Anomalous Transport
from
Kubo Formulas

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in collaboration with

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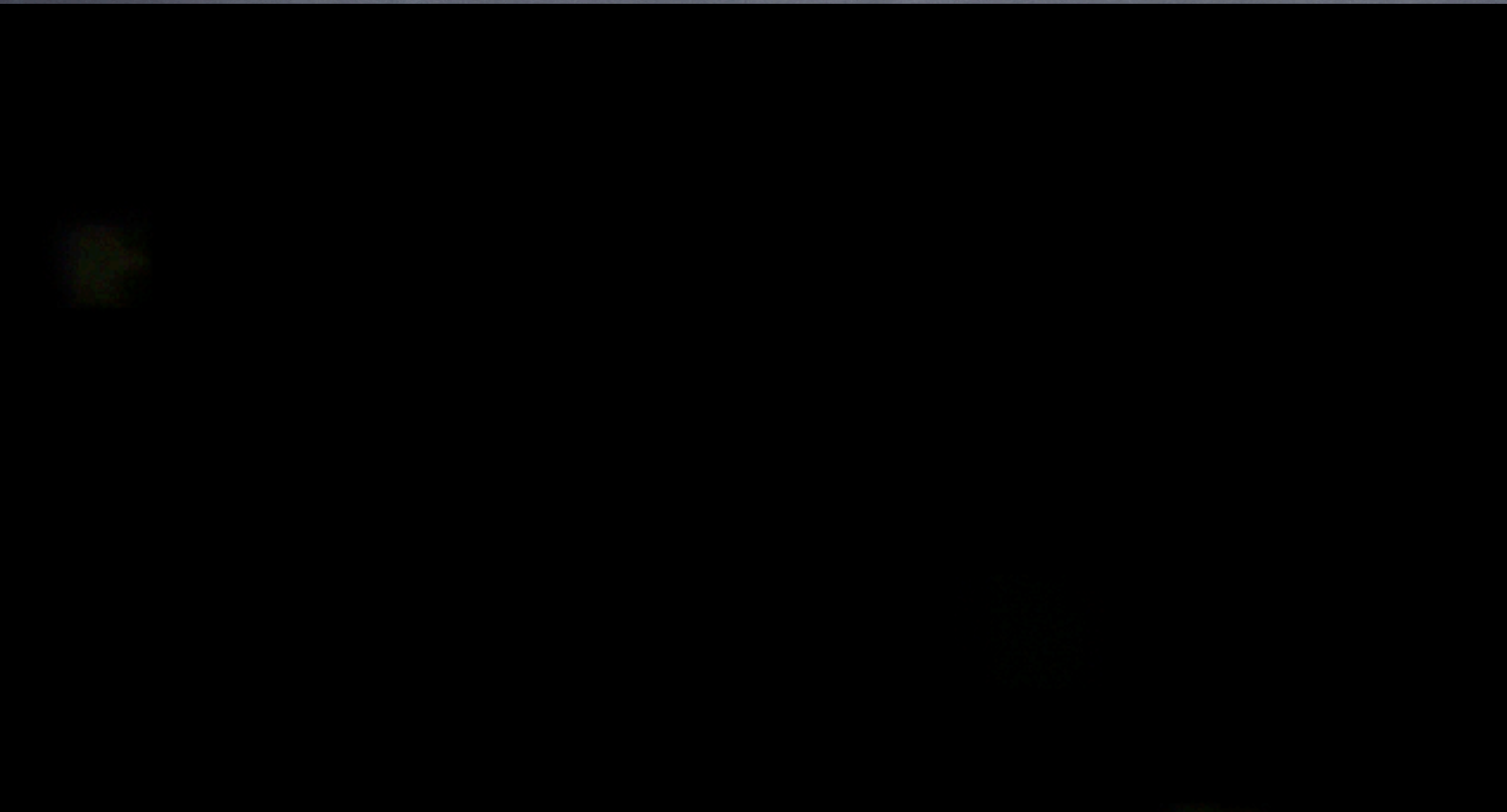
UW-Seattle, 13 March 2012

Outline

- Movie
- CME and CVE (flash review)
- Kubo formulas
- Hydrodynamics
- Chem. Potentials
- Currents and Counterterms
- Holography
- Outlook

