Preliminary results from the 4PI effective action

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Outline:

- Introduction to nPI
- Motivation - transport coefficients
- 4PI scalar theory - structure of renormalization
- Results in a toy (toy) model

2PI for Scalar Theories:

generating functional with local and bi-local sources

$$Z[J,K] = e^{iW[J,K]} = \int \mathcal{D}\hat{\phi} e^{i(S[\hat{\phi}]+J_i\hat{\phi}^i+\frac{1}{2}R_{ij}\hat{\phi}^i\hat{\phi}^j)}$$

short-hand notation:

$$\int dx \int dy \, \varphi(x)R(x,y)\varphi(y) \to \varphi_i R_{ij}\varphi_j \to R\varphi^2$$

Legendre transform:

$$
\begin{aligned}
\Gamma[\phi, D] &= W[J,K] - J_i\phi^i - \frac{1}{2}\phi^i R_{ij}\phi^j \\
&= S_{\rm cl}[\phi] + \frac{i}{2}\mathrm{Tr}\ln D^{-1} + \frac{i}{2}\mathrm{Tr}\, D_0^{-1}(D - D_0) + \Gamma_2[\phi, D]
\end{aligned}
$$

$\Gamma[\phi, D]$ is a function of the 1- and 2-point functions

ϕ and D are determined self-consistently from the equations of motion

variational principle (in the absence of sources)

$$
\frac{\delta\Gamma}{\delta\phi} = \frac{\delta\Gamma}{\delta D} = 0
$$

$$
\Rightarrow \quad D^{-1} = D_0^{-1} - \Sigma[\phi, D]\,, \qquad \Sigma[\phi, D] := 2i\frac{\delta\Gamma_2}{\delta D}
$$

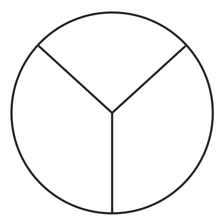
Compare to $\Gamma[\phi]$ = 1PI effective action:

- $\Gamma[\phi, D]$ depends on the self consistent propagator

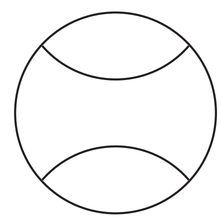
$\longrightarrow$ truncated $\Gamma[\phi, D]$ includes an infinite resummation of diagrams

$\longrightarrow$ non-perturbative

- $\Gamma[\phi, D]$ is 2PI - no double counting

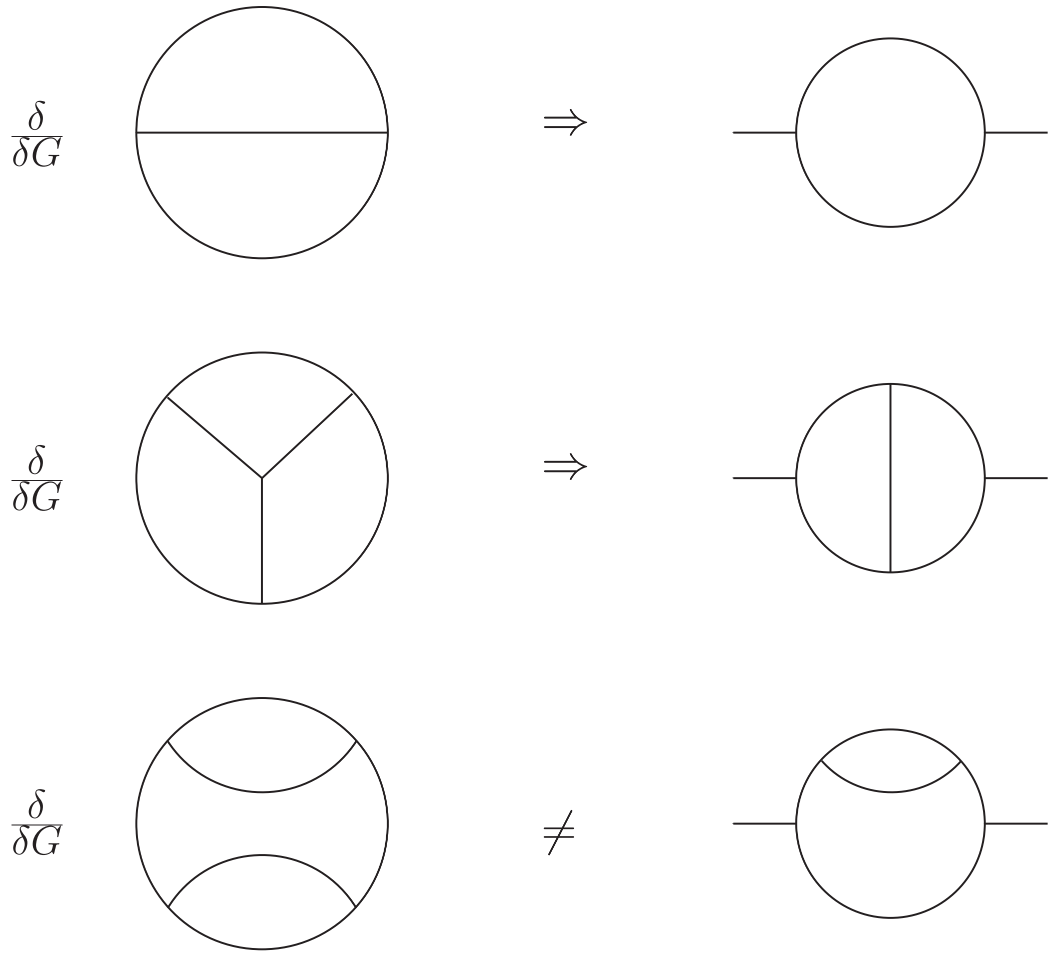


$2PI$



$2PR$

example: consider contributions to $\Sigma = 2i\delta\Gamma_2/\delta G$



note: expect difficulties with the ward identity

nPI effective action

nPI Γ is a functional of n-point functions

3PI $\Gamma[\phi, D, U]$, 4PI $\Gamma[\phi, D, U, V] \cdots$

n-point functions determined self-consistently from the equations of motion

$\Rightarrow$ hierarchy of coupled equations - no exact solution method is available

$\Rightarrow$ use approximation techniques: truncate the effective action

goal: improve convergence relative to standard perturbation theory

compare: HTL effective theory (high Temperature gauge theories)

in principle nPI is valid arbitrarily far from equilibrium

can apply formalism to the calculation of transport coefficients

Transport Coefficients:

idea: a far from equilibrium system will eventually equilibrate and thermalize

look at final stages of evolution and study the approach to equilibrium

transport coefficients:

- efficiency to transport conserved quantities over large distances
- characterise linear deviations from equilibrium
- calculated from moments of the phase space distribution or from the kubo formulae

