Quantum simulating real-time dynamics in gauge theories

Collaborators:

Pascal Stebler, Philippe Widmer, Uwe-Jens Wiese (Bern)

Markus Müller (Madrid)

Enrique Rico-Ortega, Marcello Dalmonte, Peter Zoller (Innsbruck)

Debasish Banerjee

Institute for Theoretical Physics, Bern, Switzerland

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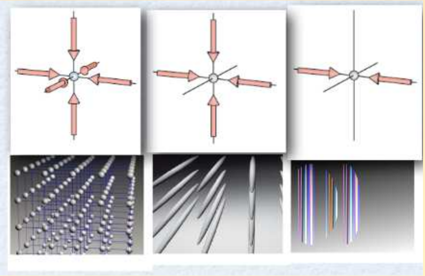
Sign Problem in it's various incarnations

- Finite density: finite μ_B QCD prime example. Simpler examples (e.g. scalar theories) can be constructed $\Rightarrow$ Complex actions $\Rightarrow$ certain cases can be solved with reformulations (world line, worm) or complex Langevin
- Fermions: repulsive fermionic Hubbard model away from half filling prime example. Meron cluster algorithms can be used to solve certain sign problems.
- Non-zero θ angle: QCD at non-zero θ. Again, simpler models at finite-θ (e.g. 2d O(3), CP(N-1)) can be solved with meron cluster methods
- Geometrically frustrated anti-ferromagnets: Meron cluster method can again help in certain cases
- Real time evolution: No method known to solve any simple non-trivial model at sufficently large lattices. Quantum Simulation with cold atoms/molecules and/or Rydberg ions

In it's most general form sign problem is NP hard Troyer, Wiese (2005).
General solution applicable to all problems unlikely

Quantum Simulation: Analogue

Basic idea: system of interest $\rightarrow$ model Hamiltonian (usually Hubbard type) $\rightarrow$ implement via optical lattices



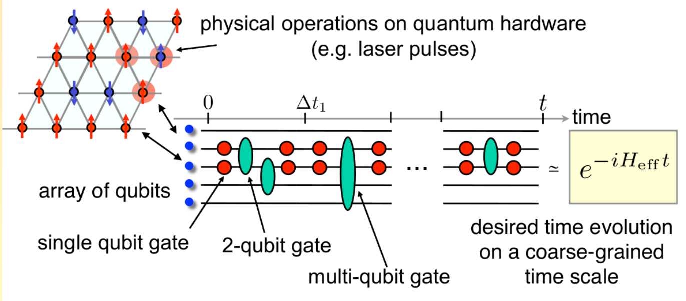
cold atoms in optical lattices realize Bosonic and Fermionic Hubbard models

$$H = -\kappa(t)\sum_{i,j}(b_i^\dagger b_j + b_j^\dagger b_i) + U(t)\sum_i \hat{n}_i(\hat{n}_i - 1) - \mu\sum_i \hat{n}_i$$

Example: Observation of Mott-insulator (disordered) to superfluid (ordered) phase. Excitation spectrum probed. Greiner et. al. (2002)

Quantum Simulation: Digital

Ions confined in an ion-trap; but interactions between individual ions can be controlled using gates. Engineering Hamiltonian not required. More control over interactions; complicated interactions can be programmed in; Main challenge: scalability to large systems



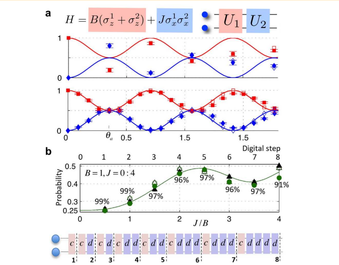
Small-scale prototype of quantum computer