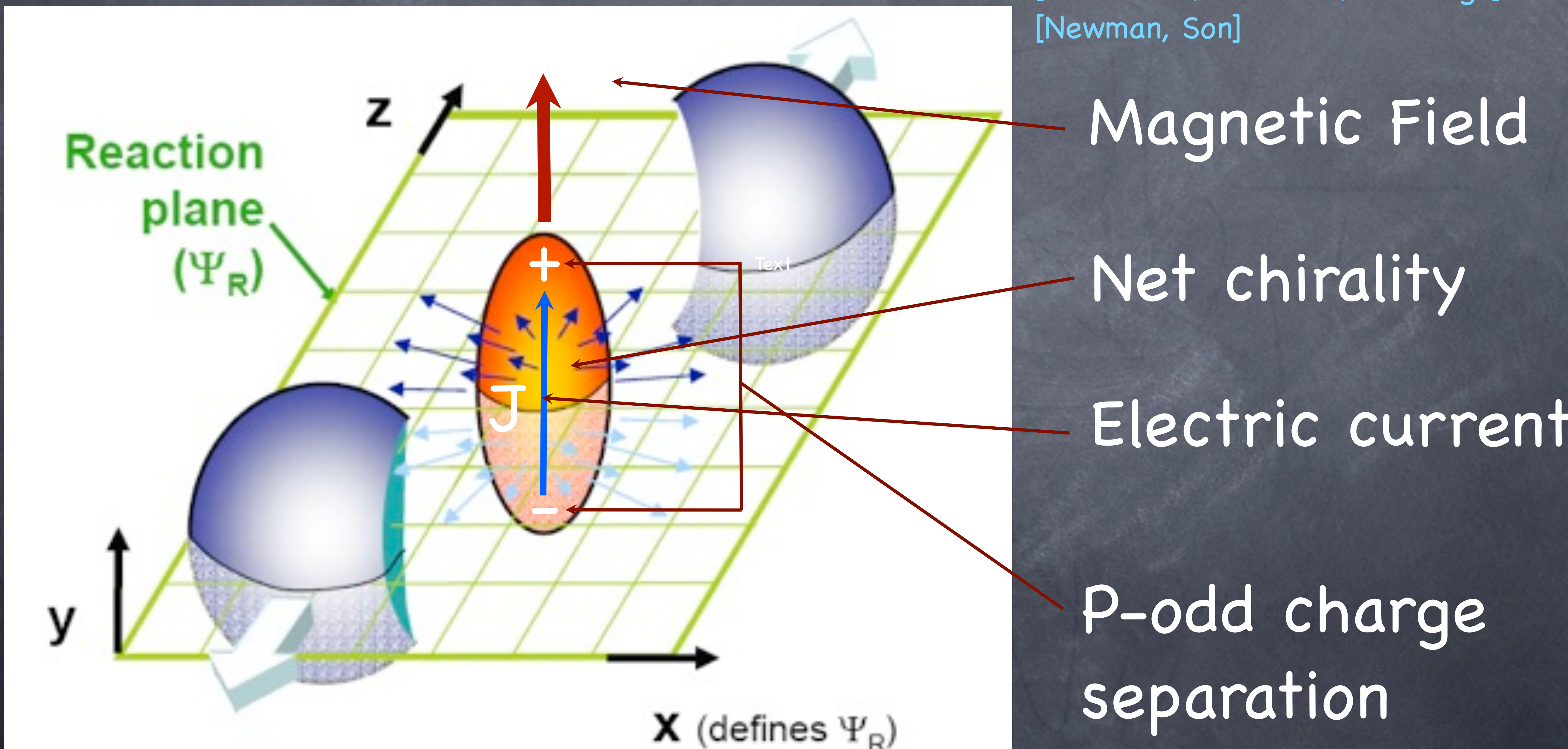
RHIC’s hot quark soup



www.youtube.com/watch?v=kXy5EvYu3fw

The CME

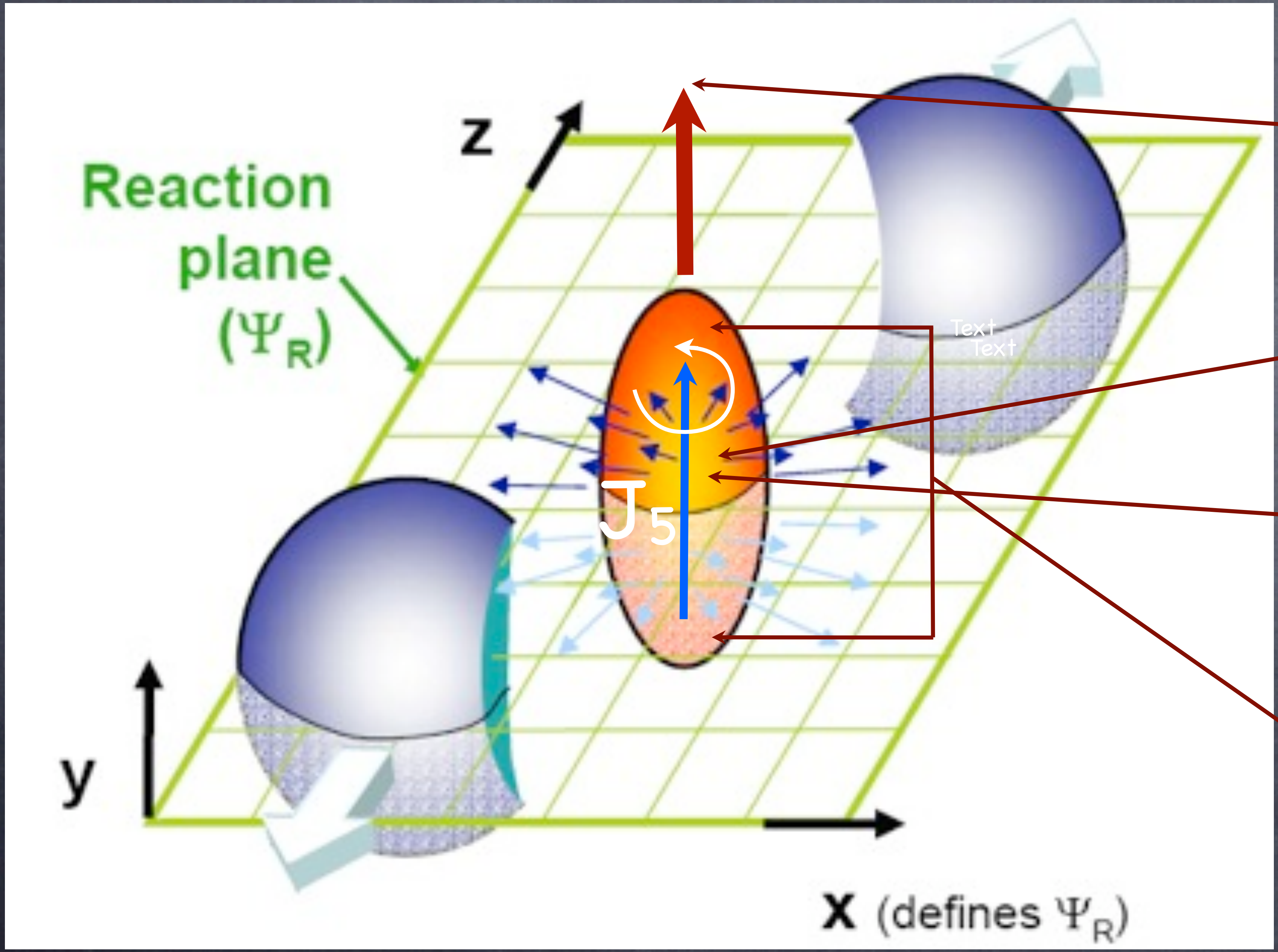
[Kharzeev, McLarren, Warringa],
[Fukushima, Kharzeev, Warringa]
[Newman, Son]



[parity violating currents: Vilenkin '80, Giovannini, Shaposhnikov '98, Alekseev, Chaianov, Fröhlich '98]

The CVE

[Banerjee et al, Erdmenger et al.]



Angular Momentum

Temperature

Axial Current

P-odd axial charge separation

[Vilenkin '80, CVE Kharzeev&Son '10, Keren-Zur&Oz '11]

Kubo formulas

- Chiral magnetic conductivity

$$\vec{J} = \sigma \vec{B}$$

[Kharzeev, Warringa]

$$J_i = \sigma \epsilon_{ijk} (i p_j) A_k$$

- Kubo formula, general symmetry group

$$[T^A, T^B] = i f_C^{AB} T^C$$

$$\sigma^{AB} = \lim_{p_j \to 0} \sum_{i,k} \frac{i}{2 p_j} \epsilon_{ijk} \langle J_i^A J_k^B \rangle \Big|_{\omega=0}$$

