Hadronic light-by-light scattering in the muon $g-2$: Chiral approach and resonance dominance

Andreas Nyffeler

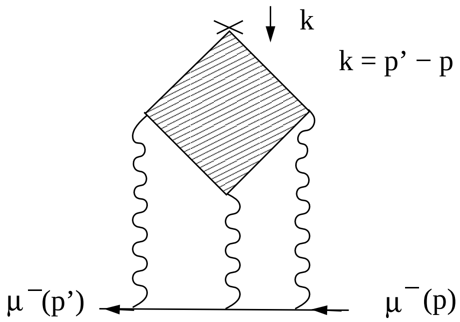
Regional Centre for Accelerator-based Particle Physics
Harish-Chandra Research Institute
Allahabad, India
nyffeler@hri.res.in

Contents

February 28, 2011
INT Workshop on “The Hadronic Light-by-Light Contribution to the Muon Anomaly”
Institute for Nuclear Theory, University of Washington, Seattle, WA, USA

1. Pion-pole versus pion-exchange in $\langle VVVV \rangle$ and in $a_\mu^{\text{LbyL;had}}$

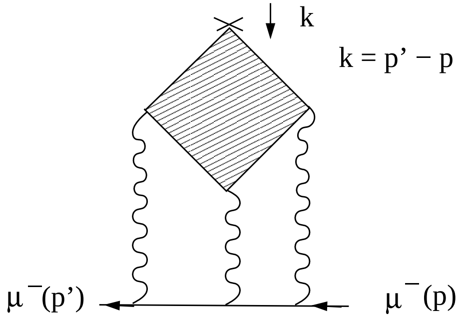
What we need to calculate is a higher order $\mathcal{O}(\alpha^3)$ hadronic contribution to the muon $g-2$:



Four-point function $\langle VVVV \rangle$ projected onto the a_μ (with soft external photon: $k \to 0$).

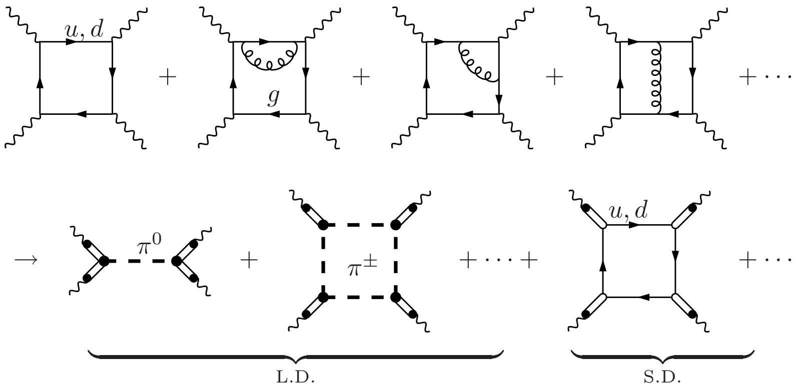
1. Pion-pole versus pion-exchange in $\langle VVVV \rangle$ and in $a_\mu^{\text{LbyL;had}}$

What we need to calculate is a higher order $\mathcal{O}(\alpha^3)$ hadronic contribution to the muon $g-2$:



Four-point function $\langle VVVV \rangle$ projected onto the a_μ (with soft external photon: $k \to 0$).

Look at "underlying" hadronic Green's function $\langle VVVV \rangle$ in QCD (all photon legs off-shell).
Can evaluate it in quark-gluon or hadronic picture (global duality), e.g. for u, d quark sector:

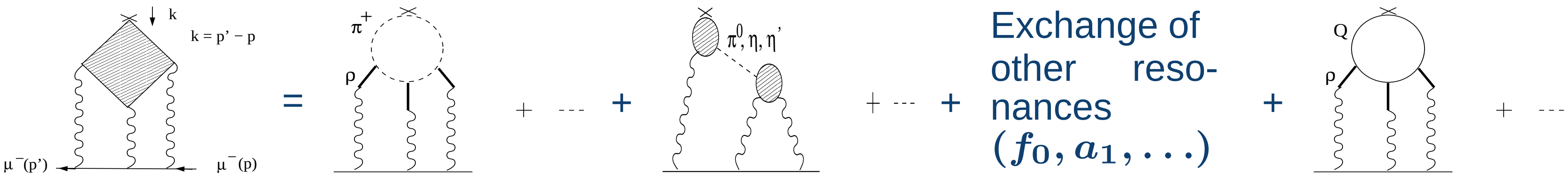


Usually, one uses some hadronic model at low energies (L.D. = long-distances) with exchanges of resonances and loops of resonances and some form of (dressed) "quark loop" at high energies (S.D. = short-distances).
Since the four-point function $\langle VVVV \rangle$ depends on several invariant momenta, the distinction between low and high energies is not as easy as for two-point function $\langle VV \rangle$ (had. vac. pol.).

Current approach to had. LbyL scattering

Classification of de Rafael '94

Use chiral counting p^2, derived from Chiral Perturbation Theory, and large-N_C counting as guideline to classify contributions (in general, all higher orders in p^2 and N_C will contribute):



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Relevant scales in $\langle VVVV \rangle$ (off-shell !): $0 - 2$ GeV, i.e. much larger than m_μ ! No direct relation to experimental data, in contrast to hadronic vacuum polarization in $g-2$
$\rightarrow$ need hadronic (resonance) model (or lattice QCD)

Reduce model dependence by imposing experimental and theoretical constraints on form factors and $\langle VVVV \rangle$, e.g. from QCD short-distances (operator product expansion (OPE)) to get better matching with perturbative QCD for high momenta

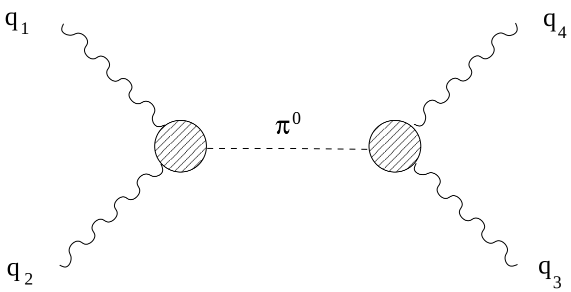
de Rafael '94: last diagram can be interpreted as irreducible contribution to 4-point function $\langle VVVV \rangle$. Appears as short-distance complement of low-energy hadronic models

Pseudoscalars: numerically dominant contribution to had. LbyL scattering

Exchange of lightest state π^0 yields largest contribution $\rightarrow$ warrants special attention

Pion-pole in $\langle VVVV \rangle$ versus pion-exchange in had. LbyL in a_μ

Example: to uniquely identify contribution of exchanged neutral pion π^0 in Green's function $\langle VVVV \rangle$, we need to pick out pion-pole:



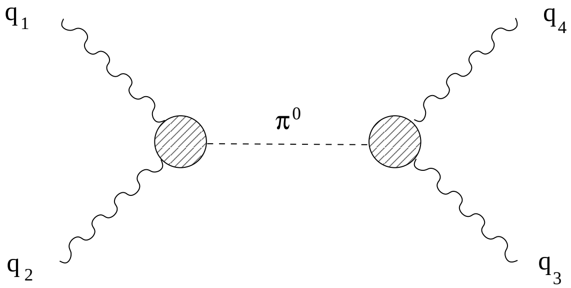
+ crossed diagrams

$$\lim_{(q_1+q_2)^2 \to m_\pi^2} ((q_1 + q_2)^2 - m_\pi^2) \langle VVVV \rangle$$

Residue of pole: on-shell vertex function $\langle 0|VV|\pi \rangle \to$ on-shell form factor $\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2)$

Pion-pole in $\langle VVVV \rangle$ versus pion-exchange in had. LbyL in a_μ

Example: to uniquely identify contribution of exchanged neutral pion π^0 in Green's function $\langle VVVV \rangle$, we need to pick out pion-pole:

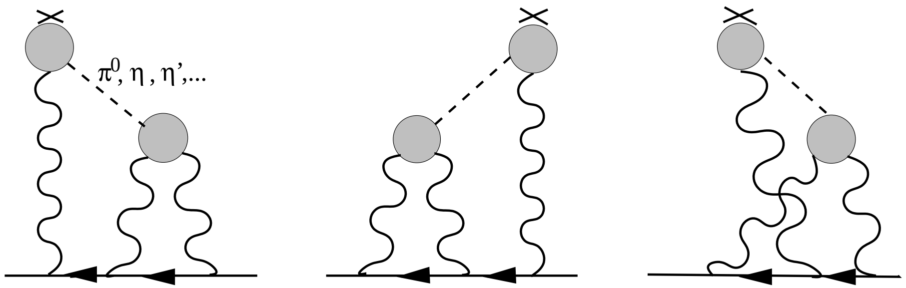


+ crossed diagrams

$$\lim_{(q_1+q_2)^2 \to m_\pi^2} ((q_1+q_2)^2 - m_\pi^2) \langle VVVV \rangle$$

Residue of pole: on-shell vertex function $\langle 0 | VV | \pi \rangle \to$ on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

But in contribution to the muon $g-2$, we have to evaluate Feynman diagrams, integrating over the photon momenta with exchanged off-shell pions. In general, for all the pseudoscalars:



Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where $\mathrm{PS} = \pi^0, \eta, \eta', \pi^{0\prime}, \ldots$
The off-shell form factors are either inserted "by hand" starting from the constant, pointlike Wess-Zumino-Witten (WZW) form factor or using e.g. some resonance Lagrangian.

2. The off-shell pion form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ from $\langle VVP \rangle$

- Following Bijnens, Pallante, Prades '95, '96; Hayakawa, Kinoshita, Sanda '95, '96; Hayakawa, Kinoshita '98, we can define off-shell form factor for π^0 as follows:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2 \cdot y)} \langle 0 | T\{ j_\mu(x) j_\nu(y) P^3(0)\} | 0 \rangle$$

$$= \quad \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta \frac{i\langle \overline{\psi}\psi\rangle}{F_\pi} \frac{i}{(q_1+q_2)^2 - m_\pi^2} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) + \ldots$$

Up to small mixing effects of P^3 with η and η' and neglecting exchanges of heavier states like $\pi^{0\prime}, \pi^{0\prime\prime}, \ldots$

$j_\mu =$ light quark part of the electromagnetic current: $j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x)$

$$\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \hat{Q} = \text{diag}(2,-1,-1)/3$$

$P^3 = \overline{\psi} i \gamma_5 \frac{\lambda^3}{2}\psi = \left(\overline{u} i\gamma_5 u - \overline{d} i \gamma_5 d\right)/2$, $\quad \langle \overline{\psi}\psi \rangle =$ single flavor quark condensate

Bose symmetry: $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_2^2, q_1^2)$

- Note: for off-shell pions, instead of $P^3(x)$, we could use any other suitable interpolating field, like $(\partial^\mu A_\mu^3)(x)$ or even an elementary pion field $\pi^3(x)$!

Integral representation for pion-exchange contribution

Projection onto the muon $g-2$ leads to (Knecht + Nyffeler '02; Jegerlehner '07, '08: use off-shell form factors):

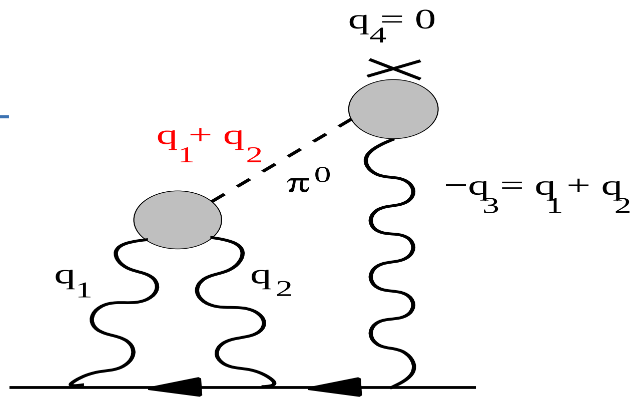
$$
\begin{aligned}
a_\mu^{\mathrm{LbyL};\pi^0} &= -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]} \\
&\times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, q_1^2, (q_1+q_2)^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_2^2, q_2^2, 0)}{q_2^2 - m_\pi^2} T_1(q_1, q_2; p) \right. \\
&\left. + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)\ \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)}{(q_1+q_2)^2 - m_\pi^2} T_2(q_1, q_2; p) \right]
\end{aligned}
$$

$$
\begin{aligned}
T_1(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_2)^2 q_1^2 \\
&- \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + 8(p\cdot q_2) q_1^2 q_2^2 - \frac{16}{3}(p\cdot q_2)(q_1\cdot q_2)^2 \\
&+ \frac{16}{3} m_\mu^2 q_1^2 q_2^2 - \frac{16}{3} m_\mu^2 (q_1\cdot q_2)^2 \\
T_2(q_1, q_2; p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_1)^2 q_2^2 \\
&+ \frac{8}{3}(p\cdot q_1)(q_1\cdot q_2) q_2^2 + \frac{8}{3}(p\cdot q_1) q_1^2 q_2^2 \\
&+ \frac{8}{3} m_\mu^2 q_1^2 q_2^2 - \frac{8}{3} m_\mu^2 (q_1\cdot q_2)^2
\end{aligned}
$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon).

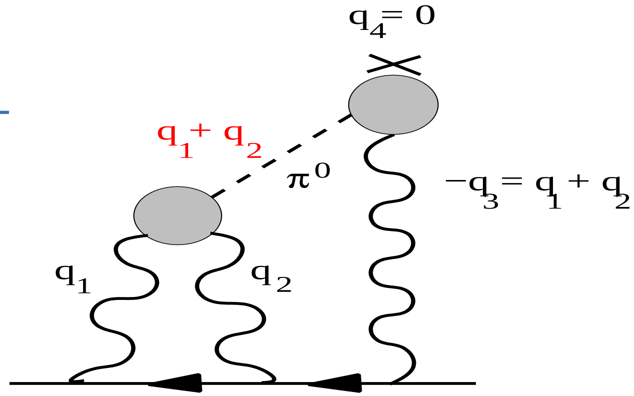
Off-shell versus on-shell form factors

- As stressed before, off-shell form factors have been used to evaluate the pion-exchange contribution in BPP '96, HKS '96, HK '98. "Rediscovered" by Jegerlehner in '07, '08. Consider diagram:



$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

Off-shell versus on-shell form factors



- As stressed before, off-shell form factors have been used to evaluate the pion-exchange contribution in BPP ’96, HKS ’96, HK ’98. “Rediscovered” by Jegerlehner in ’07, ’08. Consider diagram:

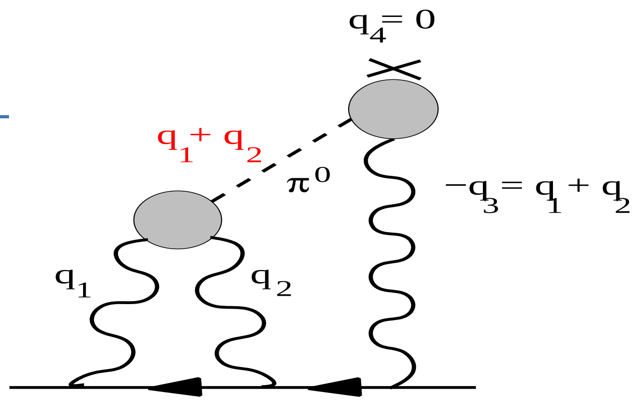
$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

- On the other hand, Bijnens, Persson ’01, Knecht, Nyffeler ’02 used on-shell form factors:

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following misleading notation was used: $\mathcal{F}_{\pi^0\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$

Off-shell versus on-shell form factors



- As stressed before, off-shell form factors have been used to evaluate the pion-exchange contribution in BPP '96, HKS '96, HK '98. "Rediscovered" by Jegerlehner in '07, '08. Consider diagram:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

- On the other hand, Bijnens, Persson '01, Knecht, Nyffeler '02 used on-shell form factors:

$$\mathcal{F}_{\pi^{0}\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0}\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$$

- But form factor at external vertex $\mathcal{F}_{\pi^{0}\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !
  Often the following misleading notation was used: $\mathcal{F}_{\pi^{0}\gamma^*\gamma^*}((q_1+q_2)^2, 0) \equiv \mathcal{F}_{\pi^{0}\gamma^*\gamma^*}(m_\pi^2, (q_1+q_2)^2, 0)$
- Melnikov, Vainshtein '04 had already observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^{0}\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0}\gamma\gamma}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the Wess-Zumino-Witten term
- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution !
- In view of the identification of the pion-pole contribution in $\langle VVVV \rangle$ as discussed earlier: should one also go "on-shell" in those additional factors $T_{1,2}$ in the two-loop integral for the pion-pole contribution ? This has not been done in the literature.