Kubo Formulae:

electrical conductivity and shear viscosity:

$$\sigma = \frac{1}{6}\frac{\partial}{\partial\omega}\rho^{ii}(\omega,0)|_{\omega=0} \qquad \eta = \frac{1}{20}\frac{\partial}{\partial\omega}\rho_{\pi\pi}(\omega,0)|_{\omega=0}$$

$$\rho_{\mu\nu}(\omega,\vec{p}) = \int d^4x\; e^{ipx}\langle[j^\mu(x), j^\nu(0)]\rangle_{eq}$$

$$\rho_{\pi\pi}(\omega,\vec{p}) = \int d^4x\; e^{ipx}\langle[\pi_{ij}(x), \pi_{ij}(0)]\rangle_{eq}$$

tc's $\sim$ slope of the current-current spectral functions at $\omega = 0$

$\Rightarrow$ related to the equilibrium 2-point vertex function: $\lim_{q_0\to 0}\Sigma(q_0,0)$

$\Rightarrow$ are calculated using methods of equilibrium field theory

Transport Coefficients and Singularities

calculation is complicated by the presence of singularities

$\longrightarrow$ there are infinite sets of terms that contribute at leading order

$\longrightarrow$ we need a resummation

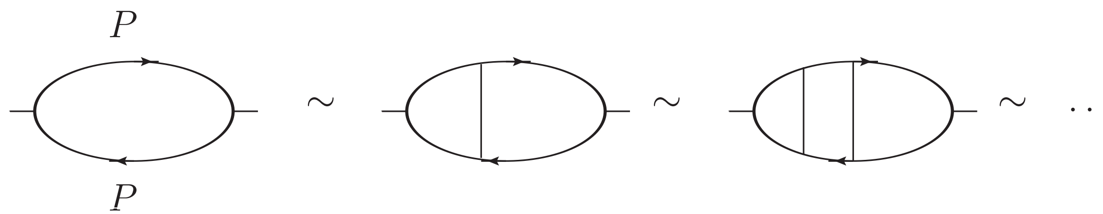
IDEA:

eom's from nPI formalism resum diagrams that contribute at a given order

(1) pinch singularities

consider the zero frequency limit in the kubo formulae

$\lim_{q_0 \to 0}$



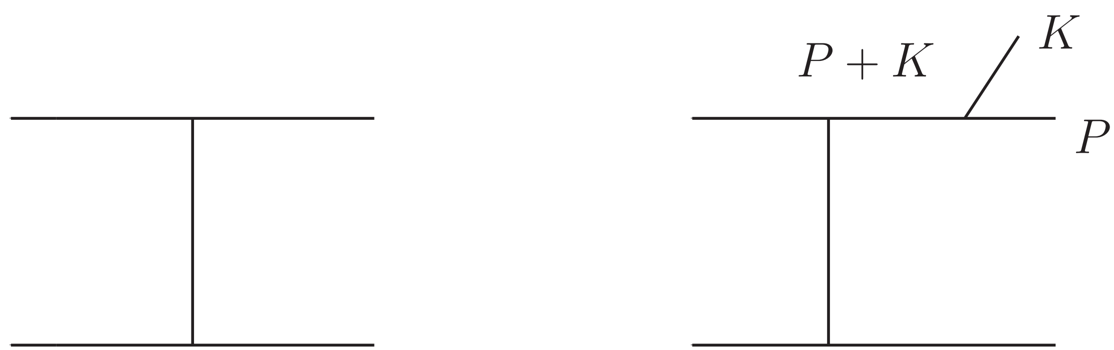
pairs of ret/adv props with the same momenta: $\int dp_0 D^{ret}(P) G^{adv}(P)$
$\longrightarrow$ a divergence called a ‘pinch singularity’

regulate using resummed propagators: $D^{ret} G^{adv} \sim \frac{\rho}{2\mathrm{Im}\Sigma}$
$\longrightarrow$ extra factors of coupling in the denominators

$\longrightarrow$ infinite set of graphs with pinching pairs - all need to be resummed

(2) colinear singularities

compare $2 \leftrightarrow 2$ scattering and $2 \leftrightarrow 3$ scattering



2nd is formally higher order BUT colinear singularity $\longrightarrow$ enhancement

$$(P+K)^2 = \underbrace{P^2 + K^2}_{zero} \pm 2pk(1-\cos\theta)$$

$\longrightarrow \infty$ series of colinear singularities must be resummed (LPM effect)

leading order transport coefficients:

need 2 coupled integral equations that resum pinch and colinear singularities

very subtle power-counting arguments to see what contributes even at lo

Use nPI to calculate of transport coefficients:

integral equations that resum singularities are produced in a natural way

without any power-counting arguments

$\Rightarrow$ a natural framework to work beyond leading order

Results:

3PI: can reproduce lo kinetic theory results = direct connection to field theory

P. Arnold, G.D. Moore and L. G. Yaffe, JHEP **0305**, *051 (2003).*

MEC and E. Kovalchuk, Phys. Rev. D **76**, *045019 (2007) - σ qed.*

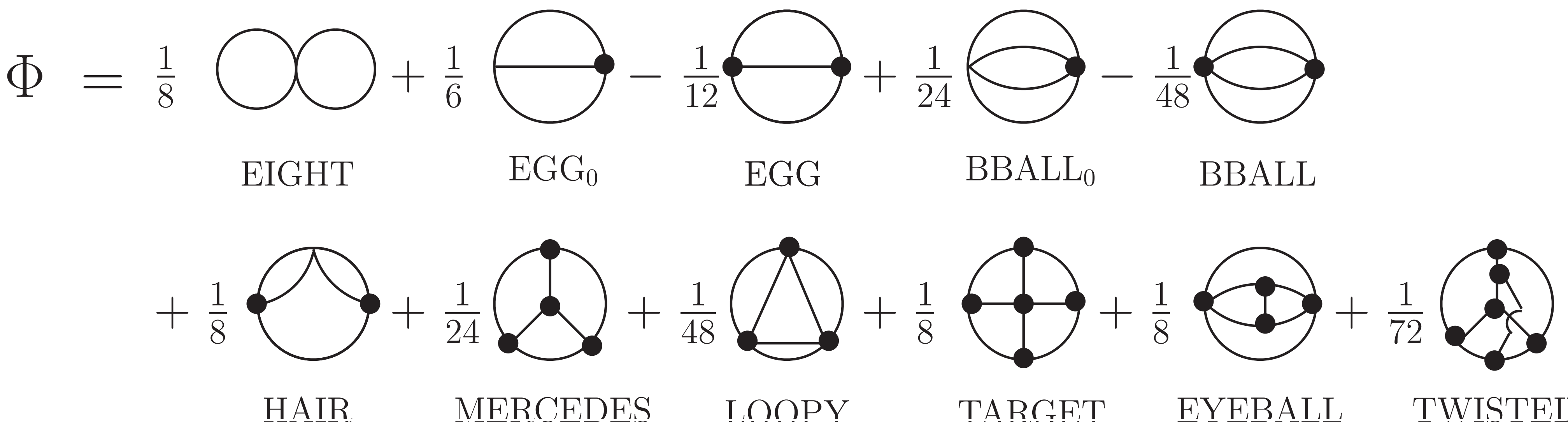
MEC and E. Kovalchuk, Phys. Rev. D **80**, *085013 (2009) - η qcd.*

