The Realization of the Sharpe-Singleton Scenario

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Gauge Field Dynamics In and Out of Equilibrium

INT, Seattle, March 9, 2012

What The phase structure of lattice QCD with Wilson fermions

Aoki VS Sharpe-Singleton

Why Extract continium physics from the lattice

How Wilson Chiral Perturbation Theory ($a \neq 0$)

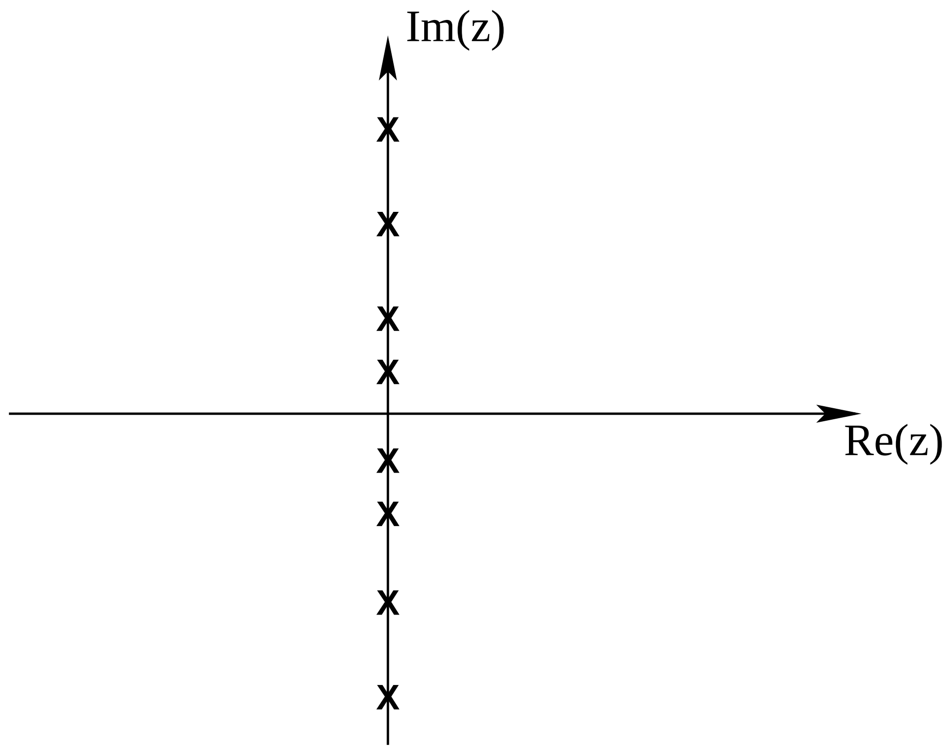
New Link to the spectrum of the QCD Wilson Dirac operator

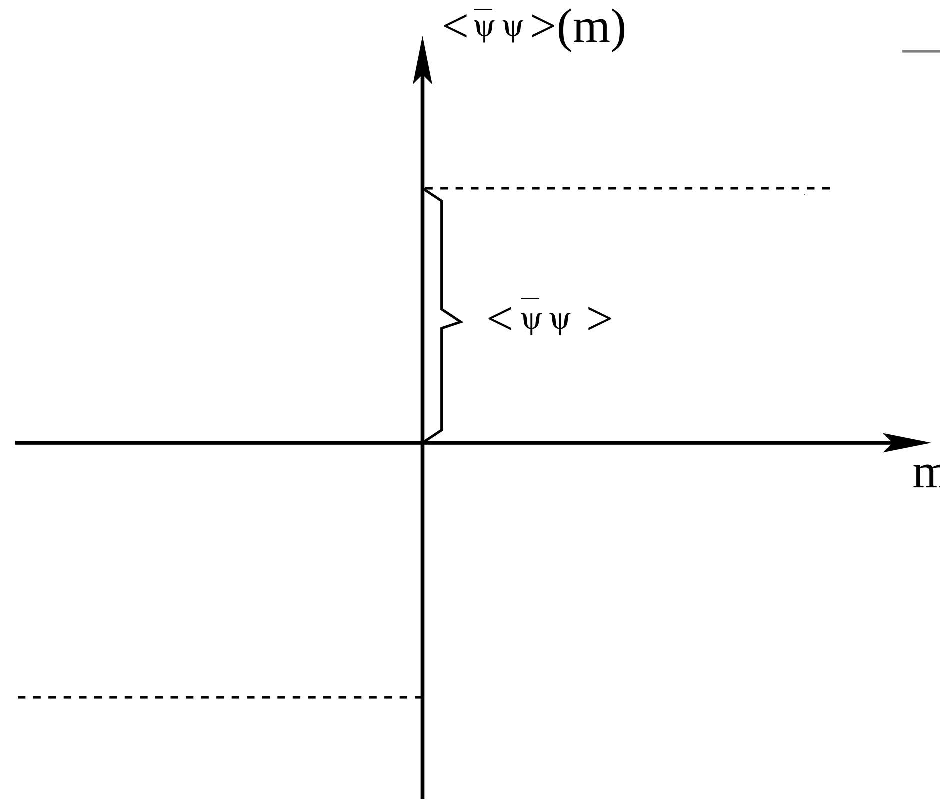
puzzle

Warm up: Zero a

$a = 0$

Banks Casher





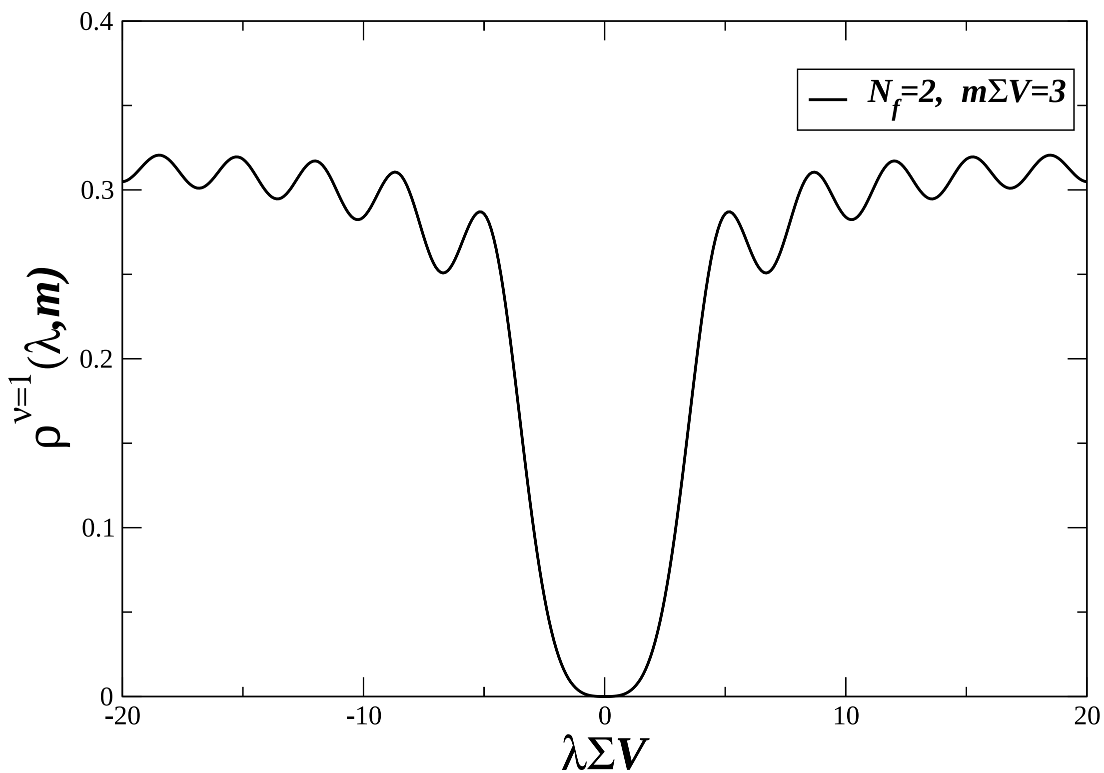
$$\langle \bar{\psi}\psi \rangle = \frac{\pi}{V}\rho(0)$$

Banks Casher NPB 169 (1980) 103

Eigenvalue density at $a = 0$:

$$\gamma_5 D = -D\gamma_5 \qquad \text{eigenvalues in pairs } (i\lambda, -i\lambda)$$

ν zero ev's



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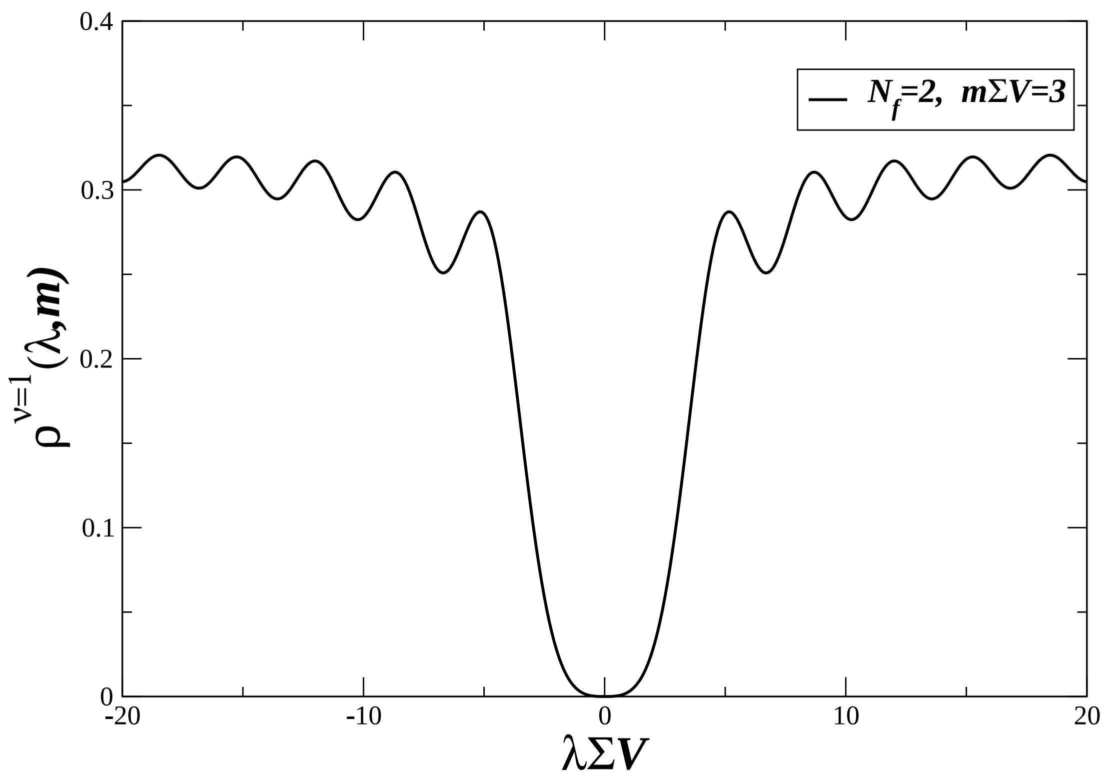


One fit parameter Σ

Verbaarschot Wettig Ann.Rev.Nucl.Part.Sci. 50 (2000) 343, hep-ph/0003017

CPT in the ϵ-regime $m\Sigma V \sim 1$ $\quad a = 0$

The partition function in a sector of topological charge ν

$$Z^{\nu}_{N_f}(m; a = 0) \;=\; \int_{U(N_f)} dU \; \det{}^{\nu}(U) e^{\frac{m}{2}\Sigma V \, \mathrm{Tr}(U + U^{\dagger})}$$

A group integral (*not a path integral*)

Σ is the chiral condensate

Gasser, Leutwyler, PLB **188**(1987) 477; NPB **307** (1988) 763

Leutwyler, Smilga, PRD 46 (1992) 5607

New: non zero lattice spacing a

Discretization effects depend on the discretization

Here: Wilson fermions

$$\gamma_5 D_W \neq -D_W \gamma_5$$
$$D_W^\dagger \neq -D_W$$

γ_5-hermiticity

$$D_W^\dagger = \gamma_5 D_W \gamma_5$$

Discretization effects depend on the discretization

Here: Wilson fermions

$$\gamma_5 D_W \neq -D_W \gamma_5$$
$$D_W^\dagger \neq -D_W$$

γ_5-hermiticity

$$D_W^\dagger = \gamma_5 D_W \gamma_5$$

Eigenvalues, z, of D_W

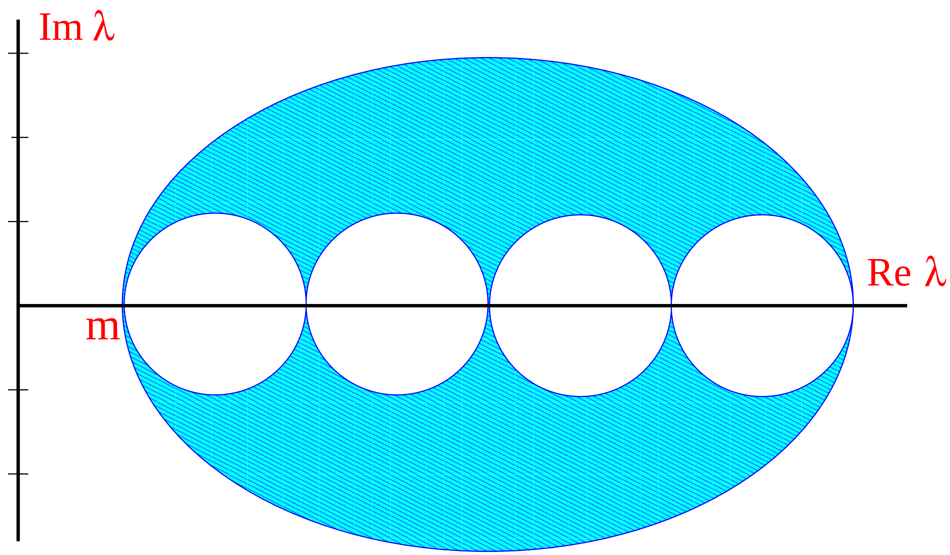
- complex conjugate pairs (z, z^*)

- exact real eigenvalues

Itho, Iwasaki, Yoshie, PRD 36 (1987) 527

Free Wilson fermions.