Off-shell versus on-shell form factors (cont.)

- In general, any evaluation e.g. using some resonance Lagrangian, will lead to off-shell form factors at both the vertices in the Feynman integral.
- Note: strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell (or nearly on-shell).

  If one is (far) off the mass shell of the exchanged particle, it is not possible to unambiguously separate different contributions to the muon $g-2$, unless one uses some particular model where elementary pions can propagate.

  Although the contribution in a particular channel will then be model dependent, the sum of all off-shell contributions in all channels will lead, at least in principle, to a model-independent result (Knecht '04, unpublished notes).
- The pion-exchange contribution is therefore model dependent, but the whole calculation is anyhow done in a specific model, unless we could do it from first principles on the lattice.

Off-shell versus on-shell form factors (cont.)

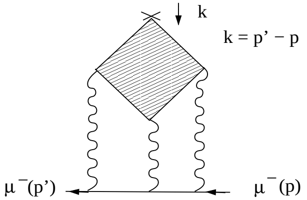
- In general, any evaluation e.g. using some resonance Lagrangian, will lead to off-shell form factors at both the vertices in the Feynman integral.
- Note: strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell (or nearly on-shell).

  If one is (far) off the mass shell of the exchanged particle, it is not possible to unambiguously separate different contributions to the muon $g-2$, unless one uses some particular model where elementary pions can propagate.

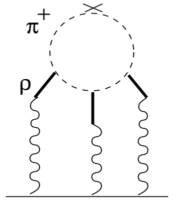
  Although the contribution in a particular channel will then be model dependent, the sum of all off-shell contributions in all channels will lead, at least in principle, to a model-independent result (Knecht '04, unpublished notes).
- The pion-exchange contribution is therefore model dependent, but the whole calculation is anyhow done in a specific model, unless we could do it from first principles on the lattice.

- But the prescription to put the form factors in the Feynman diagram on-shell to get the “pion-pole” contribution, seems also arbitrary to me (even if it is not model-dependent).
- Actually, since we can do the calculation of the $g-2$ in Euclidean space (after a Wick rotation), there is not even a “real” pole, which gives an enhanced contribution (we simply calculate $F_2(0)$, magnetic form factor at zero momentum transfer). This in contrast to a resonance pole in a cross-section, e.g. from Z-boson exchanged in s-channel at LEP.
- Similar statements apply to exchanges of other resonances (pseudoscalars, axial-vectors, scalars) or for pion/Kaon loop contribution (two-particle intermediate states).

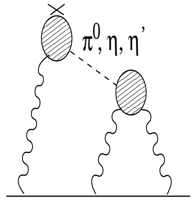
Had. LbyL scattering: Summary of results



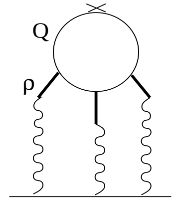
=



+ ··· +



+ ··· + Exchange of other reso- nances $(f_0, a_1, \ldots)$ +



+ ···

Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

BPP: +83 (32)	-19 (13)	+85 (13)	-4 (3) $[f_0, a_1]$	+21 (3)
HKS: +90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) $[a_1]$	+10 (11)
KN: +80 (40)		+83 (12)		
MV: +136 (25)	0 (10)	+114 (10)	+22 (5) $[a_1]$	0
2007: +110 (40)				
PdRV: +105 (26)	-19 (19)	+114 (13)	+8 (12) $[f_0, a_1]$	+2.3 [c-quark]
N,JN: +116 (40)	-19 (13)	+99 (16)	+15 (7) $[f_0, a_1]$	+21 (3)
FGW: +191 (81)		+84 (13)		+107 (48)
	ud.: -45	ud.: +∞		ud.: +60

ud. = undressed, i.e. point vertices without form factors

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09; N,JN = Nyffeler '09; Jegerlehner, Nyffeler '09; FGW = Fischer, Goecke, Williams '10, '11 (total includes estimate of "other contributions" $= 0\ (20)$).

Had. LbyL scattering: Summary of results (cont.)

- Evaluations of full had. LbyL scattering contribution:
  - Bijnens, Pallante, Prades ’95, ’96, ’02
    Use mainly Extended Nambu-Jona-Lasinio (ENJL) model; but for some contributions also other models (in particular for pseudoscalars)
  - Hayakawa, Kinoshita, Sanda ’95, ’96; Hayakawa, Kinoshita ’98, ’02
    Use mainly Hidden Local Symmetry (HLS) model; often HLS = VMD
- Selected partial evaluations:
  - Knecht, Nyffeler ’02: use large-N_C QCD
  - Melnikov, Vainshtein ’04: use large-N_C QCD
  - Fischer et al. ’10, ’11: use Dyson-Schwinger equations
- Prades, de Rafael, Vainshtein ’09: Analyzed results obtained by different groups and suggested new estimates for some contributions (shifted central values, enlarged errors). No dressed light quark loops ! Assumed to be taken into account by short-distance constraint of MV ’04 on pseudoscalar-pole contribution. Added errors in quadrature.
- Nyffeler ’09, Jegerlehner, Nyffeler ’09: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on pion-exchange. Combined with MV (for axial-vectors) + BPP (rest of contributions). Added errors linearly.
- Fischer, Goecke, Williams ’10, ’11: New approach with Dyson-Schwinger equations. Is there some double-counting between their dressed quark loop (largely enhanced !) and the pseudoscalar exchanges ? Added errors linearly.

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$

- 2-loop integral $\rightarrow$ 8-dim. integral. Integration over 3 angles can be done easily
- 5 non-trivial integrations: 2 moduli: $|q_1|$, $|q_2|$, 3 angles: $p \cdot q_1$, $p \cdot q_2$, $q_1 \cdot q_2$ (recall $p^2 = m_\mu^2$).
- Observation: $p \cdot q_1$, $p \cdot q_2$ do not appear in the model-dependent form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$
- Can perform those two angular integrations by averaging expression for $a_\mu^{\mathrm{LbyL};\pi^0}$ over the direction of p (hyperspherical approach) $\Rightarrow$ 3-dimensional integral representation for general form factors ! (Jegerlehner, Nyffeler '09)

In Euclidean space:

$$a_\mu^{\mathrm{LbL};\pi^0} = -\frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt\, \sqrt{1-t^2}\, Q_1^3\, Q_2^3$$

$$\times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_2^2, -Q_1^2, -Q_3^2)\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_2^2, -Q_2^2, 0)}{(Q_2^2 + m_\pi^2)} I_1(Q_1, Q_2, t) \right.$$

$$\left. + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_3^2, -Q_1^2, -Q_2^2)\, \mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_3^2, -Q_3^2, 0)}{(Q_3^2 + m_\pi^2)} I_2(Q_1, Q_2, t) \right]$$

Integration variables: Q_1^2, Q_2^2 and angle θ between Q_1 and Q_2: $Q_1 \cdot Q_2 = |Q_1||Q_2| \cos\theta$.
$t = \cos\theta$, $\quad Q_3^2 = (Q_1 + Q_2)^2$

3. Chiral Perturbation Theory approach to had. LbyL scattering

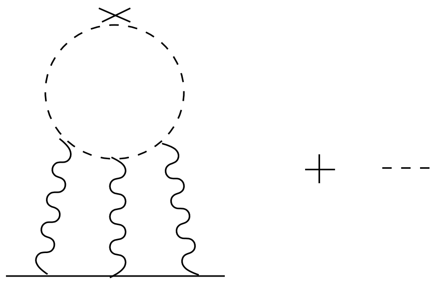
Effective Field Theory (EFT) for $E \ll 1$ GeV with pions, photons and muons

[de Rafael '94; Knecht, Nyffeler, Perrottet, de Rafael, '02; Ramsey-Musolf, Wise '02; based on chiral Lagrangians given earlier in literature]

Note: chiral counting below refers to contribution in a_μ with $m_\mu, m_\pi, e = \mathcal{O}(p)$.
Differs from counting in de Rafael '94 !