4PI: next-to-leading order transport coefficients

MEC and E. Kovalchuk, Phys. Rev. D **81**, *065017 (2010).*

nlo - 4loop 4PI - scalar theory with cubic and quartic interaction

$$
\begin{aligned}
Z[J,R,R_3,R_4] &= \mathrm{Exp}[iW[J,R,R_3,R_4]] \\
&= \int D\hat{\phi}\; \mathrm{Exp}[i(S_{\mathrm{cl}}[\hat{\phi}] + J\hat{\phi} + \frac{1}{2!}R\hat{\phi}^2 + \frac{1}{3!}R_3\hat{\phi}^3 + \frac{1}{4!}R_4\hat{\phi}^4)] \\
\Gamma[\phi,D,U,V] &= S_{\mathrm{cl}}[\phi] + \frac{i}{2}\mathrm{Tr}\,\mathrm{Ln}D_{12}^{-1} \\
&\quad + \frac{i}{2}\mathrm{Tr}\left[(D_{12}^{0}(\phi))^{-1}\left(D_{21} - D_{21}^{0}(\phi)\right)\right] - i\Phi[\phi,D,U,V]
\end{aligned}
$$



Problem

n-point functions do not satisfy standard symmetry constraints

ex: in scalar theories w/ sb $O(N)$ symmetry - 2-pt fcn has no goldstone mode

ex: qed photon polarisation tensor isn't transverse in momentum space

related issues:

- renormalizability
- gauge invariance of results

simple example: 2PI effective theory $\Gamma[\phi, G]$
$\longrightarrow$ *corrected propagators but not corrected vertices*
but wi depend on cancellations between different topologies
(vertex corrections and self energy corrections)

resummed effective action

equations of motion

$$\frac{\delta\Gamma[\phi, D, U, V]}{\delta X_i} = 0\,, \qquad X_i \in \{\phi, D, U, V\}$$

self-consistent solutions $\tilde{X}_i \in \{\tilde{D}[\phi],\, \tilde{U}[\phi],\, \tilde{V}[\phi]\}$

$\Rightarrow$ resummed action- depends only on field expectation:

$$\tilde{\Gamma}[\phi] = \Gamma[\phi, \tilde{D}[\phi], \tilde{U}[\phi], \tilde{V}[\phi]]$$

definitions of n-point functions

theory defined in terms of self-consistently determined n-point fcns

additional (different) definitions are possible

- equivalent if the action is not truncated
- related through integral equations

will consider only the symmetric theory: $\tilde{\phi} = \tilde{U} = 0$

(1) Kernels: $\Lambda_{nml} = G^{-4l} 4!^l 2^m \frac{\delta^{l+m+n}\Phi}{\delta V^l \delta G^m \delta \phi^n}|_{\tilde{X}}$

$x = 4l + 2m + n$ is the total number of legs

example: $\Lambda_{020} = 4\frac{\delta^2\Phi}{\delta^2 G}$ - a 4-point function

(2) Resummed/Mixed: $M_n^{(x)} = \frac{\delta \tilde{X}^{x-n}}{\delta \tilde{\phi}^n}, \qquad M_n^{(2+n)} = \frac{\delta^n \tilde{\Sigma}}{\delta \tilde{\phi}^n}$

n derivs wrt ϕ of sc vertex $\tilde{X}^{x-n}$ w/ $x - n$ legs $\longrightarrow$ M has x legs

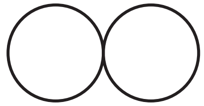
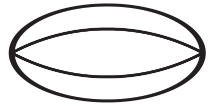
ex: $\frac{\delta^2 \tilde{\Sigma}}{\delta^2 \phi}$ = another 4-point function

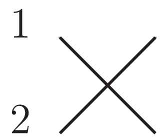
(3) Connected: $M_k^c = \langle \phi\phi \cdots \rangle = -(-i)^{k+1} \frac{\delta^k W}{\delta J^k}$

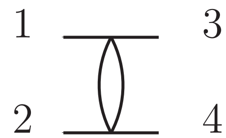
ex: M_k^c = connected 4-point function

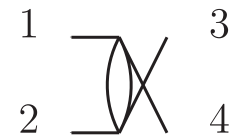
M vertices satisfy integral equations w Λ functions as kernels (chain rule)

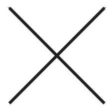
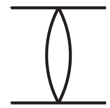
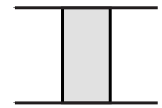
2PI example:

$\Phi = \frac{1}{8}$  $+\frac{1}{48}$ 

$\Lambda_{020} = 4\frac{\delta^2\Phi}{G_{12}G_{34}} =$ 

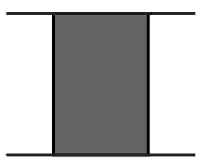
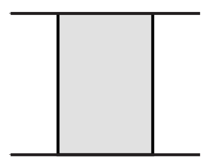
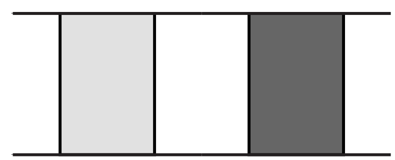
$+\frac{1}{2}$ 

$+\frac{1}{2}$ 

$=$  $+(2)\frac{1}{2}$  $=$ 

$M = D^{-4}M_4^c$ (amputated-cntd 4-point fcn) - resumms Λ in the s-channel

$$M = \Lambda_{020} + \frac{1}{2}\Lambda_{020}G^2M$$

 $=$  $+$ 

Renormalization:

previous m, λ, G and V were bare quantities (unrenormalized)

bare quantities related to renormalized ones:

$$\delta m^2 = Z m_B^2 - m^2 \,, \quad \delta\lambda = Z^2 \lambda_B - \lambda \,, \quad \delta Z = Z - 1$$