Spectrum of $D_W + m$



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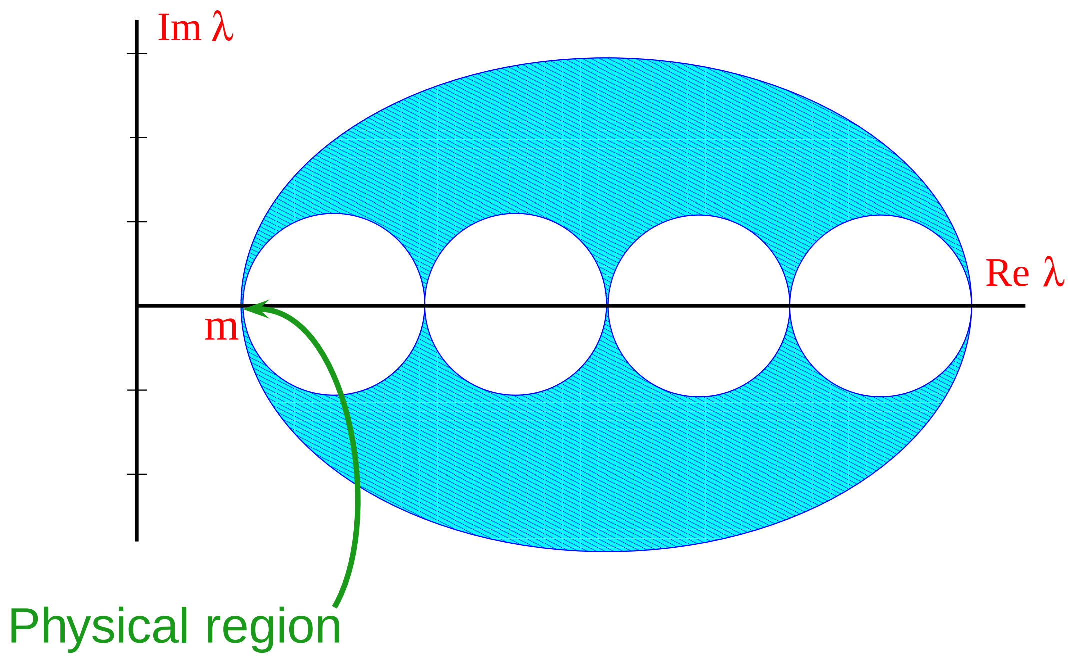
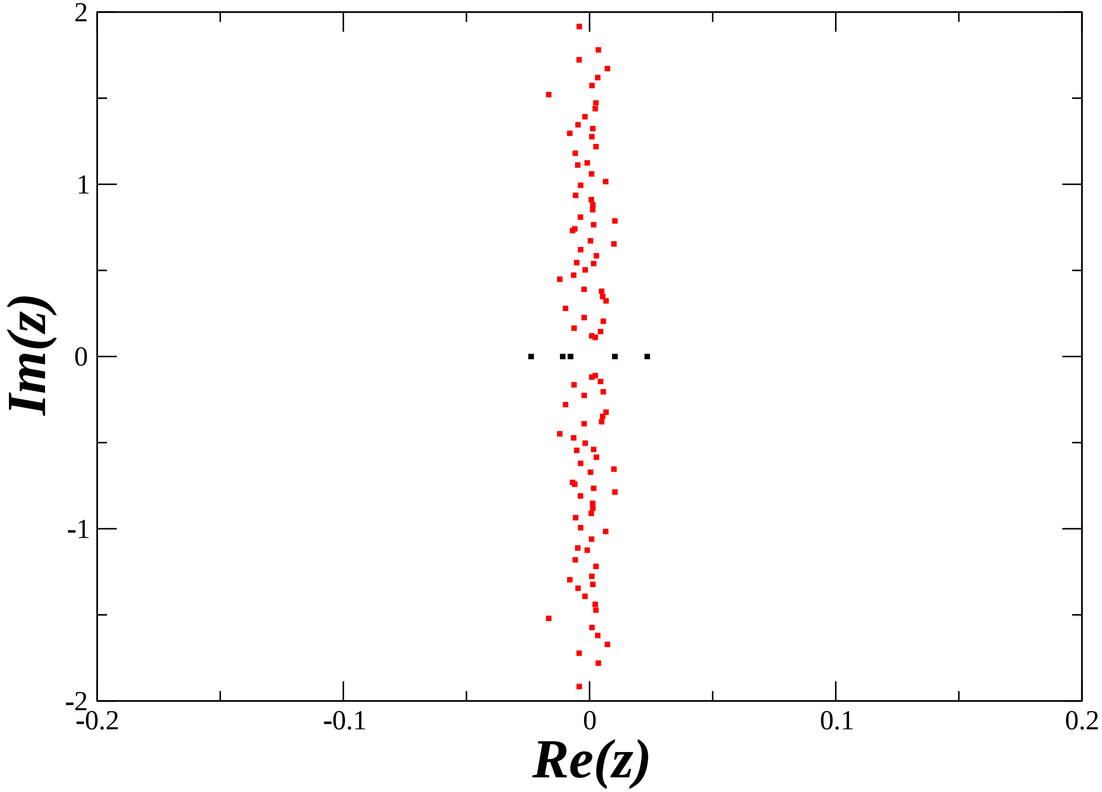
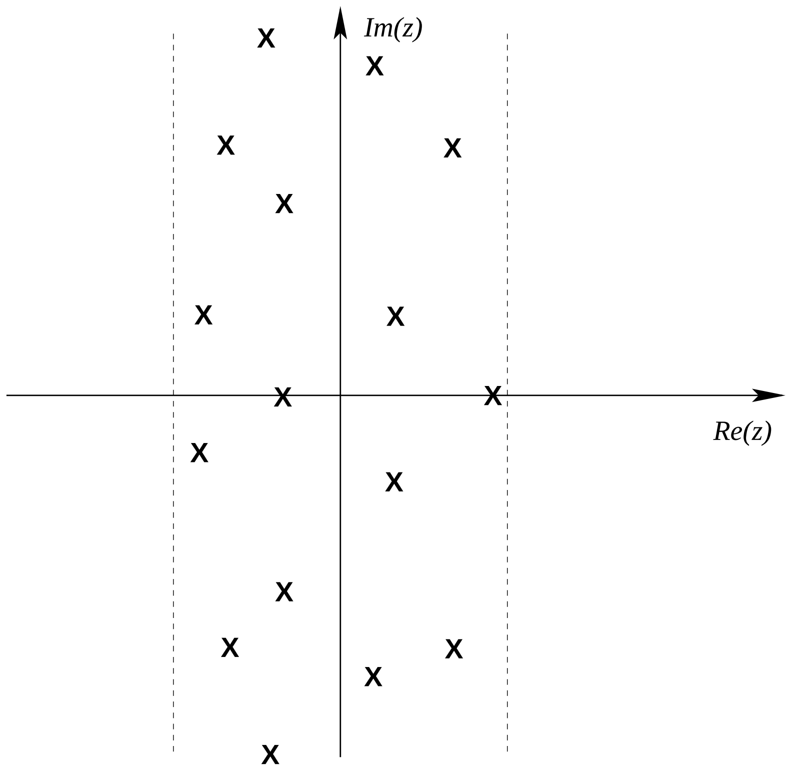


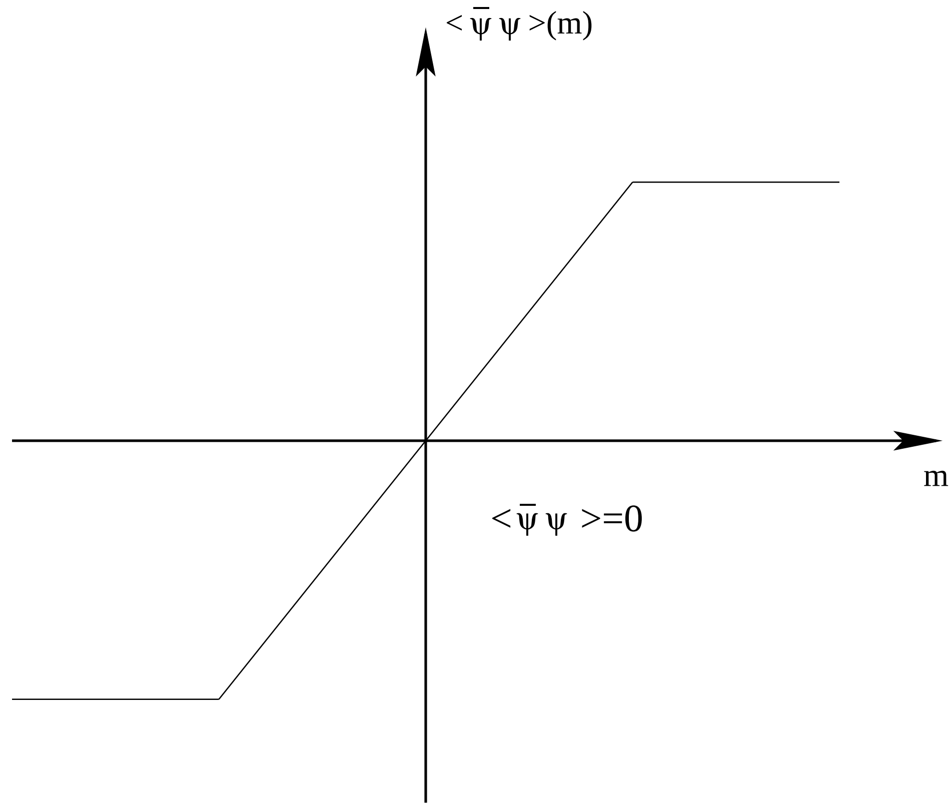
Creutz Annals Phys.322:1518-1540,2007

Eigenvalues, z, of D_W (*illustration*)



$a \neq 0$ Aoki phase (parity broken phase)





Electrostatic analogy:

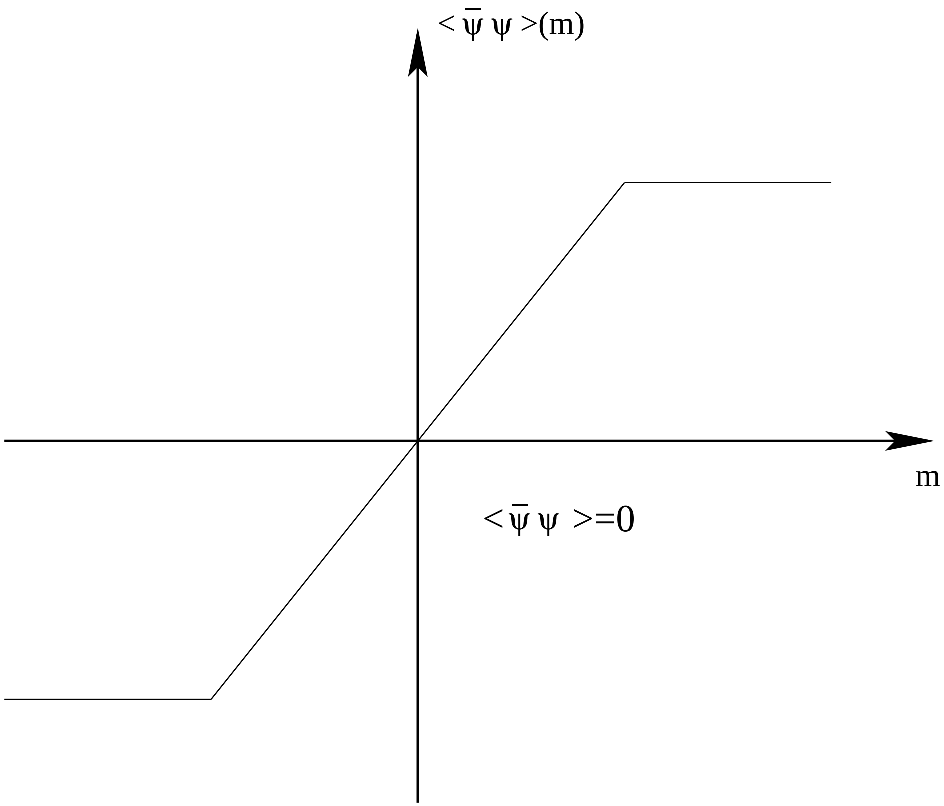
Eigenvalues = charges, quark mass = test charge

Aoki PRD **30** 2653 (1984)

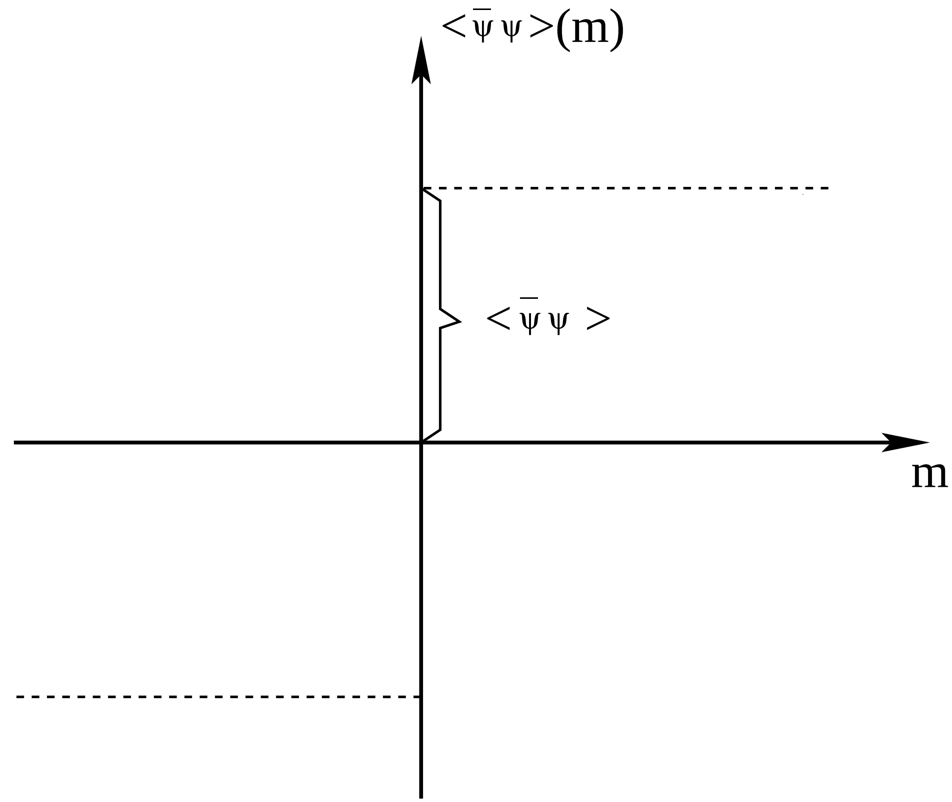
Barbour et al. NPB 275 (1986) 296 (nonzero μ)

Phases of Wilson fermions

Aoki (2nd order)



Sharpe Singleton (1st order)



Aoki PRD **30** 2653 (1984)

Sharpe Singleton PRD **58**, 074501 (1998)

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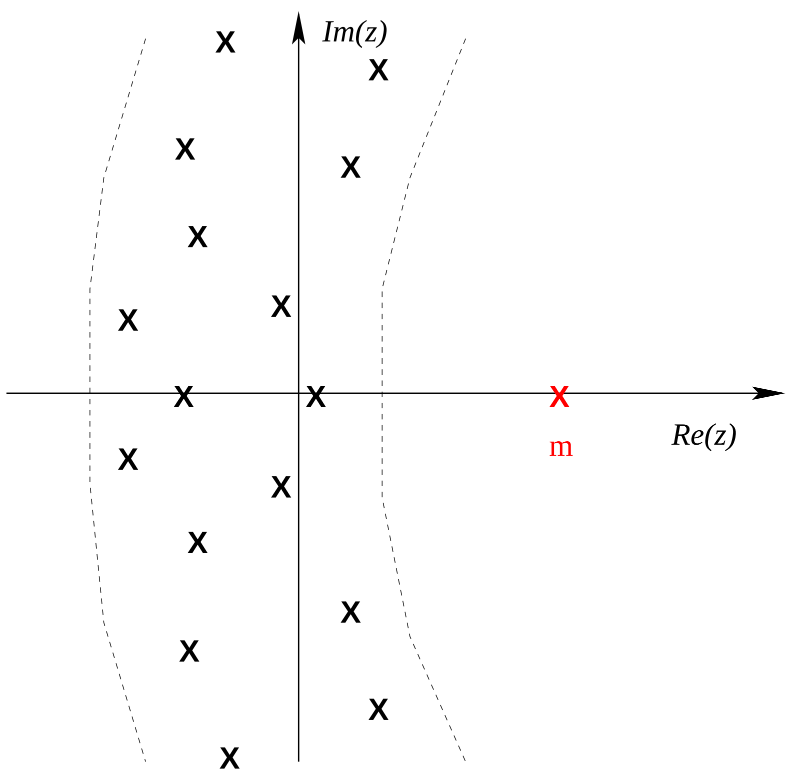

