Numerical simulation of complex SDEs driven by real Brownian motion

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First, the connection with the Fokker-Planck equation: for

$$dX = b(X)dt + \sigma(X)d\omega(t),$$

where $X(0) = x$, the distribution function $p(x, 0; X, t)$ for X satisfies the forward Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial X}\{b(X)p\} + \frac{1}{2}\frac{\partial^2}{\partial X^2}\{a(X)p\},$$

where $a(X) = \sigma(X)\sigma(X)^T$. This can be easily generalized to include a potential $V(X)$, indeed even quasi-linear potentials, via the Feyman-Kac formula. All of this follows from Itô's formula because $d\omega(t)d\omega(t) = dt$.

Example numerical methods: Euler-Maruyama

$$z_{k+1} = z_k + b(z_k)h + \sigma(z_k)\Delta\omega$$

is **strong** order $1/2$, **weak** order 1. Milstein's method

$$\begin{aligned} z_{k+1} =& z_k + b(z_k)h + \sigma(z_k)\Delta\omega \\ &+ \frac{1}{2}\sigma(z_k)\sigma'(z_k)((\Delta\omega)^2 - h) \end{aligned}$$

is **strong** order $\beta = 1$, and **weak** order 1. Higher order weak methods require modeling

$$I_{ij} = \int_0^h \omega_i d\omega_j, \qquad I_{i0} = \int_0^h \omega_i(s) ds,$$

$$I_{ijk} = \int_0^h \omega_i \omega_j d\omega_k, \qquad I_{ii0} = \int_0^h \omega_i^2 ds.$$

A very simple $O(h^2)$ accurate model for $\Delta\omega$ is

$$\begin{aligned} \xi &= \sqrt{3h} && \text{with probability } 1/6, \\ &= -\sqrt{3h} && \text{with probability } 1/6, \\ &= 0 && \text{with probability } 2/3. \end{aligned}$$

Important facts about these bounded increments:

- they introduce Fourier spectra with wave vectors $= \mathbf{k}\sqrt{3h}$, where $\mathbf{k} \in \mathbb{Z}^d$.
- in $d > 1$ dimensions, $\Delta\mathbf{w}$ is not isotropic.

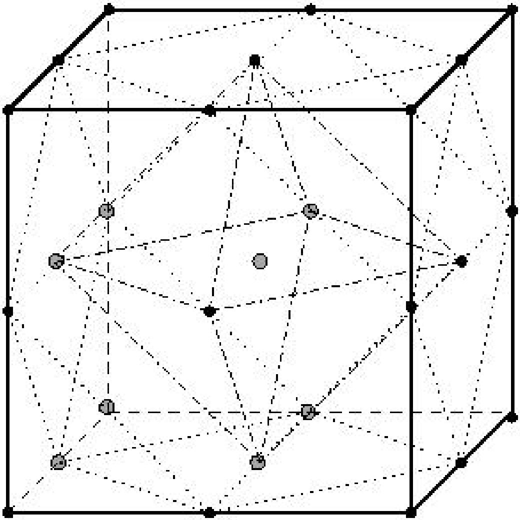

Figure: 3-D distribution of bounded increments

We can make these isotropic using Pete Stewart's random rotations: $S = diag(sign(u_1))$ (for each $\mathbf{u}$ below)

$$U = SU_0 U_1 \ldots U_{N-2}$$

where

$$U_k = \begin{pmatrix} I_k & \\ & H_{N-k} \end{pmatrix}$$

$H_j =$ Householder transforms,

$$H_j = I_j - 2\frac{\mathbf{u}\mathbf{u}^T}{||\mathbf{u}||^2}$$

with $j-$length vectors $\mathbf{u}$

$$\mathbf{u} = \mathbf{x} - ||\mathbf{x}||\mathbf{e}_1,$$

each $x_i \in \mathcal{N}(0,1)$, $i = 1, \ldots, j$.

A quasi-symplectic method in three stages beginning at x_0,

$$x_{1/2} = x_0 + \frac{h}{4}(b(x_{1/2}) + b(x_0)), \text{ solve for } x_{1/2}, \tag{1a}$$

$$x_1 = x_{1/2} + \int_0^h \sigma(x(s))d\omega(s), \text{ starting at } x_{1/2}, \tag{1b}$$

$$x_h = x_1 + \frac{h}{4}(b(x_h) + b(x_1)), \text{ solve for } x_h. \tag{1c}$$

Two simple complex **scalar** cases where the martingale stage $x_{1/2} + \int_0^h \sigma(x(s))d\omega(s)$ is easy to compute:

- $\sigma =$ constant, then

$$x_1 = x_{1/2} + \sigma\Delta\omega,$$

  when $\sigma = 1$, we'll call this **N1**.

- $\sigma(x) = \gamma x$, then

$$x_1 = x_{1/2} e^{\gamma\Delta\omega - \gamma^2 h/2},$$

  when $\gamma = 1$, call this **N2**.