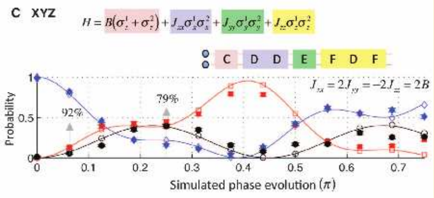
An example of real-time evolution

Use the Trotter-Suzuki decomposition

$$e^{-iHt} \simeq e^{-iH_1 t} e^{-iH_2 t} e^{[H_1, H_2] t^2/2}$$

to study the real time evolution of 2-Ising spins





Time-dependent variation of parameters possible

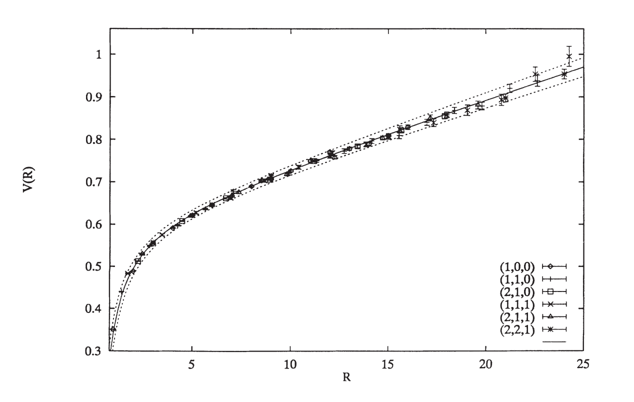
Trotter errors known and bounded; gate errors under control;

Implementation with upto 6 ions/spins Lanyon et. al. 2011

What to *see* in *real* time?

Confinement in QCD is phenomenologically explained by a “string”

$$V(r) = A + \frac{B}{r} + \sigma r$$



G.S. Bali, K. Schilling (1992); Bali et. al. (2005).

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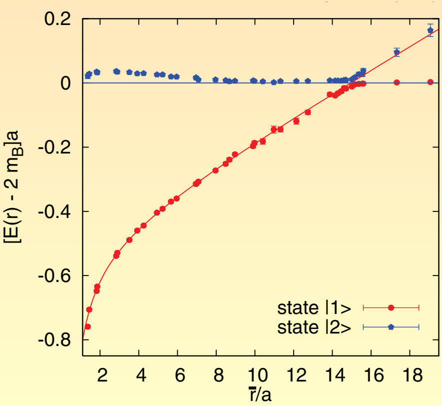

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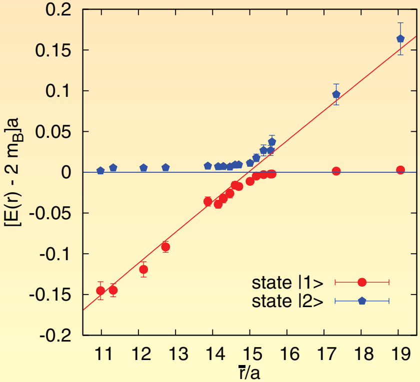

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The importance of being 'Quantum'

- Wilson formulation has continuum valued fields at each link and site; Unsuitable for AMO realisation
- Quantum Link Models (QLM) Chandrasekharan Wiese (1996) reformulate LGT to have discrete Hilbert spaces at each link, but generate continuum valued GT
- Example: the U(1) Quantum Link model
- Recall Wilson's formulation:

$$
\begin{aligned}
u_{x,\mu} &= e^{i\phi_{x,\mu}} = \cos(\phi_{x,\mu}) + i\sin(\phi_{x,\mu}) \\
S &= -\frac{J}{2}\sum_{x,\mu>\nu}(u_{x,\mu}u_{x+\mu,\nu}u^\dagger_{x+\nu,\mu}u^\dagger_{x,\nu} + h.c)
\end{aligned}
$$

Classical action → simulating a classical stat. mech. system

- Quantum counterpart → quantum H operator

$$
S = -\frac{J}{2}\sum_{x,\mu>\nu}(U_{x,\mu}U_{x+\mu,\nu}U^\dagger_{x+\nu,\mu}U^\dagger_{x,\nu} + h.c)
$$