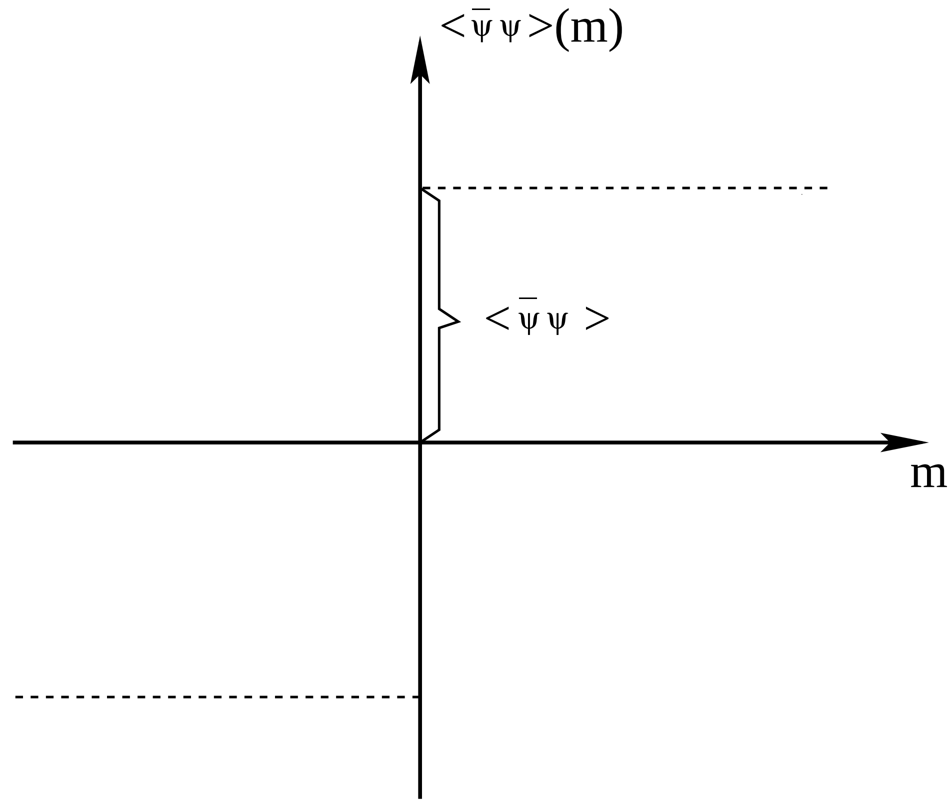

Kieburg Splittorff Verbaarschot arXiv:1202.0620

Method: *Wilson Chiral Perturbation Theory*

Sharpe PRD 74 (2006) 014512

Wilson CPT

The chiral Lagrangian for Wilson fermions has new terms

$$\begin{aligned}\mathcal{L} &= \frac{F_\pi^2}{4}\mathrm{Tr}\left(d_\mu U d_\mu U^\dagger\right) + \frac{m}{2}\Sigma\mathrm{Tr}(U+U^\dagger) \\ &-a^2 W_6[\mathrm{Tr}\left(U+U^\dagger\right)]^2 - a^2 W_7[\mathrm{Tr}\left(U-U^\dagger\right)]^2 \\ &-a^2 W_8\mathrm{Tr}(U^2+{U^\dagger}^2)\end{aligned}$$

with new constants W_6, W_7 and W_8

Sharpe Singleton PRD **58**, 074501 (1998)
Rupak Shoresh PRD **66**, 054503 (2002)
Aoki PRD 68:054508,2003
Bar Rupak Shoresh PRD **70**, 034508 (2004)
Sharpe Wu PRD **70**, 094029 (2004)
Aoki Baer PRD 70 (2004) 116011
Golterman Sharpe Singleton PRD **71**, 094503 (2005)
Del Debbio Frandsen Panagopoulos Sannino JHEP0806:007 (2008)
Shindler PLB 672, 82 (2009)
Bar Necco Schaefer JHEP 0903, 006 (2009)
Bar Necco Shindler JHEP 1004:053,2010

Wilson CPT in the ϵ-regime $(m\Sigma V \sim a^2 V W_i \sim 1)$

The partition function in a sector ν

$$Z_{N_f}^{\nu} = \int_{U(N_f)} dU \, \det{}^{\nu}(U) \, e^S$$

with

$$\begin{aligned} S \quad = \quad & +\frac{m}{2}\Sigma V \operatorname{Tr}(U + U^\dagger) \\ & -a^2 V W_6 [\operatorname{Tr}\left(U + U^\dagger\right)]^2 - a^2 V W_7 [\operatorname{Tr}\left(U - U^\dagger\right)]^2 \\ & -a^2 V W_8 \operatorname{Tr}(U^2 + U^{\dagger\,2}) \end{aligned}$$

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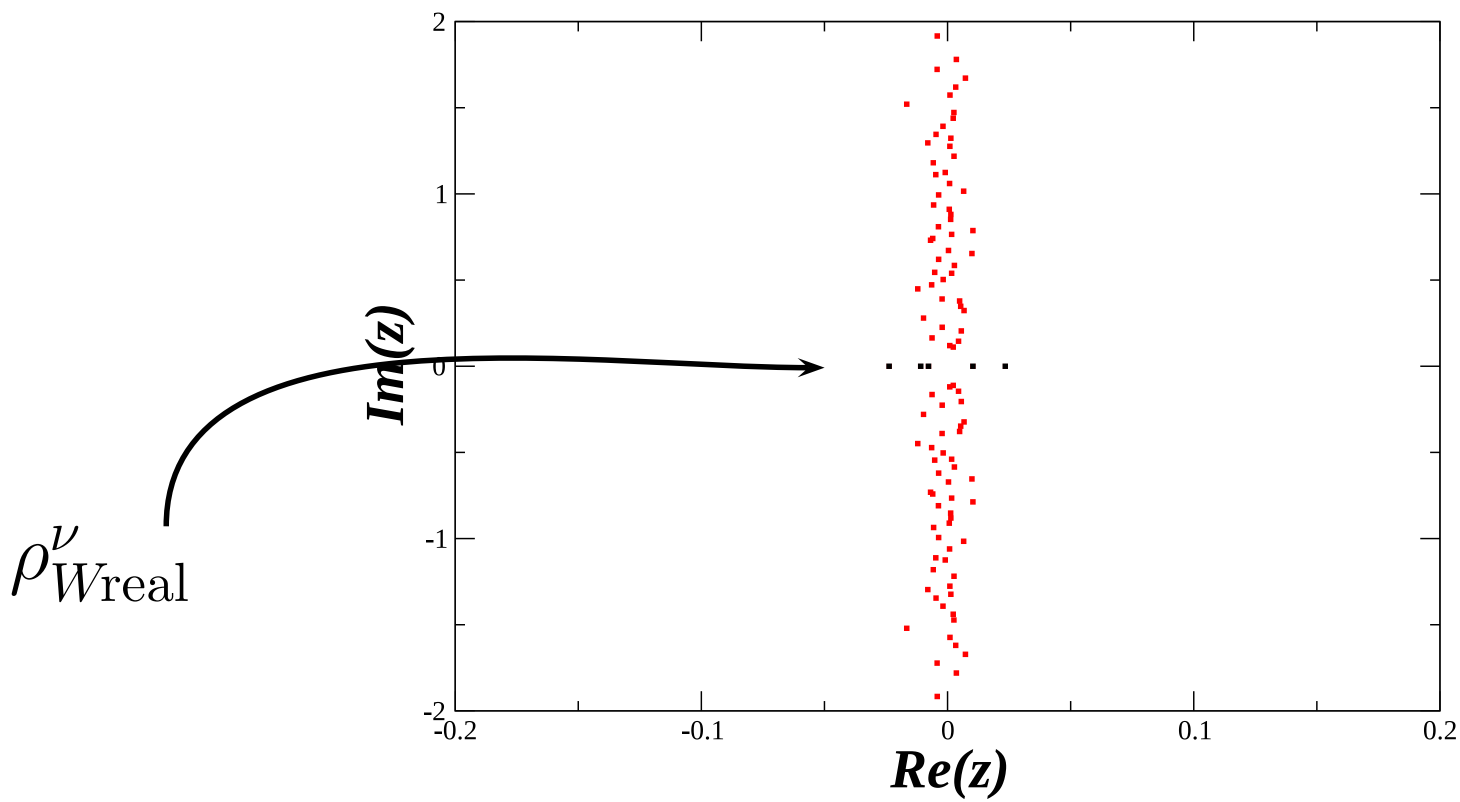
Non trivial fact: In sector ν the Wilson Dirac operator D_W has index ν

index $= \sum_k \mathrm{sign}(\langle k|\gamma_5|k\rangle)$

Damgaard Splittorff Verbaarschot PRL 105:162002, 2010

Akemann, Damgaard, Splittorff, Verbaarschot, PRD 83:085014, 2011

Eigenvalues, z, of D_W (*illustration*)



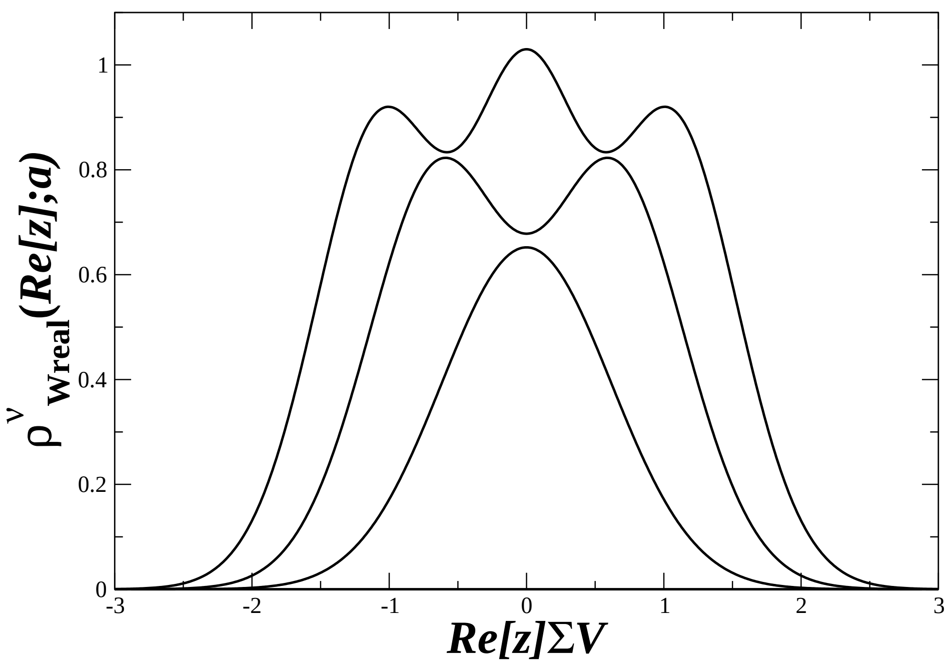
Quenched microscopic density of D_W

The real eigenvalues of D_W in sector $\nu = 0, 1, 2, 3$

$N_f = 0$

$a\sqrt{W_8 V} = 0.2$

$W_6 = W_7 = 0$



Gattringer Hip Lang NPB 508 (1997) 329

Hernandez NPB 536 (1998) 345

Damgaard Splittorff Verbaarschot PRL 105:162002,2010

Kieburg, Verbaarschot, Zafeiropoulos PRL 108, 022001 (2012)

Unquenched microscopic density of D_W

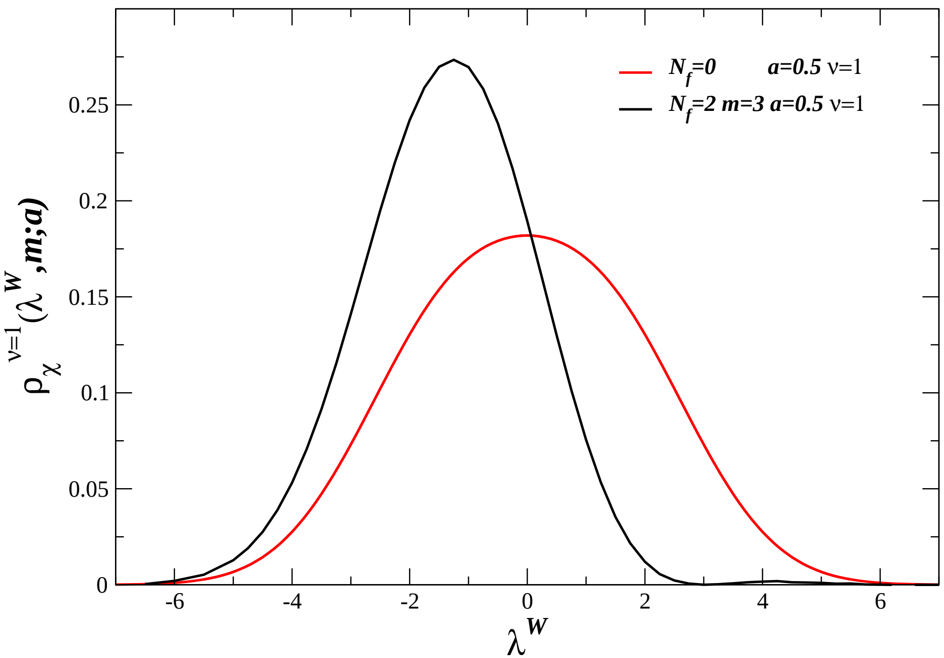
The real eigenvalues of D_W in sector $\nu = 1$

$N_f = 2$

$m\Sigma V = 3$

$a\sqrt{W_8 V} = 0.5$

$W_6 = W_7 = 0$



Splittorff Verbaarshot PRD 84, 065031 (2011)

Unquenched microscopic density of D_W

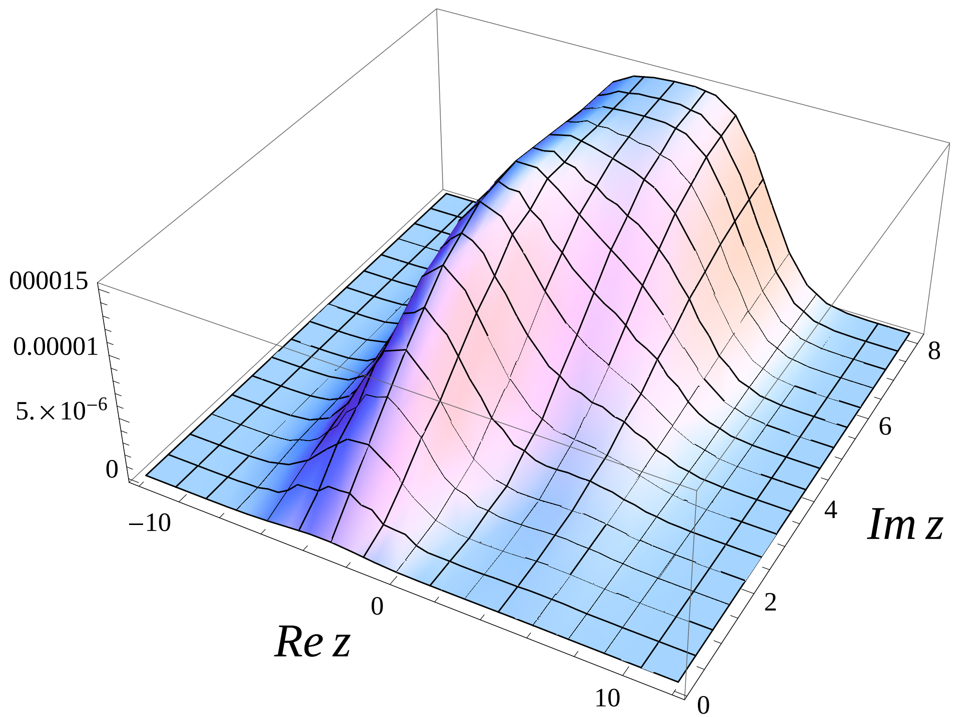
The complex eigenvalues of D_W in sector $\nu = 0$