$$G_B = Z G \,, \quad V_B = Z^{-2} V \,,$$

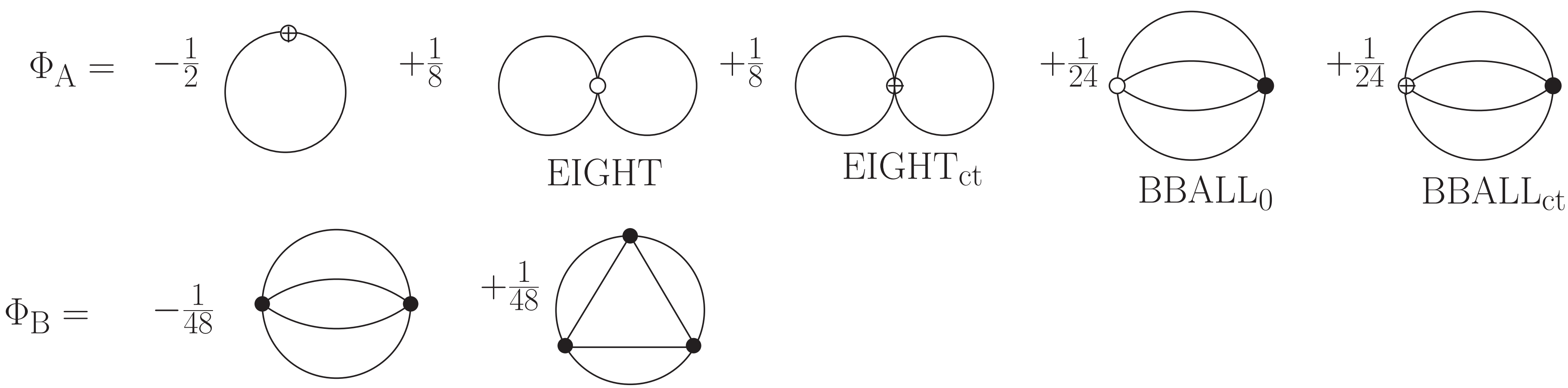
$$Z G_{0B}^{-1} = G_0^{-1} + \delta G_0^{-1} \,, \quad \delta G_0^{-1} = i(\delta Z \Box + \delta m^2)$$

Γ in terms of renormalized quantities:

$$i\Gamma[G,V] = -\frac{1}{2}\mathrm{Tr}\ln G^{-1} - \frac{1}{2}\mathrm{Tr}G_0^{-1}G + \Phi_A[G,V] + \Phi_B[G,V]$$

$$\Phi_A = -\frac{1}{2}\mathrm{Tr}\delta G_0^{-1}G + \frac{1}{8}(\lambda + \delta\lambda_{\mathrm{et}})G^2 + \frac{1}{4!}(\lambda + \delta\lambda_{\mathrm{bb}})G^4 V$$

$$\Phi_B = -\frac{1}{2}\frac{1}{4!}V^2G^4 + \frac{1}{48}V^3G^6$$



Key idea:

allowed:
finite set of $\vec{p}$-independent ct with same structure as terms in orig action

$\Rightarrow$ only one coupling constant counter-term $\delta\lambda_{\rm bb} = \delta\lambda_{\rm et}$

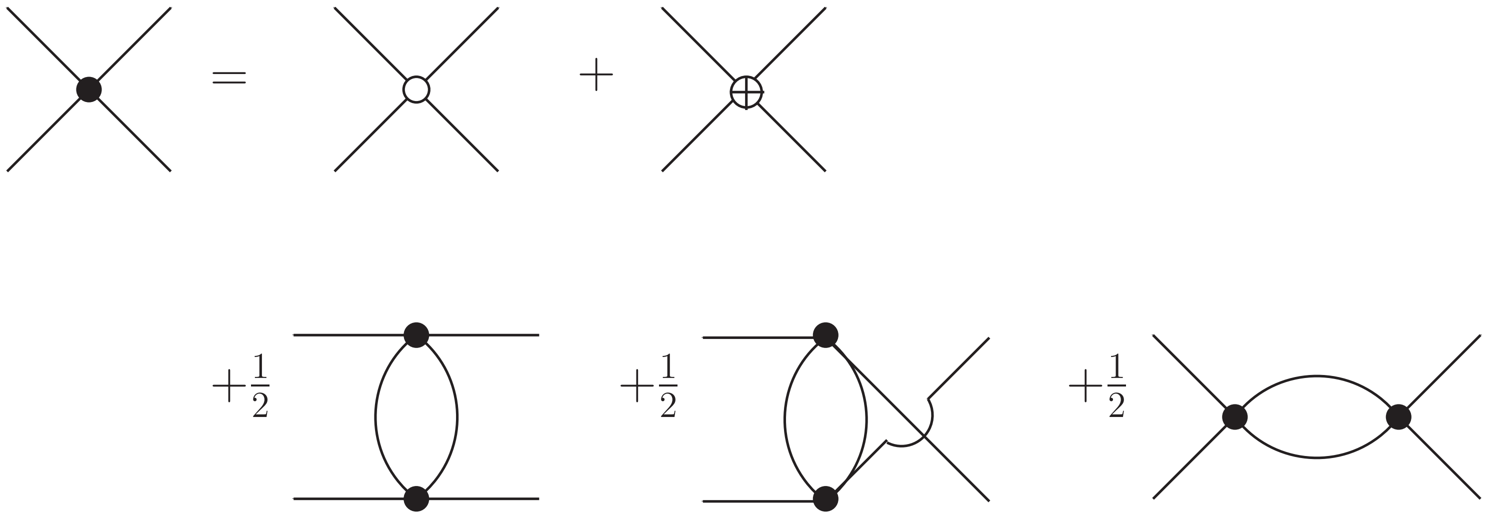
must show different n-point fcns can be renormalized w/ same counter-terms

I concentrate on the functions V and Σ $[M]$

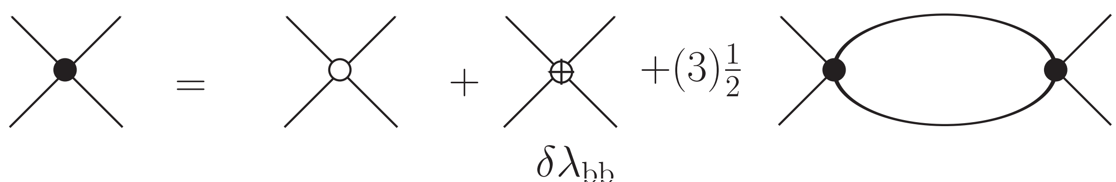
will show: M and V both renormalized by $\delta\lambda$ *up to the order of the truncation*

2PI renormalization:
J-P Blaizot, E. Iancu, U. Reinosa, Nucl. Phys. A736, 149 (2004).
H. Hees, J. Knoll, Phys. Rev. D65, 025010 (2002); D65 105005 (2002); D66 025028 (2002).
J. Berges, S. Borsanyi, U. Reinosa, J. Serreau, Annals Phys. 320, 344 (2005); JHEP 0607, 028, (2006).

