Effective hot and dense QCD from strong coupling expansions

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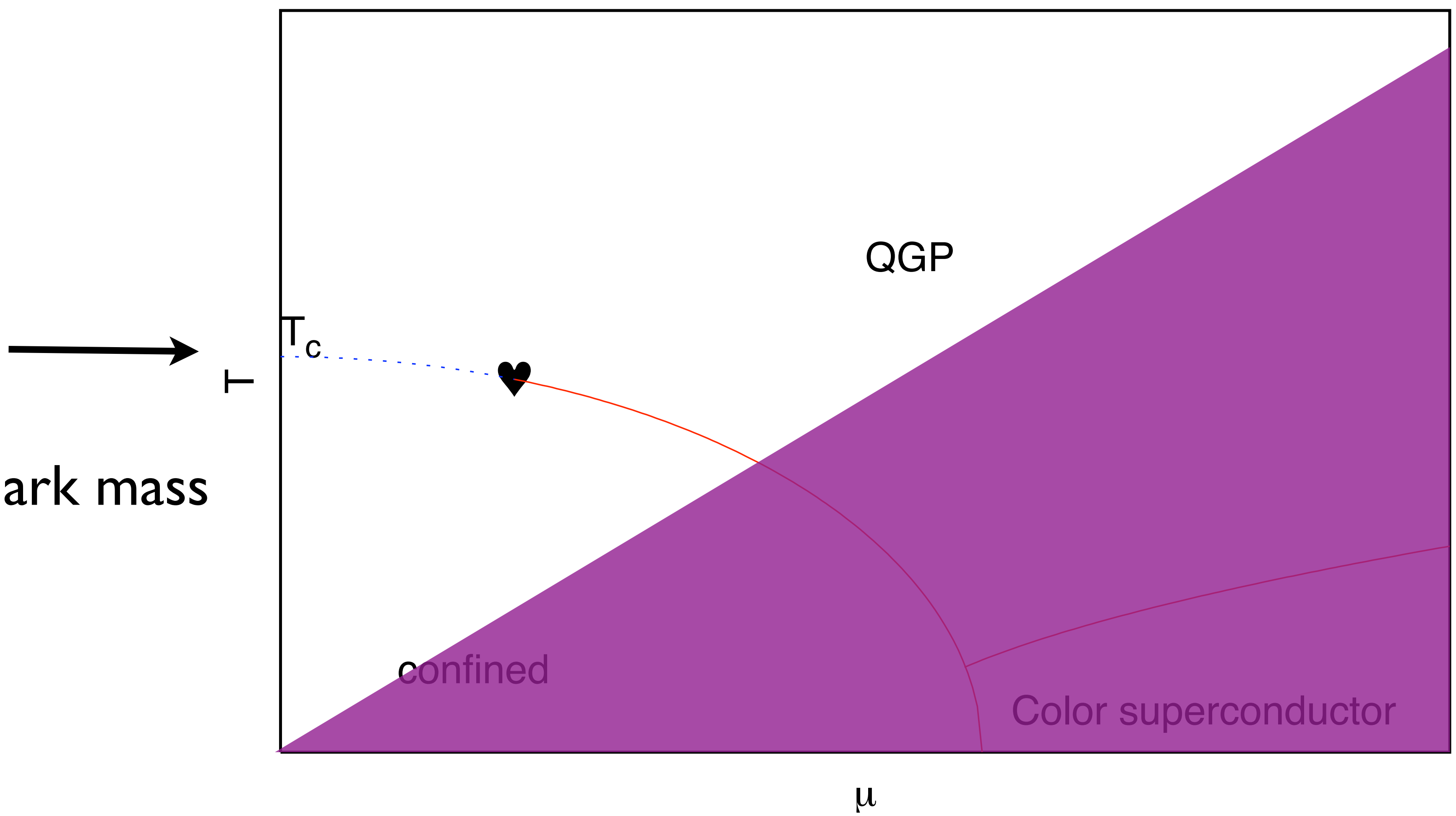
- Introduction: The QCD phase diagram
- The deconfinement transition in Yang-Mills theory
- The deconfinement transition in QCD with heavy dynamical quarks

in collaboration with M. Fromm, J. Langelage, S. Lottini

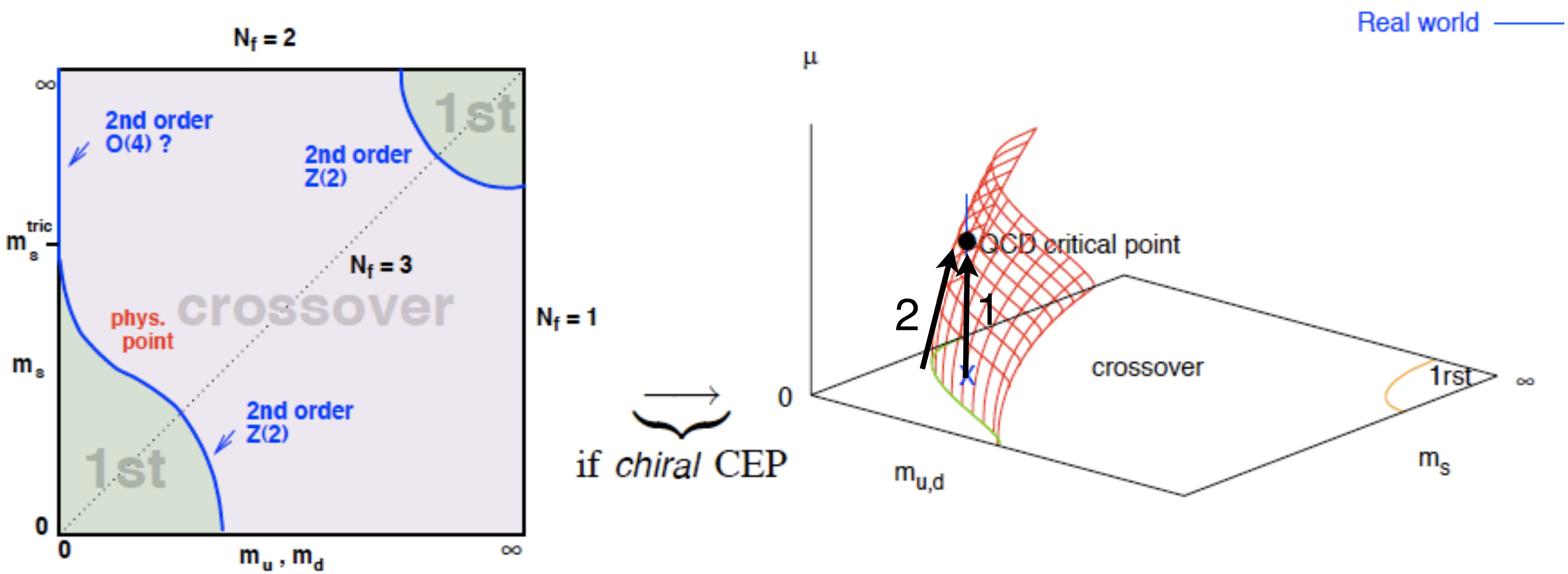
The (lattice) calculable region of the phase diagram

order of p.t.
at zero density
depends on Nf, quark mass



- Sign problem prohibits direct simulation, circumvented by approximate methods: reweigthing, Taylor expansion, imaginary chem. pot., need $\mu/T \lesssim 1 \quad (\mu = \mu_B/3)$
- Upper region: equation of state, screening masses, quark number susceptibilities etc. under control
- Here: phase diagram itself, so far based on models, most difficult!

Much harder: is there a QCD critical point?



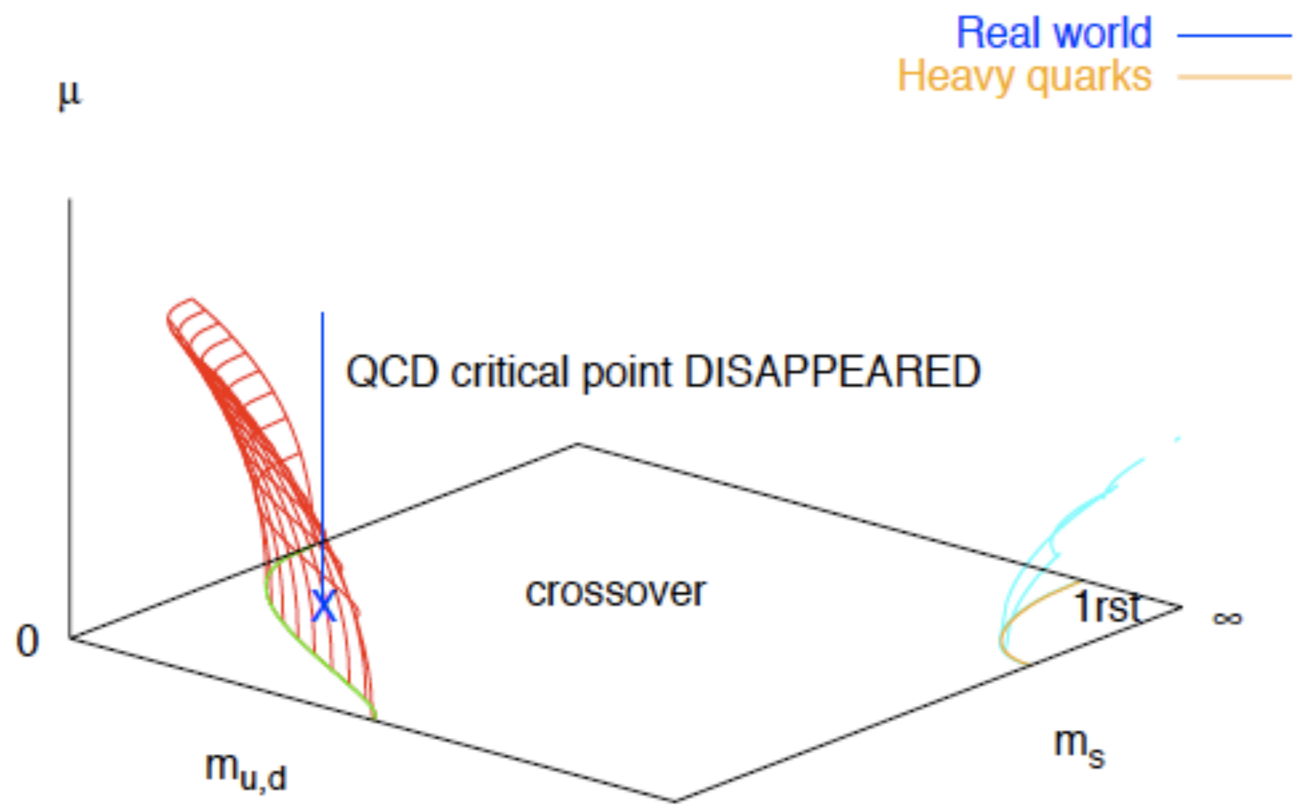
Two strategies:

1 follow vertical line: $m = m_{\text{phys}}$, turn on μ

2 follow critical surface: $m = m_{\text{crit}}(\mu)$

Some methods trying (1) give indications of critical point, but systematics not yet controlled

On coarse lattice exotic scenario: no chiral critical point at small density



de Forcrand, O.P. 08,09

Weakening of p.t. with chemical potential also for:

-Heavy quarks — Fromm, Langelage, Lottini, O.P. 11

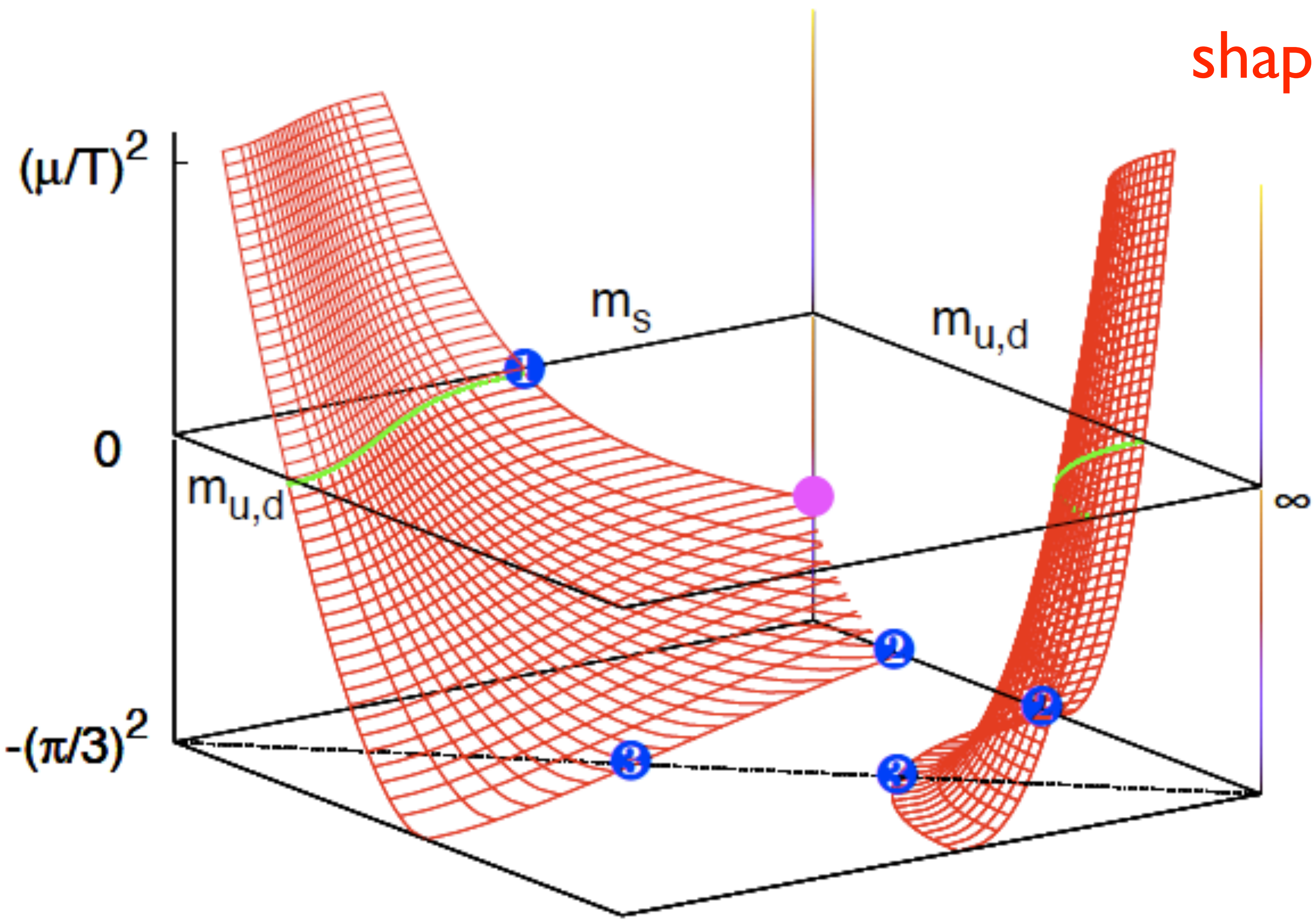
-Light quarks with finite isospin density — Kogut, Sinclair 07

-Electroweak phase transition with finite lepton density — Gynther 03

Continuation to imaginary chemical potential

No sign problem, computable by straightforward Monte Carlo

shape determined by tric. scaling!