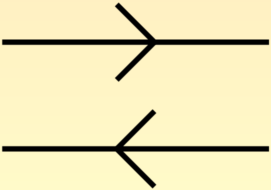
- $U_{x,\mu}$: operator acting on a discrete Hilbert space

The U(1) Quantum Link model

- Gauss Law generates gauge transformations. $[H, G_x] = 0$:

$$U'_{x,\mu} = \left[\prod_m \exp(-i\alpha_m G_m)\right] U_{x,\mu} \left[\prod_n \exp(i\alpha_n G_n)\right] = \exp(i\alpha_x) U_{x,\mu}$$

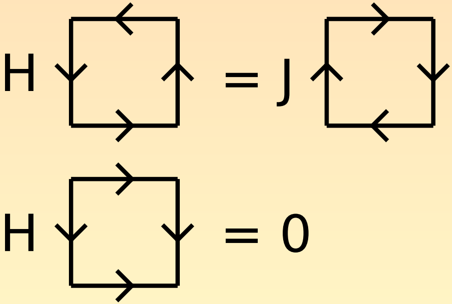
  Commutation relations $[G_x, U_{y,\mu}] = (\delta_{x,y+\mu} - \delta_{x,y}) U_{y,\mu}$ and $[G_x, U^\dagger_{y,\mu}] = (\delta_{x,y} - \delta_{x,y+\mu}) U^\dagger_{y,\mu}$ ensures Gauge invariance
- GT generated by electric fields: $G_x = \sum_\mu (E_{x,\mu} - E_{x-\mu,x})$
- Consider the spin-1/2 representation. Then, $E_{x,\mu} = \sigma^z$ Eigenvalues: $\pm\frac{1}{2}$
- Pictorially:



- Gauss' Law: $G_x |\psi\rangle = 0$

The U(1) QLM: Hamiltonian

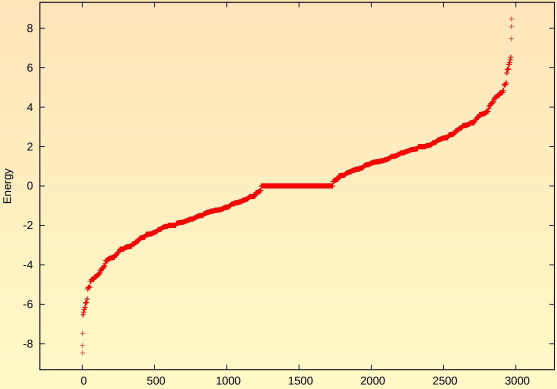
- In the spin-1/2 representation, $U = S^{+}; U^{\dagger} = S^{-}$
- The Hamiltonian acts by flipping a plaquette:



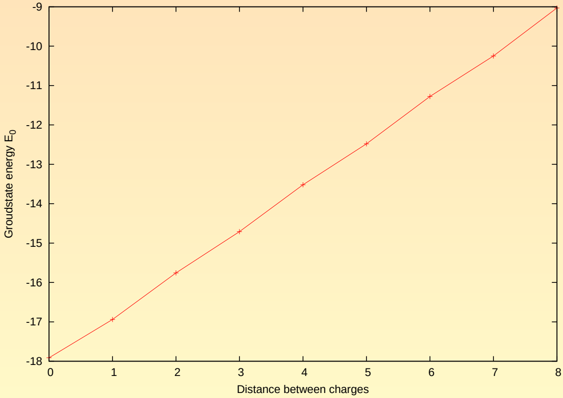
- Related to the Rokshar-Kivelson model, a candidate for exhibiting the spin-liquid phase

$$H_{RK} = H_{QLM} - \lambda \sum_{x} \left(\delta_{\square,flippable}\right)$$

The U(1) QLM: Spectrum



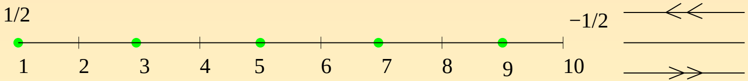
The U(1) QLM: Confinement



Schwinger model

Stick to simple models in 1-d having the same qualitative features for validating the quantum simulator. Schwinger model with QL in spin-1 representation and staggered fermions:

$$H = -\sum_x \left[\psi_x^\dagger U_{x,x+1} \psi_{x+1} + h.c.\right] + m \sum_x (-1)^x \psi_x^\dagger \psi_x + \frac{g^2}{2} \sum_x E_{x,x+1}^2 - 2\lambda \sum_x (-1)^x E_{x,x+1}$$