Contributions to $a_\mu^{\text{LbyL;had}}$

$\mathcal{O}(p^6)$: charged pion loop
(finite, subleading in $1/N_C$)

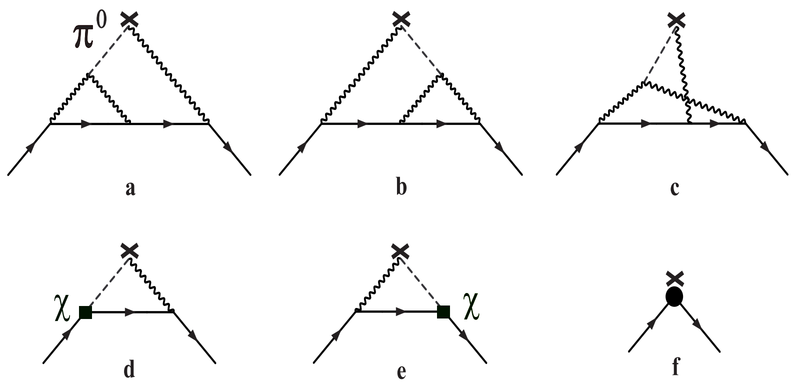


$\mathcal{O}(p^8)$: pion-pole (leading in $1/N_C$)

Divergent 2-loop contribution (a)+(b)

$\longrightarrow$ need counterterms

1. One-loop graphs with insertion of χ (■) = coupling $\overline{\psi}\,\gamma_\mu\gamma_5\,\psi\,\partial^\mu\pi^0$
2. Local counterterm (●)



$\Rightarrow a_\mu^{\text{LbyL;had}}$ cannot be obtained in (pure) EFT framework

$\longrightarrow$ resonance models for form factors (sensitivity to higher energy scales around $1-2$ GeV)

ChPT approach: Large log's

Renormalization group in EFT $\Rightarrow$ "large" logarithm $\ln^2(\mu_0/m_\mu)$ (assume $m_\mu \approx m_\pi$)

$$a_{\mu;\text{ChPT}}^{\text{LbyL;had}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f\left(\frac{m_{\pi^\pm}}{m_\mu}, \frac{m_{K^\pm}}{m_\mu}\right) \right. \quad \text{(loops with pions and kaons)}$$

$$\left. + \overbrace{N_C \left(\frac{m_\mu^2}{16\pi^2 F_\pi^2} \frac{N_C}{3}\right)}^{\mathcal{C} \approx 0.025 \text{ (universal)}} \left[\ln^2 \frac{\mu_0}{m_\mu} + \overbrace{c_1}^{\chi(\mu_0)\ (\pi^0 \to e^+e^-)} \ln \frac{\mu_0}{m_\mu} + c_0 \right] + \ldots \right\}$$

$f = -0.038$, formally $\mathcal{O}(1)$; $\mu_0 \sim M_\rho$: hadronic scale, $\ln \frac{M_\rho}{m_\mu} \sim 2$

$\ln^2$ behavior first predicted by Melnikov '01; explicit result for $\mathcal{C}$ first given in Knecht, Nyffeler '02; Knecht et al. '02.

EFT analysis shows that modelling of had. LbyL using a constituent quark loop is not consistent with QCD as there is no large $\ln^2$:
$a_{\mu;\text{CQM}}^{\text{LbyL;had}} \sim (\alpha/\pi)^3 N_C (m_\mu^2/M_Q^2) + \ldots$ for $M_Q \gg m_\mu$. $\mathcal{O}(p^8)$, same as for π^0-exchange, differs from de Rafael '94.

ChPT approach: Large log's

Renormalization group in EFT $\Rightarrow$ "large" logarithm $\ln^2(\mu_0/m_\mu)$ (assume $m_\mu \approx m_\pi$)

$$a_{\mu;\text{ChPT}}^{\text{LbyL;had}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f\left(\frac{m_{\pi^\pm}}{m_\mu}, \frac{m_{K^\pm}}{m_\mu}\right) \right. \quad \text{(loops with pions and kaons)}$$

$$\left. + \overbrace{N_C \left(\frac{m_\mu^2}{16\pi^2 F_\pi^2}\frac{N_C}{3}\right)}^{\mathcal{C}\approx 0.025 \text{ (universal)}} \left[\ln^2\frac{\mu_0}{m_\mu} + \overbrace{c_1}^{\chi(\mu_0)\ (\pi^0\to e^+e^-)} \ln\frac{\mu_0}{m_\mu} + c_0\right] + \ldots \right\}$$

$f = -0.038$, formally $\mathcal{O}(1)$; $\quad \mu_0 \sim M_\rho$: hadronic scale, $\ln\frac{M_\rho}{m_\mu} \sim 2$

$\ln^2$ behavior first predicted by Melnikov '01; explicit result for $\mathcal{C}$ first given in Knecht, Nyffeler '02; Knecht et al. '02.

EFT analysis shows that modelling of had. LbyL using a constituent quark loop is not consistent with QCD as there is no large $\ln^2$:

$a_{\mu;\text{CQM}}^{\text{LbyL;had}} \sim (\alpha/\pi)^3 N_C (m_\mu^2/M_Q^2) + \ldots$ for $M_Q \gg m_\mu$. $\mathcal{O}(p^8)$, same as for π^0-exchange, differs from de Rafael '94.

Problem: π^0-exchange $\to$ large cancellation between $\ln^2$ and $\ln$:

$$a_{\mu;\text{VMD}}^{\text{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C}\left[\ln^2\frac{M_\rho}{m_\mu} + c_1 \ln\frac{M_\rho}{m_\mu} + c_0\right]$$

$$\stackrel{\text{Fit}}{=} \left(\frac{\alpha}{\pi}\right)^3 \mathcal{C}\ [3.94 - 3.30 + 1.08] = [123 - 103 + 34] \times 10^{-11} = 54 \times 10^{-11}$$

Compare with analytic result as expansion in $(m_\mu/M_\rho)^2$ and $\delta = (m_\pi^2 - m_\mu^2)/m_\mu^2$ in Blokland et al. '02 (also contains corrections $(m_\mu/M_\rho)^2$ to coefficient $\mathcal{C}$ of $\ln^2$ term):

$$a_{\mu;\text{VMD}}^{\text{LbyL};\pi^0} = (136 - 112 + 30) \times 10^{-11} = 54 \times 10^{-11}$$

ChPT approach: Limitations

Ramsey-Musolf, Wise '02 calculated: $c_1 = -2\chi(\mu_0)/3 + 0.237 = -0.93^{+0.67}_{-0.83}$
with our conventions (scheme) for χ and $\chi(M_\rho)_{\rm exp} = 1.75^{+1.25}_{-1.00}$ from Ametller '02.

Ramsey-Musolf, Wise then obtain for the logarithmically enhanced terms and the full ChPT result (adding the charged pion loop $a_{\mu;\,\text{sQED}}^{\text{LbyL};\pi^\pm} = -44 \times 10^{-11}$):

$$a_{\mu;\text{log}}^{\text{LbyL};\pi^0} = (57^{+50}_{-60}) \times 10^{-11}$$
$$a_{\mu;\text{ChPT}}^{\text{LbyL; had}} = (13^{+50}_{-60} + 31\, c_0) \times 10^{-11}$$

Even if one could more precisely determine $\chi(\mu_0)$ in c_1 (see also the recent papers by Dorokhov et al. '08 – '10), since $c_0 = \mathcal{O}(1) \Rightarrow \pm 30 \times 10^{-11}$ in had. LbyL, the ChPT approach does not seem to lead much further.