The vertex V obtained from eom:



short-hand notation:



Vertex Splitting:

Method to isolate $\vec{p}$-independent divergences: vertex splitting

group factors in integrand $f(\{P_i\}, L)$

- L an integration variable and $L \notin \{P_i\}$

split the function f:

$$f(\{P_i\}, L) = \Delta_L f + f(0, L) \quad \text{with} \quad \Delta_L f = f(\{P_i\}, L) - f(0, L)$$

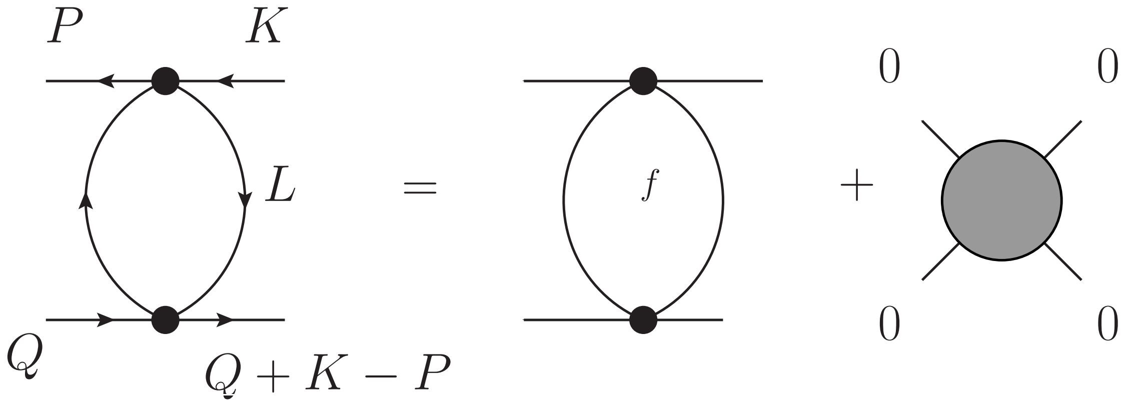
$$\text{large } L \quad \rightarrow \quad \Delta_L f \sim \frac{1}{L} f$$

Weinberg's theorem

- can show integrals we need at most logarithmically divergent

$\Rightarrow$ term with a factor $\Delta_L f \sim \frac{1}{L} f$ is finite when integrated over L

Example:



$$I_t^{VV}(-P,Q,K,-Q_t) = \int dL\, V(Q,-Q_t,L,-L_t) G_L G_{L_t} V(-L,L_t,-P,K)$$

define: $f_1(\{P_i\},L) = V(Q,-Q_t,L,-L_t)G_L \qquad f_2(\{P_i\},L) = G_{L_t}V(-L,L$

$$I_t^{VV}(-P,Q,K,-Q_t) = \int dL[(\Delta_L f_1 + f_1(0,L))(\Delta_L f_2 + f_2(0,L))]$$

terms with least one Δ_L are finite

remaining term is momentum independent:

$$I_t^{VV}(-P,Q,K,-Q_t) = I_t^{(VV\,\mathrm{fin})}(-P,Q,K,-Q_t) + I_1(V_0,V_0)$$

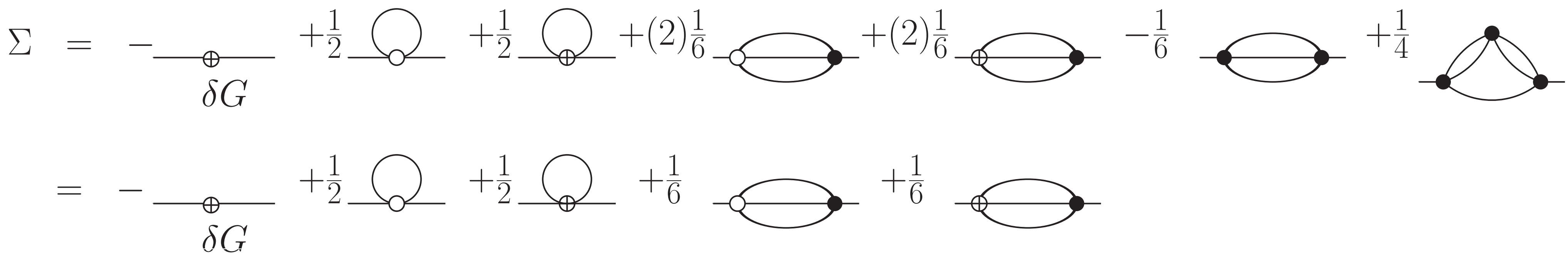
$$I_1(V_0,V_0) = \int dL\, V(0,0,L,-L) G_L^2 V(L,-L,0,0)$$

RESULT: $\delta_{\mathrm{bb}}^{(1)} = -\frac{3}{2} I_1(V_0,V_0)$

renormalization of the 2-point function

KEY 1: eom for Σ simplifies to SD form using higher eom's

(MEC and Yun Guo, Phys. Rev. D83, 016006 (2011))



$$\Sigma(p) = i(\delta Z P^2 - \delta m^2) + \frac{1}{2}(\lambda + \delta\lambda_{\rm et}) \int dQ\; G(Q)$$
$$+\frac{1}{6}(\lambda + \delta\lambda_{\rm bb}) \int dQ \int dK\; V(P,Q,K) G(Q) G(K) G(Q+K+P)$$

KEY 2: Σ eom can be repackaged to contain the amputated vertex M

extract the leading asymptotic behaviour:

$$\tilde{G} = G_{\rm as} + \delta G\,, \quad \Sigma(\tilde{G}) = \Sigma_{\rm as} + \Sigma_0$$

where $G_{\rm as} \sim P^{-2}$ and $\Sigma_{\rm as} \sim P^2$

$M(P,K) = \Lambda(P,K) + \frac{1}{2}\int dR\,\Lambda(P,R)G(R)^2 M(R,K)$

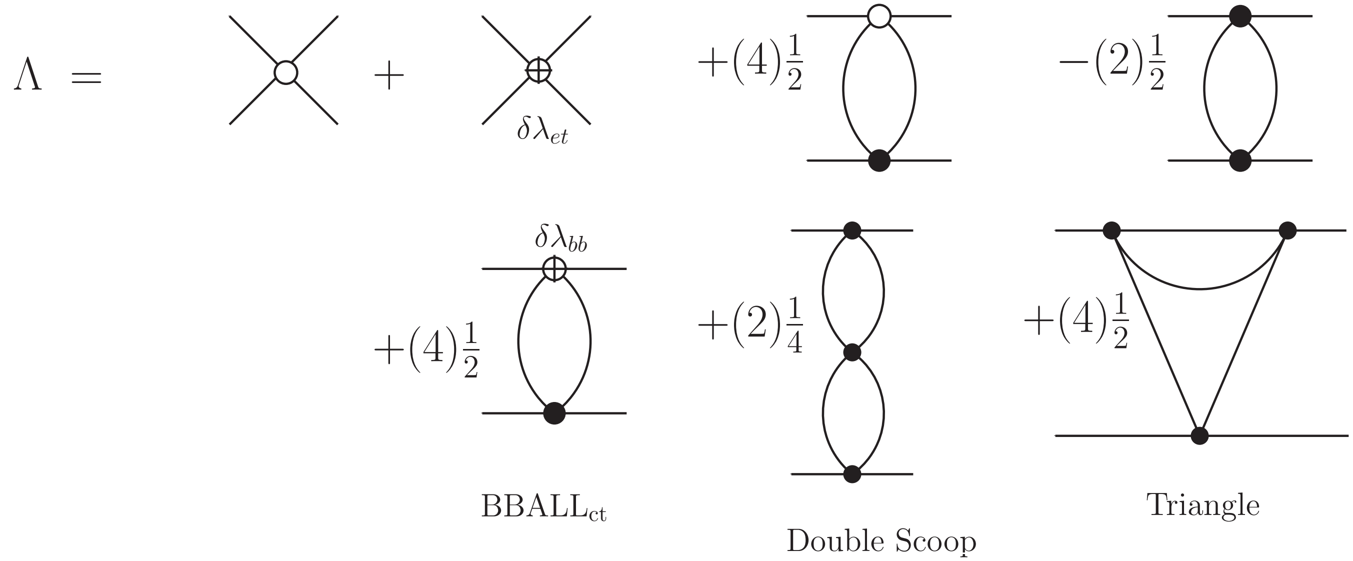
$$\Sigma_0(K) = -i\delta m^2 - \frac{i}{2}(m^2 + \delta m^2)\int dP G_{\rm as}^2(P)M(P,0) + \text{ finite terms}$$