Blue points tricritical, lower four have been calculated!

de Forcrand, O.P. 10
D'Elia, Sanfilippo 10

Large densities? Try effective theories!

- Example e.w. phase transition: success with dimensional reduction!
- Scale "separation"
  Integrate hard scale perturbatively, $g^2T < gT < 2\pi T$ y on lattice,
  valid for sufficiently weak coupling
- Does not work for the QCD transition, breaks Z(3) symmetry of Yang-Mills theory
- Bottom up construction of Z(N)-invariant theory by matching couplings: works for SU(2), not finished for SU(3)
  Vuorinen, Yaffe; de Forcrand, Kurkela; ....
- Here: solution by strong coupling expansion!

Starting point: Wilson's lattice YM action

Partition function; link variables as degrees of freedom

$$Z = \int \prod_{x,\mu} dU(x;\mu) \exp(-S_{YM}) = \int DU \exp(-S_{YM})$$

Wilson's gauge action

$$S_W = -\frac{\beta}{N} \sum_p \mathrm{ReTr}(U_p) = \sum_p S_p \qquad \beta = \frac{2N}{g^2}$$

Plaquette:

$$\square \rightarrow 1 + ia^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu} F^{\mu\nu} + O(a^6) + \ldots$$

$$U_\mu(x) = e^{-iagA_\mu(x)}$$

$$T = \frac{1}{aN_t} \qquad \text{continuum limit} \quad a \rightarrow 0, N_t \rightarrow \infty$$

Small $\beta(a) \Rightarrow$ small T

The strong coupling expansion

Expansion in irreducible characters $\chi_r(U) = \mathrm{Tr}\, D_r(U)$

$$\exp(-S_p) = c_0(\beta) \left\{ 1 + \sum_{r \neq 0} d_r c_r(\beta) \chi_r(U_p) \right\}$$

Expansion parameters $c_r(\beta)$ are combinations of modified Bessel functions (for $SU(N)$)

$$c_f \equiv u \sim \beta + \ldots$$
$$c_{ad} \sim \beta^2 + \ldots$$

Higher dimensional representations go with higher orders in β

Here: effective lattice theory, general strategy

- Start with the partition function of (3+1) dimensional lattice gauge field theory at finite temperature
- Integrate out degrees of freedom in order to have an effective action in terms of the order parameter (*here*: Polyakov loop)

$$-S_{eff} = \lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 + \ldots$$

- S_n depend only on Polyakov loops
- Find critical parameters $\lambda_{n,crit}$ and relate back to critical lattice couplings β_{crit} for different N_τ

$\longrightarrow$ Crucial to know mappings $\lambda_n(N_\tau, \beta)$

The effective theory for SU(2)

- Split temporal and spatial link integration and use character expansion ($c_r(\beta)$: expansion parameter of representation r)

$$
\begin{aligned}
Z &= \int [dW] \exp\left\{ \ln \int [dU_i] \prod_p \left[1 + \sum_{r \neq 0} d_r c_r(\beta) \chi_r(U_p) \right] \right\} \\
&\equiv \int [dW] \exp\left[-S_{eff} \right] \qquad\qquad W(\vec{x}) = \prod_{\tau=1}^{N_\tau} U_0(\tau, \vec{x})
\end{aligned}
$$

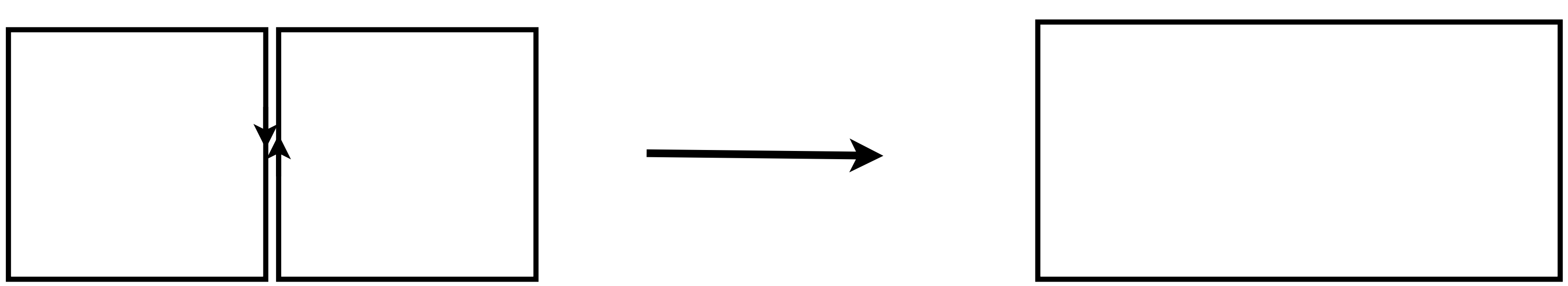
Integration rule 1

$$\int dU \chi_r(U) = \delta_{r,0}$$

Integration rule 2

$$\int dU \chi_r(UV)\chi_s(U^{-1}W) = \delta_{rs}\frac{1}{d_r}\chi_r(VW)$$

Used to perform the occurring group integrations



- Leading order graph in case of $N_\tau = 4$:

Figure: 4 plaquettes in fundamental representation lead to a 2 Polyakov loop interaction term

- Integration of spatial link variables leads to

$$-S_1 = u^{N_\tau} \sum_{\langle ij \rangle} \mathrm{tr}\, W_i \, \mathrm{tr}\, W_j$$

- Possible generalizations: larger distance, higher dimensional representations, larger number of loops involved, ...
- *Here*: Decorate LO graph with additional spatial and temporal plaquettes

- Z(2) symmetric 3 dimensional partition function

$$Z = \int [dW] \exp\left[\lambda_1 \sum_{\langle ij \rangle} \mathrm{tr}\, W_i \,\mathrm{tr}\, W_j\right]$$

- Can be further simplified by using $L \equiv \mathrm{tr}\, W$ as degrees of freedom: ordinary integration instead of group integration
- Introduces potential term: $V_{SU(2)} = \frac{1}{2}\sum_i \ln\left[4 - L_i^2\right]$

$$Z = \int [dL] \exp\left[\lambda_1 \sum_{\langle ij \rangle} L_i L_j + \frac{1}{2}\sum_i \ln\left[4 - L_i^2\right]\right]$$

$$\lambda_1(u, N_\tau \geq 5) \;=\; u^{N_\tau} \exp\left[N_\tau\left(4u^4 - 4u^6 + \frac{140}{3}u^8 - \frac{36044}{405}u^{10}\right)\right]$$

- **One** determination of $\lambda_{1,crit}$ gives all $\beta_{crit}(N_\tau)$

- Subclass of higher order interaction terms (Powers of the leading order term) arrange schematically as

$$-S_{eff} = \lambda_1(LL) - \frac{\lambda_1^2}{2}(LL)^2 + \frac{\lambda_1^3}{3}(LL)^3 - \ldots = \ln\left[1 + \lambda_1(LL)\right]$$

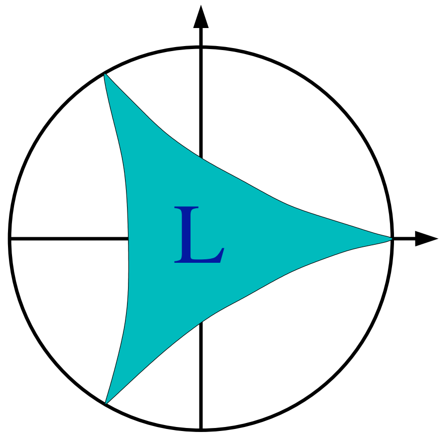
- SU(2) effective theory to be simulated

$$Z = \int [dL] \prod_i \sqrt{4 - L_i^2} \prod_{\langle ij \rangle} \left[1 + \lambda_1 L_i L_j\right]$$

Generalisation to SU(3)

- SU(3) straightforward, but: Now also with anti-fundamental representation (i.e. L_i are complex)

$$
\begin{aligned}
Z &= \int [dL] \exp\left[-S_1 + V_{SU(3)}\right] \\
&= \int [dL] \prod_{\langle ij \rangle} \left[1 + 2\lambda_1 \mathrm{Re}\left(L_i L_j^*\right)\right] * \\
&\quad * \prod_i \sqrt{27 - 18|L_i|^2 + 8\mathrm{Re} L_i^3 - |L_i|^4}
\end{aligned}
$$



- Functional form of $\lambda_1(N_\tau, u)$ and next-to-nearest-neighbour effects are analogous to SU(2)

Subleading couplings

Subleading contributions for next-to-nearest neighbours:

$$\lambda_2 S_2 \quad \propto \quad u^{2N_\tau+2} \sum_{[kl]}' 2\mathrm{Re}(L_k L_l^*) \quad \text{distance} \; = \sqrt{2}$$

$$\lambda_3 S_3 \quad \propto \quad u^{2N_\tau+6} \sum_{\{mn\}}'' 2\mathrm{Re}(L_m L_n^*) \quad \text{distance} \; = 2$$

as well as terms from loops in the *adjoint* representation:

$$\lambda_a S_a \propto u^{2N_\tau} \sum_{<ij>} \mathrm{Tr}^{(a)} W_i \mathrm{Tr}^{(a)} W_j \quad ; \quad \mathrm{Tr}^{(a)} W = |L|^2 - 1$$

Numerical evaluation of effective theories

Monte Carlo simulation of scalar model, Metropolis update

Search for criticality:

Binder cumulant: $B(|L|) = 1 - \frac{\langle |L|^4 \rangle}{3\langle |L|^2 \rangle^2} \rightarrow \lambda_{1,c}(N_s)$ is the minimum

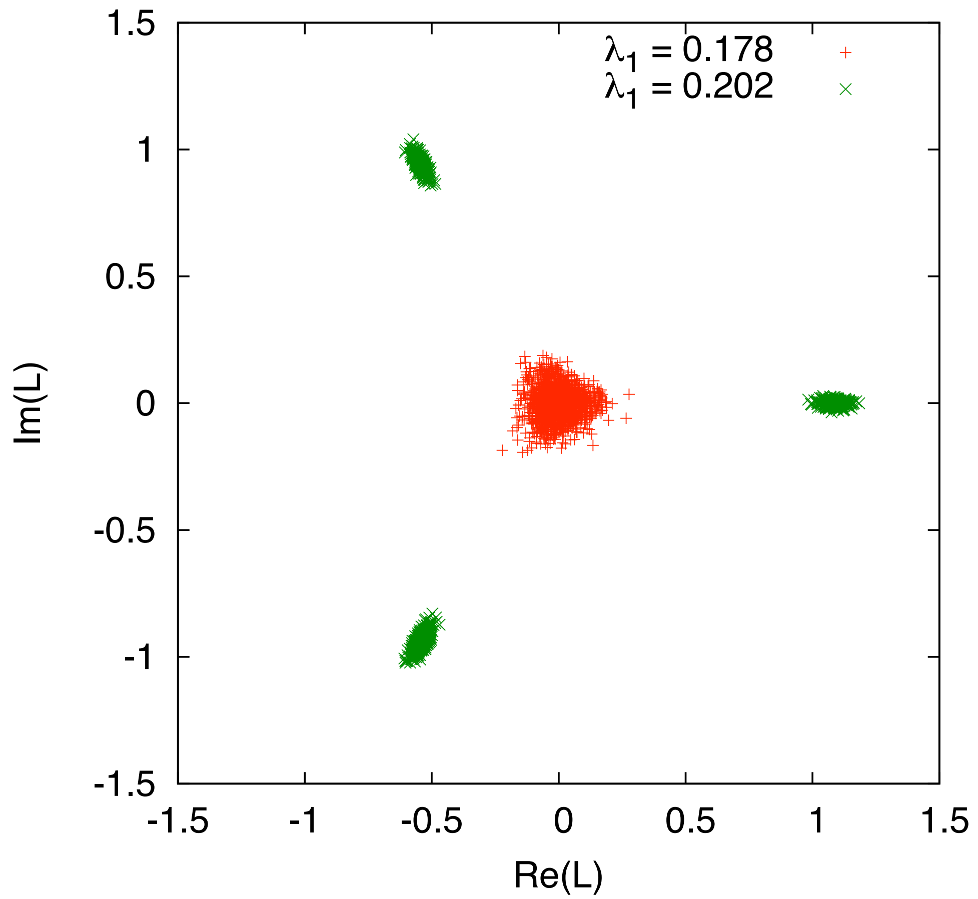
Susceptibility: $\chi(|L|) = \left\langle \left(|L| - \langle |L| \rangle \right)^2 \right\rangle \rightarrow \lambda_{1,c}(N_s)$ is the maximum