ChPT approach: Limitations

Ramsey-Musolf, Wise '02 calculated: $c_1 = -2\chi(\mu_0)/3 + 0.237 = -0.93^{+0.67}_{-0.83}$ with our conventions (scheme) for χ and $\chi(M_\rho)_{\rm exp} = 1.75^{+1.25}_{-1.00}$ from Ametller '02.

Ramsey-Musolf, Wise then obtain for the logarithmically enhanced terms and the full ChPT result (adding the charged pion loop $a_{\mu;\,\text{sQED}}^{\text{LbyL};\pi^\pm} = -44 \times 10^{-11}$):

$$
\begin{aligned}
a_{\mu;\text{log}}^{\text{LbyL};\pi^0} &= (57^{+50}_{-60}) \times 10^{-11} \\
a_{\mu;\text{ChPT}}^{\text{LbyL; had}} &= (13^{+50}_{-60} + 31\, c_0) \times 10^{-11}
\end{aligned}
$$

Even if one could more precisely determine $\chi(\mu_0)$ in c_1 (see also the recent papers by Dorokhov et al. '08 – '10), since $c_0 = \mathcal{O}(1) \Rightarrow \pm 30 \times 10^{-11}$ in had. LbyL, the ChPT approach does not seem to lead much further.

Moreover, the charged pion loop result gets drastically reduced when one includes form factors, see BPP, HKS. Therefore the chiral expansion does not seem to be a good guideline.

MV '04 studied HLS model via expansion in $(m_\pi/M_\rho)^2$ and $(m_\mu - m_\pi)/m_\pi$:

$$
\begin{aligned}
a_{\mu;\text{HLS}}^{\text{LbL};\pi^\pm} &= \left(\frac{\alpha}{\pi}\right)^3 \sum_{i=0}^{\infty} f_i \left[\frac{m_\mu - m_\pi}{m_\pi}, \ln\left(\frac{M_\rho}{m_\pi}\right)\right] \left(\frac{m_\pi^2}{M_\rho^2}\right)^i = \left(\frac{\alpha}{\pi}\right)^3 (-0.0058) \\
&= (-46.37 + 35.46 + 10.98 - 4.70 - 0.3 + \ldots) \times 10^{-11} = -4.9(3) \times 10^{-11}
\end{aligned}
$$

Large cancellation between first three terms in series. Expansion converges only very slowly. Main reason: typical momenta in the loop integral are of order $\mu = 4m_\pi \approx 550$ MeV and the effective expansion parameter is μ/M_ρ, not m_π/M_ρ.

4. Exp. and theoret. constraints on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ and $\langle VVVV \rangle$

Experimental constraints on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

- Any hadronic model of the form factor has to reproduce the $\pi^0 \to \gamma\gamma$ decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}[1 + \mathcal{O}(m_q)]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small), see also Kampf + Moussallam '09. For $F_\pi = 92.4$ MeV, this reproduces very well the decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.49)$ eV (PDG 2010, 6.3% precision). Leads to normalization:

$$\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = -\frac{N_C}{12\pi^2 F_\pi}$$

Note: Uncertainty in neutral pion contribution to had. LbyL originating from $\Gamma(\pi^0 \to \gamma\gamma)$ has not been taken into account so far ! F_π without any error attached to it is used.

Note: recently the PrimEx Collaboration presented the new measurement $\Gamma(\pi^0 \to \gamma\gamma) = (7.82 \pm 0.23)$ eV (2.8% precision).

4. Exp. and theoret. constraints on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ and $\langle VVVV \rangle$

Experimental constraints on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

- Any hadronic model of the form factor has to reproduce the $\pi^0 \to \gamma\gamma$ decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}[1 + \mathcal{O}(m_q)]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small), see also Kampf + Moussallam '09. For $F_\pi = 92.4$ MeV, this reproduces very well the decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.49)$ eV (PDG 2010, 6.3% precision). Leads to normalization:

$$\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = -\frac{N_C}{12\pi^2 F_\pi}$$

Note: Uncertainty in neutral pion contribution to had. LbyL originating from $\Gamma(\pi^0 \to \gamma\gamma)$ has not been taken into account so far ! F_π without any error attached to it is used.

Note: recently the PrimEx Collaboration presented the new measurement $\Gamma(\pi^0 \to \gamma\gamma) = (7.82 \pm 0.23)$ eV (2.8% precision).

- Information on the $\pi^0 - \gamma$ transition form factor with one on-shell and one off-shell photon from the process $e^+e^- \to e^+e^-\pi^0$

Experimental data (CELLO '90, CLEO '98) fairly well confirm the Brodsky-Lepage behavior:

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$$

Maybe with slightly different prefactor ! Note: data from BABAR '09 do not show this fall-off !

Theory: QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

Knecht, Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

- If the space-time arguments of all three currents approach each other one obtains (up to corrections $\mathcal{O}(\alpha_s)$):

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3} \frac{1}{\lambda^2} \frac{q_1^2 + q_2^2 + (q_1 + q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

Theory: QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

Knecht, Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

- If the space-time arguments of all three currents approach each other one obtains (up to corrections $\mathcal{O}(\alpha_s)$):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1+\lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3}\frac{1}{\lambda^2}\frac{q_1^2+q_2^2+(q_1+q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

- When the space-time arguments of the two vector currents in $\langle VVP \rangle$ approach each other (OPE leads to Green's function $\langle AP \rangle$):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, (\lambda q_1)^2, (q_2-\lambda q_1)^2) = \frac{2F_0}{3}\frac{1}{\lambda^2}\frac{1}{q_1^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

As pointed out in Melnikov, Vainshtein '04, higher twist corrections have been worked out in Shuryak, Vainshtein '82, Novikov et al. '84 (in chiral limit):

$$\lim_{\lambda\to\infty} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, (\lambda q_1)^2, (\lambda q_1)^2)}{\mathcal{F}_{\pi^0\gamma\gamma}(0,0,0)} = -\frac{8}{3}\pi^2 F_0^2 \left\{ \frac{1}{\lambda^2 q_1^2} + \frac{8}{9}\frac{\delta^2}{\lambda^4 q_1^4} + \mathcal{O}\left(\frac{1}{\lambda^6}\right)\right\}$$

δ^2 parametrizes the relevant higher-twist matrix element.
The sum-rule estimate in Novikov et al. '84 yielded $\delta^2 = (0.2 \pm 0.02)$ GeV^2

New short-distance constraint on form factor at external vertex

- When the space-time argument of one of the vector currents approaches the argument of the pseudoscalar density in $\langle VVP \rangle$ one obtains (Knecht, Nyffeler, EPJC '01):

$$\langle V \underbrace{VP}_{\text{OPE}} \rangle \to \langle VT \rangle \qquad \text{Vector-Tensor two-point function}$$

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + q_2)^2, (\lambda q_1)^2, q_2^2) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(q_2^2) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

The vector-tensor two-point function Π_{VT} is defined by:

$$\delta^{ab}(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = \int d^4x e^{ip\cdot x}\langle 0|T\{V_\mu^a(x)(\overline{\psi}\,\sigma_{\rho\sigma}\frac{\lambda^b}{2}\,\psi)(0)\}|0\rangle, \qquad \sigma_{\rho\sigma} = \frac{i}{2}[\gamma_\rho,\gamma_\sigma]$$

$$(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = (p_\rho\eta_{\mu\sigma} - p_\sigma\eta_{\mu\rho})\,\Pi_{\mathrm{VT}}(p^2), \qquad \text{conservation of vector current and parity invariance}$$

At the external vertex in light-by-light scattering the following limit is relevant (soft photon $q_2 \to 0$)

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(0) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

New short-distance constraint at the external vertex (cont.)