$N_f = 2$

$m\Sigma V = 2$

$a\sqrt{W_8 V} = 0.8$

$W_6 = W_7 = 0$



Kieburg Splittorff Verbaarshot arXiv:1202.0620

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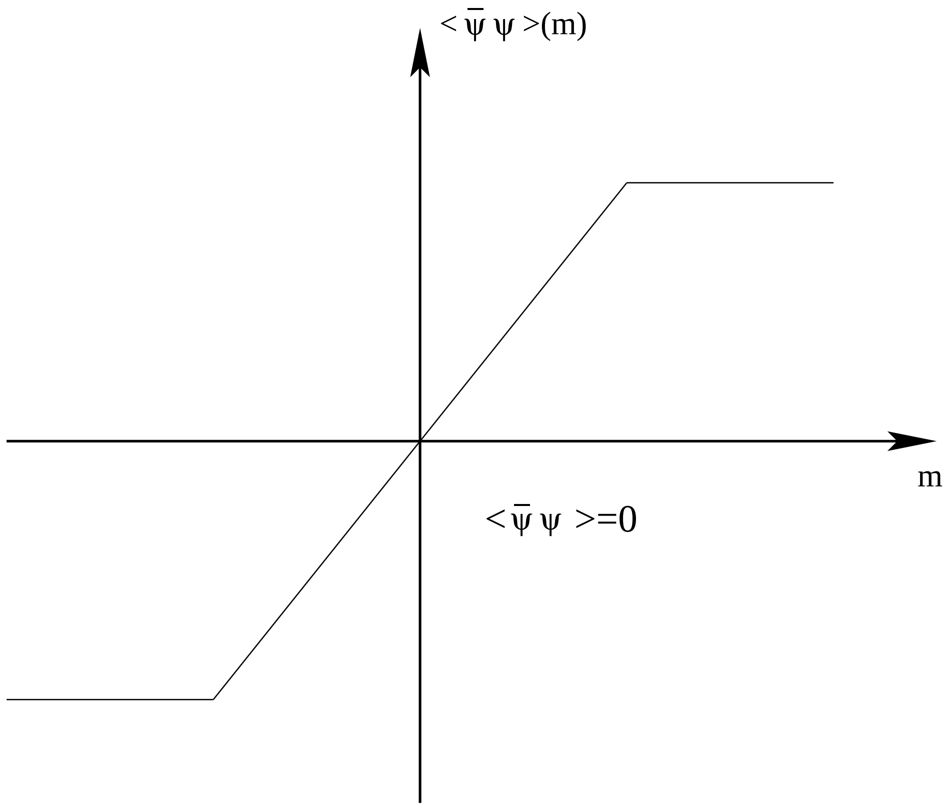
So far we used $W_6 = W_7 = 0$ *and* $W_8 > 0$

Sharpe Singleton PRD **58**, 074501 (1998)
Rupak Shoresh PRD **66**, 054503 (2002)
Aoki PRD 68:054508,2003
Bar Rupak Shoresh PRD **70**, 034508 (2004)
Sharpe Wu PRD **70**, 094029 (2004)
Aoki Baer PRD 70 (2004) 116011
Golterman Sharpe Singleton PRD **71**, 094503 (2005)
Del Debbio Frandsen Panagopoulos Sannino JHEP0806:007 (2008)
Shindler PLB 672, 82 (2009)
Bar Necco Schaefer JHEP 0903, 006 (2009)

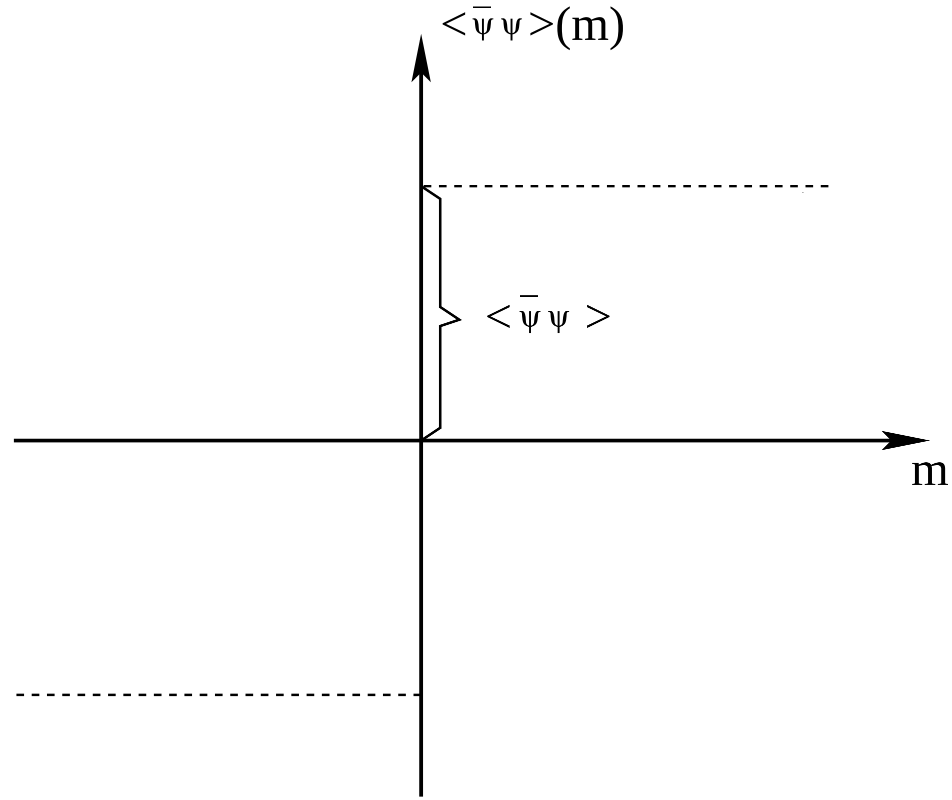
Bar Necco Shindler JHEP 1004:053,2010

Phases of Wilson fermions

Aoki ($W_8 + 2W_6 > 0$)



Sharpe Singleton ($W_8 + 2W_6 < 0$)



Sharpe Singleton PRD **58**, 074501 (1998)

The signs of W_6 and W_8

Only Wilson CPT with

$$W_6 < 0 \text{ and } W_8 > 0$$

corresponds to the γ_5-Hermitian D_W

$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - \frac{ar}{2}\nabla_\mu\nabla^*_\mu$$

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$$D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - \frac{ar}{2}\nabla_\mu\nabla_\mu^*$$

Wilson CPT with

$$W_6 > 0 \text{ and } W_8 < 0$$

corresponds to an Anti-Hermitian (not γ_5-Hermitian)

$$D_{iW} = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - i\frac{ar}{2}\nabla_\mu\nabla_\mu^*$$

sign problem

Kieburg Splittorff Verbaarschot arXiv:1202.0620
QCD ineq: Hansen and Sharpe arXiv:1111.2404, arXiv:1112.3998

Akemann Damgaard Splittorff Verbaarschot PRD 83 (2011) 085014

From Wilson CPT to the spectrum of D_W

The SUSY method

The SUSY way of writing the eigenvalue density

$$\rho_c^{N_f}(z,m;a) = \frac{1}{\pi} \lim_{\tilde{z}\to z} \frac{d}{dz}\frac{d}{dz^*} \log Z^{\nu}_{N_f+2|2}(m,z,z^*,\tilde{z},\tilde{z}^*;a)]$$

SUSY/graded *generating function* for the eigenvalue density

$$Z^{\nu}_{N_f+2|2}(m,z,z^*,\tilde{z},\tilde{z}^*;a) = \int \mathrm{d}A \det(D_W+m)^{N_f} \frac{|\det(D_W+z)|^2}{|\det(D_W+\tilde{z})|^2}\, \mathrm{e}^{-S_{\mathrm{YM}}(A)}$$

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integrate over the gauge fields

Efetov *Supersymmetry in disorder and chaos* Cambridge Uni Press 1997

The SUSY method in Wilson CPT

The SUSY way of writing the eigenvalue density

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$$\mathcal{Z}_{N_f+2|2}(m,z,z^*,\tilde{z},\tilde{z}^*;a) = \int dU\ \mathrm{Sdet}(U)^{\nu} \times e^{\frac{1}{2}\mathrm{Str}(\mathcal{M}[U+U^{-1}])-a^2W_8V\mathrm{Str}(U^2+U^{-2})}$$

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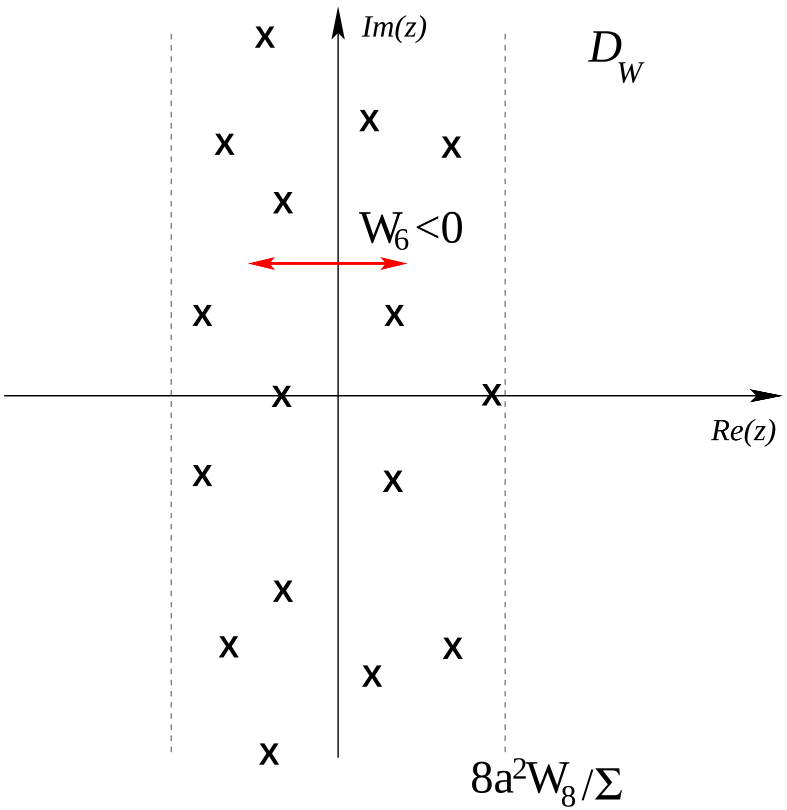
integrate over graded Goldstone manifold $Gl(N_f+2|2)$

Damgaard Osborn Toublan Verbaarschot NPB **547** 305 (1999): $a=0$

Splittorff, Verbaarschot, NPB 683 (2004) 467: $\mu \neq 0$

The effect of $W_6 < 0$ on the spectrum of D_W



$$e^{-a^2 W_6 V [\mathrm{STr}\left(U+U^\dagger\right)]^2}$$

$$\sim \int_{-\infty}^{\infty} dy\, e^{-\frac{y^2}{16|W_6 V a^2|}} e^{-\frac{y}{2}\mathrm{STr}\left(U+U^\dagger\right)}$$

Mass matrix in generating functional

$$\mathcal{M} = \mathrm{diag}\,(m-y, z-y, z^*-y, \tilde{z}-y, \tilde{z}^*-y)$$

Kieburg Splittorff Verbaarschot arXiv:1202.0620

The sign of W_6 and W_8

Constraints from γ_5-Hermiticity

$$W_6 < 0, \quad W_8 > 0$$

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Aoki (2nd order PT)

$$W_8 + 2W_6 > 0$$

Sharpe Singleton (1st order PT)

$$W_8 + 2W_6 < 0$$

The sign of W_6 and W_8

Constraints from γ_5-Hermiticity

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Aoki (2nd order PT)

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Sharpe Singleton (1st order PT)

$$W_8 + 2W_6 < 0$$

Both allowed by γ_5 hermiticity !

Both observed on the lattice

Aoki phase

Aoki Gocksch PRD **45**, 3845 (1992)
Aoki Gocksch PLB **231** (1989) 449
Aoki Gocksch PLB **243**, 409 (1990)
Jansen *et al.* [XLF Collaboration] PLB **624**, 334 (2005)
Aoki Ukawa Umemura PRL **76**, 873 (1996)
Aoki Nucl.Phys.Proc.Suppl. **60A**, 206 (1998)
Ilgenfritz *et al.* PRD **69**, 074511 (2004)

Del Debbio Giusti Luscher Petronzio Tantalo JHEP **0602**, 011 (2006)
Del Debbio Giusti Luscher Petronzio Tantalo JHEP **0702**, 056 (2007)
Del Debbio Giusti Luscher Petronzio Tantalo JHEP **0702**, 082 (2007)

Bernardoni Bulava Sommer arXiv:1111.4351
Aoki *et al.* [JLQCD Collaboration] PRD **72**, 054510 (2005)

Farchioni et al. Eur.Phys.J.C39:421 (2005)
Farchioni et al. Eur.Phys.J.C42:73 (2005)
Farchioni et al. PLB **624**, 324 (2005)
Farchioni et al. Eur.Phys.J.C47:453,2006

Baron et al. (ETM collab) JHEP08(2010)097

Sharpe-Singleton scenario

Puzzle: Quenched only observes Aoki

Aoki phase

Aoki Gocksch PRD **45**, 3845 (1992)
Aoki Gocksch PLB **231** (1989) 449
Aoki Gocksch PLB **243**, 409 (1990)
Jansen *et al.* [XLF Collaboration] PLB **624**, 334 (2005)
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Del Debbio Giusti Luscher Petronzio Tantalo JHEP **0602**, 011 (2006)
Del Debbio Giusti Luscher Petronzio Tantalo JHEP **0702**, 056 (2007)
Del Debbio Giusti Luscher Petronzio Tantalo JHEP **0702**, 082 (2007)

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Baron et al. (ETM collab) JHEP08(2010)097

Sharpe-Singleton scenario

To include $W_6 < 0$ in ρ_c

Gaussian trick

$$e^{-a^2 W_6 V [\mathrm{STr}\left(U+U^\dagger\right)]^2} = \frac{1}{4\sqrt{-\pi W_6 V a^2}} \int_{-\infty}^{\infty} dy\; e^{-\frac{y^2}{16|W_6 V a^2|}}\, e^{-\frac{y}{2}\mathrm{STr}\left(U+U^\dagger\right)}$$

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In the eigenvalue density for D_W

$$\rho^{\nu}_{c,N_f}(\hat{z},\hat{m};\hat{a}_6,\hat{a}_8) \quad = \quad \frac{1}{Z^{\nu}_{N_f}(\hat{m};\hat{a}_6,\hat{a}_8)} \int [dy] \; Z^{\nu}_{N_f}(\hat{m}-y;\hat{a}_8)\rho^{\nu}_{c,N_f}(\hat{z}-y,\hat{m}-y;\hat{a}_8)$$

where

$\hat{m} = m\Sigma V$, $\hat{z} = z\Sigma V$ and $\hat{a}_6^2 = W_6 V a^2$, $\hat{a}_8^2 = W_8 V a^2$

Kieburg Splittorff Verbaarschot arXiv:1202.0620

The mean field eigenvalue density of D_W

For $W_6 = 0$

$$\rho^{\mathrm{MF}}_{c,N_f=2}(\hat{x},\hat{m};\hat{a}_8) = \theta(8\hat{a}_8^2 - |\hat{x}|)$$

