The parameters are adjusted to give the best fit to binding energies of nuclei with a χ^2 minimization ($S_v = 31.499$ MeV and $L = 56.472$ MeV).

$$\chi^2 = \frac{1}{N}\sum(B_{ex} - B_{th})^2 \tag{9}$$

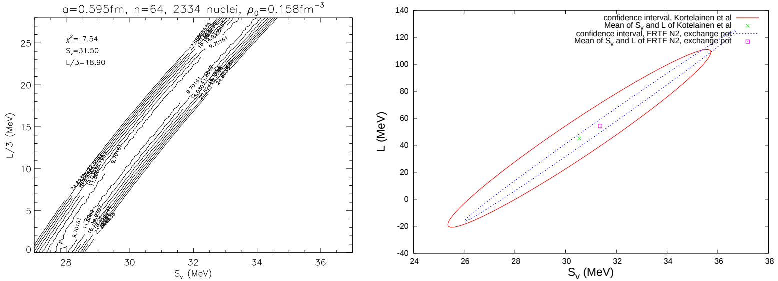


Figure: The left figure shows the correlation between $L/3$ and S_v, the right figure shows 95% confidence interval from Kortelainen et al, and FRTF N2 models.

- We can confirm the correlation ($R \sim 0.997$) between S_v and L using FRTF model.
- Simple parallel computing was used to calculate 2300 nuclei at the same time. ($2336 \times 40(S_v) \times 40(L) \sim 10^6$)

- Our result also is consistent with experimental result.

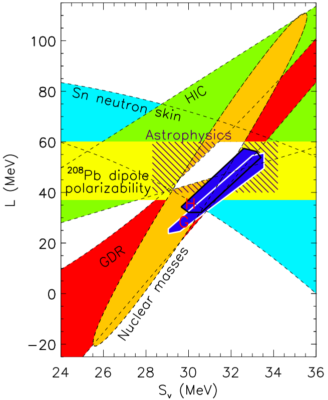


Figure: The allowed region of S_v, L from experiment and observation

The hatched rectangle shows the constraints from astrophysical modeling of M-R relation.
H and G are restricted region from neutron matter calculation.
The white area is the experimentally-allowed overlap region.

Nuclei in dense matter

- Nuclei in Dense matter
  The nuclei in dense matter exist with free gas of neutrons because $\mu_n > 0$, neutron drip (β equilibrium states $\mu_n = \mu_p + \mu_e$)

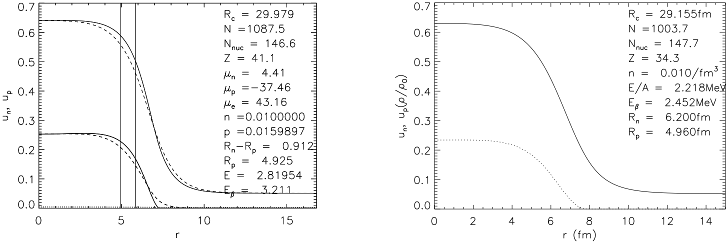


Figure: The left panel is the density profile from the Yukawa model and the right side is from the Gaussian model.

- Wigner-Seitz cell method was used to decribe the nuclear density profile.
- Electron density is almost uniform.
- The energy per baryon converges arounds -8 MeV as density decreases.
- Proton fractions becomes 0.4 as the density decrease.

Semi-infinite nuclear matter with Finite Range force

The semi-infinite nuclear matter is important to provide the surface tension formula for given proton fraction and temperature.
Liquid droplet approach need surface tension information.
In general $\omega = \omega(x, T)$, and ω is given by

$$\omega = \int_{-\infty}^{\infty} [\mathcal{E} - T(S_n + S_p) - \mu_n \rho_n - \mu_p \rho_p + p_o]\, dz = -\int_{-\infty}^{\infty} [p(z) - p_o] dz. \quad (10)$$

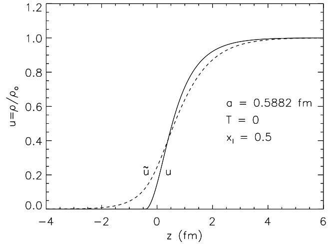


Figure: Semi-infinite nuclear matter density profile

Uniform nuclear matter with Finite Range force model

In the uniform matter, there is no finite range integrals ($\int f(r/a)d^3r = 1$).