Ioffe, Smilga '84 defined the quark condensate magnetic susceptibility χ of QCD in the presence of a constant external electromagnetic field

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e\, e_q\, \chi\, \langle\overline{\psi}\psi\rangle_0\, F_{\mu\nu}, \qquad e_u = 2/3,\; e_d = -1/3$$

Belyaev, Kogan '84 then showed that

$$\Pi_{\mathrm{VT}}(0) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\,\chi$$

New short-distance constraint at the external vertex (cont.)

Ioffe, Smilga '84 defined the quark condensate magnetic susceptibility χ of QCD in the presence of a constant external electromagnetic field

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e\, e_q\, \chi\, \langle\overline{\psi}\psi\rangle_0\, F_{\mu\nu}, \qquad e_u = 2/3,\ e_d = -1/3$$

Belyaev, Kogan '84 then showed that

$$\Pi_{\mathrm{VT}}(0) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\,\chi$$

$\Rightarrow$ New short-distance constraint on off-shell form factor at external vertex (Nyffeler '09):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3}\chi + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad (*)$$

- Note that there is no falloff in OPE in $(*)$, unless χ vanishes !
  A constituent quark model for the form factor would lead to a $1/q_1^2$ fall-off instead.
- Corrections of $\mathcal{O}(\alpha_s)$ in OPE $\Rightarrow$ χ depends on renormalization scale μ
- Unfortunately there is no agreement in the literature what the value of $\chi(\mu)$ should be !
  Range of values from $\chi(\mu \sim 0.5\ \mathrm{GeV}) \approx -9\ \mathrm{GeV}^{-2}$ (Ioffe, Smilga '84; Vainshtein '03, ..., Narison '08) to $\chi(\mu \sim 1\ \mathrm{GeV}) \approx -3\ \mathrm{GeV}^{-2}$ (Balitsky, Yung '83; Ball et al. '03; ...; Ioffe '09). Running with μ cannot explain such a difference.

Estimates for the quark condensate magnetic susceptibility χ

Authors	Method	$\chi(\mu)$ [GeV]$^{-2}$	Footnote
Ioffe, Smilga '84	QCD sum rules	$\chi(\mu = 0.5\ \text{GeV}) = -\left(8.16^{+2.95}_{-1.91}\right)$	[1]
Narison '08	QCD sum rules	$\chi = -(8.5 \pm 1.0)$	[2]
Vainshtein '03	OPE for $\langle VVA \rangle$	$\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$	[3]
Gorsky, Krikun '09	AdS/QCD	$\chi = -(2.15 N_C)/(8\pi^2 F_\pi^2) = -9.6$	[4]
Dorokhov '05	Instanton liquid model	$\chi(\mu \sim 0.5 - 0.6\ \text{GeV}) = -4.32$	[5]
Ioffe '09	Zero-modes of Dirac operator	$\chi(\mu \sim 1\ \text{GeV}) = -3.52\ (\pm 30 - 50\%)$	[6]
Buividovich et al. '09	Lattice	$\chi = -1.547(6)$	[7]
Balitsky, Yung '83	LMD for $\langle VT \rangle$	$\chi = -2/M_V^2 = -3.3$	[8]
Belyaev, Kogan '84	QCD sum rules for $\langle VT \rangle$	$\chi(0.5\ \text{GeV}) = -(5.7 \pm 0.6)$	[9]
Balitsky et al. '85	QCD sum rules for $\langle VT \rangle$	$\chi(1\ \text{GeV}) = -(4.4 \pm 0.4)$	[9]
Ball et al. '03	QCD sum rules for $\langle VT \rangle$	$\chi(1\ \text{GeV}) = -(3.15 \pm 0.30)$	[9]

[1]: QCD sum rule evalation of nucleon magnetic moments.
[2]: Recent reanalysis of these sum rules for nucleon magnetic moments. At which scale μ ?
[3]: Probably at low scale $\mu \sim 0.5$ GeV, since pion dominance was assumed in derivation.
[4]: From derivation in holographic model it is not clear what is the relevant scale μ.
[5]: The scale is set by the inverse average instanton size ρ^{-1}.
[6]: Study of zero-mode solutions of Dirac equation in presence of arbitrary gluon fields (à la Banks-Casher).
[7]: Again à la Banks-Casher. Quenched lattice calculation for $SU(2)$. μ dependence is not taken into account. Lattice spacing corresponds to 2 GeV.
[8]: The leading short-distance behavior of Π_{VT} is given by (Craigie, Stern '81)

$$\lim_{\lambda\to\infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2} \frac{\langle \overline{\psi}\psi \rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

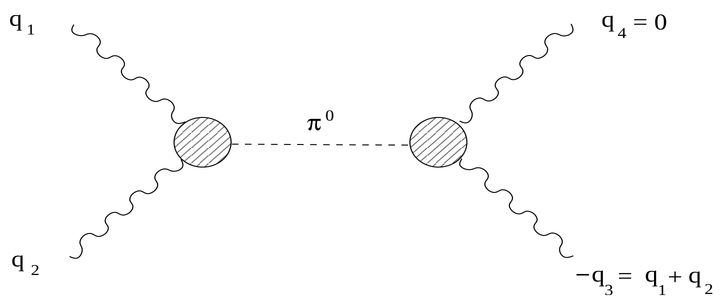
Assuming that the two-point function Π_{VT} is well described by the multiplet of the lowest-lying vector mesons (LMD) and satisfies this OPE constraint leads to the ansatz (Balitsky, Yung '83, Belyaev, Kogan '84, Knecht, Nyffeler, EPJC '01)

$$\Pi_{VT}^{\text{LMD}}(p^2) = -\langle \overline{\psi}\psi \rangle_0 \frac{1}{p^2 - M_V^2} \quad \Rightarrow \quad \chi^{\text{LMD}} = -\frac{2}{M_V^2} = -3.3\ \text{GeV}^{-2}$$

Not obvious at which scale. Maybe $\mu = M_V$ as for low-energy constants in ChPT.
[9]: LMD estimate later improved by taking more resonance states $\rho', \rho'', \ldots$ in QCD sum rule analysis of $\langle VT \rangle$.
Note that the last value by Ball et al. is very close to original LMD estimate !

QCD short-distance constraint on $\langle VVVV \rangle$ in $g-2$

- Melnikov, Vainshtein '04 found QCD short-distance constraint on whole 4-point function:



$$\langle \underbrace{VV}_{\text{OPE}} V|\gamma\rangle \overset{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

- From this they deduced for the LbyL scattering amplitude for finite $q_1^2, q_2^2, -q_3 = q_1 + q_2$ (Eq. (18) in MV '04, using our normalization for form factor and Minkowski space notation):

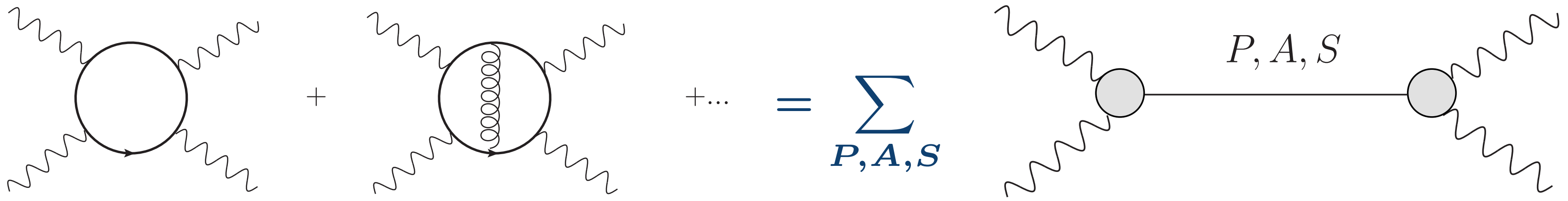
$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu} \tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$ = field strength tensors of internal photons with polarization vectors ϵ_i, for external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$. Except in $\tilde{f}_{\rho\sigma}$, $q_4 \to 0$ is understood in $f_3^{\sigma\rho}$ and in pion propagator.