The mean field eigenvalue density of D_W

For $W_6 = 0$

$$\rho^{\rm MF}_{c,N_f=2}(\hat{x},\hat{m};\hat{a}_8) = \theta(8\hat{a}_8^2 - |\hat{x}|)$$

With W_6

$$\rho^{\rm MF}_{c,N_f=2}(\hat{x},\hat{m};\hat{a}_6,\hat{a}_8) = \frac{1}{Z_2^{\rm MF}(\hat{m};\hat{a}_6,\hat{a}_8)}\int dy\, e^{-y^2/16|\hat{a}_6^2|} Z_2^{\rm MF}(\hat{m}-y;\hat{a}_8)\theta(8\hat{a}_8^2 - |\hat{x}-y|)$$

(2

where

$$Z_2^{\rm MF}(\hat{m};\hat{a}_8) = e^{2\hat{m}-4\hat{a}_8^2} + e^{-2\hat{m}-4\hat{a}_8^2} + \theta(8\hat{a}_8^2 - |\hat{m}|)e^{\hat{m}^2/8\hat{a}_8^2+4\hat{a}_8^2}$$

Kieburg Splittorff Verbaarschot arXiv:1202.0620

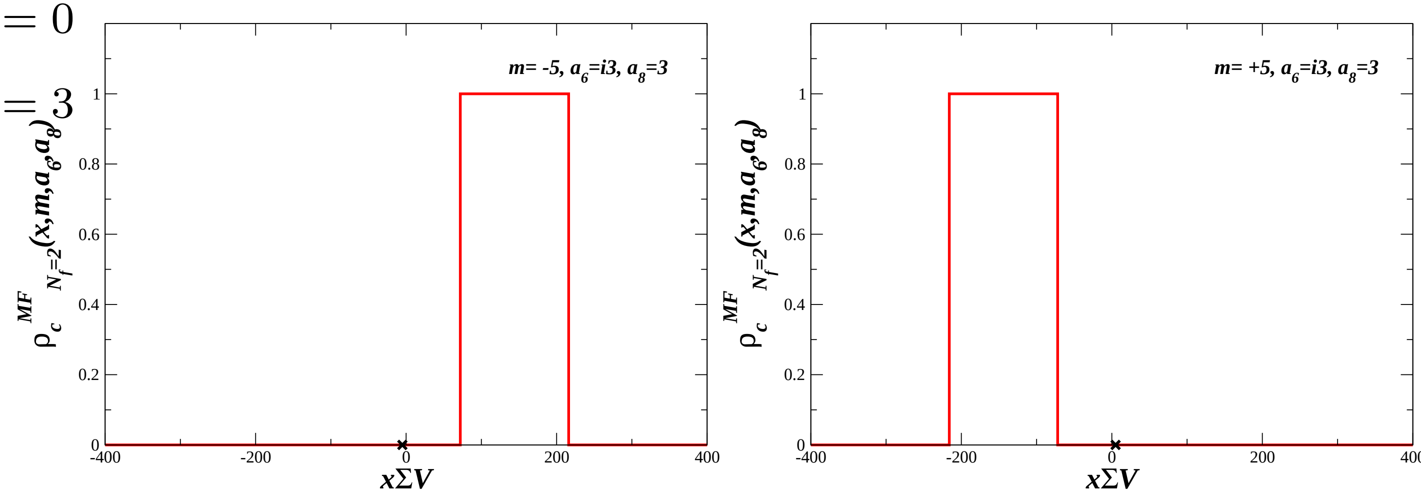
Sharpe Singleton $W_8 + 2W_6 < 0$

$m < 0$ $\qquad$ $0 < m$

$a\sqrt{W_6 V} = i3$

$a\sqrt{W_7 V} = 0$

$a\sqrt{W_8 V} = 3$

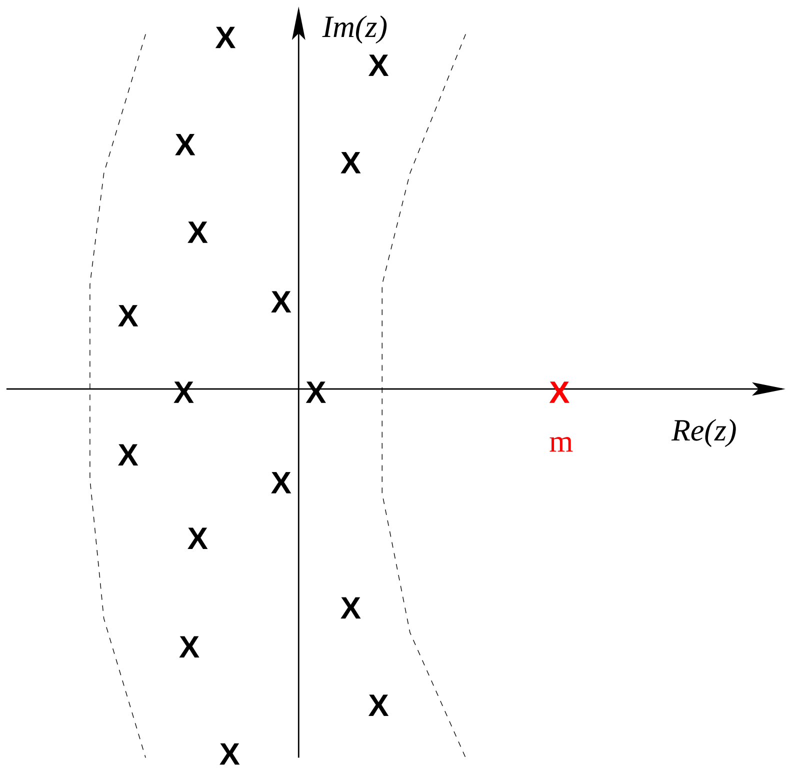


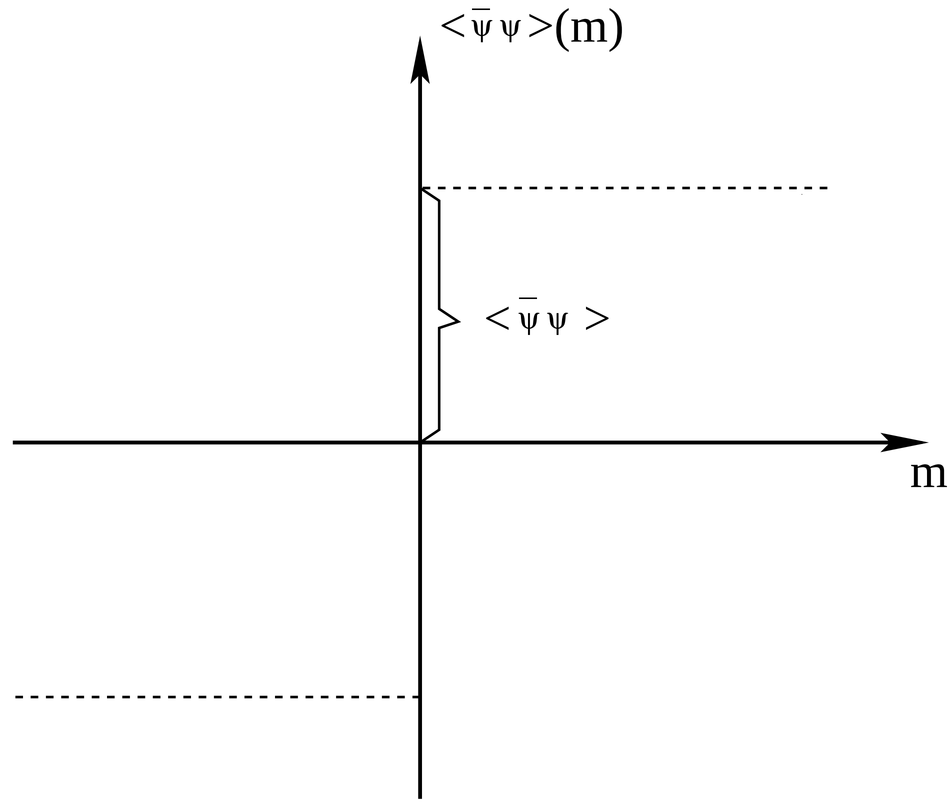
Gap and pion mass

$$\frac{m_\pi^2 F_\pi^2}{2} = |m|\Sigma - 8(W_8 + 2W_6)a^2$$

Kieburg Splittorff Verbaarschot arXiv:1202.0620

Sharpe Singleton $W_8 + 2W_6 < 0$





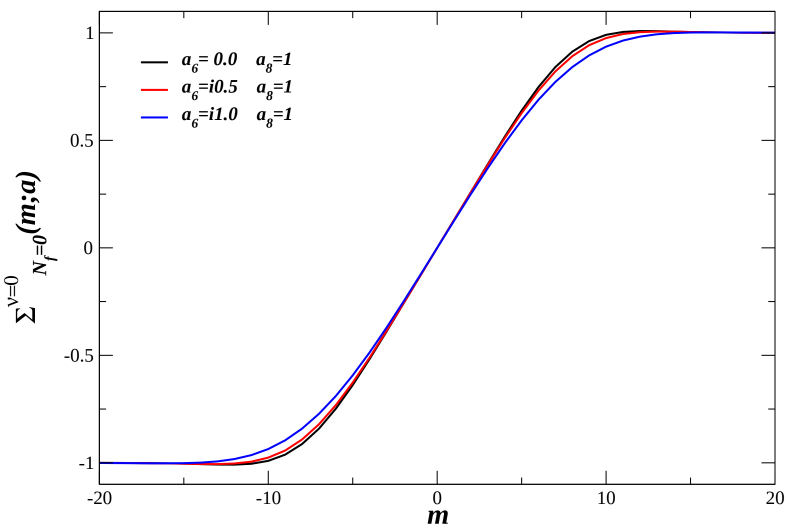
$W_6 < 0$ prefers the same signs of all $Re[z]$

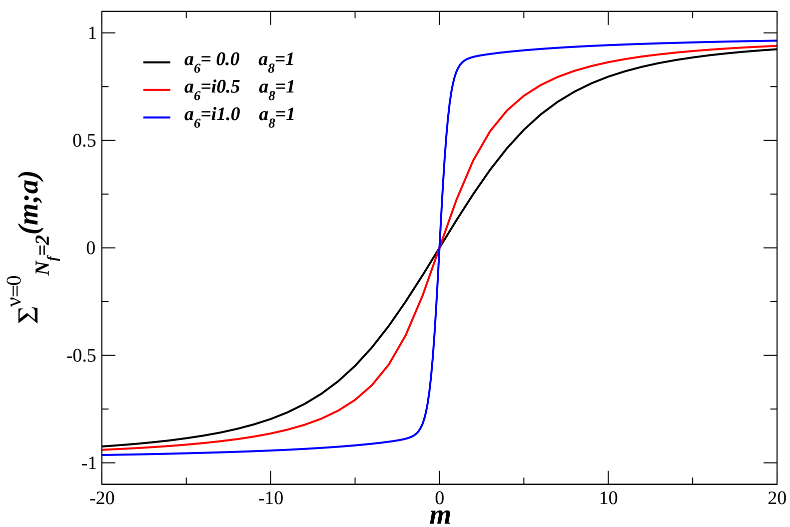
Kieburg Splittorff Verbaarschot arXiv:1202.0620

Quenched and unquenched condensate

$W_8 + 2W_6 > 0$ $\quad$ $W_8 + 2W_6 = 0$ $\quad$ $W_8 + 2W_6 < 0$

Quenched $\quad$ $N_f = 2$





Sharpe Singleton $\quad$ only for $N_f > 0$

Kieburg Splittorff Verbaarschot arXiv:1202.0620

Conclusions

Derived the microscopic eigenvalue density from WCPT

- for the real and complex eigenvalues of D_W

in sectors with fixed index of D_W

Conclusions

Derived the microscopic eigenvalue density from WCPT

- for the real and complex eigenvalues of D_W

in sectors with fixed index of D_W

Constraints on the parameters of WCPT from γ_5*-Hermiticity*

Conclusions

Derived the microscopic eigenvalue density from WCPT

- for the real and complex eigenvalues of D_W

in sectors with fixed index of D_W

Constraints on the parameters of WCPT from γ_5-Hermiticity

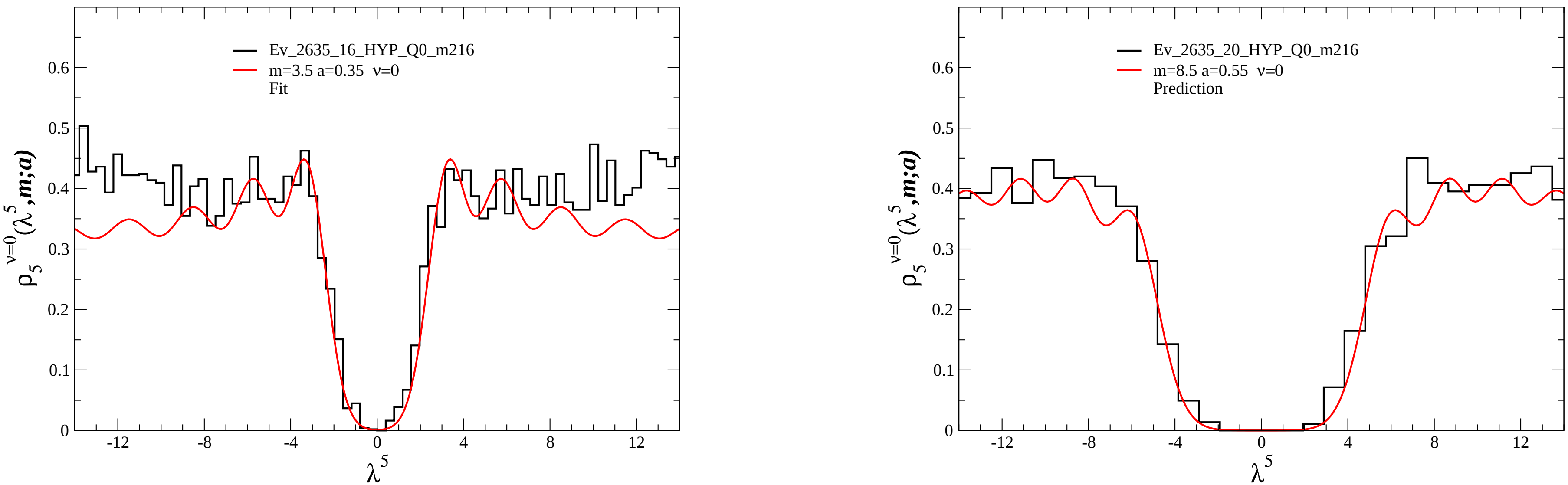
The realization of the Sharpe Singleton scenario

- does not occur quenched

Does it work ?