Translation symmetry by even number of lattice spacings

Gauss' Law: $[H, G_x] = 0$; $G_x = \psi_x^\dagger \psi_x + \frac{(-1)^x - 1}{2} - (E_{x,x+1} - E_{x-1,x})$

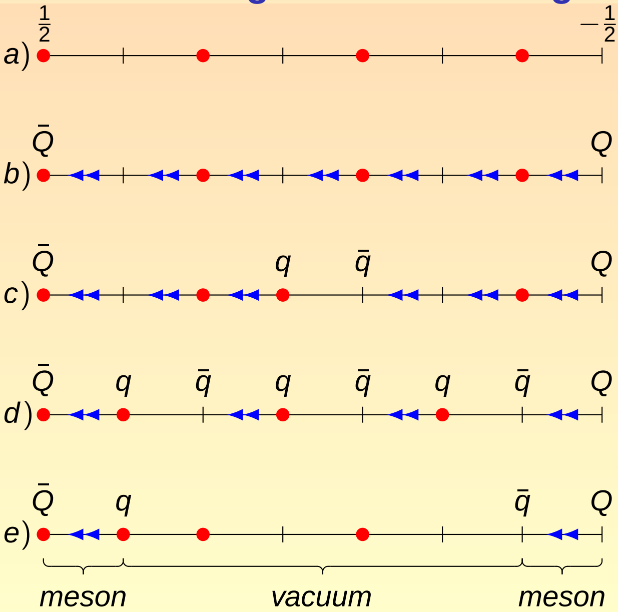
Parity: ${}^P\psi_x \to \psi_{-x}$, ${}^P\psi_x^\dagger \to \psi_{-x}^\dagger$; ${}^P U_{x,x+1} \to U^\dagger_{-x-1,-x}$, ${}^P E_{x,x+1} \to -E_{-x-1,-x}$

Charge Conjugation: ${}^C\psi_x \to (-1)^{x+1} \psi_{x+1}^\dagger$, ${}^C\psi_x^\dagger \to (-1)^{x+1} \psi_{x+1}$; ${}^C U_{x,x+1} \to U^\dagger_{x+1,x+2}$, ${}^C E_{x,x+1} \to -E_{x+1,x+2}$

Discrete **Chiral Symmetry:** broken by the mass term

${}^\chi\psi_x \to \psi_{x+1}$, ${}^\chi\psi_x^\dagger \to \psi_{x+1}^\dagger$; ${}^\chi U_{x,x+1} \to U_{x+1,x+2}$, ${}^\chi E_{x,x+1} \to E_{x+1,x+2}$