Kubo formulas

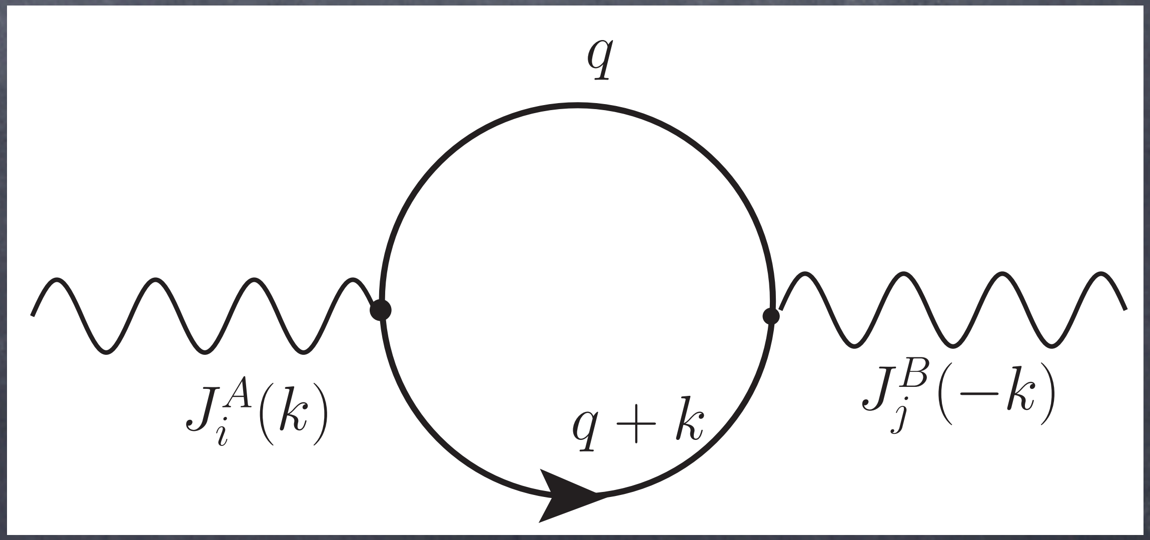
- chiral fermions

$$J_i^A = \sum_{f,g=1}^{N} (T^A)^g{}_f \bar{\Psi}_g \gamma_i P_+ \Psi^f$$

- chemical potentials and Cartan generators

$$H_A = q_A^f \delta^f{}_g \qquad \mu^f = \sum_A q_A^f \mu_A$$

- 1-loop graph



Kubo formulas

$$G^{AB} = \frac{1}{2}\sum_{f,g} T^{g}_{A\ f} T^{f}_{B\ g} \frac{1}{\beta}\sum_{\tilde{\omega}^f}\int \frac{d^3q}{(2\pi)^3}\epsilon_{ijn}\mathrm{tr}\left[S^{f}{}_{f}(q)\gamma^i S^{f}{}_{f}(q+k)\gamma^j\right]$$

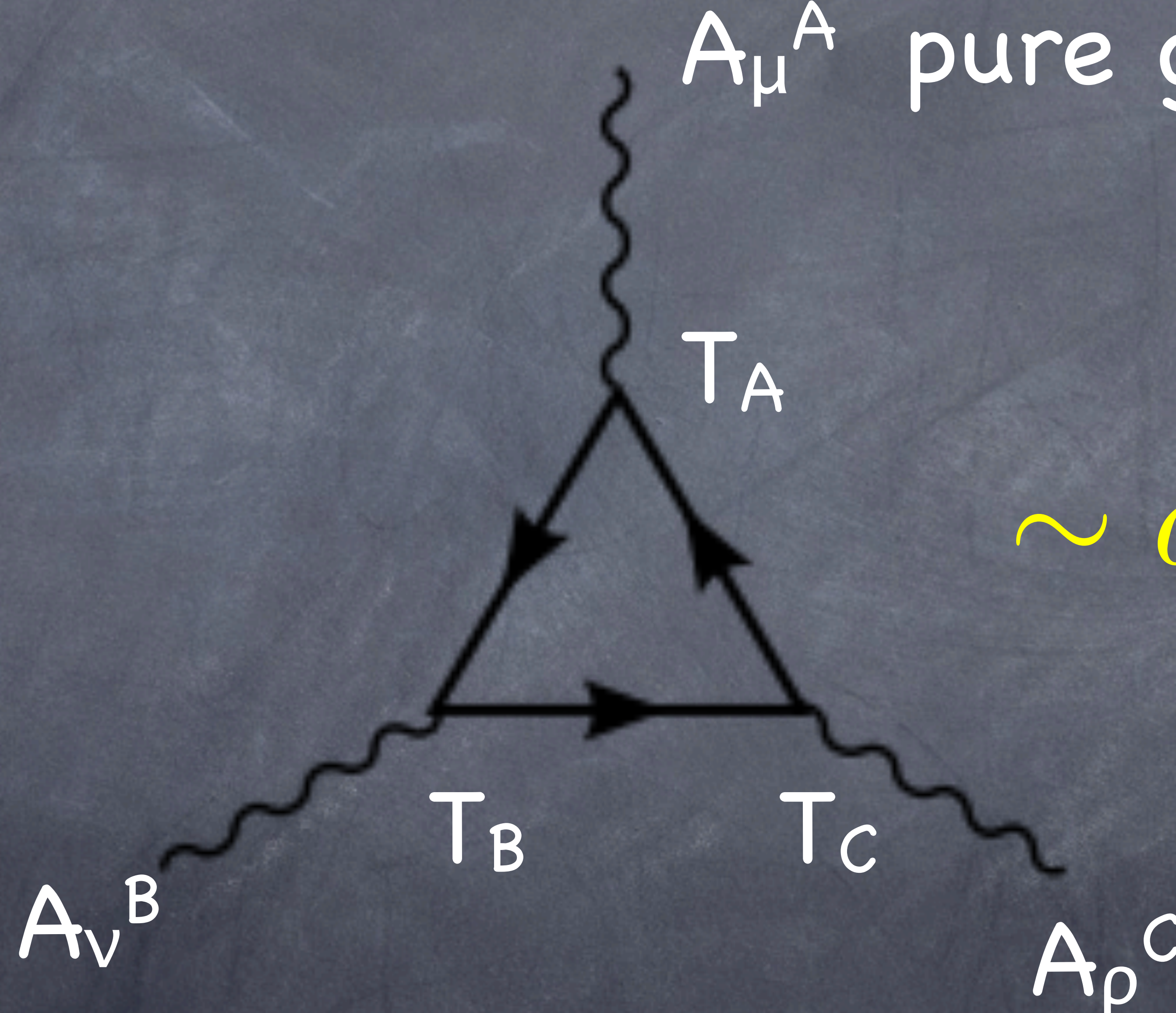
$$S(q)^{f}{}_{g} = \frac{\delta^{f}{}_{g}}{2}\sum_{t=\pm}\Delta_t(i\tilde{\omega}^f,\vec{q})\mathcal{P}_+\gamma_\mu \hat{q}^\mu_t$$

$$\Delta_t(i\tilde{\omega}^f, q) = \frac{1}{i\tilde{\omega}^f - tE_q} \qquad \tilde{\omega}^f_n = (2n+1)\pi T - i\mu^f$$