Spectrum of $D_5 \equiv \gamma_5(D_W + m)$ on 16^4 and 20^4 lattice

Histograms: lattice Curves: WCPT



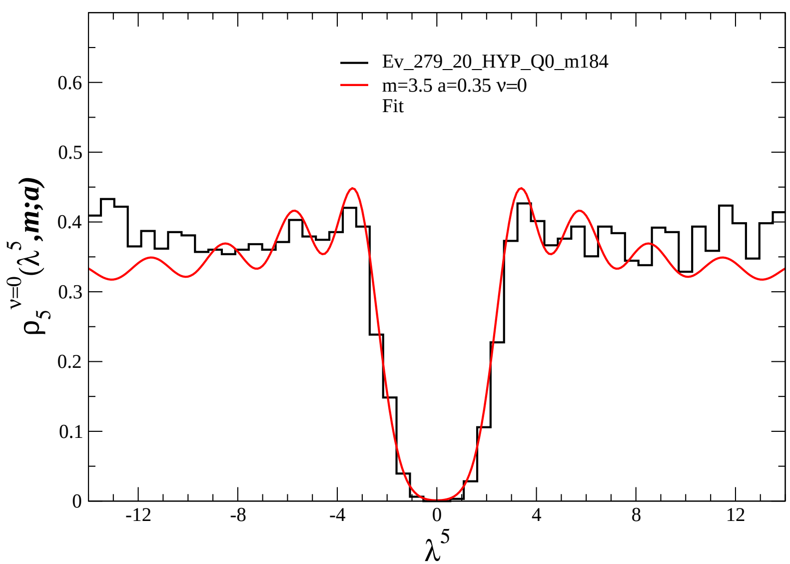
LHS fit (ΣV, $m\Sigma V$ and a_8) RHS prediction: Volume scaling

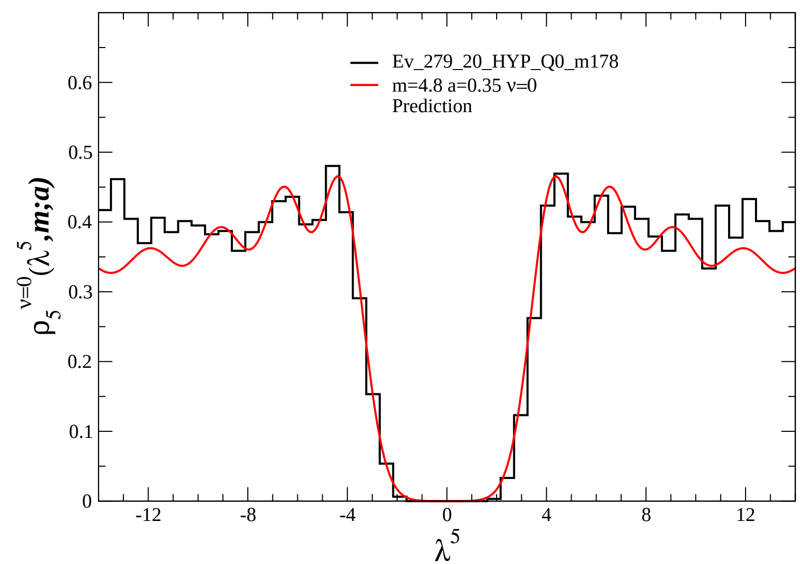
$$N_f = 0$$

Damgaard Heller Splittorff Phys.Rev. D85 (2012) 014505

Spectrum of D_5 on 20^4 lattice smaller coupling

Histograms: lattice Curves: WCPT





LHS fit (ΣV, $m\Sigma V$ and a_8) RHS prediction: mass scaling

$$N_f = 0$$

Damgaard Heller Splittorff Phys.Rev. D85 (2012) 014505

Deuzeman Wenger Wuilloud JHEP 1112 (2011) 109

Additional slides

The Hermitian Wilson Dirac operator D_5

Introduce

$$D_5 \equiv \gamma_5 (D_W + m)$$

The Hermitian Wilson Dirac operator D_5

Introduce

$$D_5 \equiv \gamma_5(D_W + m)$$

γ_5-Hermiticity of D_W | Hermiticity of D_5

$$D_W^\dagger = \gamma_5 D_W \gamma_5 \qquad \Rightarrow \qquad D_5^\dagger = D_5$$

The Hermitian Wilson Dirac operator D_5

Introduce

$$D_5 \equiv \gamma_5(D_W + m)$$

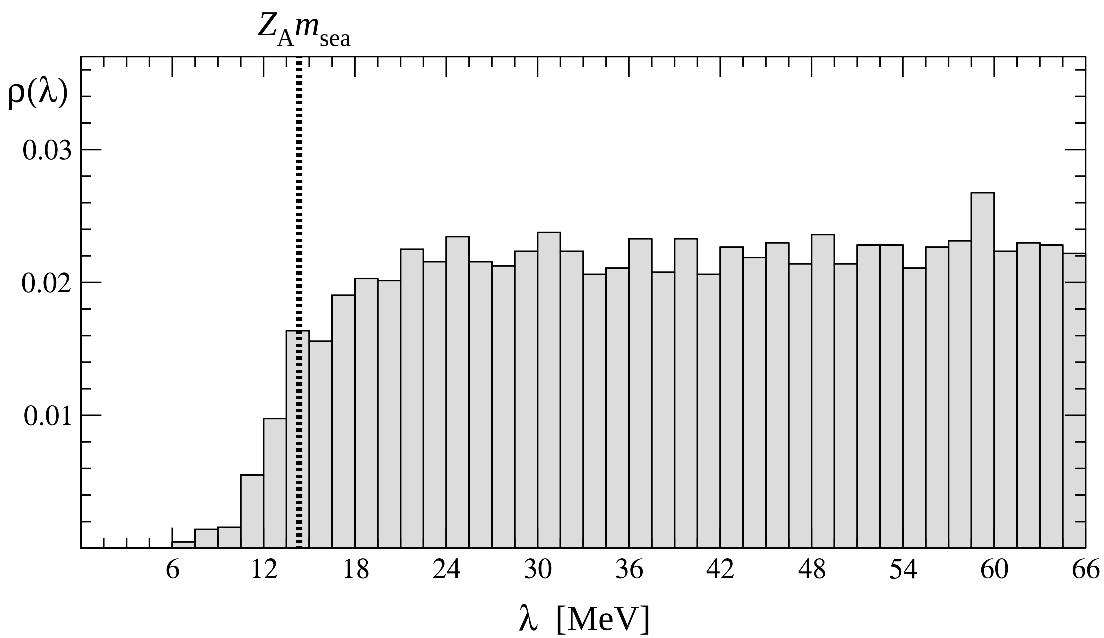
γ_5-Hermiticity of D_W Hermiticity of D_5

$$D_W^\dagger = \gamma_5 D_W \gamma_5 \qquad \Rightarrow \qquad D_5^\dagger = D_5$$

D_5 is hermitian but spectrum *not* symmetric: *not* $(\lambda^5, -\lambda^5)$

Lattice

Spectrum of D_5 for $N_f = 2$



- Aoki phase when gap closes

Lüscher JHEP0707:081,2007
Del Debbio Giusti Lüscher Petronzio Tantalo JHEP0702:082,2007
Aoki PRD 30 (1984) 2653

Bitar Heller Narayanan PLB **418** 167 (1998)

From Wilson CPT to the spectrum of D_5

The SUSY method

The SUSY way of writing the eigenvalue density

$$\rho_5^{N_f}(\lambda^5, m; a) = \frac{1}{\pi}\mathrm{Im}[\lim_{z'\to z}\frac{d}{dz}Z^{\nu}_{N_f+1|1}(m, m, z, z'; a)]$$

SUSY/graded *generating function* for the eigenvalue density

$$Z^{\nu}_{N_f+1|1}(m, m, z, z'; a) = \int \mathrm{d}A\,\det(D_W + m)^{N_f}\,\frac{\det(D_W + m + z\gamma_5)}{\det(D_W + m + z'\gamma_5)}\ \mathrm{e}^{-S_{\mathrm{YM}}(A)}$$

The SUSY method

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integrate over the gauge fields

Efetov *Supersymmetry in disorder and chaos* Cambridge Uni Press 1997

The SUSY method in Wilson CPT

The SUSY way of writing the eigenvalue density

$$\rho_5^{N_f}(\lambda^5, m; a) = \frac{1}{\pi}\mathrm{Im}[\lim_{z'\to z}\frac{d}{dz}Z^{\nu}_{N_f+1|1}(m, m, z, z'; a)]_{z=\lambda^5}$$

SUSY/graded *generating function* for the eigenvalue density

$$\mathcal{Z}_{N_f+1|1}(m, m, z, z'; a) =$$
$$\int dU\ \mathrm{Sdet}(U)^{\nu}$$
$$\times e^{i\frac{1}{2}\mathrm{Str}(\mathcal{M}[U-U^{-1}])+i\frac{1}{2}\mathrm{Str}(\mathcal{Z}[U+U^{-1}])+a^2W_8V\mathrm{Str}(U^2+U^{-2})}$$

The SUSY method in Wilson CPT

The SUSY way of writing the eigenvalue density

$$\rho_5^{N_f}(\lambda^5, m; a) = \frac{1}{\pi}\mathrm{Im}[\lim_{z'\to z} \frac{d}{dz} Z^{\nu}_{N_f+1|1}(m, m, z, z'; a)]_{z=\lambda^5}$$

SUSY/graded *generating function* for the eigenvalue density

$$\mathcal{Z}_{N_f+1|1}(m, m, z, z'; a) = \int dU \ \mathrm{Sdet}(U)^{\nu} \times e^{i\frac{1}{2}\mathrm{Str}(\mathcal{M}[U-U^{-1}])+i\frac{1}{2}\mathrm{Str}(\mathcal{Z}[U+U^{-1}])+a^2W_8V\mathrm{Str}(U^2+U^{-2})}$$

integrate over graded Goldstone manifold $Gl(N_f+1|1)$

Damgaard Osborn Toublan Verbaarschot NPB **547** 305 (1999): $a = 0$

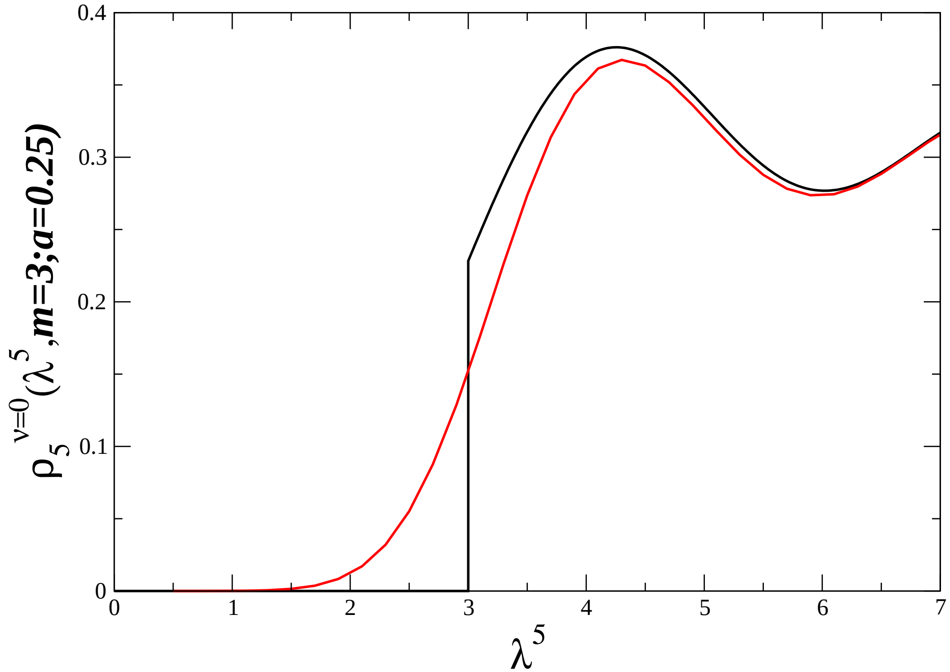
Splittorff, Verbaarschot, NPB 683 (2004) 467: $\mu \neq 0$

Spectrum of D_5 $N_f = 2$

$m\Sigma V = 3$

$\nu = 0$

$a\sqrt{VW_8} = 0.25$



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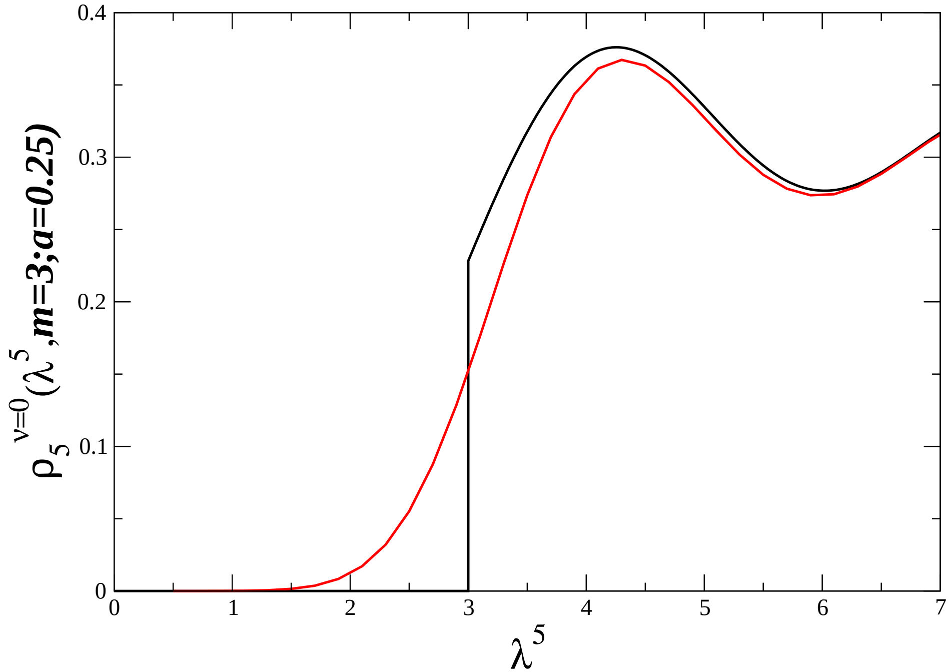

Explains $1/\sqrt{V}$ scaling of width of smallest eigenvalues

Del Debbio Giusti Lüscher Petronzio Tantalo JHEP0702:082,2007
Damgaard Splittorff Verbaarschot PRL 105:162002,2010
Akemann Damgaard Splittorff Verbaarschot PoS LATTICE2010 (2010) 079

Splittorff Verbaarschot PRD 84 (2011) 065031

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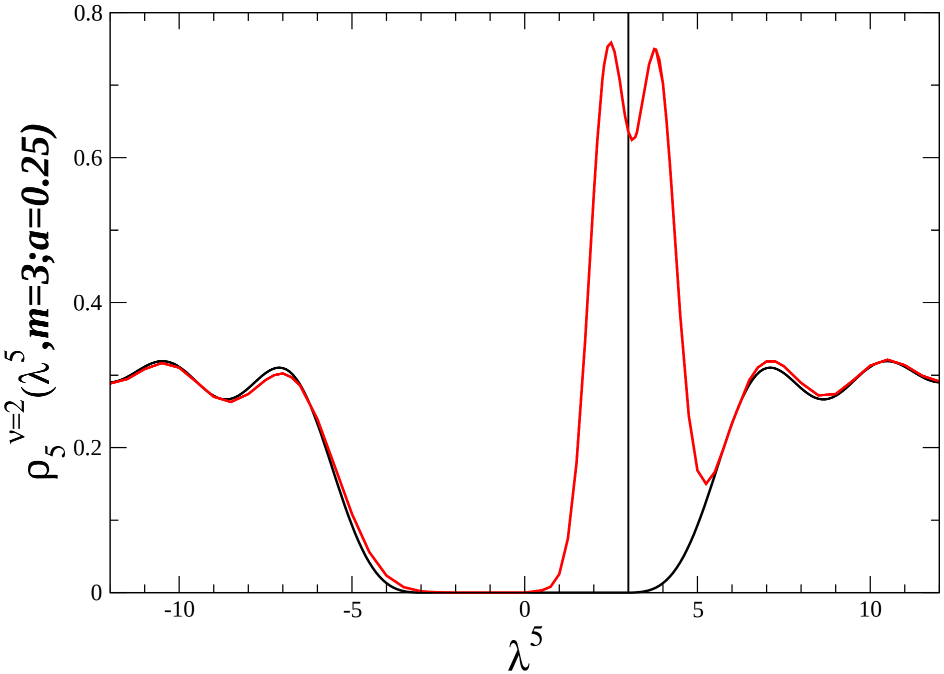