Finite size scaling: $\lambda_{1,c}(N_s) = \lambda_{1,c} + bN_s^{-1/\nu}$

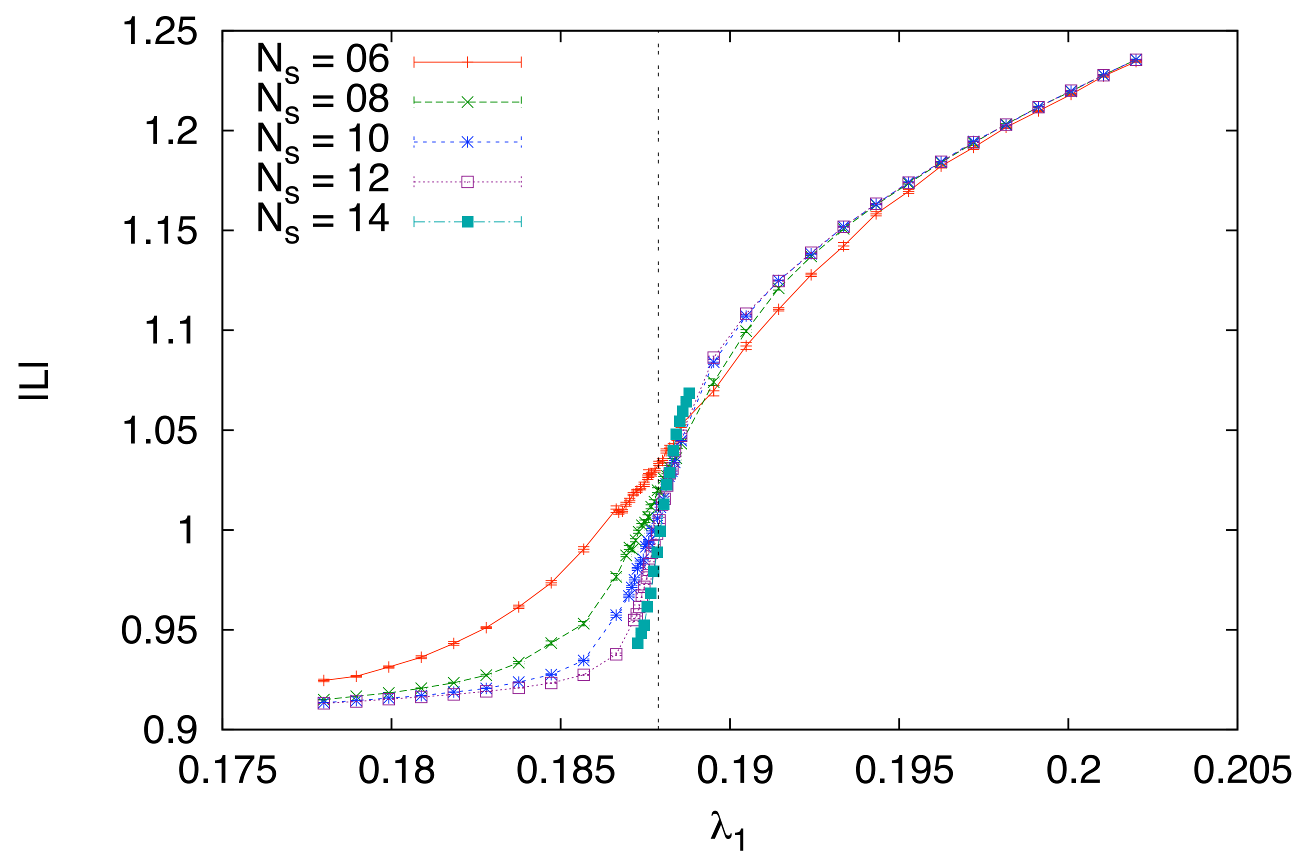
$\nu = 1/3$ for the 1st order $SU(3)$, ν_{Ising3D} for $SU(2)$.

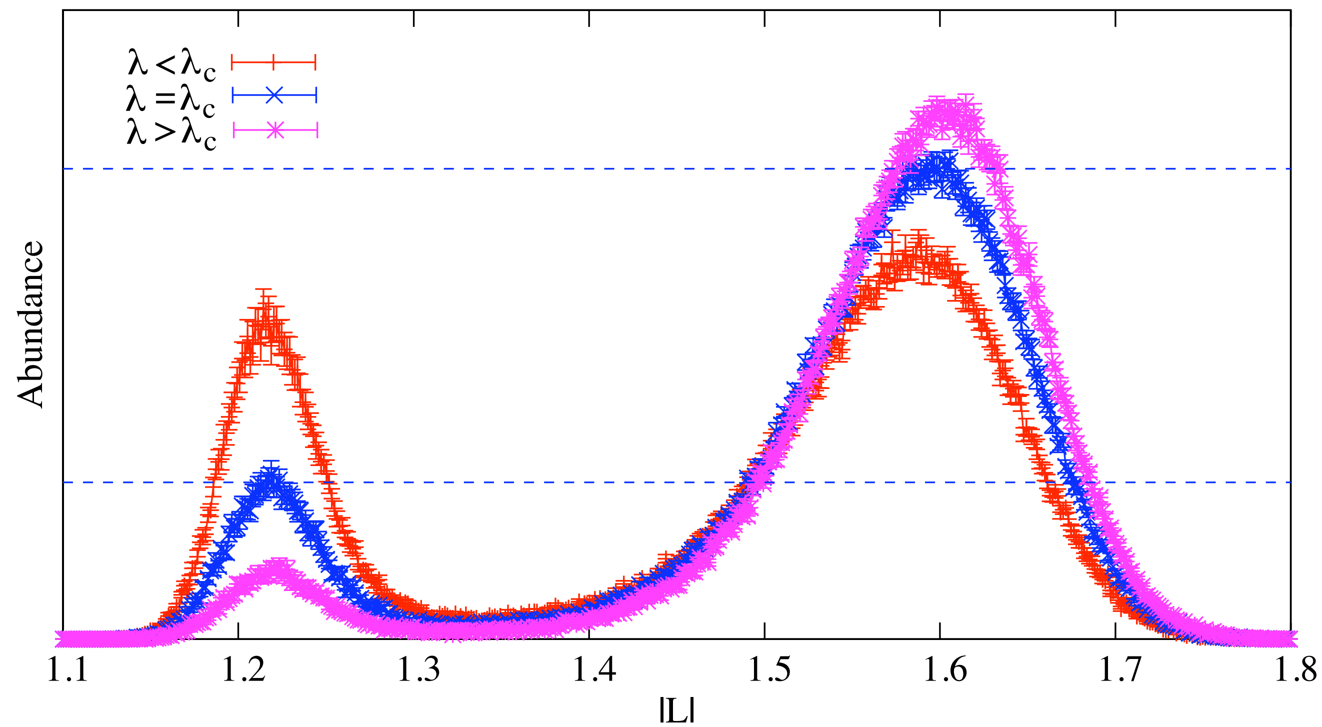
Typical sizes range from $N_s = 6$ to 16; time needed is of order **a few days** on an ordinary PC.

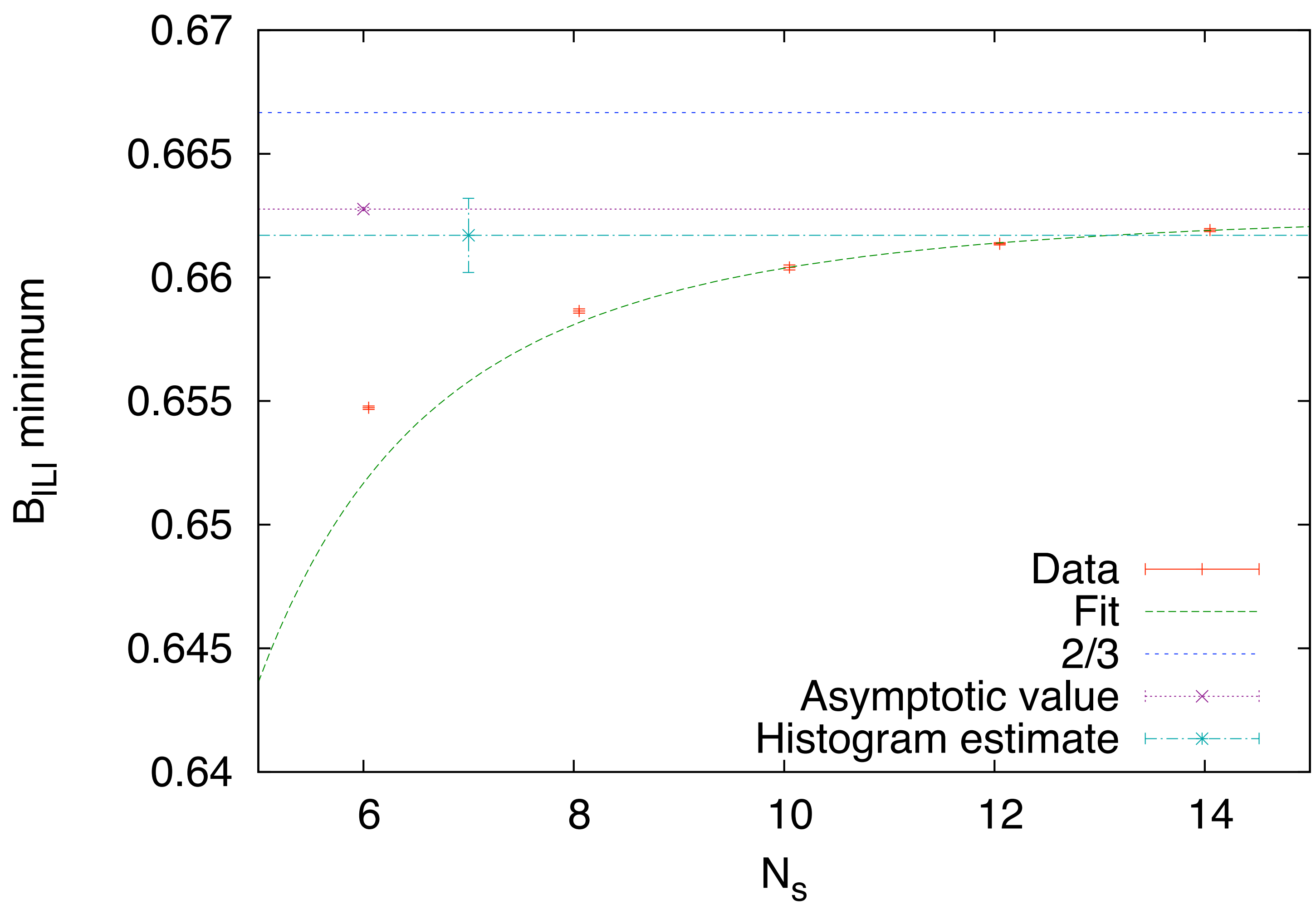
Numerical results for SU(3)