Anomalycoeff

$$\sigma_{AB} = \frac{1}{8\pi^2}\sum_C \mathrm{tr}\left(T^A\{T^B, H^C\}\right)\mu_C = \frac{1}{4\pi^2} d^{ABC}\mu_C$$

Kubo formulas



$\sim d_{ABC}$

Kubo formulas

- finite density: charge transport => energy transport

$$\delta T_{0i} = \mu \delta J_i = \mu \delta \sigma B_i$$

- energy flux sourced by magnetic fields

$$\frac{i}{2p_j} \sum_{i,k} \epsilon_{ijk} \left. \langle T_{0i} J_k \rangle \right|_{\omega=0} = \int \mu d\sigma + \text{const.}$$

[Neiman, Oz], [Loganayagam], [Kharzeev, Yee]

- at ω=0 reverse order of operators

$$\sigma_V = \frac{i}{2p_j} \sum_{i,k} \epsilon_{ijk} \left. \langle J_i T_{0k} \rangle \right|_{\omega=0} = \int \mu d\sigma + \text{const.}$$

conductivity ?

Kubo formulas

- $T_{\mu\nu}$ sourced by metric

$$ds^2 = -(1-2\Phi)dt^2 + 2\vec{A}_g dt d\vec{x} + (1+2\Phi)d\vec{x}^2$$

- A_g "gravitomagnetic field" -> chiral gravitomagnetic effect

$$\vec{J} = \sigma_V \vec{B}_g$$

- chiral vortical effect: fluid velocities

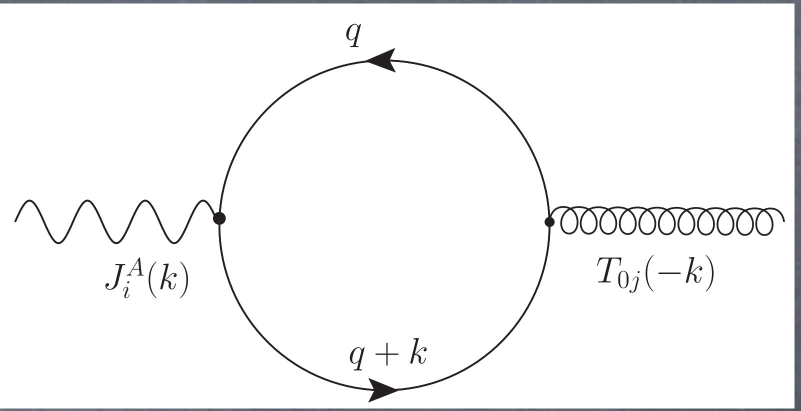
$$u^\mu = (1,0,0,0) \qquad u_\mu = (-1, \vec{A}_g)$$

$$J^i = \sigma_V \epsilon^{ijk} \partial_j u_k$$

Kubo formulas

- as before: general symmetry group

$$T^{0i} = \frac{i}{2}\sum_{f=1}^{N} \bar{\Psi}_f(\gamma^0\partial^i + \gamma^i\partial^0)P_+\Psi^f$$

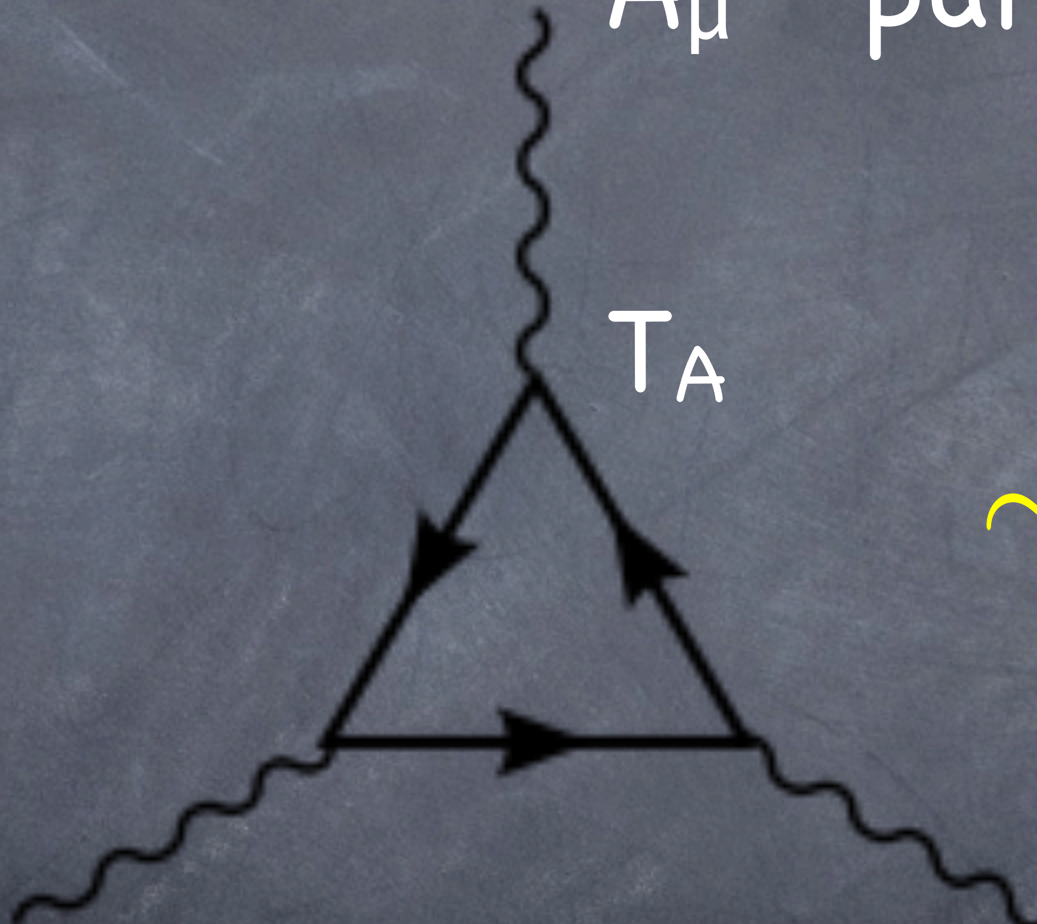


$$\sigma_V^A = \frac{1}{8\pi^2}\sum_{f=1}^{N}(T^A)^f{}_f\left[(\mu^f)^2 + \frac{\pi^2}{3}T^2\right]$$

$$= \frac{1}{8\pi^2}\sum_{B,C} d^{ABC}\mu_B\,\mu_C + \frac{T^2}{24}\mathrm{tr}\left(T^A\right)$$

Integration constant -> gravitational anomaly!