Spectrum of D_5 $\qquad N_f = 2$

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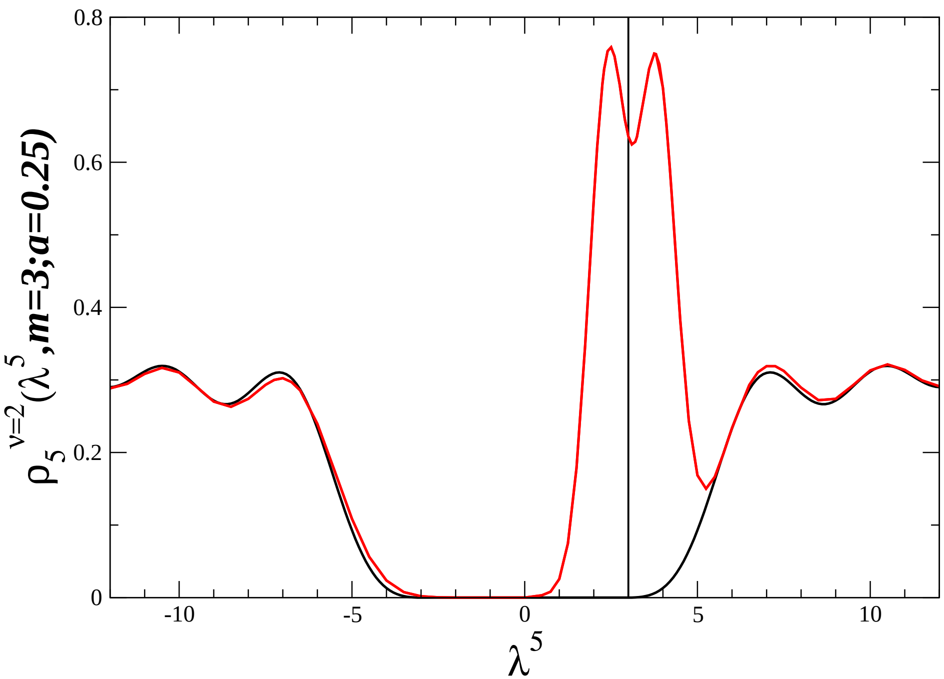


Unquenched: $\rho_5(\lambda^5 = 0, m; a) = 0$ since

$$\det{}^2(D_W + m) = \det{}^2 D_5(m) = \prod_j \lambda_j^5(m)^2$$

Damgaard Splittorff Verbaarschot PRL 105:162002,2010

Akemann Damgaard Splittorff Verbaarschot PoS LATTICE2010 (2010) 079

Splittorff Verbaarschot PRD 84 (2011) 065031

Twisted mass

$$\det(D_W + m + iz_t\gamma_5\tau_3) = \det(D_5(m) + iz_t\tau_3)$$
$$= \prod_j(\lambda_j^5(m) + iz_t)(\lambda_j^5(m) - iz_t) = \prod_j(\lambda_j^5(m)^2 + z_t^2)$$

Twisted mass

$$\det(D_W + m + iz_t\gamma_5\tau_3) = \det(D_5(m) + iz_t\tau_3)$$
$$= \prod_j (\lambda_j^5(m) + iz_t)(\lambda_j^5(m) - iz_t) = \prod_j (\lambda_j^5(m)^2 + z_t^2)$$

Maximal twist ($m = 0$):

spectrum of $D_5(m = 0)$

$$\frac{d}{dz_t} \log Z(m = 0, z_t; a) = \int d\lambda^5 \frac{2z_t}{{\lambda^5}^2 + z_t^2}\ \rho_5(\lambda^5, m = 0, z_t; a)$$

Banks-Casher relation

$$\Sigma = \frac{\pi\rho_5(\lambda^5 = 0; a)}{V}$$

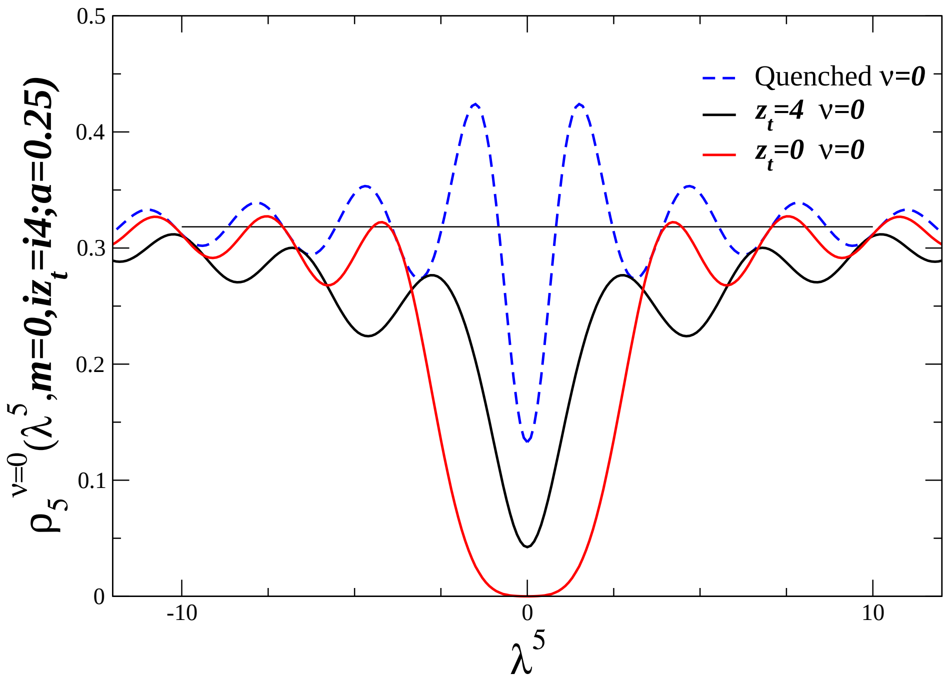
Frezzotti Grassi Sint Weisz JHEP **0108**, 058 (2001)

Banks Casher NPB 169, 103 (1980)

Spectrum of D_5 at maximally twisted mass

$\nu = 0$

$a\sqrt{VW_8} = 0.25$



$\prod_j (\lambda_j^5(m=0)^2 + z_t^2)$

Splittorff Verbaarschot arXiv:1201.1361

Quenched microscopic density of $D_5 = \gamma_5(D_W + m)$

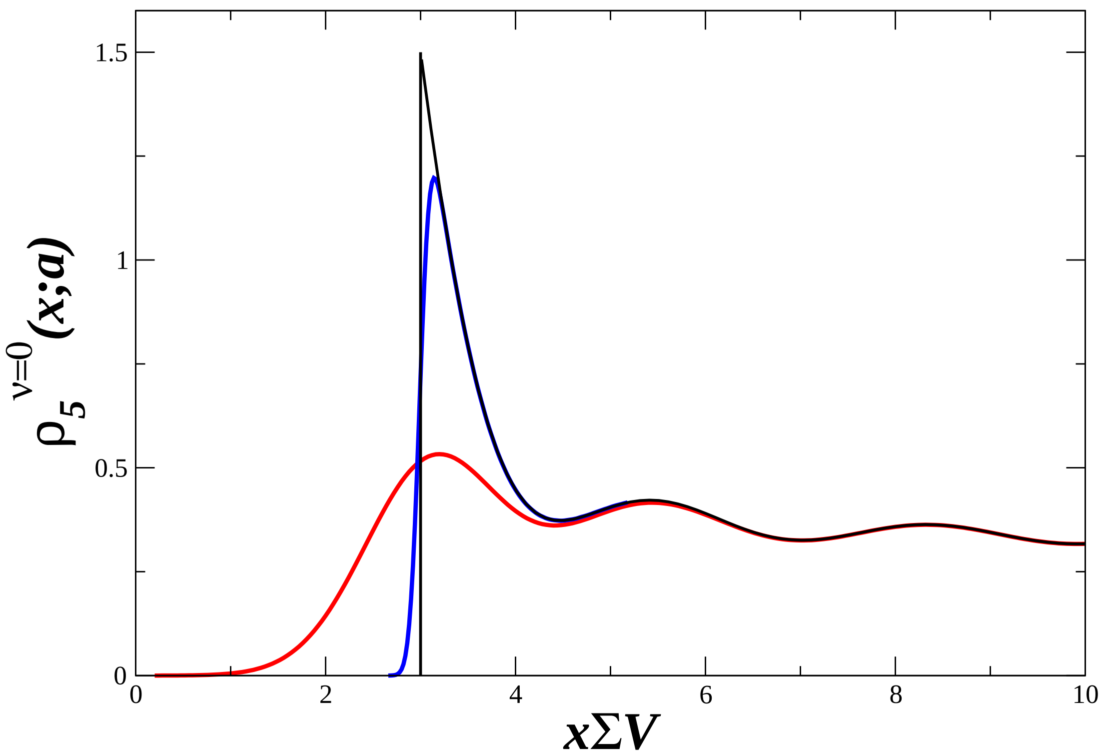
Sector $\nu = 0$

$m\Sigma V = 3$

$a\sqrt{W_8 V} = 0$

$a\sqrt{W_8 V} = 0.03$

$a\sqrt{W_8 V} = 0.250$



For $\nu = 0$ the density is symmetric: $\rho_5^{\nu=0}(x; a) = \rho_5^{\nu=0}(-x; a)$

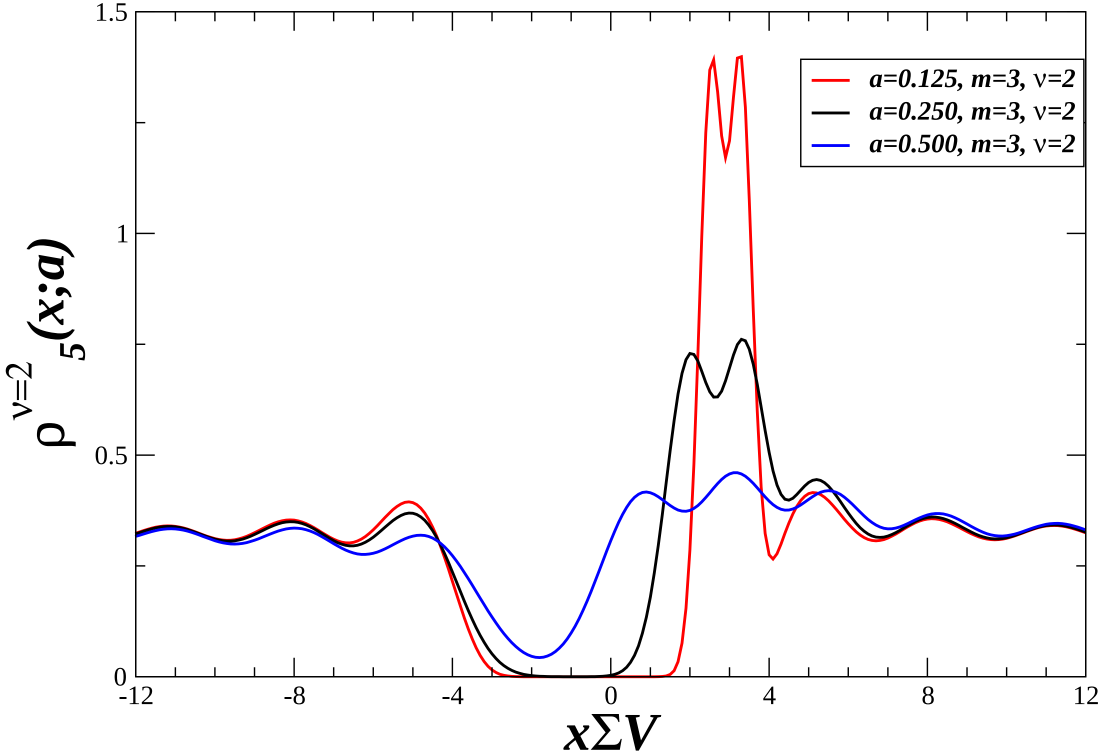
Damgaard Splittorff Verbaarschot PRL 105:162002,2010

Akemann Damgaard Splittorff Verbaarschot PRD 83 (2011) 085014

Quenched microscopic density of $D_5 = \gamma_5(D_W + m)$

Sector $\nu = 2$ increasing $a\sqrt{W_8 V}$

$m\Sigma V = 3$

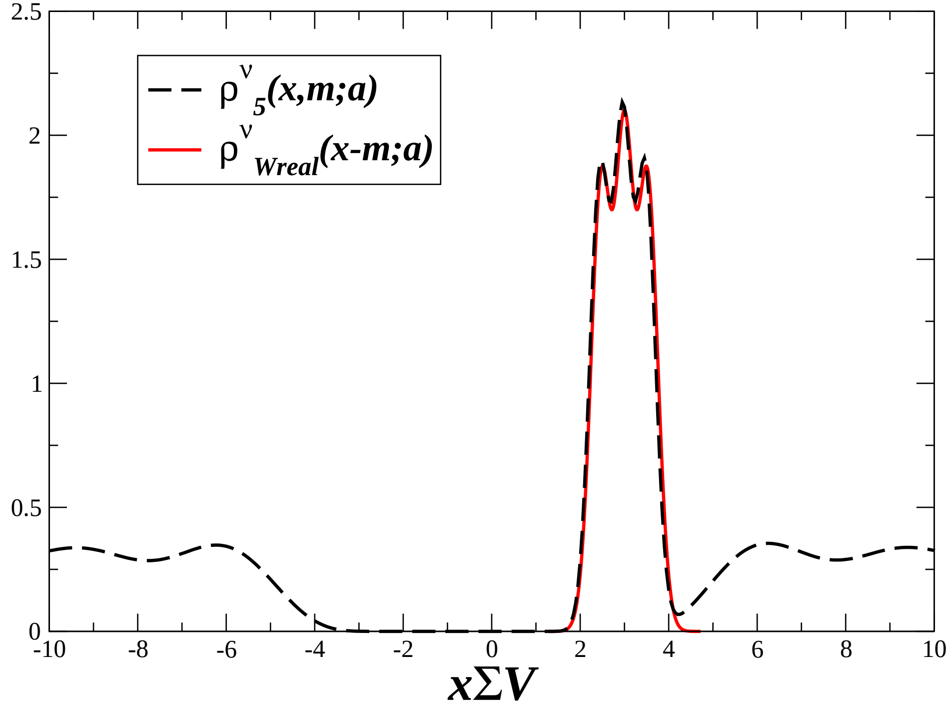


Quenched microscopic density of D_5 and D_W

Sector $\nu = 3$:

$a\sqrt{W_8 V} = 0.1$

$m\Sigma V = 3$



The real modes, ϕ, of D_W are almost chiral: $\phi^\dagger \gamma_5 \phi \simeq 1$

Itho Iwasaki Yoshie PRD 36 (1987) 527
Gattringer Hip Lang NPB 508 (1997) 329
Gattringer Hip NPB 536 (1998) 363
Hernandez NPB 536 (1998) 345