Order-disorder transition



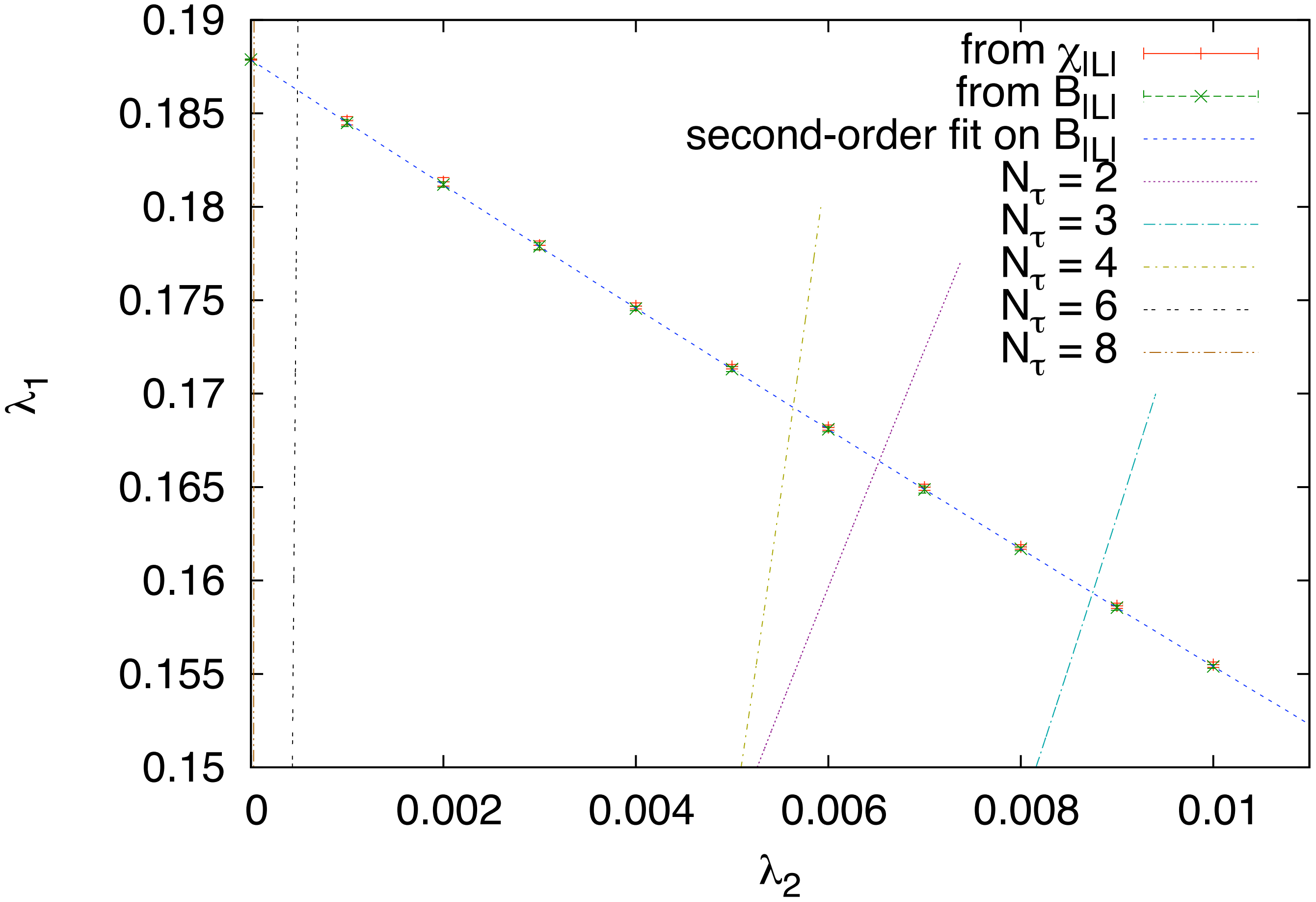


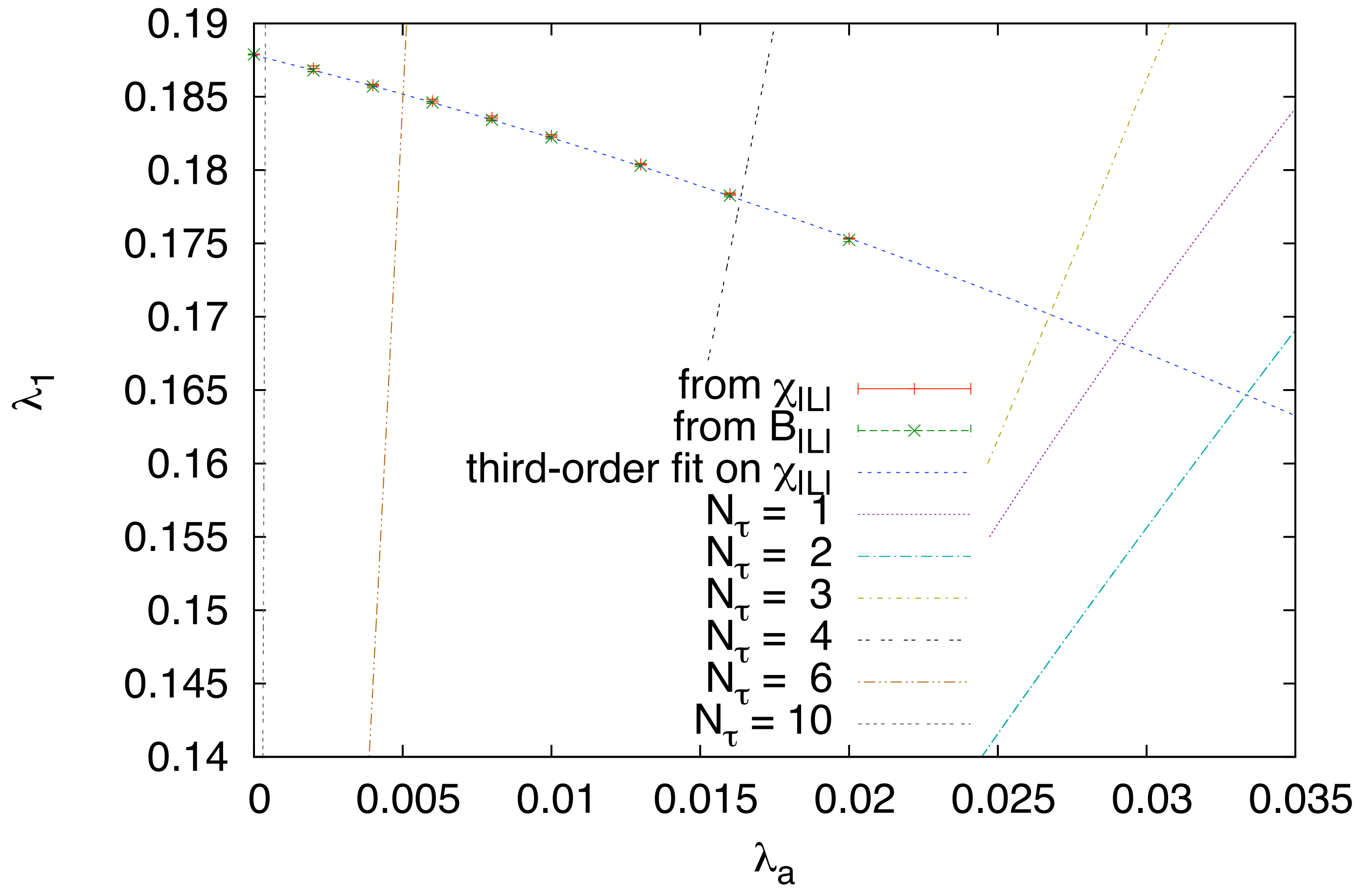


First order phase transition for SU(3) in the thermodynamic limit!

The influence of a second coupling

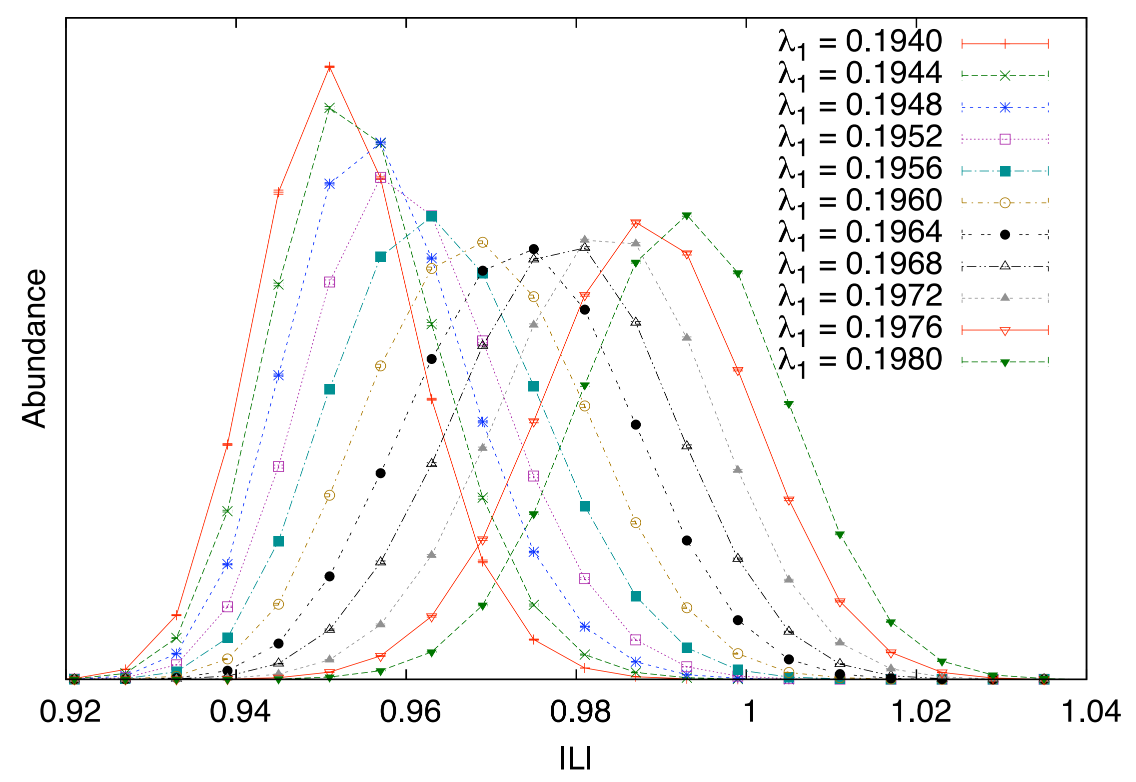
NLO-couplings: next-to-nearest neighbour, adjoint rep. loops

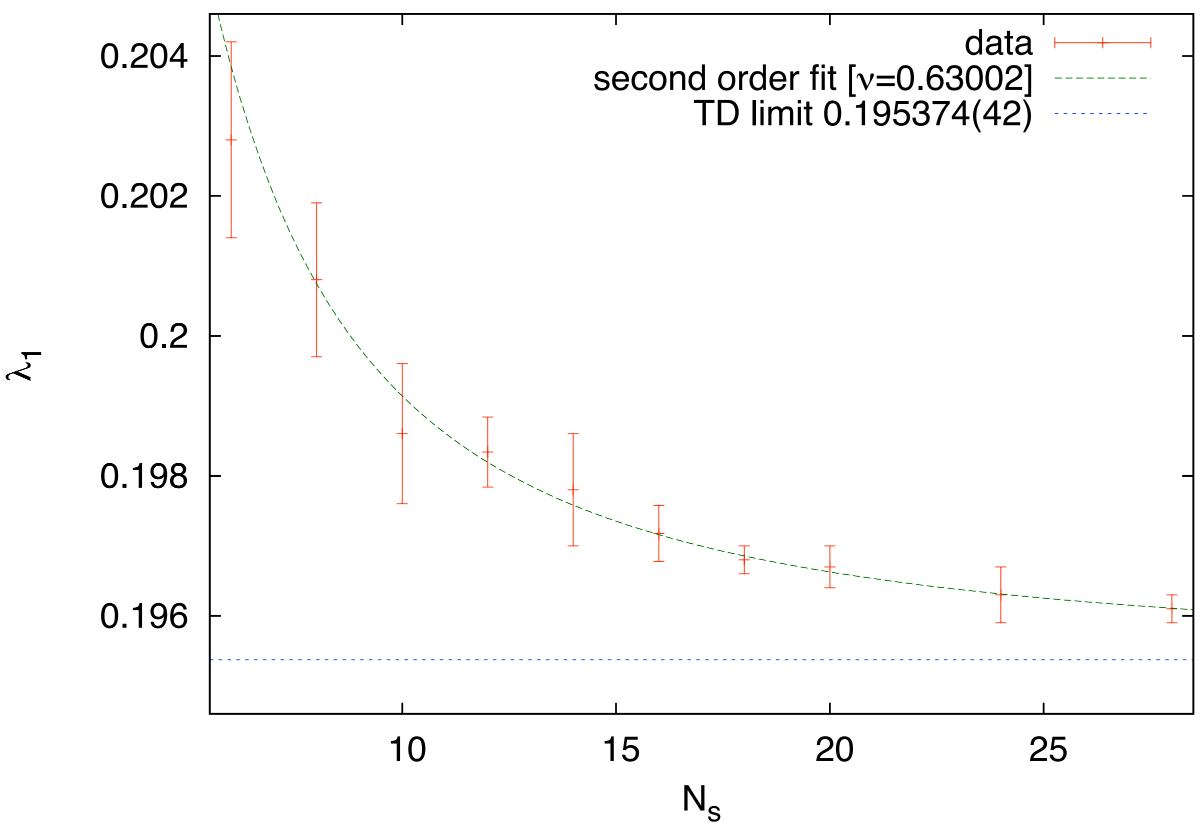




...gets very small for large N_τ !

Numerical results for SU(2), one coupling





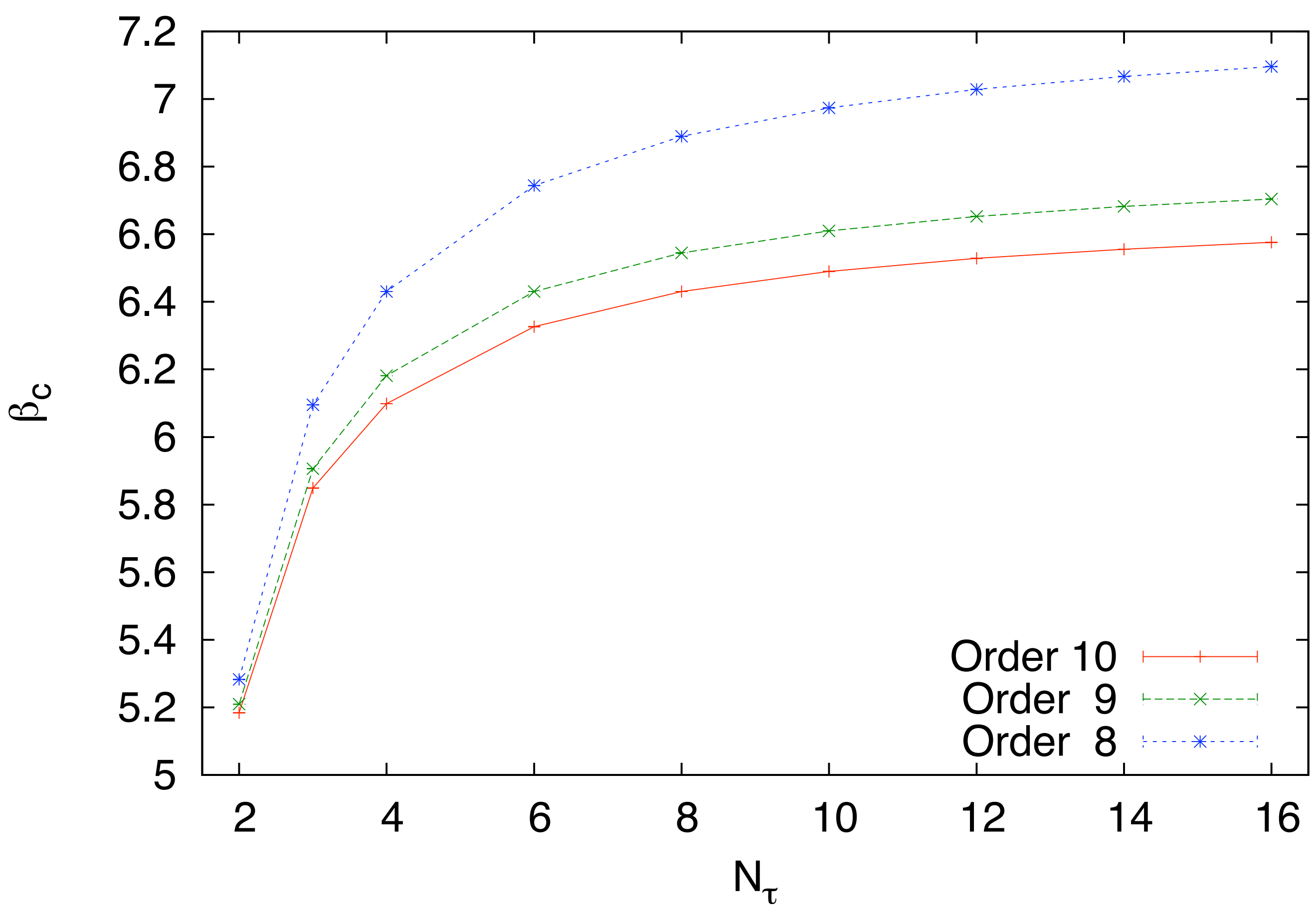
Second order (3d Ising) phase transition for SU(2) in the thermodynamic limit!