The String and its breaking



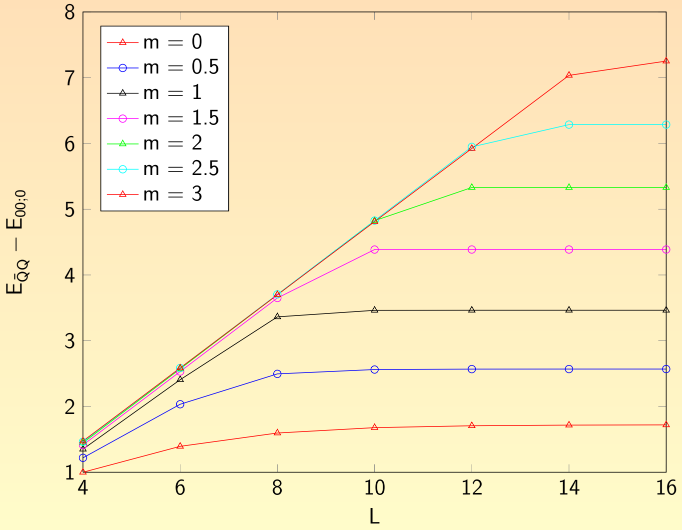
Energetics

- Energetics at $t \to 0$ straightforward to calculate. The λ term is redundant. String formation and breaking involves the gauge coupling.
- Vacuum state: $E_0 = -m\frac{L}{2}$
- Energy for the unbroken string state:
  $E_{\text{string}} - E_0 = \frac{g^2}{2}(L-1)$
- Energy for the two meson state: $E_{\text{mesons}} - E_0 = 2(\frac{g^2}{2} + m)$
- Energy difference: $E_{\text{string}} - E_{\text{mesons}} = \frac{g^2}{2}(L-3) - 2m$
- Critical distance for string breaking obtained from:

$$E_{\text{string}} - E_{\text{mesons}} = 0 \Longrightarrow L = \frac{4m}{g^2} + 3$$

- Sites and links next to the two boundaries do not change during the whole evolution. Quantum simulation of system size L needs $L + (L-1) - 4 = 2L - 5$ ions

Static Properties



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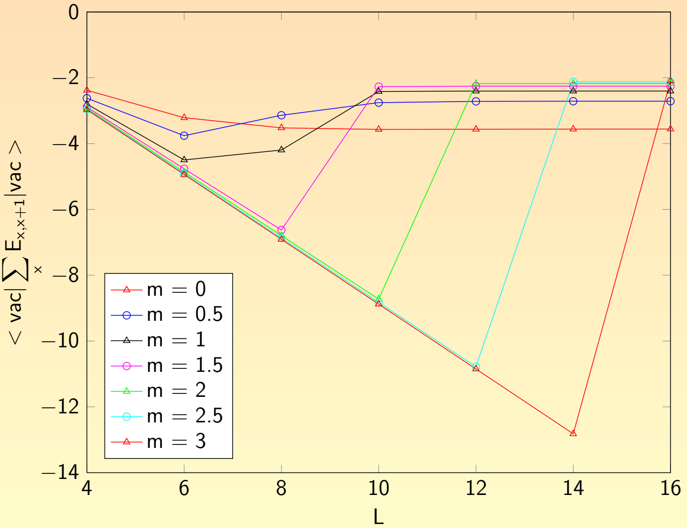
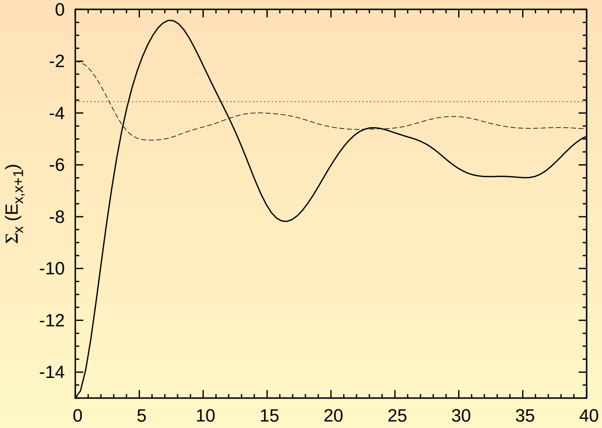


Time evolution

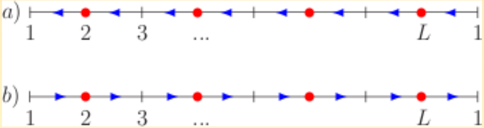


Dotted dashed: $\langle 0|\sum_x E_{x,x+1}|0\rangle$; Dashed: $\langle \text{br(t)}|\sum_x E_{x,x+1}|\text{br(t)}\rangle$; Continuous: $\langle \text{unbr(t)}|\sum_x E_{x,x+1}|\text{unbr(t)}\rangle$