Kubo formulas



$$\sim \mathrm{tr}\,(T_A)$$

Kubo formulas

- also energy flux

$$\lim_{p_j \to 0} \frac{i}{2p_j} \sum_{i,k} \epsilon_{ijk} \langle T_{0i} T_{0k} \rangle |_{\omega=0} \neq 0$$

- 4 types of conductivities:

$$\sigma_{\mathcal{B}}^{AB} = \frac{1}{4\pi^2} d^{ABC} \mu_C$$

$$\sigma_{\mathcal{V}}^{A} = \sigma_{\mathcal{B}}^{\epsilon,A} = \frac{1}{8\pi^2} d^{ABC} \mu_B \mu_C + b^A \frac{T^2}{24}$$

$$\sigma_{\mathcal{V}}^{\epsilon} = \frac{1}{12\pi^2} d^{ABC} \mu_A \mu_B \mu_C + b^A \mu_A \frac{T^2}{12}$$

Kubo formulas

- Compare with electric conductivity

$$\sigma_{\rm el} = \lim_{\omega\to 0} \frac{-i}{\omega} \langle J_i J_i \rangle|_{k=0}$$

$$\sigma^{AB} = \lim_{p_j\to 0} \sum_{i,k} \frac{i}{2p_j} \epsilon_{ijk} \langle J_i^A J_k^B \rangle \Big|_{\omega=0}$$

- Rate of dissipation

$$\frac{dW}{dt} = \frac{\omega}{2} f_I(-\omega) \rho^{IJ}(\omega) f_J(\omega)$$

- Not in spectral density and at zero frequency
- Dissipationless transport!

Hydrodynamics

[Son,Surowka] [Neiman, Oz], [Loganayagam]

- hydrodynamics:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}_{\text{diss}} + \tau^{\mu\nu}_{\text{anom}}$$

$$J^{\mu,A} = \rho^A u^\mu + \nu^{\mu,A}_{\text{diss}} + \nu^{\mu,A}_{\text{anom}}$$

$$\nu^{\mu,A}_{\text{anom}} = \sigma^{AB}_{\mathcal{B}} \mathcal{B}^\mu_B + \sigma^A_{\mathcal{V}} \omega^\mu$$

$$Q^\mu_{\text{anom}} = \sigma^{A,\epsilon}_{\mathcal{B}} \mathcal{B}^\mu_A + \sigma^\epsilon_{\mathcal{V}} \omega^\mu$$

$$\tau^{\mu\nu}_{\text{anom}} = Q^\mu u^\nu + Q^\nu u^\mu$$

$$\mathcal{B}^\mu_A = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} u_\nu F_{A,\rho\lambda}$$

$$\omega^\mu = \epsilon^{\mu\nu\rho\lambda} u_\nu \partial_\rho u_\lambda$$

dissipative terms: shear- and bulk viscosities, diffusivity and electric conductivity

Hydrodynamics

- Landau frame: $u^\mu \to u^\mu + \delta u^\mu$

  such that $u_\mu T^{\mu\nu} = 0$

- anomalous terms only in current

$$\nu^\mu_{\rm anom} = \xi_B \mathcal{B}^\mu + \xi_V \omega^\mu \qquad\qquad Q^\mu = 0$$

- ξ coefficients are different from σ's

$$\xi_B = \lim_{k_n \to 0} \frac{-i}{2k_n} \sum_{k.l} \epsilon_{nkl} \left(\langle J^k J^l \rangle - \frac{n}{\epsilon + P} \langle T^{tk} J^l \rangle \right)\Big|_{\omega=0}$$

$$\xi_V = \lim_{k_n \to 0} \frac{-i}{2k_n} \sum_{k.l} \epsilon_{nkl} \left(\langle J^k T^{tl} \rangle - \frac{n}{\epsilon + P} \langle T^{tk} T^{tl} \rangle \right)\Big|_{\omega=0}$$

Chemical Potentials

[Landsmann, Waert] [T.S. Evans]

- Textbook: $H \to H - \mu Q$

$$Z = \sum_n \langle n | e^{-\beta(H-\mu Q)} | n \rangle = \sum_n \langle n | e^{-\beta H} e^{\beta \mu q_n} | n \rangle$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\psi(t - i\beta) = \pm \psi(t) \qquad\qquad \psi(t - i\beta) = \pm e^{\beta \mu q} \psi(t)$$

- Alternative: deform state space instead of Hamiltonian (state vs. coupling)

- Tacitly assumed: [H,Q] =0 !

Chemical Potentials

- Keldysh-Schwinger:

- define initial (equilibrium) state through

$$\psi(t_i - i\beta) = \pm e^{\beta\mu q}\psi(t_i)$$

- but do (real) time evolution with H′ where [H′,Q] <>0!
- non-equilibrium setup (quantum quench)

Chemical Potentials

Formalism	Hamiltonian	Boundary conditions
(A)	$H - \mu Q$	$\Psi(t_i) = -\Psi(t_i - i\beta)$
(B)	H	$\Psi(t_i) = -e^{\beta\mu}\Psi(t_i - i\beta)$

- What's the difference? Gauge trafo!

$$A_0 = \mu \qquad \delta A_\mu = \partial_\mu \chi \qquad \chi = -\mu t$$

- Gauge invariant: Parallel transport
- Anomalous charge: (B) seems better suited, b
- (A) not anymore equivalent to (B), ANOMALY!

Chemical Potentials

- Formalism (A') (with anomaly):

$$\delta S = \int d^4x \chi\, \epsilon^{\mu\nu\rho\lambda}\left(C_1 F_{\mu\nu}F_{\rho\lambda} + C_2 R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\lambda\rho}\right)$$

$$S_\Theta = \int d^4x \Theta\, \epsilon^{\mu\nu\rho\lambda}\left(C_1 F_{\mu\nu}F_{\rho\lambda} + C_2 R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\lambda\rho}\right)$$