Mapping back to 4d finite T Yang-Mills

Inverting

$$\lambda_1(N_\tau, \beta) \rightarrow \beta_c(\lambda_{1,c}, N_\tau)$$

...points at reasonable convergence

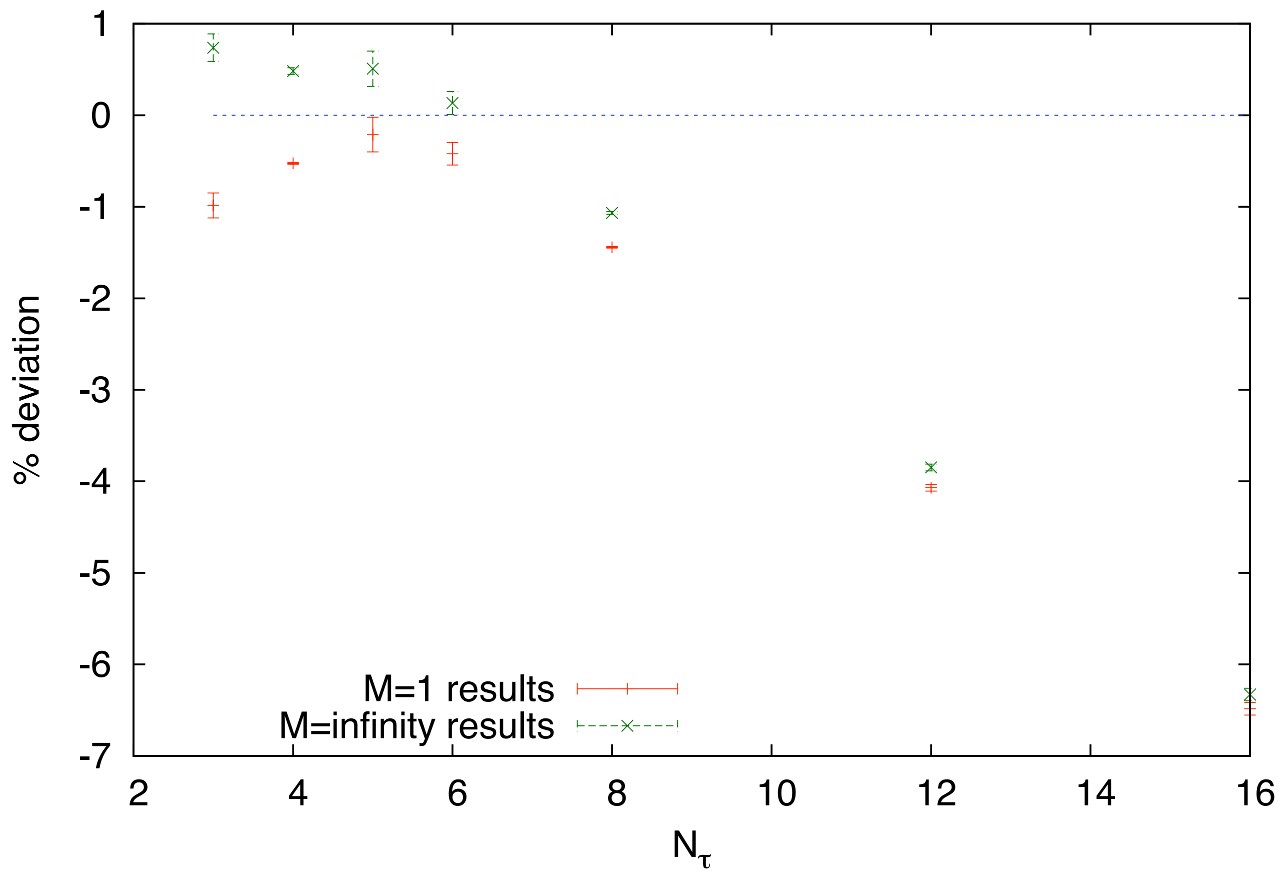


SU(3)

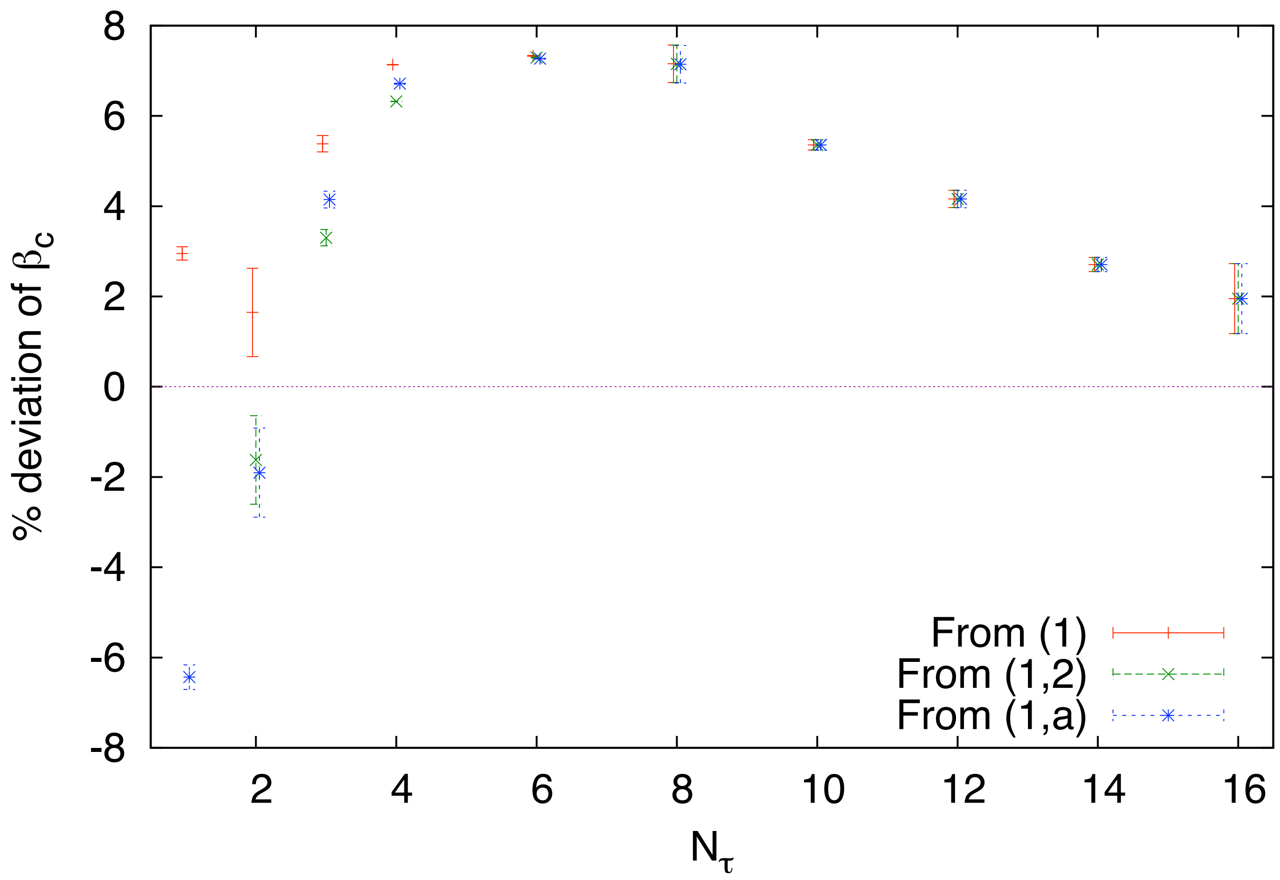
Comparison with 4d Monte Carlo

Relative accuracy for β_c compared to the full theory

SU(2)

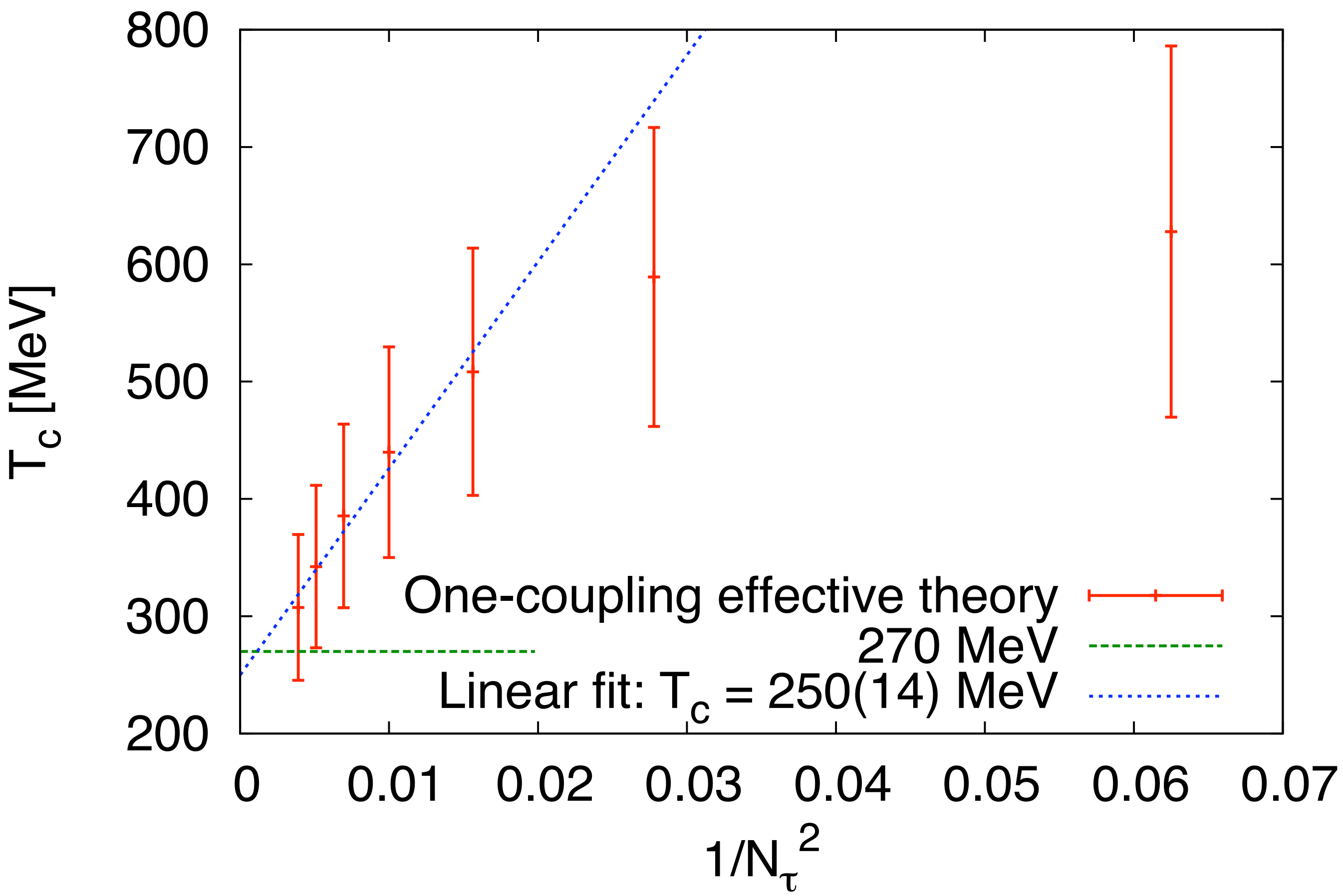


SU(3)



Note: influence of some couplings checked explicitly!

Continuum limit feasible!



-error bars: difference between last two orders in strong coupling exp.

-using non-perturbative beta-function (4d T=0 lattice)

-all data points from one single 3d MC simulation!

Including heavy, dynamical Wilson fermions

N_f (degenerate) fermions $\Longrightarrow \quad S = S_{\rm gauge} + S_q[U, \psi, \overline{\psi}]$

$$S_q = \sum_{x,y;f} \overline{\psi}_{f,y} \Big(\mathbb{1} - \kappa H[U] \Big)_{yx} \psi_{f,x} \quad , \quad H[U]_{yx} = \sum_{\pm\mu} \delta_{y,x+\hat{\mu}} (\mathbb{1} + \gamma_\mu) U_{x,\mu}$$

Integrate the Grassmann variables $\psi, \overline{\psi}$:

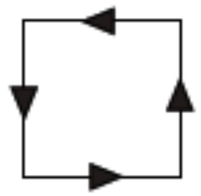
$$S = S_{\rm gauge} - N_f \mathrm{Tr} \log(\mathbb{1} - \kappa H)$$

Expand in the *hopping parameter* $\kappa = 1/(2aM + 8)$: [*]

$$S = S_{\rm gauge} + N_f \sum_{\ell=1}^{\infty} \frac{\kappa^\ell}{\ell} \mathrm{Tr} H[U]^\ell$$

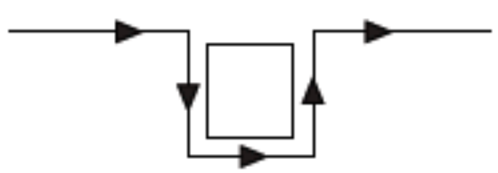
Similar to de Pietri, Feo, Seiler, Stamatescu 07

Links along imaginary time gain $\exp(\pm\mu a)$



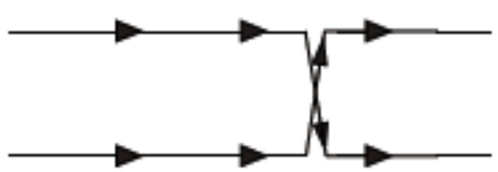
reabsorbed in gauge part: $\left\{ \begin{array}{l} \beta \to \beta + \mathcal{O}(\kappa^4) \\ u(\beta) \to u(\beta, \kappa) \end{array} \right.$

LO Polyakov "magnetic" term $\sim \left\{ \begin{array}{l} \underbrace{(2\kappa e^{+a\mu})^{N_\tau}}_{h_1} L \\ \underbrace{(2\kappa e^{-a\mu})^{N_\tau}}_{\overline{h}_1} L^* \end{array} \right.$



higher corrections to the above:

$$h_1 = (2\kappa e^{a\mu})^{N_\tau} \left[1 + \mathcal{O}(k^2) f(u) + \ldots \right]$$



other (suppressed) terms, such as $h_2(L_x L_{x+\hat{i}})$,

$$h_2 \sim (2\kappa e^{a\mu})^{2N_\tau} \kappa^2 \ldots$$

In general the model becomes (with $\overline{h}_i(\mu) = h_i(-\mu)$)

$$-S_{\rm eff} = \sum_i \lambda_i(u,\kappa,N_\tau) S_i^{\rm S} - 2N_f \sum_i \left[h_i(u,\kappa,\mu,N_\tau) S_i^{\rm A} + \overline{h}_i(u,\kappa,\mu,N_\tau) S_i^{\dagger \rm A} \right]$$