False Vacuum Decay

- Consider the spin-1/2 representation. Then, the g^2 term trivial
- Useful to consider the term: $-2\lambda \sum_x (-1)^x E_{x,x+1}$
- C, P exact symmetries with the C & P inv ground state for $m < \lambda (\lambda > 0)$

- $m < \lambda$: two competing ground states exist, CP partners of each other



- Mimics $\theta = \pi$ in Schwinger model
- Decay of false C and P invariant vacuum by bubble nucleation into true vacuum with spontaneous breaking of C and P is another interesting exercise in real-time evolution of a quantum system, that cannot be done by classical computers

Microscopic and Effective theory

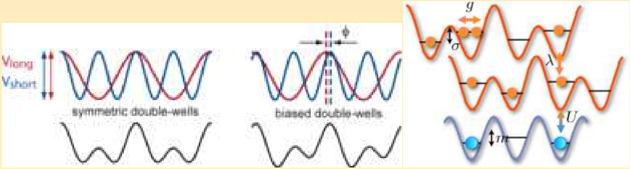
- The Schwinger model acts as an **effective** theory denote it by $\mathcal{H} = \{\mathcal{H}_F, \mathcal{H}_\mathcal{S}\}$ induced at low-energies by a microscopic **Hubbard** type model denoted by $H = \{H_F, H_S\} = H_{ph} \oplus H_{unph}$
- Idea: Clear separation of energy scales between H_{ph} and H_{unph}
- $\mathcal{H}$ *should* satisfy Gauss Law while acting in H_{ph}. States in H_{unph} need not.
- one-to-one correspondence between states in $\mathcal{H}$ and the physical Hilbert space of and H, i.e.,

$$\mathcal{H}|\Psi\rangle = \epsilon_\Psi|\Psi'\rangle \Rightarrow H|\Phi\rangle = \epsilon_\Phi|\Phi'\rangle \forall |\Phi\rangle \in H_{ph}$$

- ϵ_Ψ and ϵ_Φ same upto shifts

Microscopic Model

- Each term of the Schwinger model can be implemented via a Bose-Fermi Hubbard model and using superlattices
- What is a superlattice? – optical potential created by superposition of different harmonics



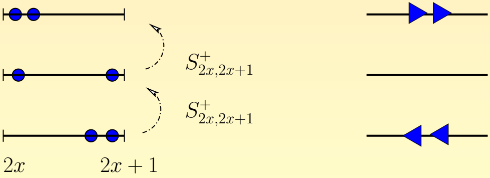
- Fermion mass can be directly implemented with super-lattice
- For link fields need Schwinger boson(s)

Microscopic Model

Label bosonic d.o.f on even and odd links as a and b

$$
\begin{aligned}
U_{2x,2x+1} = S^{+}_{2x,2x+1} &= a^{\dagger}_{2x} a_{2x+1} \\
U_{2x-1,2x} = S^{+}_{2x-1,2x} &= b^{\dagger}_{2x-1} b_{2x} \\
E_{2x,2x+1} = S^{z}_{2x,2x+1} &= (n^{a}_{2x} - n^{a}_{2x+1})/2 \\
E_{2x-1,2x} = S^{z}_{2x-1,2x} &= (n^{b}_{2x-1} - n^{b}_{2x})/2 \\
n^{a}_{2x} + n^{a}_{2x+1} = 2S &= n^{b}_{2x-1} + n^{b}_{2x}
\end{aligned}
$$