A complex example:

$$dX = (a_0 + a_1 X)dt + (c_0 + c_1 X)d\omega(t), \quad \text{initially} \;\; X(0) = X_0. \;\; (2)$$

A formal solution to this problem is

$$\begin{aligned} X(t) \;\; &= \;\; X_0\Phi(t) + (a_0 - c_0 c_1)\Phi(t)\int_0^t \Phi^{-1}(s)ds \\ &\qquad + c_0\Phi(t)\int_0^t \Phi^{-1}(s)d\omega(s), \end{aligned}$$

where

$$\Phi(t) = e^{c_1\omega(t) + (a_1 - c_1^2/2)t}.$$

Its inverse is $\Phi^{-1}(t) = e^{-c_1\omega(t) - (a_1 - c_1^2/2)t}$. To see how our method works, we need a test statistic

$$\mathbf{E}[X(t)] = \left(X_0 + \frac{a_0}{a_1}\right)e^{a_1 t} - \frac{a_0}{a_1}.$$

Notice that $\mathbf{E}[X(t)]$ is independent of coefficients c_0, c_1, but the distributions for $X(t)$ are very different. For our examples, $a_1 = i$, so $\mathbf{E}[X(t)]$ rotates.

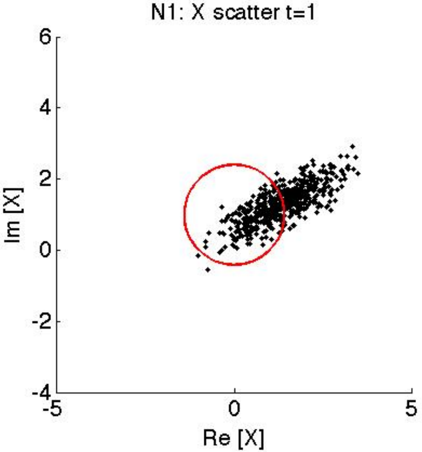

Figure: Distribution for $X(t = 1)$ when $a_0 = 1, a_1 = i, c_0 = 1$ and $c_1 = 0$

For this *additive noise* case ($c_0 = 1$, $c_1 = 0$), the variance grows linearly with t.

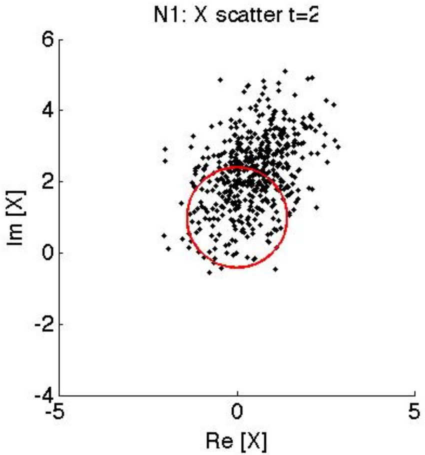

Figure: Distribution for $X(t = 2)$ when $a_0 = 1, a_1 = i, c_0 = 1$ and $c_1 = 0$

For a *multiplicative noise* case ($c_0 = 0, c_1 = 1$), the variance grows exponentially in t.

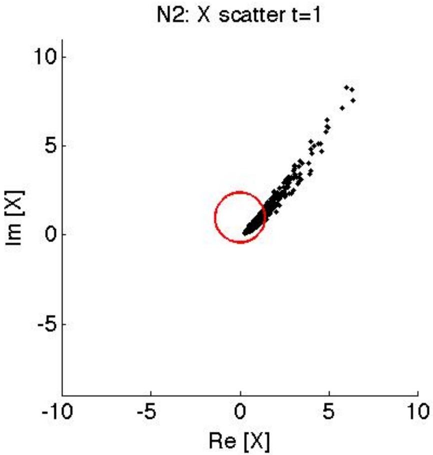

Figure: Distribution for $X(t = 1)$ when $a_0 = 1, a_1 = i, c_0 = 0$ and $c_1 = 1$

Again, the *multiplicative noise* case ($c_0 = 0, c_1 = 1$), at $t = 2$:

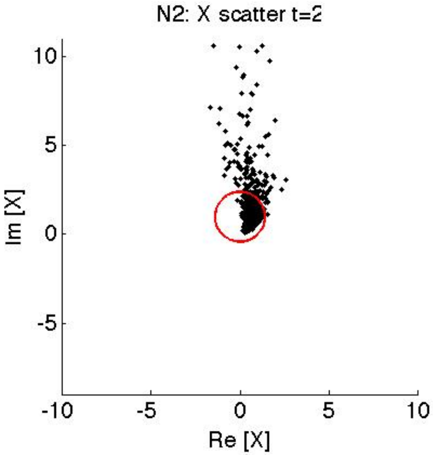

Figure: Distribution of $X(t = 2)$ when $a_0 = 1, a_1 = i, c_0 = 0$ and $c_1 = 1$

For the additive noise $c_1 = 0$ case, our test statistic $\mathbf{E}[X(t)]$ for $0 \le t \le 6\pi$ is:

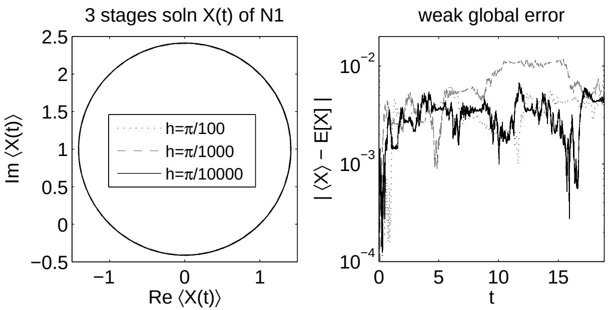

Figure: $\mathbf{E}[X(t)]$ when $a_0 = 1, a_1 = i, c_0 = 1$ and $c_1 = 0$

For the additive noise $c_1 = 0$ case, the test statistic $\mathbf{E}[X(t)]$ for $0 \leq t \leq 6\pi$ using Higham, Mao, and Stuart's 2-stage split-step backward Euler method:

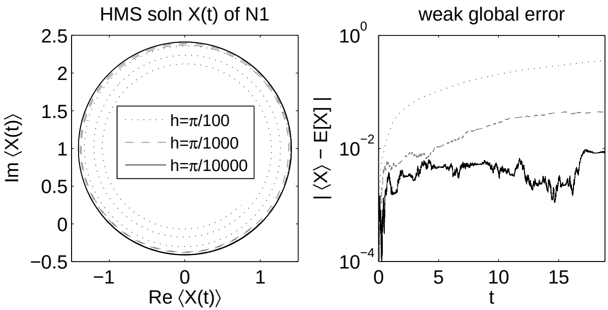

Figure: $\mathbf{E}[X(t)]$ when $a_0 = 1, a_1 = i, c_0 = 1$ and $c_1 = 0$

For the additive noise $c_1 = 0$ case, the test statistic $\mathbf{E}[X(t)]$ for $0 \le t \le 6\pi$ using the forward Euler method,

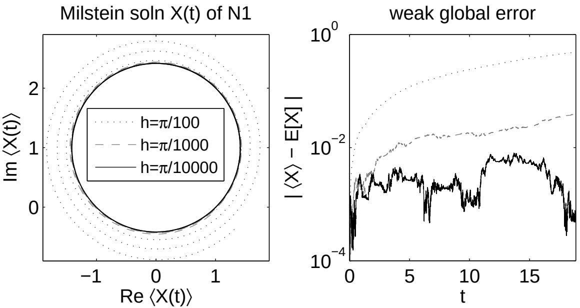

Figure: $\mathbf{E}[X(t)]$ when $a_0 = 1, a_1 = i, c_0 = 1$ and $c_1 = 0$

Our test statistic $\mathbf{E}[X(t)]$ for the multiplicative noise case and $0 \leq t \leq 2\pi$:

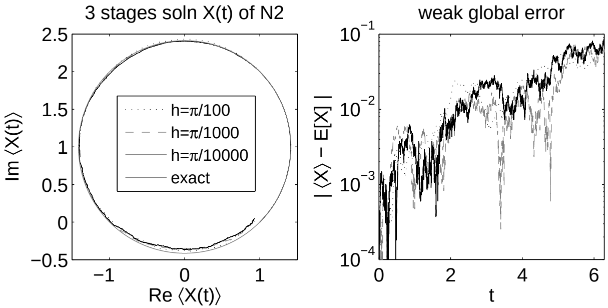

Figure: $\mathbf{E}[X(t)]$ when $a_0 = 1, a_1 = i, c_0 = 0$ and $c_1 = 1$

A simple example with drift:

$$dX = -ib\,X\,dt + \sqrt{\mu i}\,X\,d\omega(t).$$

The explicit (but formal) Itô solution is

$$X(t) = X_0 e^{-i(b+\mu/2)t + \sqrt{\mu i}\,\omega(t)},$$

with explicit mean

$$\langle X(t) \rangle = \mathbf{E}[X(t)] = X_0 e^{-ibt}.$$

The awkward part is the variance: using the modulus $|X(t)| = |X_0| \exp\left(\sqrt{\frac{\mu}{2}}\omega(t)\right)$,

$$\mathbf{E}[|X(t) - \mathbf{E}[X(t)]|^2] = |X_0|^2 (e^{\mu t} - 1),$$

which grows exponentially.

The simplest example, with complex Brownian motion:

$$dz = z\, d\omega$$

where

$$d\omega = \frac{1}{\sqrt{2}}(d\omega_1 + i d\omega_2).$$

Here each ω_1, ω_2 are real and uncorrelated. The formal solution is a conformal martingale,

$$z(t) = z_0 e^{\omega(t)} = z_0 \exp\left(\frac{1}{\sqrt{2}}(\omega_1 + i\omega_2)\right).$$

Compare this to the real case $dx = x\, d\omega_1(t)$, whose solution is $x_0 \exp(\omega_1(t) - t/2)$. Both have constant mean but growing variances:

$$\mathbf{E}[x(t)] = x_0, \qquad \mathbf{E}[z(t)] = z_0, \quad \text{and,}$$

$$\mathbf{E}[|x(t) - \mathbf{E}[x(t)]|^2] = |x_0|^2(e^t - 1), \quad \mathbf{E}|z(t) - \mathbf{E}[z(t)]|^2 = |z_0|^2(e^t - 1).$$

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