- From expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ it is obvious that Melnikov and Vainshtein only consider the pion-pole contribution !
- No 2nd form factor at ext. vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by constant WZW form factor $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0) \approx \mathcal{F}_{\pi^0\gamma\gamma}(0, 0)$!

Matching with the quark-loop

- If one then studies the behavior for large q_3^2 one obtains from the pion propagator an overall $1/q_3^2$ behavior (apart from $f_3^{\sigma\rho}$).
- According to MV '04 this agrees exactly with behavior of quark-loop in perturbative QCD for large momenta (first $q_1^2, q_2^2 \gg q_3^3$, then large q_3^2).
- But: from quark-hadron duality in large-N_C QCD it follows that only the sum of all resonance exchanges matches with quark-loop ! Including all gluonic corrections to the one quark loop.



- Why should already the pion-pole contribution alone match with the quark-loop ?

5. Evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD
(Peris et al. '98, ...)

- In leading order in N_C, an infinite tower of narrow resonances contributes in each channel of a particular Green's function.
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD)

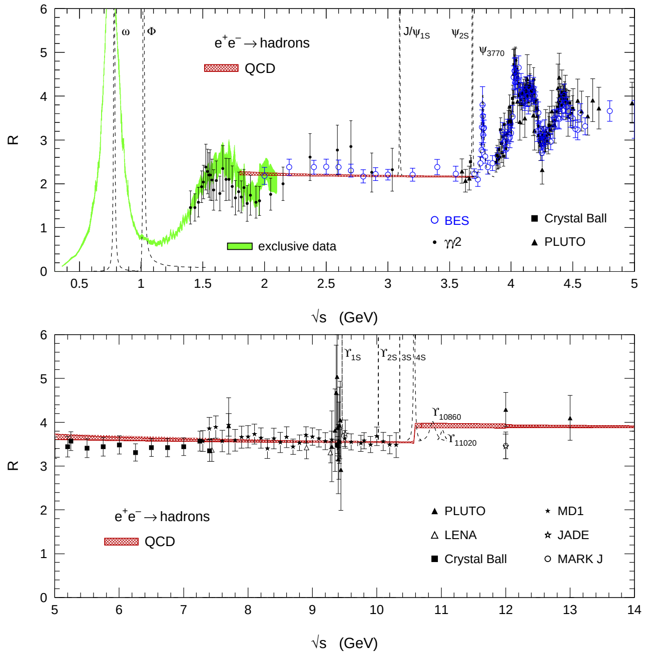
5. Evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD (Peris et al. '98, ...)

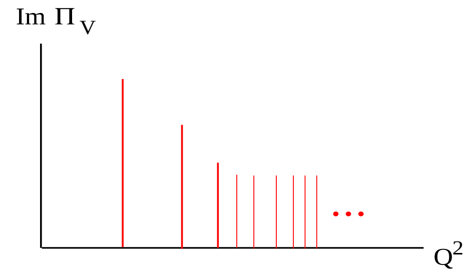
- In leading order in N_C, an infinite tower of narrow resonances contributes in each channel of a particular Green's function.
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD)

Example: 2-point function $\langle VV \rangle \to$ spectral function Im$\Pi_V \sim \sigma(e^+e^- \to$ hadrons)

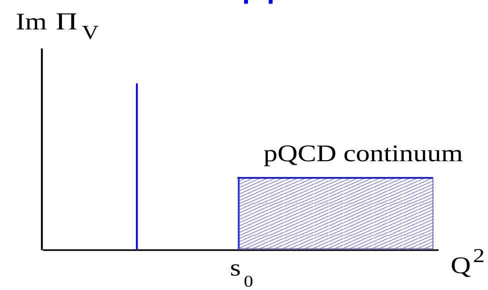
Real world (Davier et al., '03)



Large-N_C QCD ('t Hooft '74)



Minimal Hadronic Approximation (MHA)



Scale s_0 fixed by the OPE

Off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in large-N_C QCD

Knecht, Nyffeler, EPJC '01

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all QCD short-distance (OPE) constraints
- Reproduces Brodsky-Lepage behavior (confirmed by CLEO, but not by recent BABAR data):

$$\lim_{Q^2 \to \infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$$

- Normalized to decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.49)$ eV

Off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in large-N_C QCD

Knecht, Nyffeler, EPJC ’01

- Ansatz for $\langle VVP \rangle$ and thus $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ with 1 multiplet of lightest pseudoscalars (Goldstone bosons) and 2 multiplets of vector resonances, ρ, ρ' (lowest meson dominance (LMD) + V)
- $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ fulfills all QCD short-distance (OPE) constraints
- Reproduces Brodsky-Lepage behavior (confirmed by CLEO, but not by recent BABAR data):

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$$

- Normalized to decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.49)$ eV

Off-shell LMD+V form factor:

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)}$$

$$P_H^V(q_1^2, q_2^2, q_3^2) = h_1\,(q_1^2 + q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2 + q_2^2)\, q_3^2 + h_4\, q_3^4$$
$$+ h_5\,(q_1^2 + q_2^2) + h_6\, q_3^2 + h_7, \qquad q_3^2 = (q_1 + q_2)^2$$

$F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV

We view our evaluation as being a part of a full calculation of the hadronic light-by-light scattering contribution using a resonance Lagrangian along the lines of the Resonance Chiral Theory (Ecker et al. ’89, . . .), which also fulfills all the relevant QCD short-distance constraints.

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\rm LMD+V}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\rm LMD+V}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to recent BABAR data: $h_1 = (-0.17 \pm 0.02)$ GeV^2,
$h_5 = (6.51 \pm 0.20)$ GeV$^4 - h_3 m_\pi^2$ with χ^2/dof $= 15.0/15 = 1.0$

Fixing the LMD+V model parameters h_i

h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0) \sim 1/Q^2$)
- $h_2 = -10.63$ GeV^2 (Melnikov, Vainshtein '04: Higher twist corrections in OPE)
- $h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$)
- $h_7 = -N_C M_{V_1}^4 M_{V_2}^4 / (4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$
  $= -14.83$ GeV$^6 - h_6 m_\pi^2 - h_4 m_\pi^4$ (normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

Fit to recent BABAR data: $h_1 = (-0.17 \pm 0.02)$ GeV^2,
$h_5 = (6.51 \pm 0.20)$ GeV$^4 - h_3 m_\pi^2$ with χ^2/dof $= 15.0/15 = 1.0$

h_3, h_4, h_6 are unknown / less constrained.

h_6:
Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to h_6

Nyffeler '09: assume that LMD/LMD+V estimates of low-energy constants from chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$ are self-consistent. Assume 100% error on estimate for the relevant, presumably small low-energy constant $\Rightarrow h_6 = (5 \pm 5)$ GeV^4

Fixing the LMD+V model parameters (cont.)

h_3, h_4:

$$\Pi_{\mathrm{VT}}^{\mathrm{LMD+V}}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{p^2 + c_{\mathrm{VT}}}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)}, \qquad c_{\mathrm{VT}} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2}$$

(Knecht + Nyffeler, EPJC '01; Nyffeler '09)

OPE constraint for form factor leads to relation:

$$h_1 + h_3 + h_4 = 2c_{\mathrm{VT}} = M_{V_1}^2 M_{V_2}^2 \chi \qquad (*)$$

Fixing the LMD+V model parameters (cont.)

h_3, h_4:

$$\Pi_{\rm VT}^{\rm LMD+V}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{p^2 + c_{\rm VT}}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)}, \qquad c_{\rm VT} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2}$$

(Knecht + Nyffeler, EPJC '01; Nyffeler '09)

OPE constraint for form factor leads to relation:

$$h_1 + h_3 + h_4 = 2c_{\rm VT} = M_{V_1}^2 M_{V_2}^2 \chi \qquad (*)$$

LMD ansatz for $\langle VT \rangle \Rightarrow \chi^{\rm LMD} = -2/M_V^2 = -3.3$ GeV^{-2} (Balitsky, Yung '83)