Now, keep only $\lambda_1 S_1^{\rm S}$ and $h_1 S_1^{\rm A} + \overline{h}_1 S_1^{\dagger \rm A}$ (now called just λ, h)

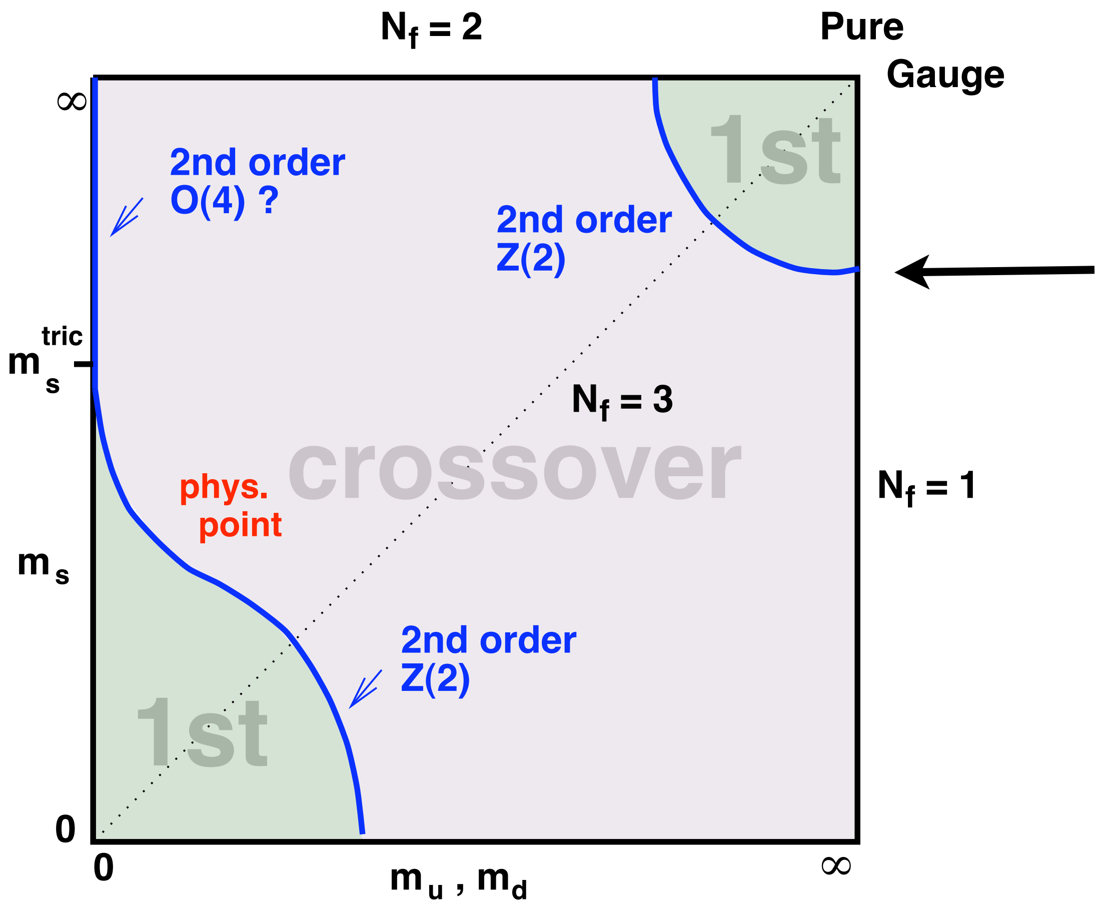
Higher powers of loops are resummed into a determinant:

$$\begin{aligned} Z_{\rm eff}(\lambda_1, h_1, \overline{h}_1; N_\tau) &= \int [\mathrm{d}L] \Big(\prod_{<ij>} [1 + 2\lambda_1 \mathrm{Re} L_i L_j^*] \Big) \\ &\quad \Big(\prod_x \underbrace{\det[(1 + h_1 W_x)(1 + \overline{h}_1 W_x^\dagger)]^{2N_f}}_{\equiv Q(L_x, L_x^*)^{N_f}} \Big) \end{aligned}$$

No chemical potential ($h = \overline{h}$): the full (λ, h)-model has then

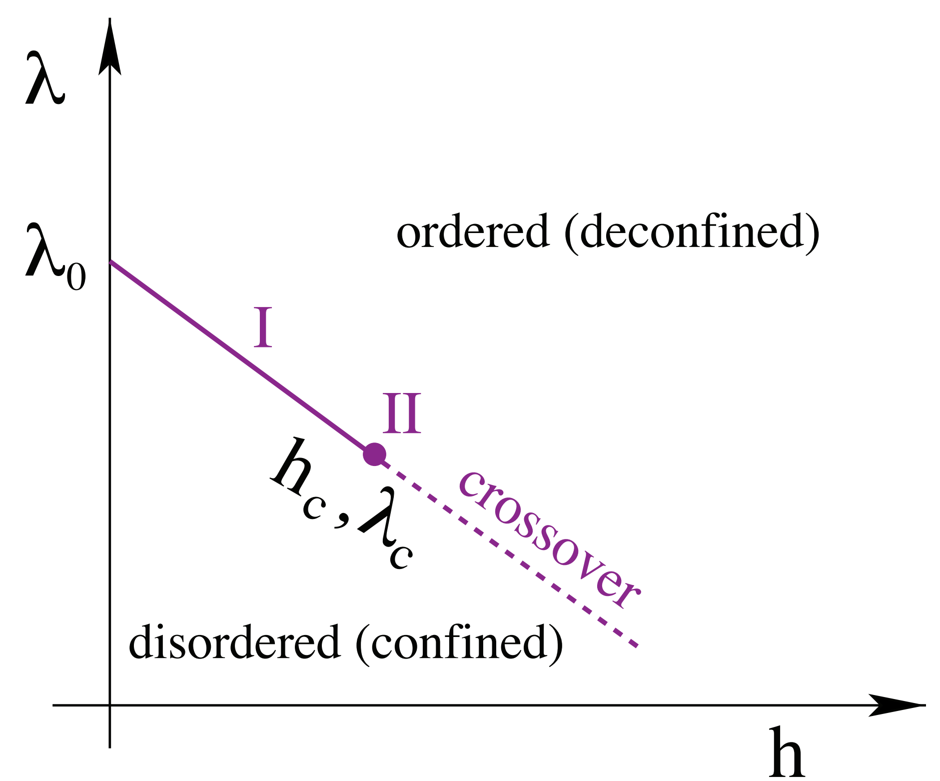
- a “spin-spin” interaction between neighbour Polyakov loops
- a “magnetic-field” term acting on sites

QCD: first order deconfinement transition region



deconfinement p.t.:
breaking of global $Z(3)$ symmetry;
explicitly broken by quark masses
transition weakens

Phase diagram in eff. theory:



Phase boundary at zero density

First order transition: coexistence of phases

Free energies in the two phases: $f_d(\lambda, h)$, $f_c(\lambda, h)$.
small $h, \lambda - \lambda_0 \Rightarrow$ expand: [Alford, Chandrasekharan, Cox, Wiese, '01]

$$\begin{aligned} f(\lambda, h) &= f(\lambda_0, 0) + \left(\partial_h f|_{\lambda_0,0}\right) \cdot h + \left(\partial_\lambda f|_{\lambda_0,0}\right) \cdot (\lambda - \lambda_0) \\ f_c &= f_0 + (\partial_\lambda f_c)\Delta\lambda \\ f_d &= f_0 + \langle L \rangle h + (\partial_\lambda f_d)\Delta\lambda \end{aligned}$$

Pseudo-critical λ_{pc} means $f_c = f_d$:

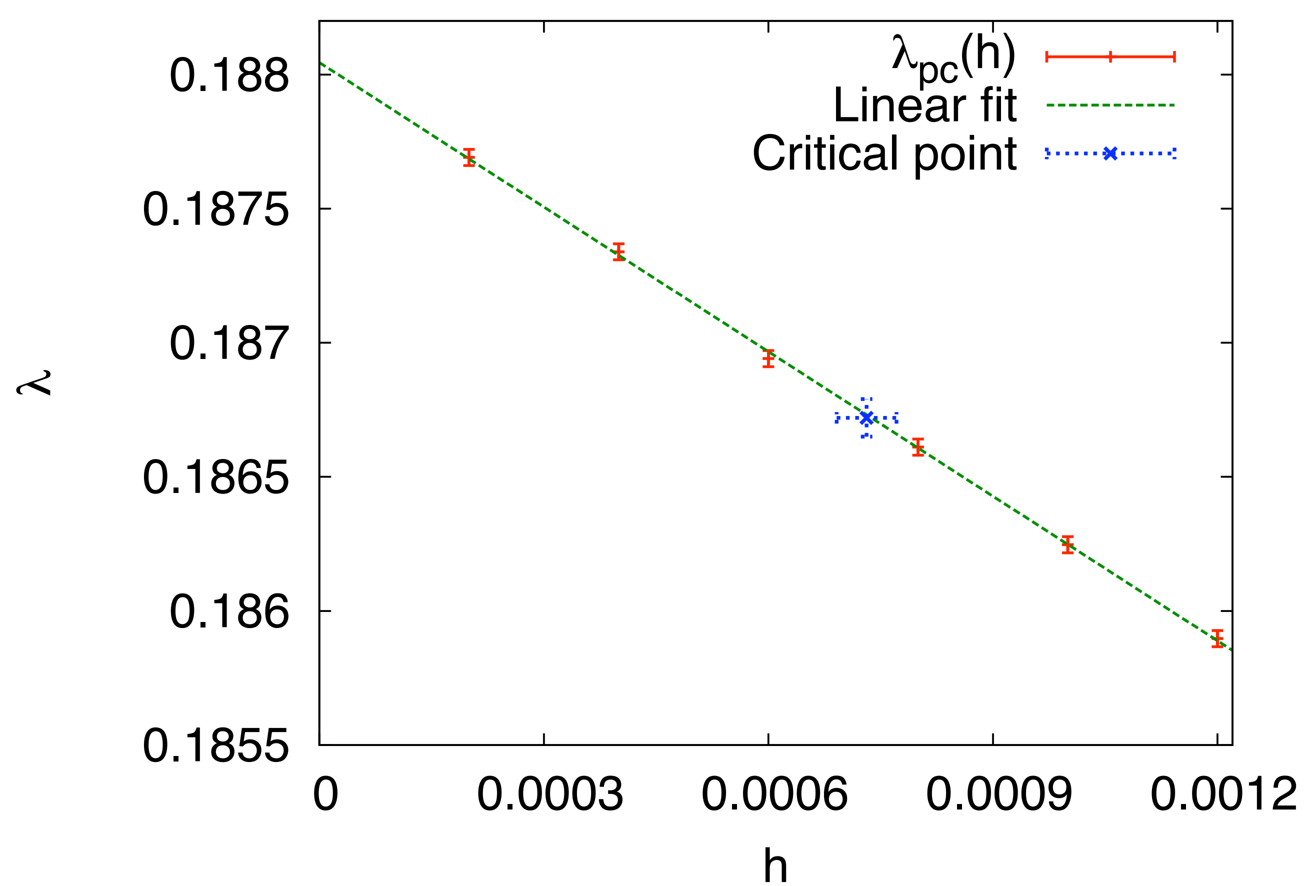
$$\langle L \rangle h = (\lambda_{pc} - \lambda_0)[\partial_\lambda (f_c - f_d)|_{\lambda_0,0}]$$

expectation: $\lambda_{pc}(h) = \lambda_0 - a_1 h$

Phase boundary, numerically

To find $\lambda_{pc}(h)$, λ-scans were performed at various fixed h

- peak in $\chi_O = \langle O^2 \rangle - \langle O \rangle^2$
- dip in $B_O = 1 - \frac{\langle O^4 \rangle}{3\langle O^2 \rangle^2}$