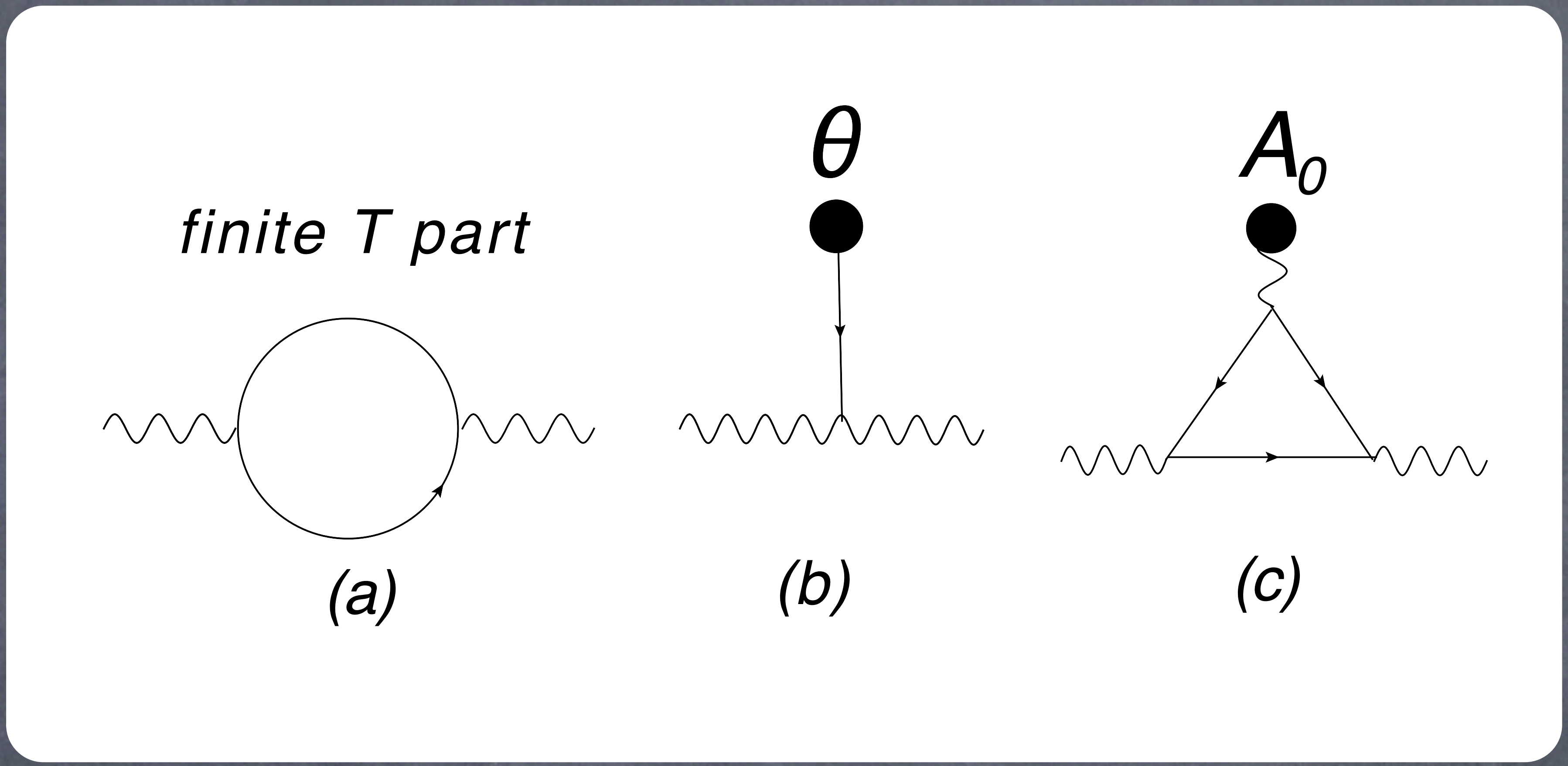
$$\delta_\Theta = -\chi \qquad\qquad \delta(S + S_\Theta) = 0$$

- Spurious "axion": (B) -> (A') switches on θ !

$$H - \mu(Q + 4\int d^3x \left(C_1 \epsilon^{0ijk} A_i \partial_j A_k + C_2 K^0\right)$$

$$K^\mu = \epsilon^{\mu\nu\rho\lambda}\Gamma^{\alpha}_{\beta\nu}\left(\partial_\rho \Gamma^{\beta}_{\alpha\lambda} + \frac{2}{3}\Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\beta}_{\sigma\rho}\right) \qquad \vec{J}_\Theta = 4 C_1 \mu \vec{\mathcal{B}}$$

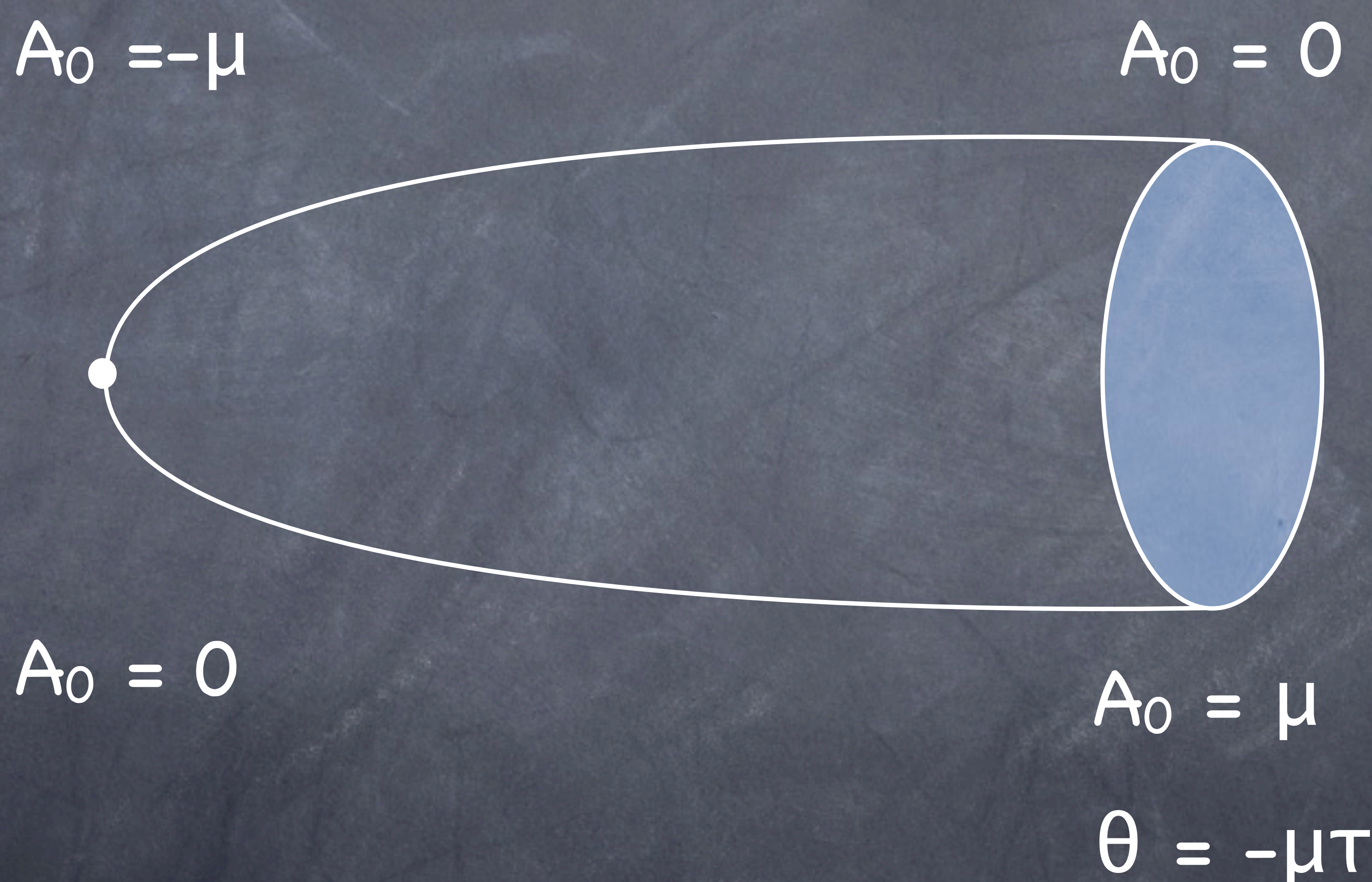
Chemical Potentials



- (A′) equivalent to (B) (b) and (c) cancel each other

Chemical Potentials

- Holography: (B) and (A′)