- Coexistence
  - The coexistence curve shows the phase equilibrium bewteen dense and dilute phase.
  - This is the simplified case of heavy nuclei with nucleon fluids.

$$p_I = p_{II}, \quad \mu_{nI} = \mu_{nII}, \quad \mu_{pI} = \mu_{pII} \tag{11}$$

  - Under critical temperature coexistence is possible.

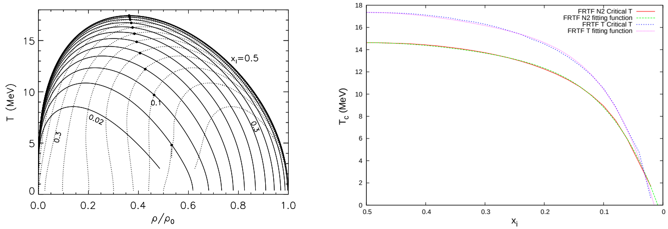
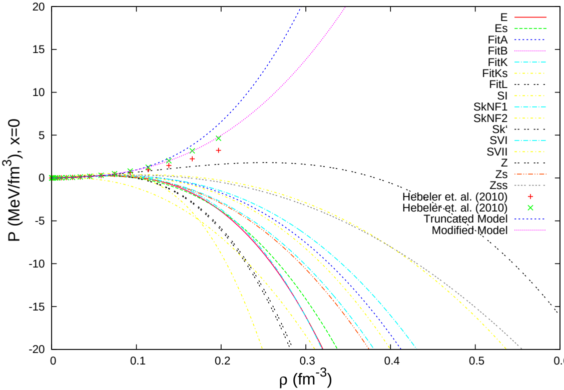


Figure: The left panel shows the coexistence curve for given Y_p and the right panel show the critical temperature as a function of Y_p.

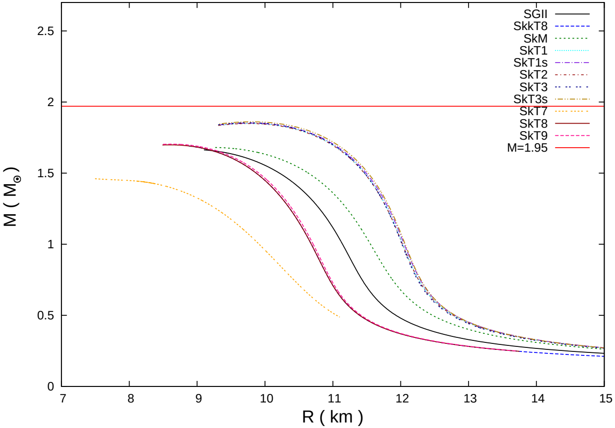
Constraints to nuclear force model

Good nuclear force models should be consistent with nuclear phenomena such as energies and radii, as well as with neutron matter and astrophysics data
60 Skyrme force (E, Es, FitA, ..., Z, Zs, Zss) $\Rightarrow$ only a few survive

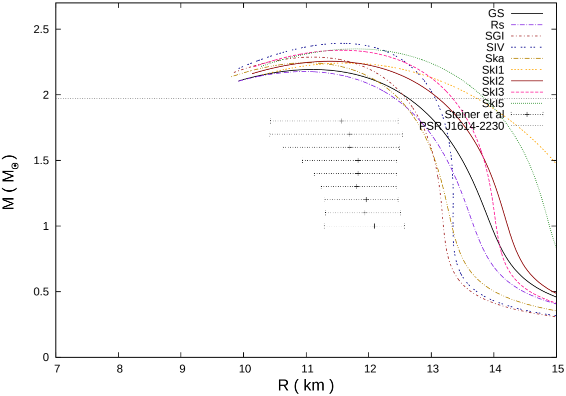
- Pure neutron matter
  - The pressure from the pure neutron matter should increase.