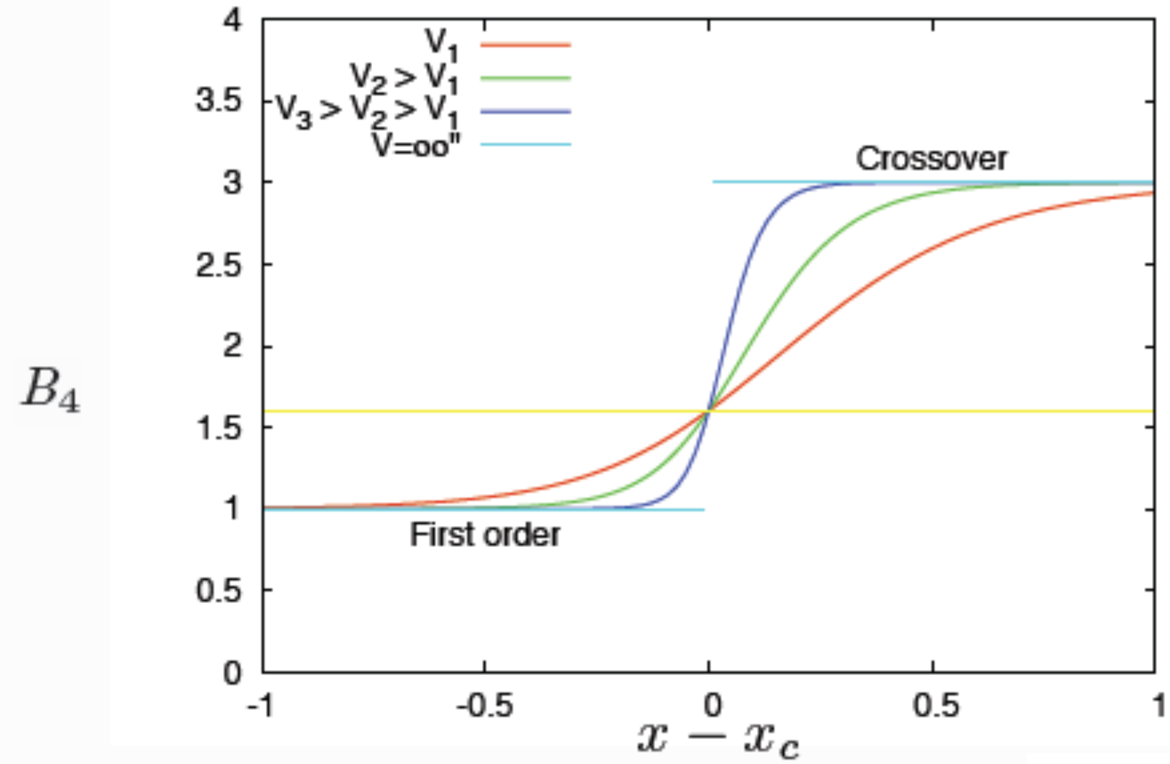


$$\lambda_{pc}(h) = 0.18805 - 1.797 \cdot h$$

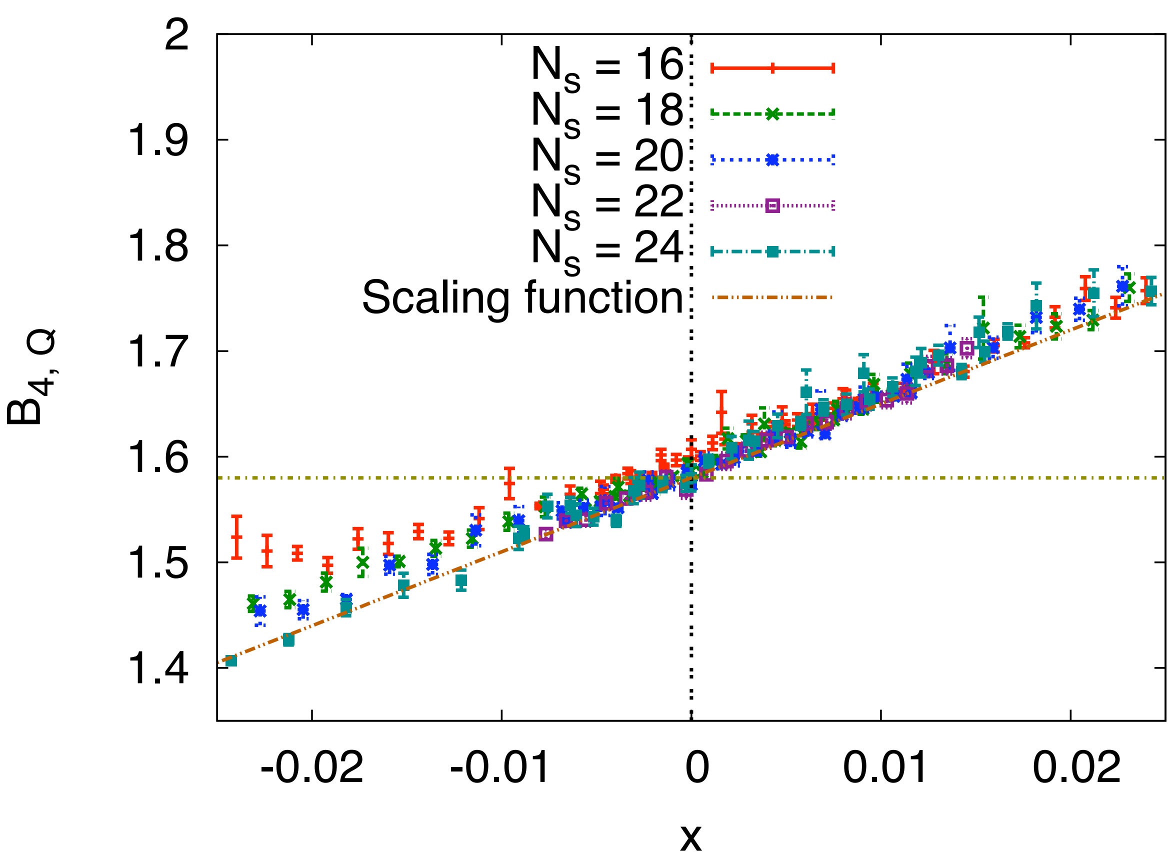
Observable to identify order of p.t.:

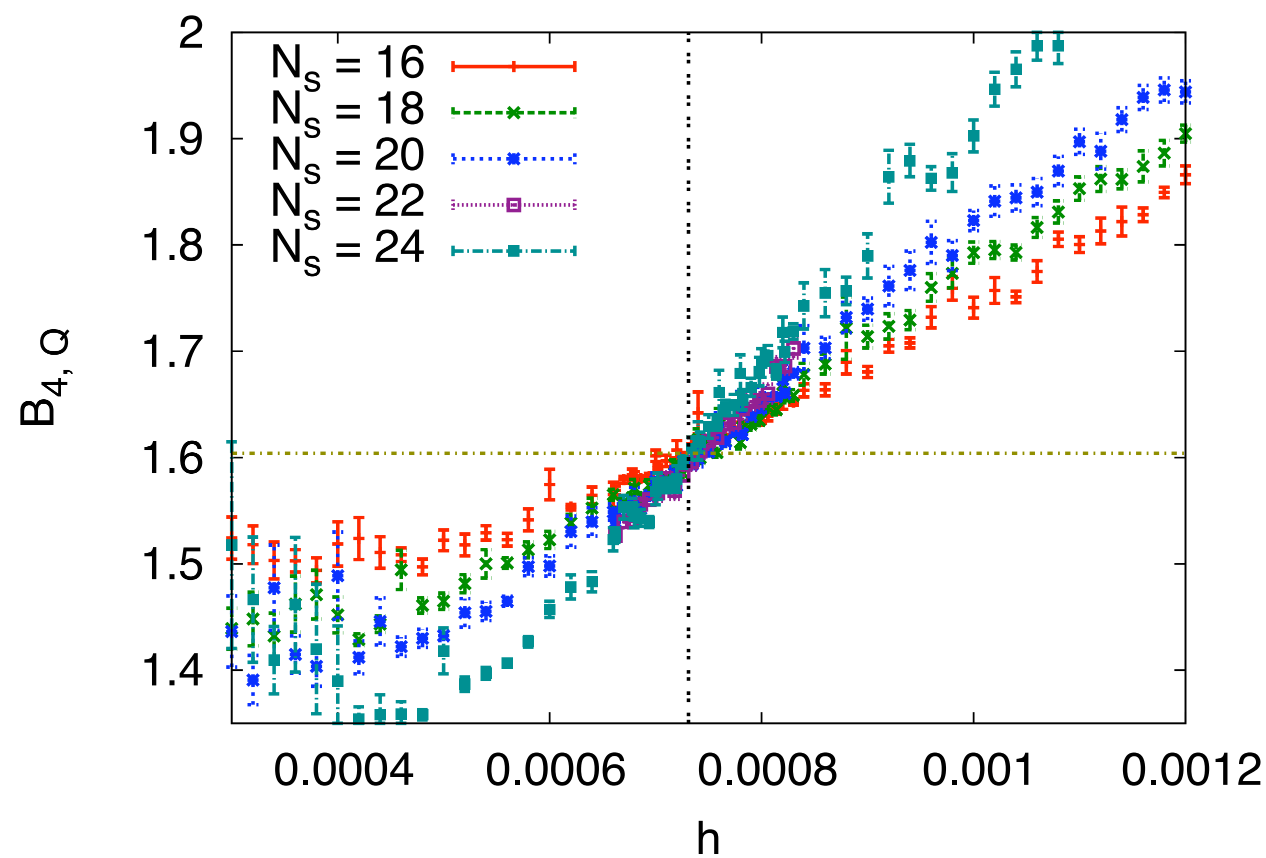
$$\delta B_Q = B_4(\delta Q) = \frac{\langle(\delta Q)^4\rangle}{\langle(\delta Q)^2\rangle^2}$$

$$B_4(x) = 1.604 + bL^{1/\nu}(x - x_c) + \ldots$$



parameter along phase boundary





The critical point

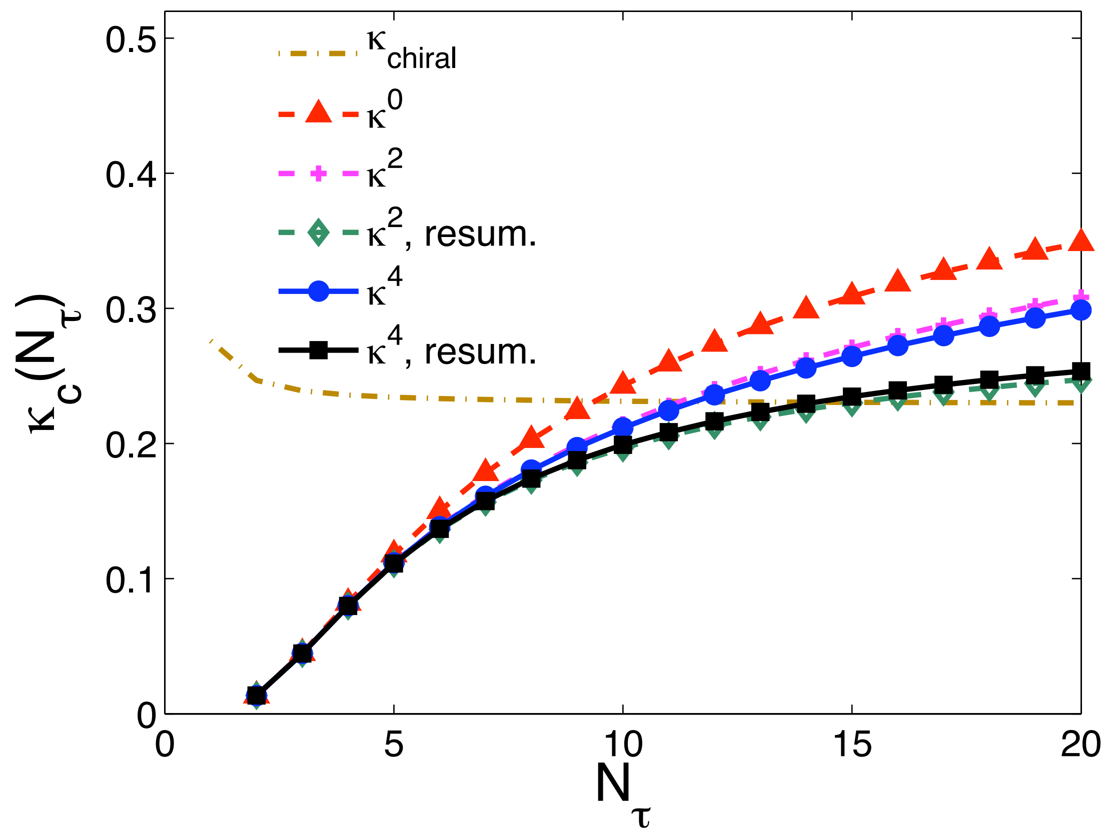
$\lambda_c = 0.18672(7), h_c = 0.000731(40)$

Mapping back to QCD:

N_f	M_c/T	$\kappa_c(N_\tau = 4)$	$\kappa_c(4)$, Ref. [23]	$\kappa_c(4)$, Ref. [22]
1	7.22(5)	0.0822(11)	0.0783(4)	~ 0.08
2	7.91(5)	0.0691(9)	0.0658(3)	–
3	8.32(5)	0.0625(9)	0.0595(3)	–

$e^{-M/T} \simeq h/N_f$ [linear approximation in $h \ll 1 \ldots$]

Convergence properties:



Finite density: sign problem!

- Metropolis algorithm: Mild sign problem; $\frac{\mu}{T} \lesssim 3$
- Worm algorithm: No sign problem cf. Gattringer et al.

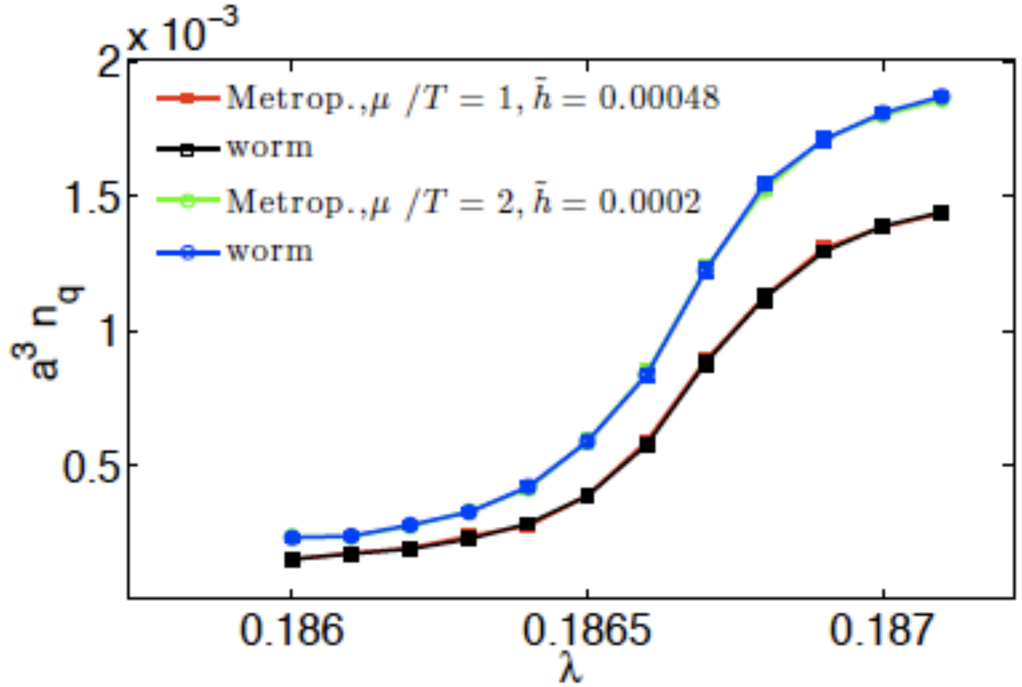


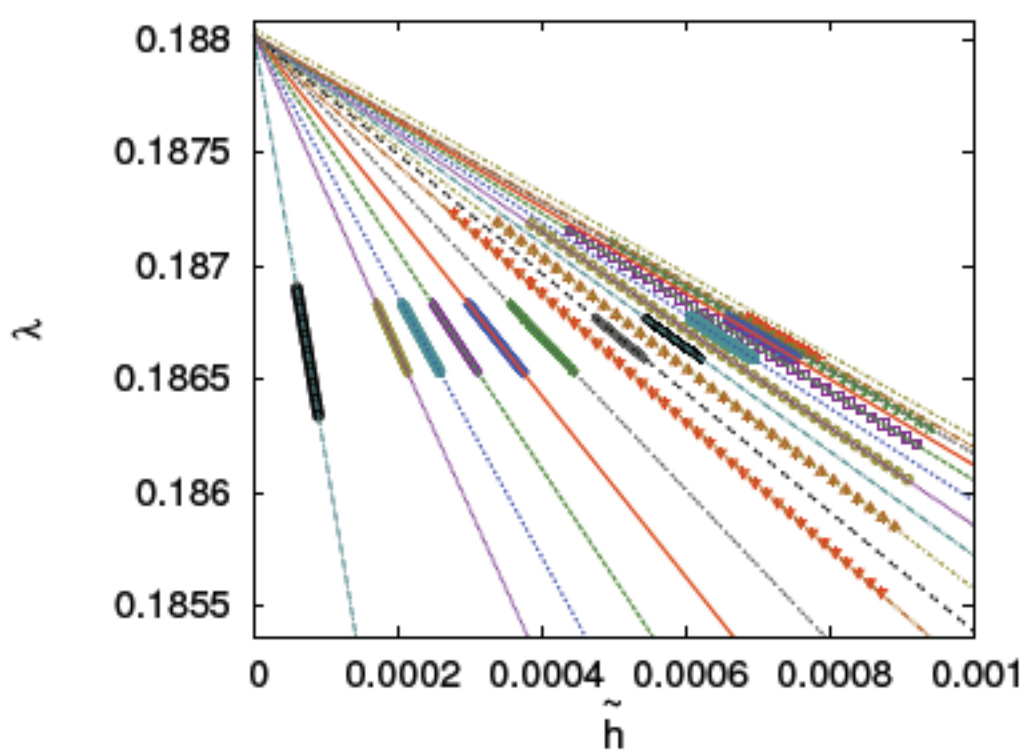
Figure: Quark density calculated with Z_{eff} from Metropolis or worm algorithm on 24^3 lattices for $\frac{\mu}{T} = 1$ and 2.