For a spin-1 representation, for example,



Imposing *effective* Gauge Invariance

Consider the following interaction term for the Hubbard hamiltonian:

$$
\begin{aligned}
H_U &= U\sum_x \left[(n_x^a)^2 + (n_x^b)^2 + 2n_x(n_a+n_b) + 2n_a n_b - (-1)^x(n_x + n_x^a + n_x^b)\right] \\
&= 2U\sum_x \left[(S^z_{x,x+1})^2 + (n_x - n_{x+1})S^z_{x,x+1} - S^z_{x-1,x}S^z_{x,x+1} - (-1)^x(S^z_{x,x+1} + n_x/2)\right] \\
&= U\sum_x (G_x)^2
\end{aligned}
$$

where we used

$$
\begin{aligned}
4(S^z_{2x,2x+1})^2 &= n^a_{2x} + n^a_{2x-1} - 2n^a_{2x}n^a_{2x-1} \\
4(S^z_{2x-1,2x})^2 &= n^b_{2x} + n^b_{2x-1} - 2n^a_{2x}n^a_{2x-1} \\
(n^a_{2x})^2 + (n^a_{2x+1})^2 &= 4S^2 - 2n^a_{2x}n^a_{2x+1}
\end{aligned}
$$

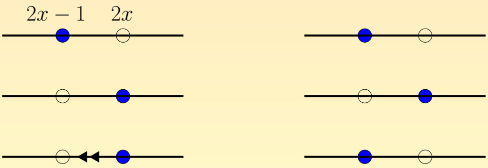
and ignored constants.
Violating GI costs energy $\mathcal{O}$(U)
In the limit $U \to \infty$ GI exact!

Low energy physics

Low energy physics induced by H_{pert}:

$$H_{pert} = t_F \sum_x (\psi_x^\dagger \psi_{x+1} + h.c) + m \sum_x (-1)^x n_x + \frac{g^2}{4} \sum_x \left[(n_x^a)^2 + (n_x^b)^2 \right]$$
$$+ \frac{J}{2} \sum_{x \in odd} (b_x^\dagger b_{x+1} + h.c) + \frac{J}{2} \sum_{x \in even} (a_x^\dagger a_{x+1} + h.c)$$

Other possible GI states are also generated: in particular, the fermion-gauge coupling is generated in 2^{nd} order PT

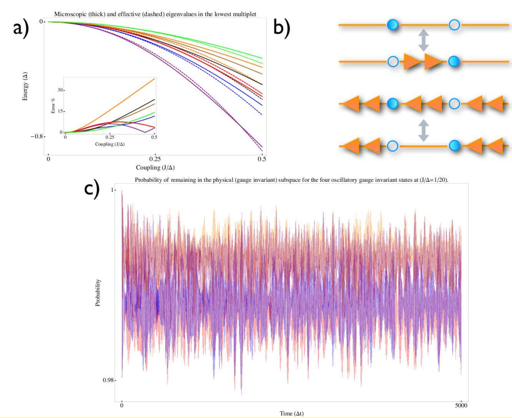


$\sim -\frac{t_F J}{2U}$ $\psi_{2x}^\dagger S_{2x-1,2x}^- \psi_{2x-1}$

$\sim \frac{t_F^2}{U}$

The other contribution supressed by $\frac{t_F}{J}$

How good is the approximation?



Prob of remaining in the GI subspace better than 98% for $\frac{J}{U} = \frac{1}{20}$!
Analysis of string breaking in the effective model for spin-1 under progress...

The SO(3) QLM

- The Hamiltonian of the model is

$$H = -J \sum_{x,\mu\neq\nu} \mathrm{Tr}(U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\mu} + \mathrm{h.c.})$$

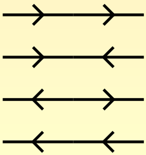
- In the SO(3) representation, $U_{x,\mu}$ are 3×3 matrices. ($U_{i,j},\ i = 1, 2, 3;\ j = 1, 2, 3$)
- Left ($\vec{L}$) and the right ($\vec{R}$) generators of GT distinct, satisfy the following commutation relations:

$$[R^a, U_{ij}] = -2i\epsilon_{akj} U_{ik}; \quad [L^a, U_{ij}] = 2i\epsilon_{aik} U_{kj}$$

An elegant representation can be obtained for all operators in terms of the σ-matrices

$$L^a = \sigma^a_L; \quad R^a = \sigma^a_R; \quad U_{ij} = \sigma^i_L \sigma^j_R$$