- The maximum mass of cold neutron star
  - Recent discovery of $1.97M_{\odot}$ neutron star (PSR J1614-2230)



- Mass-Radius region - Steiner et al, (2010)
  X-ray burst data, atmosphere model

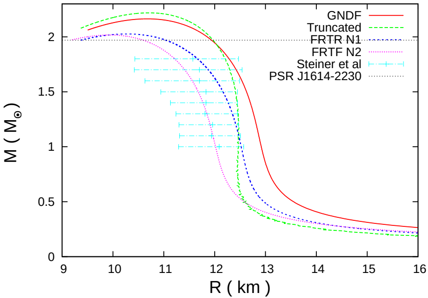


Finally, the models passed test are
SKM′, SkI4, SkI6, $M^* = 0.811$, $M^* = 0.9$, $M^* = 1.0$, SLy0 ... SLy10.

Finite Range Thomas Fermi model explains

1) the finite nuclei very well. ($\chi^2 = 4.75\text{MeV}^2$)

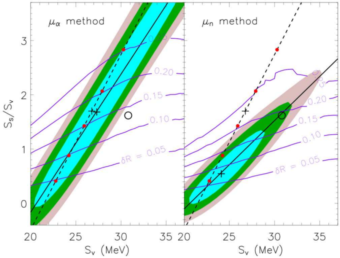
2) astrophysical observation.



$\Rightarrow$ FRTF N2 is the best parameter set for our choice.

Numerical Techniques for E.O.S

- Liquid Droplet Approach
  - It is the only approach which can generate consistent second-order derivatives without numerical differentiation.
  - HF and TF tables are relatively sparse and not smooth enough, perhaps because of convergence issues.
  - The liquid droplet approach is also much faster so you can generate several tables with different nuclear parameters. That is too difficult with HF or TF approaches.



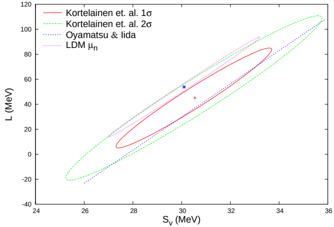


Figure: The left panel shows μ_α and μ_n LDM model (Steiner et al 2005). The right panel shows the correlation between L and S_v.

- Liquid droplet model (μ_n method).

This is a schematic model to show how we can minimize the energy from LDM.
The total binding energy of the finite nuclei at $T = 0$ MeV is given,

$$E = (-B + S_v\delta^2)(A - N_s) + 4\pi R^2\sigma + \mu_n N_s + E_c + E_{pair} + E_{shell} \tag{12}$$

where $\delta = (A - 2Z - N_s)/(A - N_s)$ and N_s is number of neutrons in the neutron skin.
Energy minimization

$$\begin{aligned} \frac{\partial E}{\partial \mu_n} &= 4\pi R^2 + N_s = 0 \\ \frac{\partial E}{\partial N_s} &= B - S_v\delta^2 + 2S_v\delta(A - N_s)\frac{\partial \delta}{\partial N_s} + \mu_n = 0 \end{aligned} \tag{13}$$

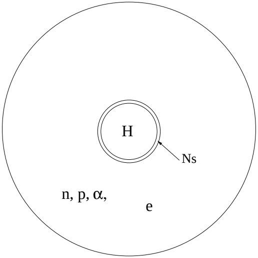
gives

$$\begin{aligned} E(A,Z) = &- BA + 4\pi r_0^2\sigma_0 A^{2/3} + S_v A I^2 \left[1 + \frac{S_s}{S_v A^{1/3}}\right]^{-1} \\ &+ \frac{3e^2Z^2}{5r_0A^{1/3}} - \frac{\pi^2 Z^2 e^2 d^2}{2r_0^3 A} - \frac{3Z^{4/3}e^2}{4r_0A^{1/3}}\left(\frac{3}{2\pi}\right)^{2/3} \\ &+ a\Delta A^{-1/2} \end{aligned} \tag{14}$$

where $4\pi r_0^3\rho_0 = 1$ and $S_s = 4\pi r_0^2\sigma_\delta$.