Currents and Countertems

- A Tale of two Symmetries: V vs. A

- Symmetric d^{ABC}

	V	A
Ψ_1	1	1
Ψ_2	-1	1

$d^{AAA} = 2$

$d^{AVV} = d^{VAV} = d^{VVA} = 2$

Indicates $\partial_\mu J^\mu_V \neq 0$

- Cure: Bardeen counterterm

[Bardeen '69], [Rebhan, Schmitt, Stricker]
[Gynther, K.L., Pena-Benitez, Rebhan]

$$S_{\rm ct} = c \int d^4x \epsilon^{\mu\nu\rho\lambda} V_\mu A_\nu F^V_{\rho\lambda}$$

$$\partial_\mu J^\mu_V = 0$$

$$\partial_\mu J^\mu_A = \frac{1}{48\pi^2} \epsilon^{\mu\nu\rho\lambda} \left(3F^V_{\mu\nu} F^V_{\rho\lambda} + F^A_{\mu\nu} F^A_{\rho\lambda}\right)$$

$$\vec{J}_V = \frac{\mu_A}{2\pi^2} \vec{\mathcal{B}}_V$$ (A'), (B)

$$\vec{J}_V = 0$$ (A)

$$\vec{J}_A = \frac{\mu}{2\pi^2} \vec{\mathcal{B}}_V$$ (A), (A'), (B)

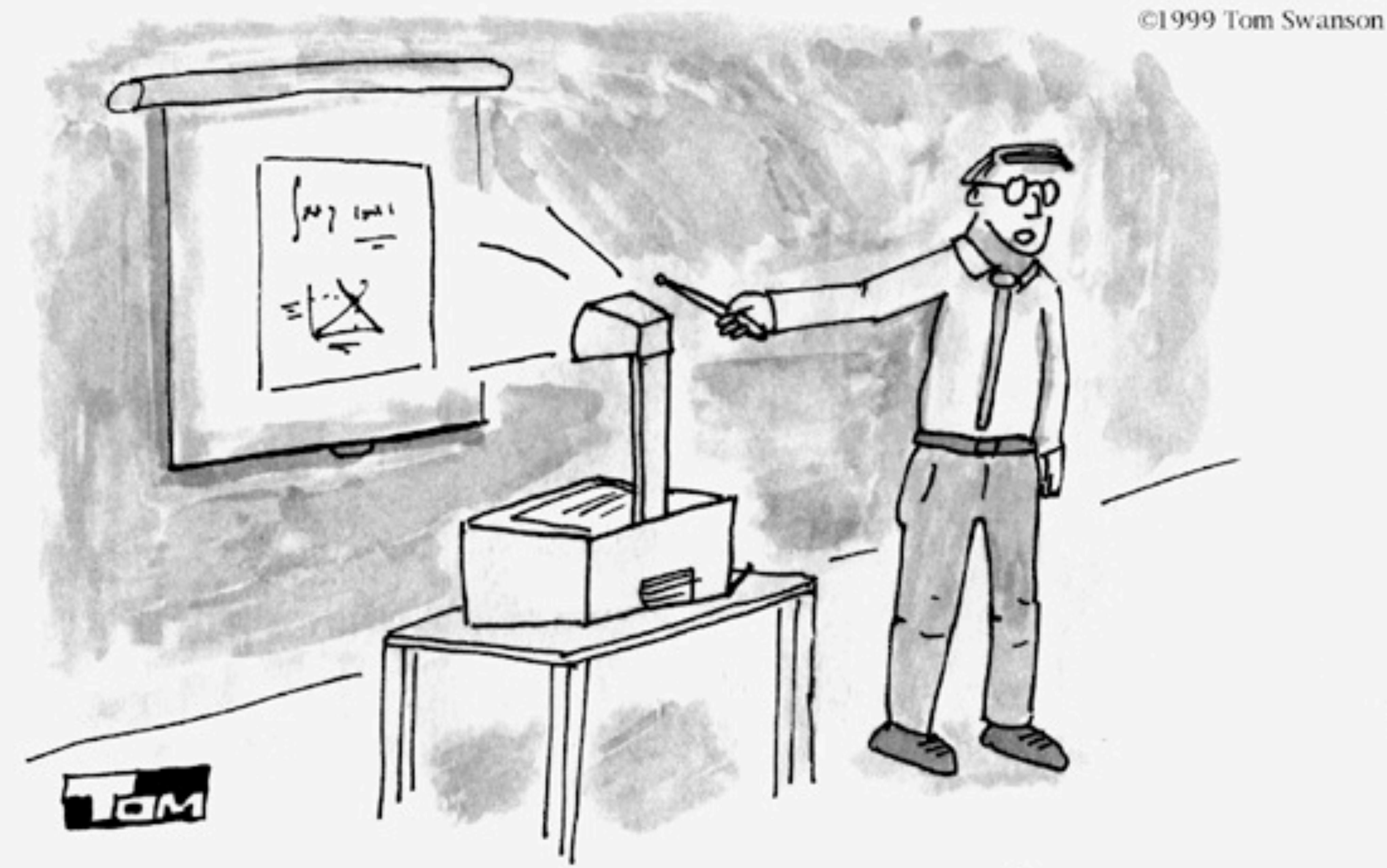
Currents and Counterterms

- Which current?
- Consistent current $J^\mu = \frac{\delta S}{\delta A_\mu}$
  - Anomaly not in covariant form
  - CS term in constitutive relation $\nu^\mu = \cdots + 3C_1 \epsilon^{\mu\nu\rho\lambda} A_\lambda F_{\rho\lambda}$
- Covariant current $J^\mu \to J^\mu + Y^\mu$ [Bardeen, Zumino]
  - Covariant form of anomaly
  - Holography: $A_\mu = A_\mu + \frac{A_\mu^{(2)}}{r^2} + \cdots$

$$\nabla_\mu J^\mu_A = \epsilon^{\mu\nu\rho\lambda}\left(\frac{d_{ABC}}{32\pi^2} F^B_{\mu\nu} F^C_{\rho\lambda} + \frac{b_A}{768\pi^2} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda}\right)$$

↓ O(4) in derivatives!