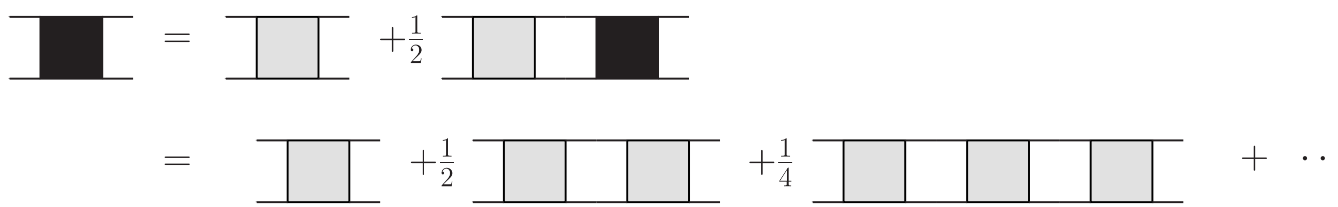
P-integral is divergent but $\vec{p}$-independent

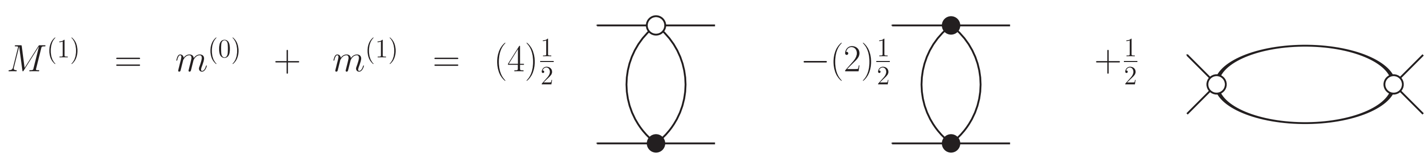
$\Rightarrow$ can be cancelled by an appropriate choice of the counter-term δm^2

$\Rightarrow$ must show that the vertex M can be renormalized with same $\delta\lambda$