For hot dense matter equation of state,

- Realistic nuclear force model
  Bulk energy ($-B + S_v \delta^2$) should be replaced.
  - we use finite range force model
- Wigner Seitz Cell method + Liquid droplet



- Heavy nuclei at the center, n, p, α, e exist outside.
- Neutron skin

Total free energy density consists of

$$F = F_N + F_o + F_\alpha + F_e + F_\gamma \tag{15}$$

where F_N, F_o, F_α, F_e, and F_γ are the free energy density of heavy nuclei, nucleons out the nuclei, alpha particles, electrons, and photons.

- $F_N = F_{bulk,i} + F_{coul} + F_{surf} + F_{trans}$
- $F_o = F_{bulk,o}$
- α particles : Non-interacting boson
- e, γ : treat separately

For $F_{bulk,i}$, $F_{bulk,o}$, and F_{surf}, we use the same force model.
F_{surf} from the semi infinite nuclear matter calculation

The is the modification of LPRL (1985), LS (1991, No skin)

Improvement - Consistent calculation of surface tension
- Coulomb diffusion and Neutron skin
- The most recent parameter set

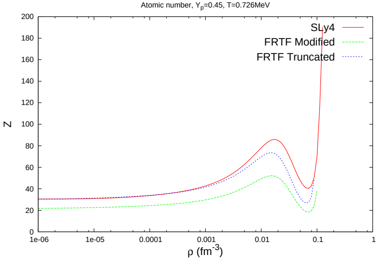
For fixed independent variables (ρ, Y_p, T), we have the 9 dependent variables $(\rho_i, x_i, r_N, x_s, \nu_n, u, \rho_o, x_o, \rho_\alpha)$.

where i heavy nuclei, o nucleons outside, x proton fraction, u filling factor, and ν_n neutron skin density.

From baryon and charge conservation, we can eliminate x_o and ρ_o.

Free energy minimization, $\frac{\partial F}{\partial \rho_i} = \frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial r_N} = \frac{\partial F}{\partial x_s} = \frac{\partial F}{\partial(\nu_n/r_N)} = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial \rho_\alpha} = 0$.

- Finally, we have 5 equations to solve and 5 unknowns. $z = (\rho_i, \ln(\rho_{no}), \ln(\rho_{po}), x_i, \ln(u))$.
- Nuclear pasta phase is considered using geometric function $\mathcal{D}(u)$.
- General Newton-Raphson method is used.
- The code (f90) is fast. $121(\rho) \times 50(T) \times 50(Y_p)$ zones < 10 min in single machine.
  - x100 more grid points needed
- SLy4 was used to compare with Finite Range force model.



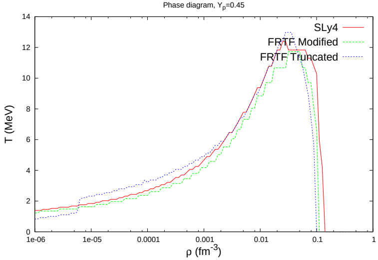


Figure: Left (right) figure shows the atomic number (phase boundaries).

- Modified model has lower atomic number since it has lower surface tension.
- Phase transition to uniform nuclear matter happens at higher densities in SLy4.
- Heavy nuclei can exists at higher temperature in the Truncated model since it has higher critical temperature.

Summary

- FR force models together with a TF unit cell calculation for nuclei
  - It is more efficient to calculate the density profiles within the cell.
  - It correctly includes Coulomb modifications to the surface energy compared to liquid droplet model.
- I used liquid droplet approach to make EOS table
  It is the only approach which can generate consistent second-order derivatives without numerical differentiation.
  - The liquid droplet approach is also much faster.
  - It now includes the neutron skin and the surface diffuseness.
- We have calibrated the FR-TF calculation to a new liquid droplet approach
- Proper Skyrme force should be used to make E.O.S table.
  - SLy series are good for making E.O.S table.