Holography



ACTUALLY, THAT ASSUMPTION ISN'T REALLY NECESSARY. WE CAN SEE HERE THAT THE POINT-COW APPROXIMATION WORKS EQUALLY WELL.

Holography

[Newman], [Banerjee et al.], [Erdmenger et al.] [Yee] [Rebhan, Schmitt, Stricker] [Khalaydzyan, Kirsch], [Hoyos, Nishioka, OBannon]

- mixed gauge gravitational Chern Simons term

$$S = S_{ME} + S_{CS} + S_{GH} + S_{CSK}$$

$$S_{EM} = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left[R + 2\Lambda - \frac{1}{4}F_{MN}F^{MN}\right]$$

$$S_{CS} = \frac{1}{16\pi G}\int d^5x\,\epsilon^{MNPQR}A_M\left(\frac{\kappa}{3}F_{NP}F_{QR} + \lambda R^A{}_{BNP}R^B{}_{AQR}\right)$$

$$S_{GH} = \frac{1}{8\pi G}\int_\partial d^4x\sqrt{-h}\,K$$

$$S_{CSK} = -\frac{1}{2\pi G}\int_\partial d^4x\sqrt{-h}\,\lambda n_M\epsilon^{MNPQR}A_N K_{PL}D_Q K^L_R$$

Holography

- mixed gauge gravitational Chern Simons term

$$G_{MN} - \Lambda g_{MN} = \frac{1}{2} F_{ML} F_N{}^L - \frac{1}{8} F^2 g_{MN} + 2\lambda \epsilon_{LPQR(M} \nabla_B \left(F^{PL} R^B{}_{N)}{}^{QR} \right)$$

$$\nabla_N F^{NM} = -\epsilon^{MNPQR} \left(\kappa F_{NP} F_{QR} + \lambda R^A{}_{BNP} R^B{}_{AQR} \right)$$

$$16\pi G J^i = -\sqrt{-\gamma} \left[E^i + \epsilon^{ijkl} \left(\frac{4\kappa}{3} A_j \hat{F}_{kl} - \frac{\lambda}{2\pi G} D_j K_{sj} D_k K_l^s \right) \right]_\partial$$

$$8\pi G T^i_j = \sqrt{-\gamma} \left[K^i_j - K \gamma^i_j + 4\lambda \epsilon^{(iklm} \left(\frac{1}{2} \hat{F}_{kl} \hat{R}_{j)m} + \nabla_n (A_k \hat{R}^n{}_{j)lm}) \right) \right]_\partial$$

$$D_i J^i = -\frac{1}{16\pi G} \epsilon^{ijkl} \left(\frac{\kappa}{3} \hat{F}_{ij} \hat{F}_{kl} + \lambda \hat{R}^m{}_{sij} \hat{R}^s{}_{mkl} \right)$$

$$D_i T^{ij} = -J_m \hat{F}^{mj} + A^j D_i J^i$$

$$\frac{-\lambda}{48\pi G} = \frac{N}{768\pi^2} \qquad \frac{-\kappa}{48\pi G} = \frac{N}{96\pi^2}$$

Holography

- Kubo formulas: fluctuations

$$a_x(z), a_y(z), h^t_x(z), h^t_y(z), h^z_x(z), h^z_y(z)$$

- background: charged AdS black hole

$$ds^2 = \frac{r^2}{L^2}\left(-f(r)dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2 f(r)}dr^2 \qquad A_{(0)} = \left(\beta - \frac{\mu\, r_{\rm H}^2}{r^2}\right)$$

- correlators are

$$\langle JJ\rangle = -ik_z\left(\frac{\mu}{4\pi^2} - \frac{\beta}{12\pi^2}\right)$$

$$\langle JT\rangle = -ik_z\left(\frac{\mu^2}{8\pi^2} + \frac{T^2}{24}\right)$$

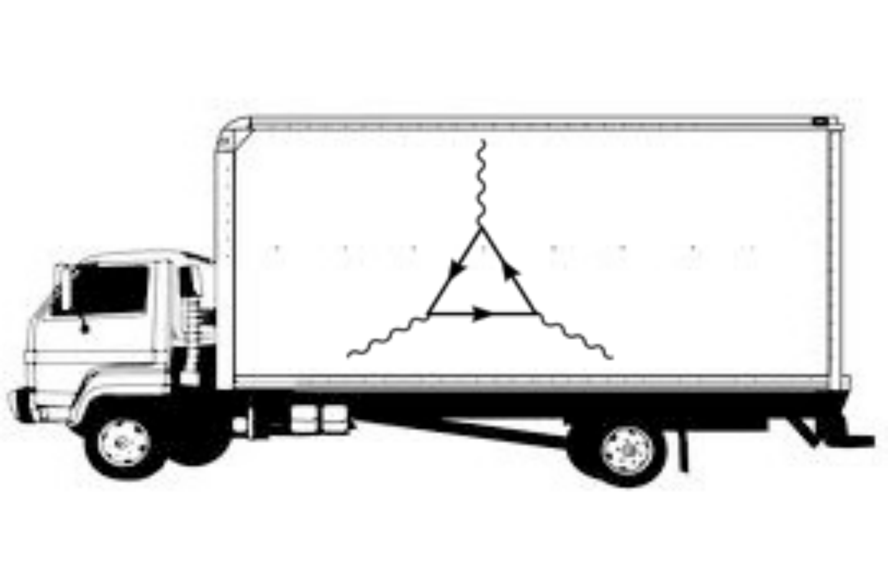
$$\langle TT\rangle = -ik_z\left(\frac{\mu^3}{12\pi^2} + \frac{\mu T^2}{12}\right)$$

same as weak coupling!
non-renormalization

(no T^3 terms)

[Neiman, Oz], [Loganayagam]

Discussion



- Anomalies -> parity violating transport
- Kubo formulas
- Surprise: mixed gauge gravitational anomaly contributes
- Holography with gravitational CS term
- (non)-renormalization?
- Higher dimension [Longanyagam], [Longanayagam, Surowka],[Kharzeev, Yee]
- Superfluids [Lin], [Neiman, Oz], [Batthacharya et al.]
- Hydrodynamics and derivative expansion? (fluid/gravity)
  [Son,Surowka], [Longanyagam], [Longanayagam, Surowka], [Chapman, Neiman, Oz]
- Observable effects?
  [Karen-Zur, Oz], [Kharzeev, Son], [Kharzeev, Yee]