Akemann Damgaard Splittorff Verbaarschot PRD 83 (2011) 085014

W_6 and W_7

The double-trace terms re-expressed as gaussian integrals

$$Z^{\nu}_{N_f}(m,x;a_6,a_8) = \frac{1}{4\sqrt{\pi}a_6}\int_{-\infty}^{\infty} dy\, e^{-\frac{y^2}{16|a_6^2|}}\, Z^{\nu}_{N_f}(m+y,x;a_6=0,a_8)$$

where $a_6 = a\sqrt{W_6 V}$ and $a_8 = a\sqrt{W_8 V}$

W_6 and W_7

The double-trace terms re-expressed as gaussian integrals

$$Z^{\nu}_{N_f}(m, x; a_6, a_8) = \frac{1}{4\sqrt{\pi}a_6}\int_{-\infty}^{\infty} dy\, e^{-\frac{y^2}{16|a_6^2|}}\, Z^{\nu}_{N_f}(m + y, x; a_6 = 0, a_8)$$

where $a_6 = a\sqrt{W_6 V}$ and $a_8 = a\sqrt{W_8 V}$

Also works for the density

$$\rho^{\nu}_5(x, m; a_6, a_8) = \frac{1}{4\sqrt{\pi}a_6}\int_{-\infty}^{\infty} dy\, e^{-\frac{y^2}{16|a_6^2|}}\, \rho^{\nu}_5(x, m + y; a_6 = 0, a_8)$$

W_6 and W_7

The double-trace terms re-expressed as gaussian integrals

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$$\rho^{\nu}_5(x, m; a_6, a_8) = \frac{1}{4\sqrt{\pi}a_6} \int_{-\infty}^{\infty} dy \, e^{-\frac{y^2}{16|a_6^2|}} \, \rho^{\nu}_5(x, m + y; a_6 = 0, a_8)$$

W_7 averaged x instead of m

RMT for Wilson Lattice QCD

Properties of the Wilson Dirac operator

γ_5-Hermiticity

$$D^\dagger = \gamma_5 D \gamma_5$$

Properties of the Wilson Dirac operator

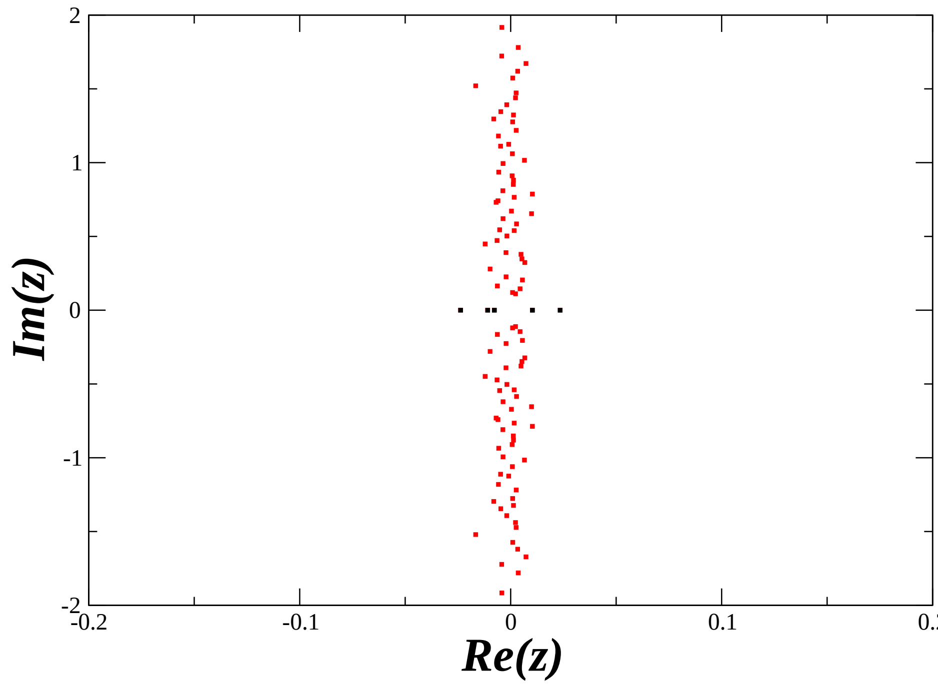
γ_5-Hermiticity

$$D^\dagger = \gamma_5 D \gamma_5$$

$$D = \begin{pmatrix} aA & iW \\ iW^\dagger & aB \end{pmatrix}$$

A ($N \times N$) and B ($(N+\nu) \times (N+\nu)$) are hermitian
W is a general complex matrix

The spectrum of one Random Matrix



Damgaard Splittorff Verbaarschot PRL 105:162002,2010

The Wilson RMT partition function

$$\mathcal{Z}_{N_f}^{\nu} \equiv \int dW \, dA dB \, \det(D+m)^{N_f} \, e^{-\frac{N}{2}\mathrm{Tr}(A^2+B^2)-N\mathrm{Tr}W^{\dagger}W}$$

where

$$D+m = \begin{pmatrix} aA+m & iW \\ iW^{\dagger} & aB+m \end{pmatrix}$$

The Wilson RMT partition function

$$\mathcal{Z}_{N_f}^{\nu} \equiv \int dW\,dA\,dB\,\det(D+m)^{N_f}\, e^{-\frac{N}{2}\mathrm{Tr}(A^2+B^2)-N\mathrm{Tr}W^\dagger W}$$

where

$$D+m = \begin{pmatrix} aA+m & iW \\ iW^\dagger & aB+m \end{pmatrix}$$

Same low energy theory in the ϵ-regime

Shuryak, Verbaarschot, NPA **560**, 306 (1993), Verbaarschot, PRL **72**, 2531 (1994)

Damgaard Splittorff Verbaarschot PRL 105:162002,2010

The Wilson RMT partition function

$$\mathcal{Z}_{N_f}^{\nu} \equiv \int dW \det(D+m)^{N_f} e^{-\frac{N}{2}\mathrm{Tr}(A^2+B^2) - N\mathrm{Tr}W^{\dagger}W}$$

where

$$D+m = \begin{pmatrix} aA+m & iW \\ iW^{\dagger} & aB+m \end{pmatrix}$$

same flavor symmetries as QCD and same breaking by m and a

$$\mathcal{Z}_{N_f}^{\nu} = \int_{U(N_f)} dU \, \det{}^{\nu}(U) \, e^{Nm\mathrm{Tr}(U+U^{\dagger}) - \frac{Na^2}{2}\mathrm{Tr}(U^2+U^{\dagger\,2})}$$

for $N \to \infty$ with mN and a^2N fixed

Shuryak Verbaarschot NPA **560**, 306 (1993), Verbaarschot PRL **72**, 2531 (1994)

Damgaard Splittorff Verbaarschot PRL 105:162002,2010

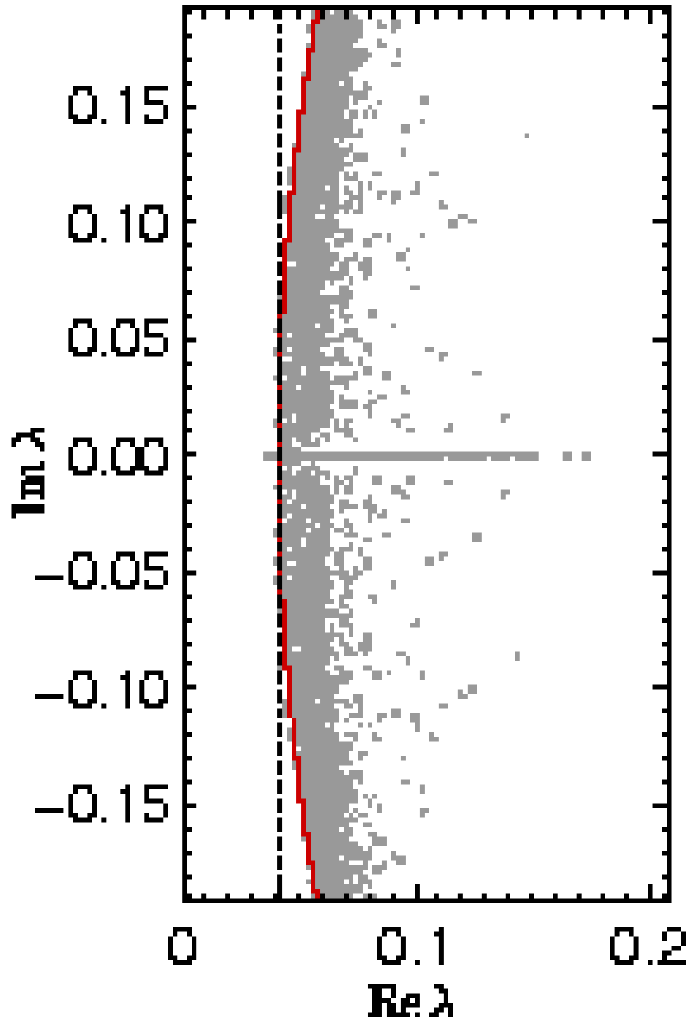
Why Wilson RMT

Usually: *easier to compute spectral correlation functions with RMT than the SUSY method*

- any N_f
- higher order correlation functions
- individual eigenvalue distributions

Splittorff Verbaarschot arXiv:to.appear

Lattice I

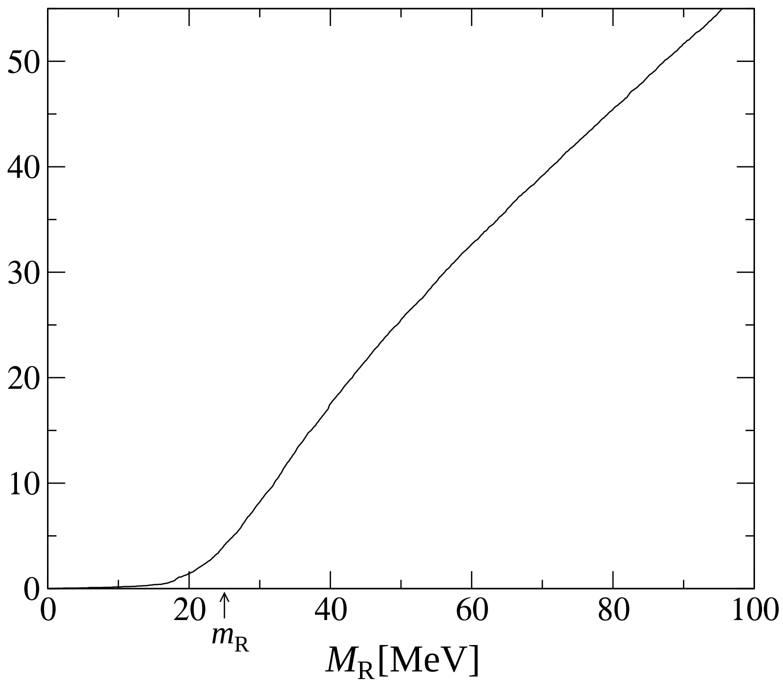


Hasenfratz Hoffmann Schaefer JHEP0705:029,2007

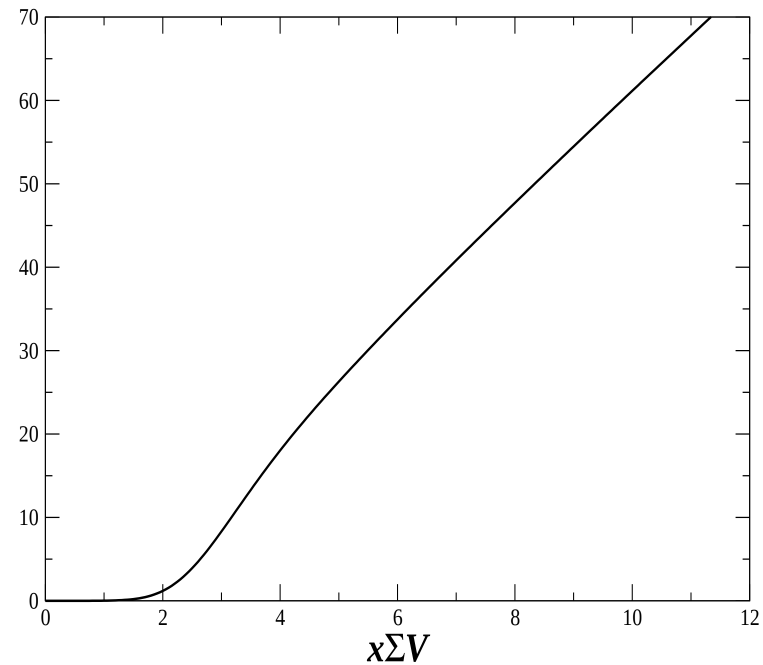
Spectrum of D_5 for $N_f = 0$

- integrated up from zero $\&$ summed over the index

Lattice $64 \times 32^3 \; a \simeq 0.07\,fm$



WCPT ($m\Sigma V = 3$, $a_8 = 0.2$)



Lüscher Palombi JHEP09(2010)110 Akemann Damgaard Splittorff Verbaarschot PRD 83 (2011) 085014

Necco Shindler arXiv:1101.1778

The sign of W_8

γ_5-Hermiticity $\Rightarrow \det^2(D_W + m) \geq 0$

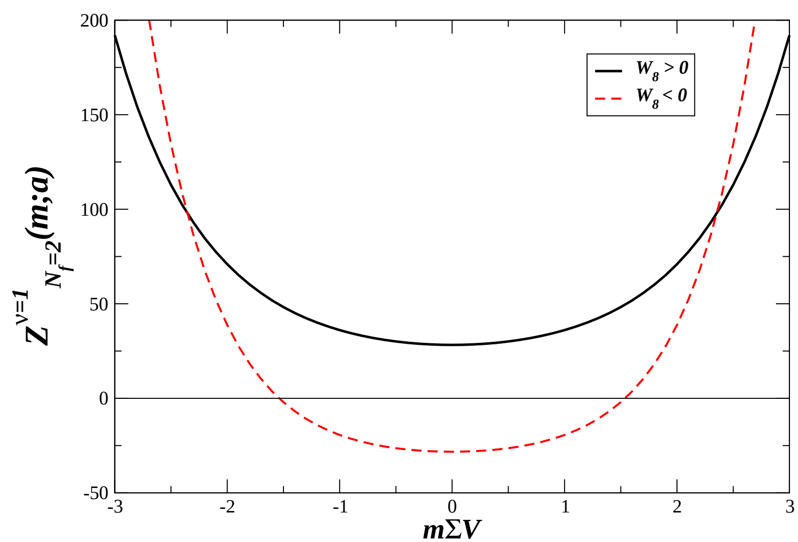
$$\text{QCD inequality} \qquad\qquad Z^{\nu}_{N_f=2}(m;a) \geq 0$$

The sign of W_8

γ_5-Hermiticity $\Rightarrow \det^2(D_W + m) \geq 0$

$$\text{QCD inequality} \qquad Z^{\nu}_{N_f=2}(m;a) \geq 0$$

Only satisfied if $W_8 > 0$ (for $W_6 = W_7 = 0$)



$a^2 V W_8 = 1$ (full) $\qquad a^2 V W_8 = -1$ (dashed)

Akemann Damgaard Splittorff Verbaarschot PRD 83 (2011) 085014

Hansen Sharpe arXiv:1111.2404

The sign of W_6 and W_7

$$W_6 < 0$$

$W_6 > 0$: lattice theory where the spectrum of D_W can fluctuate vertically

Not allowed by γ_5 hermiticity !

The sign of W_6 and W_7

$$W_6 < 0$$

$W_6 > 0$: lattice theory where the spectrum of D_W can fluctuate vertically

Not allowed by γ_5 hermiticity !

$$W_7 < 0$$

$W_7 > 0$: lattice theory where the spectrum of D_5 can fluctuate vertically

Not allowed by γ_5 hermiticity !

Kieburg Splittorff Verbaarschot arXiv:1202.0620