Phase boundary at finite density

- Introduce $\widetilde{h} \equiv h e^{-\mu/T}$ (which is $\sim (2\kappa)^{N_\tau} + \cdots$)
- Phase space is now $(\lambda, h, \overline{h})$, or: $(\lambda, \widetilde{h}, \mu/T)$

Straight pseudo-critical lines again: $\lambda_{pc}(\widetilde{h}, \mu/T) = \lambda_0 + a_1(\mu/T)\widetilde{h}$

free energy expansion $\Rightarrow \quad a_1 = C\cosh(\mu/T)$



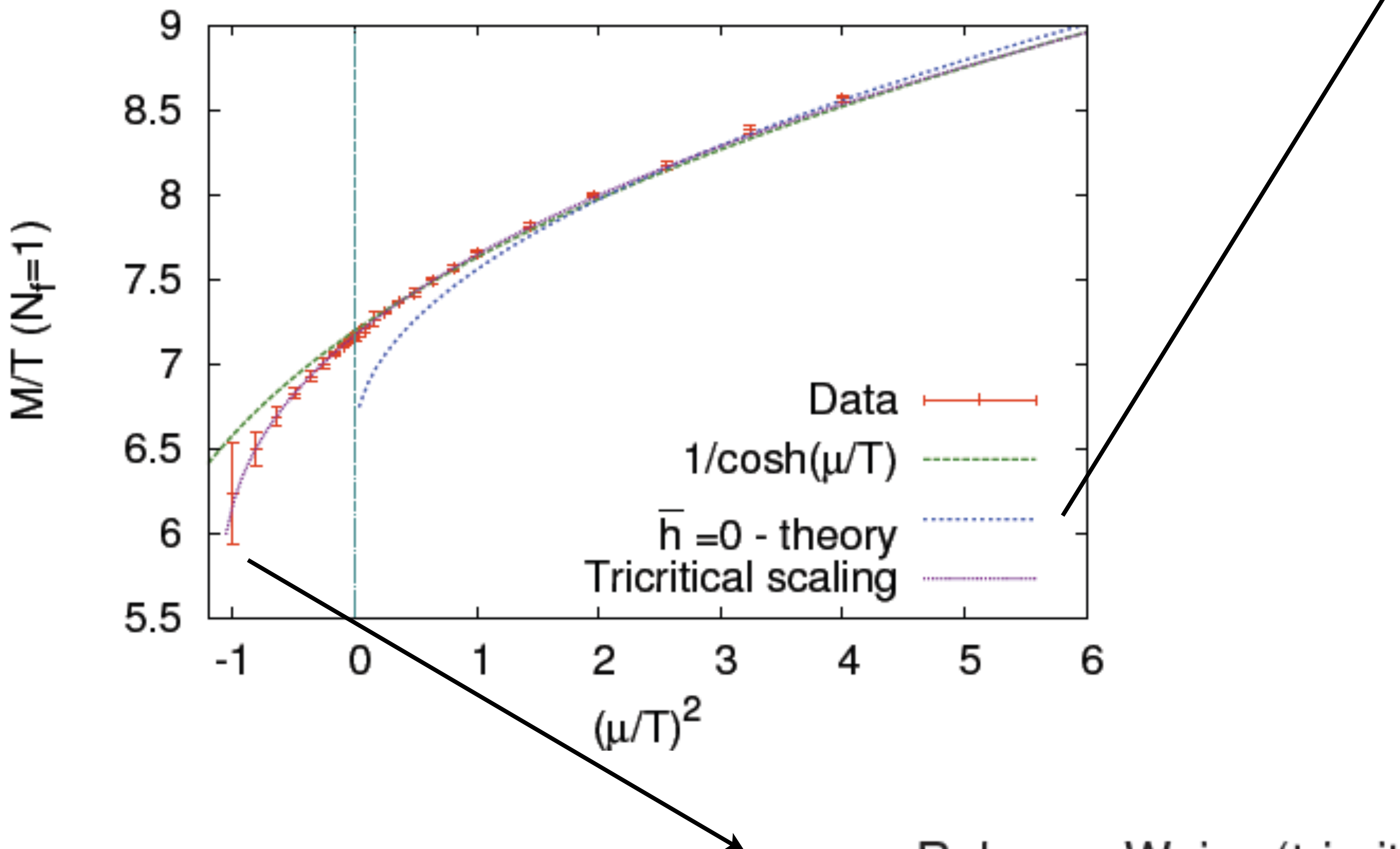
Line of critical end points:

$\lambda_c(\mu/T) = 0.18670(5)$ *for all* μ !

(not exact, but within errors)

Critical quark mass as function of chemical potential

dense massive limit $\kappa \to 0, \mu \to \infty, \kappa e^{\mu/T} = \text{constant}$

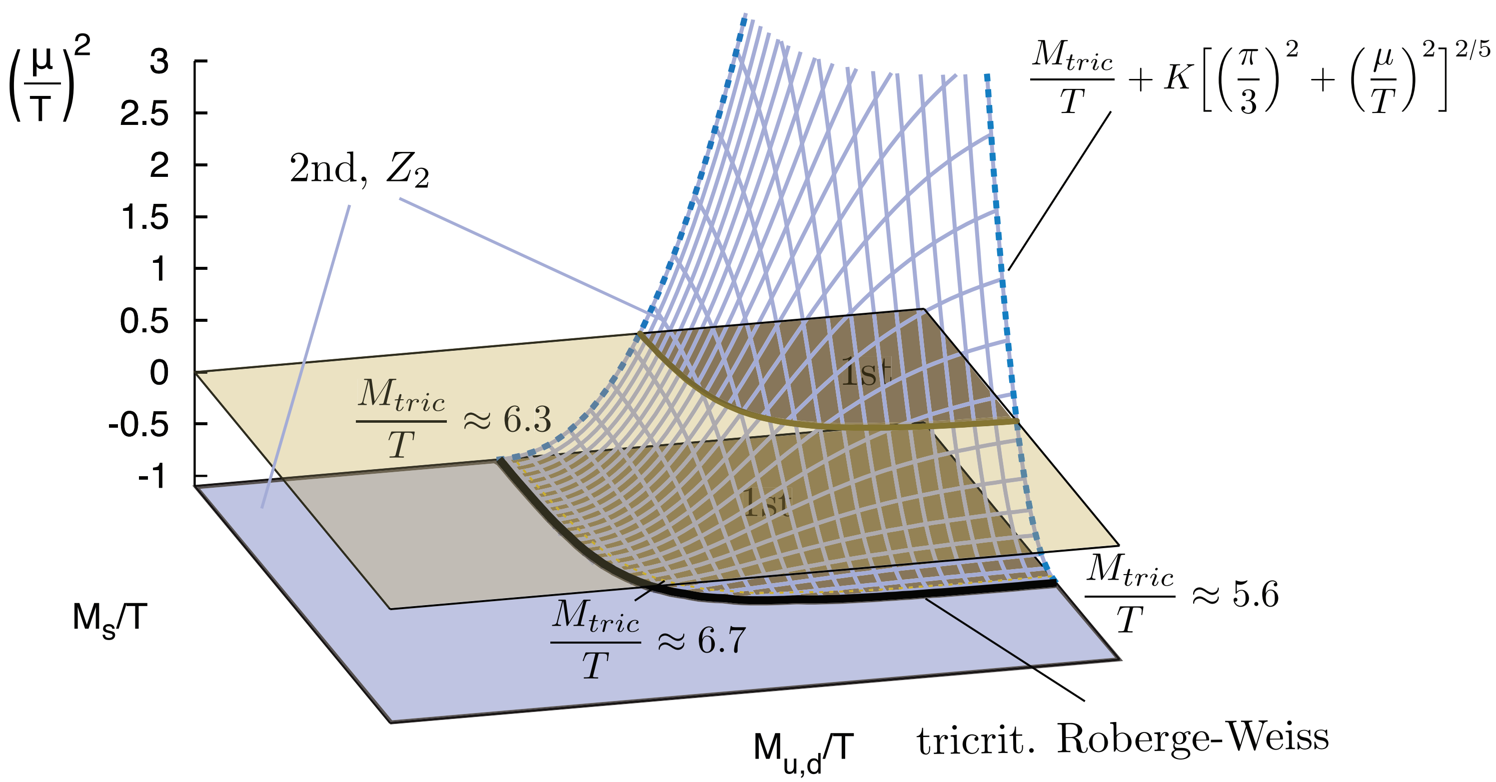


- Roberge-Weiss (tricritical) endpoint at $\mu_i/T = \pi/3$
  ($\leftrightarrow$ boundary between Z_3-sectors)

Tricritical scaling works perfectly, even well into $\mu^2 > 0$!

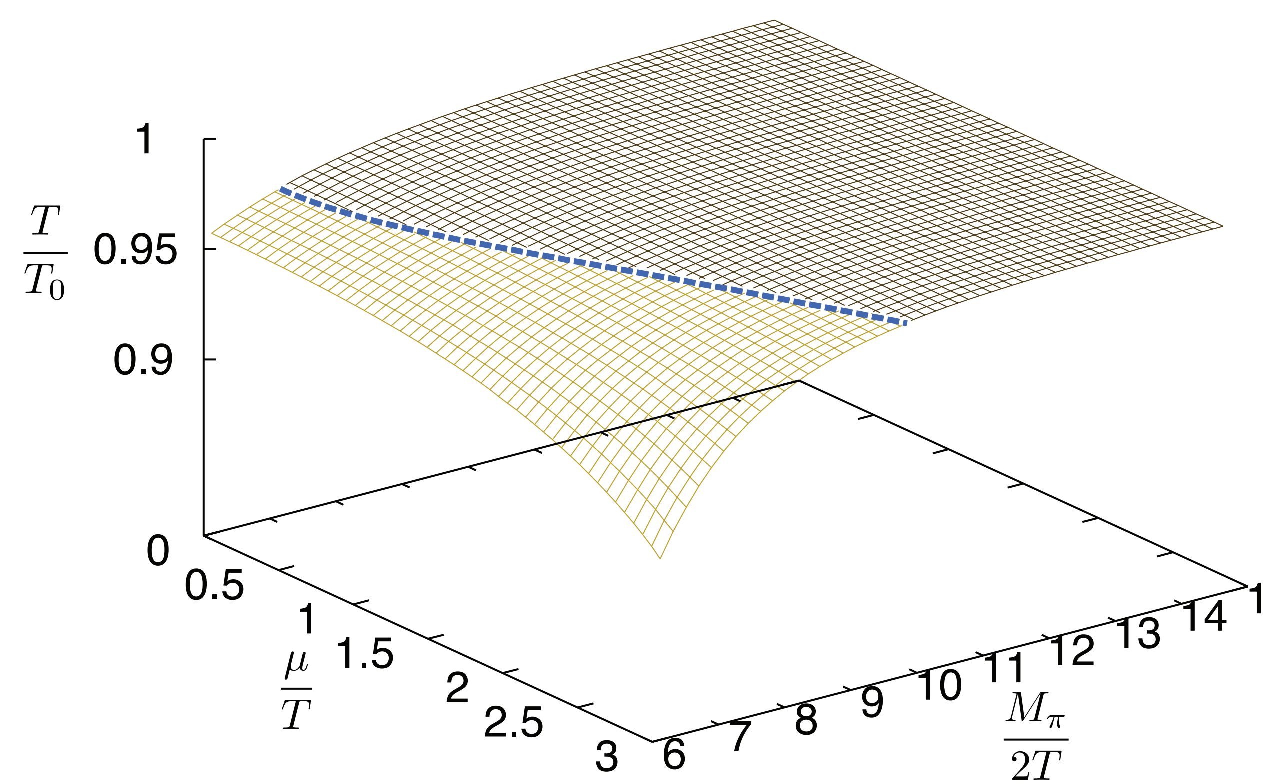
$$\frac{M_c}{T} = \frac{M_{\rm tric}}{T} + K\left[\left(\frac{\pi}{3}\right)^2 + \left(\frac{\mu}{T}\right)^2\right]^{2/5}$$

The fully calculated deconfinement transition



deconfinement critical surface

phase diagram for Nf=2, Nt=6



Conclusions

- Proposal for two-step treatment of QCD phase transition:

  I. Derivation of effective action by strong-coupling expansion
  II. Simulation of effective theory

- Z(N)-invariant effective theory for Yang-Mills, correct order of p.t., crit. temperature ~10% accurate in the continuum limit

- Deconfinement transition plus end point for heavy fermions and all chemical potentials

- Hope for finite density QCD: cold and dense regime, light fermions?