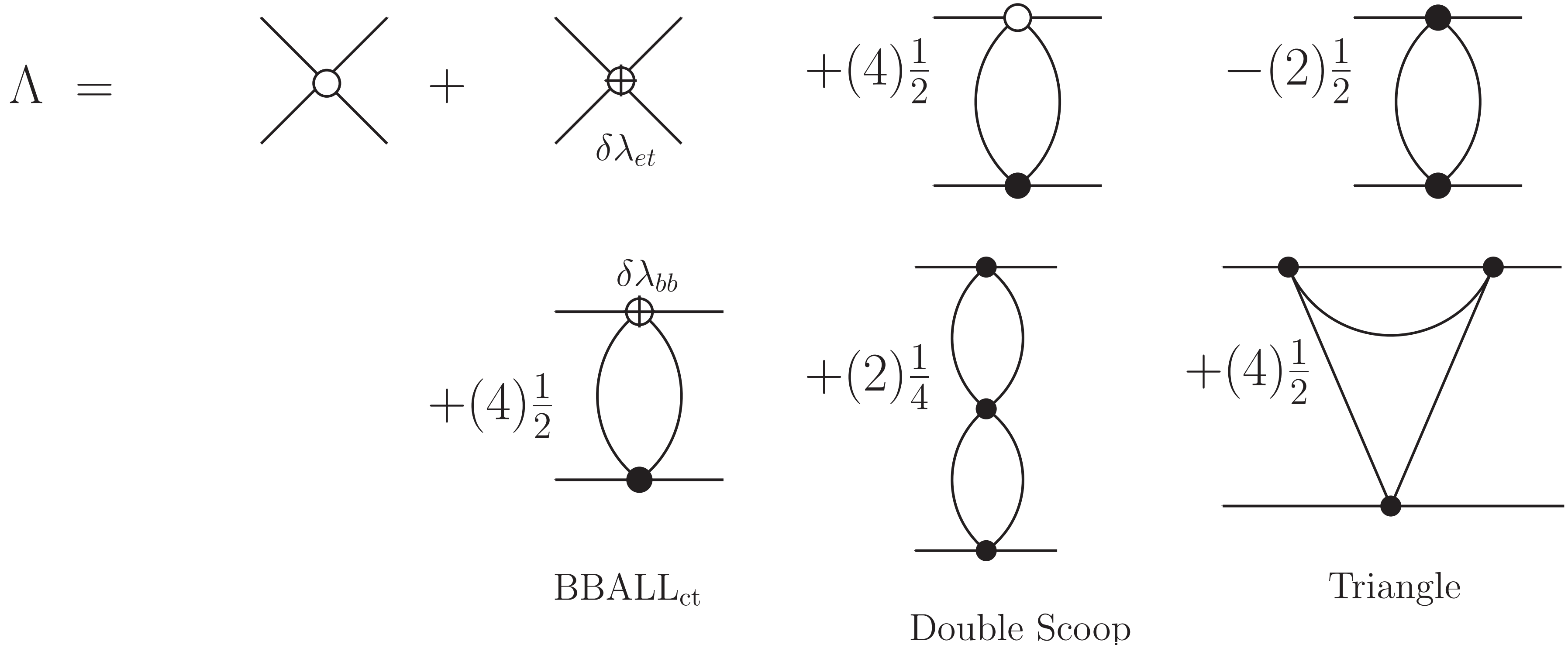
1-loop terms in M







Overlapping divergences



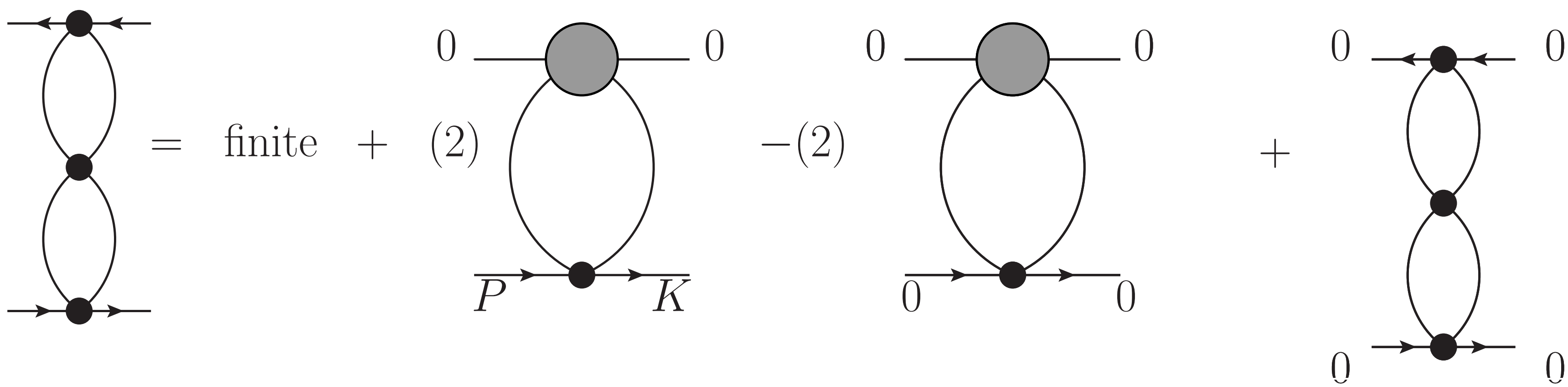
double-scoop and triangle graphs contain overlapping divergences

use $\delta\lambda_{bb}$ to cancel them

use $\delta\lambda_{\mathrm{et}}$ to cancel overall divergence vertex splitting procedure

$\longrightarrow$ isolate overlapping divergences and $\vec{p}$-independent piece

example: double-scoop diagram



Organizational trick:

2 counter-terms: $\delta\lambda_{\mathrm{bb}}$ and $\delta\lambda_{\mathrm{et}} = \delta\lambda_{\mathrm{et}}^{\mathrm{bp}} + \Delta\lambda_{\mathrm{et}}$

choose $\delta\lambda_{\mathrm{et}}^{\mathrm{bp}}$ and $\delta\lambda_{\mathrm{bb}}$ to renormalize Λ

choose $\Delta\lambda_{\mathrm{et}}$ to absorb the new divergences in M produced by iteration

$$\Lambda = \Lambda[\delta\lambda_{\mathrm{et}}^{\mathrm{bp}}, \delta\lambda_{\mathrm{bb}}] + \Delta\lambda_{\mathrm{et}} =: \Lambda_f + \Delta\lambda_{\mathrm{et}}$$

can show : $$\Delta\lambda_{\mathrm{et}} = -\frac{1}{2}\int dL\,[\Lambda_f(0,L) + \Delta\lambda_{\mathrm{et}}]\,G^2(L)M(L,0)$$

collect results:

$$\delta\lambda_{\mathrm{et}}^{\mathrm{bp}\,(1)} = -(4)\frac{1}{2}I_1(\lambda, V_0) + (2)\frac{1}{2}I_1(V_0, V_0)$$

$$\delta\lambda_{\mathrm{bb}}^{(1)} = -\frac{3}{2}I_1(V_0, V_0) \quad \text{cancel overlap in 2xScoop/triangle}$$

$$\Delta\lambda_{\mathrm{et}}^{(\mathrm{it}\,1)} = -\frac{1}{2}I_1(\Lambda_{f0}, \Lambda_{f0})$$

keep only 1-loop in the skeleton expansion:

$$\delta\lambda_{\mathrm{bb}}^{(1)} = \delta\lambda_{\mathrm{et}}^{\mathrm{bp}\,(1)} + \Delta\lambda_{\mathrm{et}}^{(1)} = -\frac{3}{2}I_1(\Lambda_0, \Lambda_0)$$

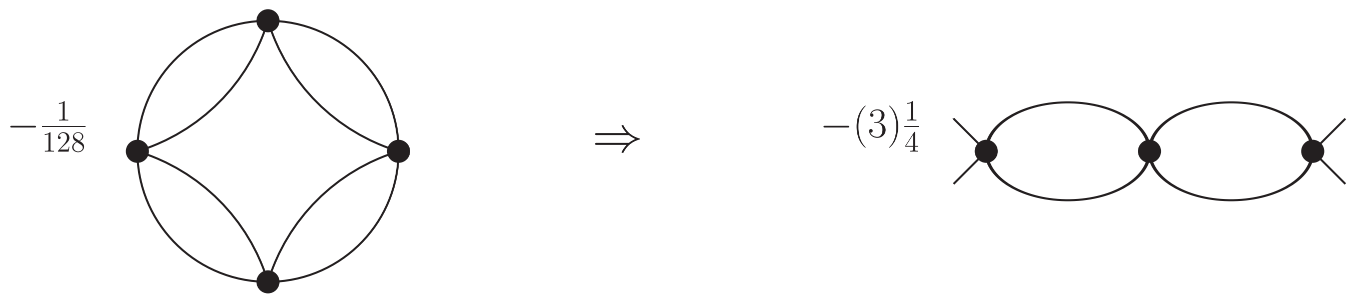
$\Rightarrow$ to 1-loop $\delta\lambda_{\mathrm{et}}$=$\delta\lambda_{\mathrm{bb}}$=$\delta\lambda$ renormalizes both M (and Σ) and V