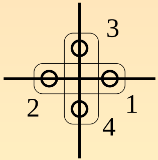
Four states per link



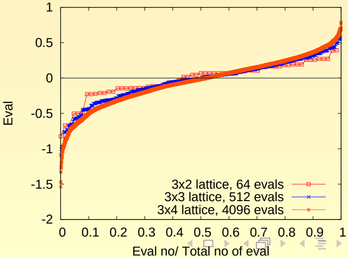
The SO(3) QLM: Spectrum

- Non-abelian Gauss' law: $\vec{G}_x = \sum_\mu (\vec{R}_{x-\hat{\mu},\mu} + \vec{L}_{x,\mu})$ requires construction of Gauge singlets
- Construct singlets out of 2d spin-1/2

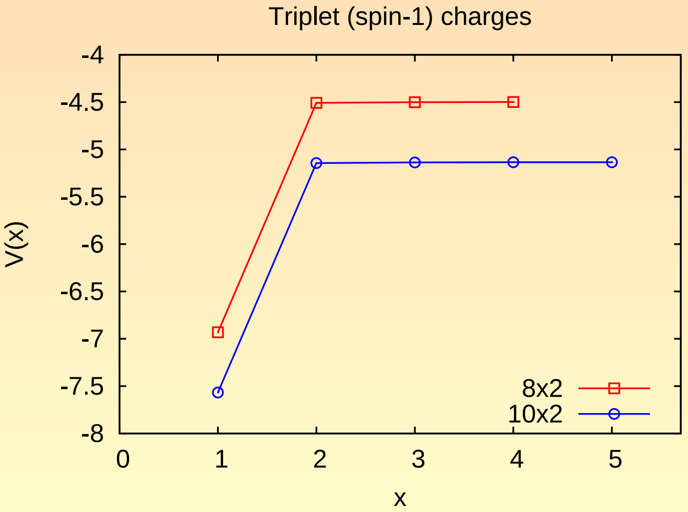
$$\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right) =$$
$$(0 \oplus 1) \otimes (0 \oplus 1) =$$
$$0 \oplus 1 \oplus 1 \oplus 0 \oplus 1 \oplus 2$$



2 gauge inv states/site: effective spin-1/2 system (again!) $\Rightarrow$ Total no of states = 2^{L^2}



The SO(3) QLM: Screening



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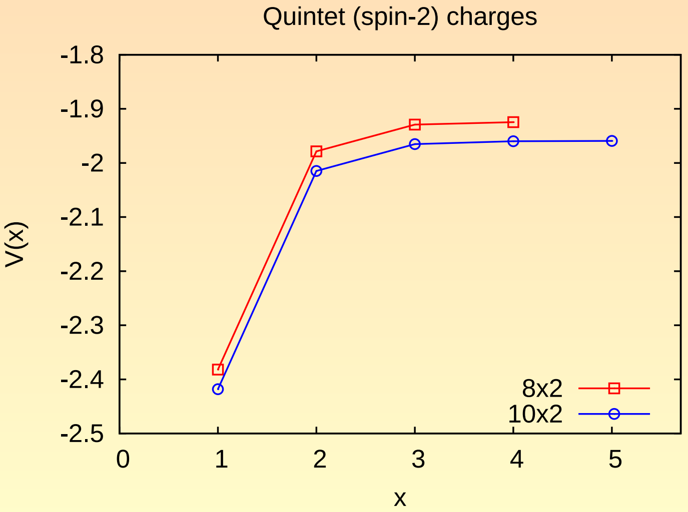

Outlook

- Disclaimer: Far from the continuum limit.
- Extension to higher dimensions in principle straightforward (note: no bosonic representation used for the fermions)
- At the starting point, even qualitative results useful
- Validation of quantum simulations will need MC simulations to check them
- Development of new algorithms ... (cluster algorithms)
- More sophisticated models ... formulation of full QCD in terms of QLMs already exist Brower Chandrasekharan Wiese (1997)