not true at 2-loops

reason: Λ and M produced at different orders in the skeleton expansion

$\delta/\delta G$ opens 1-loop
$\delta/\delta V$ opens 3-loops

can show: include contributions from 5-loop terms in effective action to Veom



$\Rightarrow$ counter-terms that renormalize V and M are the same to 2-loops

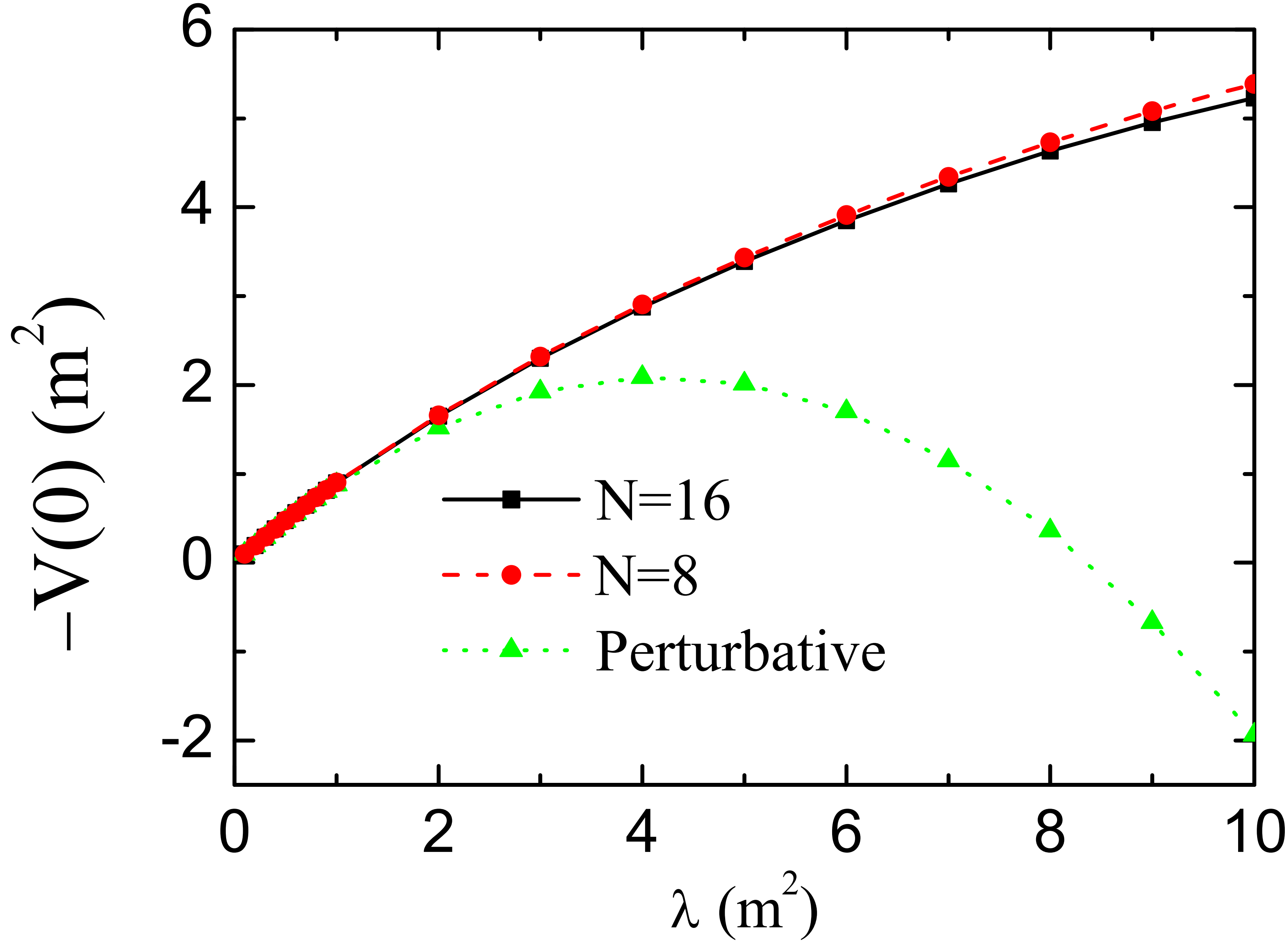
Numerical results:

work in 2-dimensions

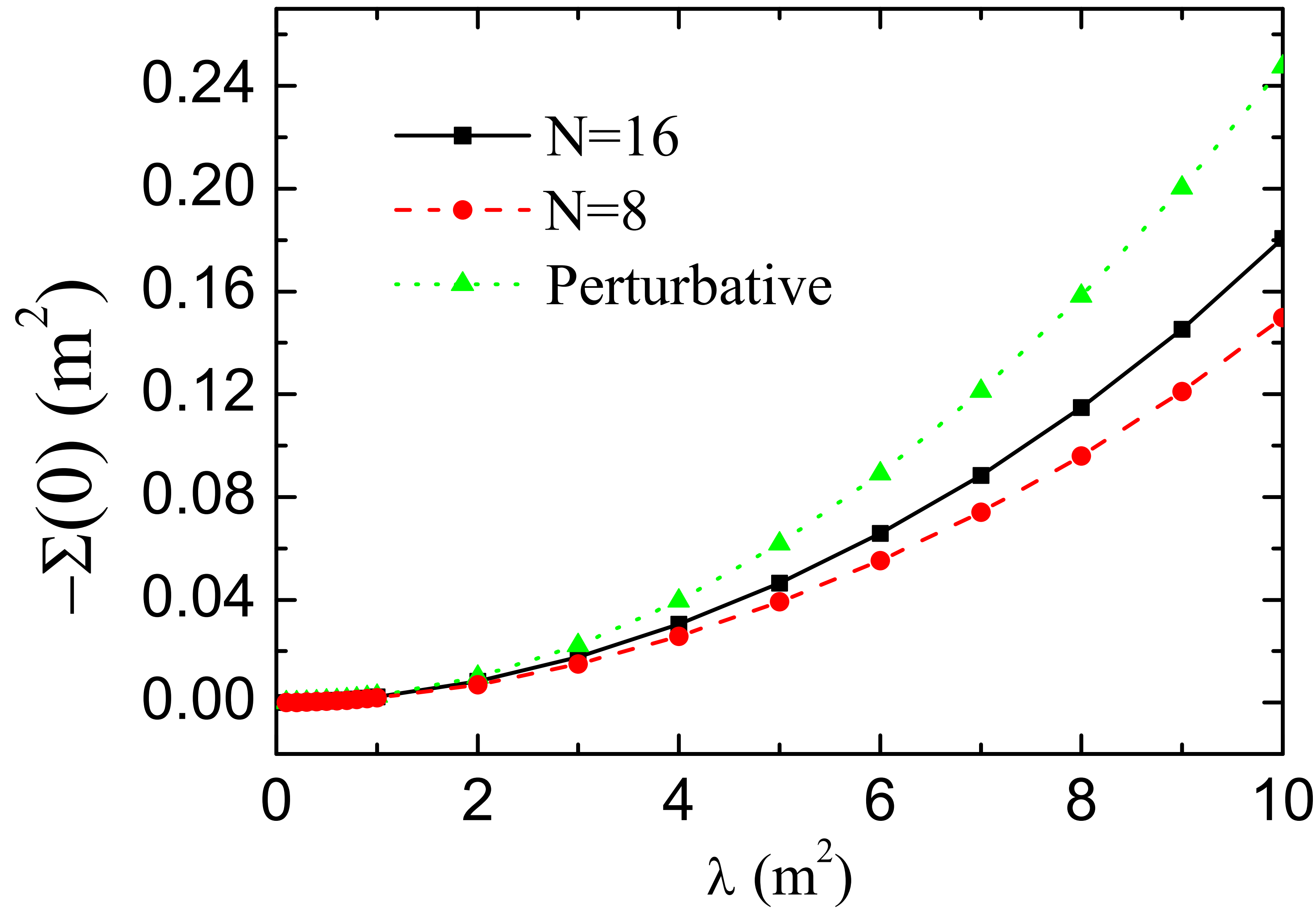
only divergence is a log divergence from tadpole diagram in 2-point function

absorb with a counter-term δm^2

$N \times N$ 2-d symmetric lattice, periodic bc, lattice spacing $a = 2\pi/(Nm)$



-V(0) (m²)
λ (m²)
N=16
N=8
Perturbative
6
4
2
0
-2
0
2
4
6
8
10



N=16
N=8
Perturbative
0.00
0.04
0.08
0.12
0.16
0.20
0.24
0
2
4
6
8
10
$-\Sigma(0)$ (m^2)
λ (m^2)

Conclusions:

- renormalization of 4PI effective action has same structure as 2PI Γ
- numerical calculations are challenging

  but preliminary results are promising