Close to $\chi(\mu = 1$ GeV$) = -(3.15 \pm 0.30)$ GeV^{-2} (Ball et al. '03)

Nyffeler '09: assume large-N_C (LMD/LMD+V) framework is self-consistent

$\Rightarrow \chi = -(3.3 \pm 1.1)$ GeV^{-2} $\quad$ ($\Rightarrow h_3 + h_4 = (-4.3 \pm 1.4)$ GeV^2)

$\Rightarrow$ vary $h_3 = (0 \pm 10)$ GeV^2 and determine h_4 from relation $(*)$ and vice versa

Fixing the LMD+V model parameters (cont.)

h_3, h_4:

$$\Pi_{\rm VT}^{\rm LMD+V}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{p^2 + c_{\rm VT}}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)}, \qquad c_{\rm VT} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2}$$

(Knecht + Nyffeler, EPJC '01; Nyffeler '09)

OPE constraint for form factor leads to relation:

$$h_1 + h_3 + h_4 = 2c_{\rm VT} = M_{V_1}^2 M_{V_2}^2 \chi \qquad (*)$$

LMD ansatz for $\langle VT\rangle \Rightarrow \chi^{\rm LMD} = -2/M_V^2 = -3.3$ GeV^{-2} (Balitsky, Yung '83)

Close to $\chi(\mu = 1$ GeV$) = -(3.15 \pm 0.30)$ GeV^{-2} (Ball et al. '03)

Nyffeler '09: assume large-N_C (LMD/LMD+V) framework is self-consistent

$\Rightarrow \chi = -(3.3 \pm 1.1)$ GeV^{-2} $\quad (\Rightarrow\ h_3 + h_4 = (-4.3 \pm 1.4)$ GeV$^2)$

$\Rightarrow$ vary $h_3 = (0 \pm 10)$ GeV^2 and determine h_4 from relation $(*)$ and vice versa

Short-distance constraint on $\langle VVVV\rangle$ by Melnikov + Vainshtein '04

First $q_1^2 \sim q_2^2 \gg (q_1 + q_2)^2 \equiv q_3^2$ and then q_3^2 large, one obtains at external vertex:

$$\frac{3}{F_\pi} \mathcal{F}_{\pi^{0*}\gamma^*\gamma}^{\rm LMD+V}(q_3^2, q_3^2, 0) \stackrel{q_3^2\to\infty}{\to} \frac{h_1 + h_3 + h_4}{M_{V_1}^2 M_{V_2}^2} = \frac{2c_{\rm VT}}{M_{V_1}^2 M_{V_2}^2} = \chi$$

With pion propagator this leads to overall $1/q_3^2$ behavior. Agrees qualitatively with M+V '04 !
With constant (WZW) form factor at external vertex we would get (in chiral limit):

$$\frac{3}{F_\pi} \mathcal{F}_{\pi^0\gamma\gamma}^{\rm LMD+V}(0,0,0) = \frac{h_7}{M_{V_1}^4 M_{V_2}^4} = -\frac{N_C}{4\pi^2 F_\pi^2} \simeq -8.9 \text{ GeV}^{-2}$$

With $\chi = -N_C/(4\pi^2 F_\pi^2)$ from Vainshtein '03, we would precisely satisfy the short-distance constraint from M+V '04. Problem: why should pion-pole alone match with quark-loop ?

Parametrization of $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ for arbitrary model parameters h_i

- The h_i enter the LMD+V form factor linearly in the numerator, therefore (Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\sum_{i=1}^{7} c_i\, \tilde{h}_i + \sum_{i=1}^{7}\sum_{j=i}^{7} c_{ij}\, \tilde{h}_i\, \tilde{h}_j\right]$$

  with dimensionless coefficients $c_i, c_{ij} \sim 10^{-4}$ (see Nyffeler '09 for the values), if we measure the h_i in appropriate units of GeV $\rightarrow \tilde{h}_i$.

  h_1, h_3, h_4 not independent, but must obey the relation $h_1 + h_3 + h_4 = M_{V_1}^2\, M_{V_2}^2\, \chi$, because of the new short-distance constraint.

- h_1, h_2, h_5, h_7 are quite well known $\rightarrow$ can write down a simplified expression with only h_3, h_4, h_6 as free parameters (up to constraint):

$$\begin{aligned} a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \Big[& 503.3764 - 6.5223\, \tilde{h}_3 - 5.0962\, \tilde{h}_4 + 7.8557\, \tilde{h}_6 \\ & + 0.3017\, \tilde{h}_3^2 + 0.5683\, \tilde{h}_3\, \tilde{h}_4 - 0.1747\, \tilde{h}_3\, \tilde{h}_6 \\ & + 0.2672\, \tilde{h}_4^2 - 0.1411\, \tilde{h}_4\, \tilde{h}_6 + 0.0642\, \tilde{h}_6^2 \Big] \times 10^{-4} \end{aligned}$$

New estimate for pseudoscalar-exchange contribution

π^0:

- Our new estimate (Nyffeler '09; Jegerlehner, Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

  With off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}$ which obeys new short-distance constraint.

- Largest uncertainty from $h_6 = (5 \pm 5)$ GeV^4 $\Rightarrow \pm 6.4 \times 10^{-11}$ in $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$

  If we would vary $h_6 = (0 \pm 10)$ GeV^4 $\Rightarrow \pm 12 \times 10^{-11}$!

- Varying $\chi = -(3.3 \pm 1.1)$ GeV^{-2} $\Rightarrow \pm 2.1 \times 10^{-11}$

  Exact value of χ not that important, but range does not include estimate by Vainshtein '03 $\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$ GeV^{-2}

- Varying $h_3 = (0 \pm 10)$ GeV^2 $\Rightarrow \pm 2.5 \times 10^{-11}$ (h_4 via $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$) (similarly for variation of h_4)

- Added errors from variation of χ, h_3 (or h_4) and h_6 linearly.

- With h_1, h_5 from fit to recent BABAR data: $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = 71.8 \times 10^{-11} \rightarrow$ result unchanged !

New estimate for pseudoscalar-exchange contribution (cont.)

η, η':

- Short-distance analysis of LMD+V form factor in Knecht + Nyffeler, EPJC '01, performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
- Simplified approach (as done in many other papers) using VMD form factors

$$\mathcal{F}^{\mathrm{VMD}}_{\mathrm{PS}^*\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \mathrm{PS} = \eta, \eta'$$

  normalized to experimental decay width $\Gamma(\mathrm{PS} \to \gamma\gamma)$

  $\Gamma(\eta \to \gamma\gamma) = 0.510 \pm 0.026$ keV $\Rightarrow F_{\eta,\mathrm{eff}} = 93.0$ MeV ($m_\eta = 547.853$ MeV)

  $\Gamma(\eta' \to \gamma\gamma) = 4.30 \pm 0.15$ keV $\Rightarrow F_{\eta',\mathrm{eff}} = 74.0$ MeV ($m_{\eta'} = 957.66$ MeV)

- Problem with the VMD form factor: damping is too strong, behaves like $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, -Q^2, -Q^2) \sim 1/Q^4$, instead of $\sim 1/Q^2$ deduced from the OPE.

  However, final result is not too sensitive to high-energy behavior (see analysis in Knecht, Nyffeler '02). It seems more important to have good description at small and intermediate energies below 1 GeV, e.g. by reproducing slope of form factor $\mathcal{F}_{\mathrm{PS}\gamma^*\gamma}(-Q^2, 0)$ at origin.

  CLEO fitted VMD ansatz for $\mathcal{F}_{\mathrm{PS}\gamma^*\gamma}(-Q^2, 0)$ with adjustable parameter Λ_{PS} instead of $M_V \Rightarrow \Lambda_\eta = 774 \pm 29$ MeV, $\Lambda_{\eta'} = 859 \pm 28$ MeV. We take these masses for our evaluation.

- With VMD form factors at both vertices (i.e. not taking pole-approximation as in Melnikov, Vainshtein '04) we get $a_\mu^{\mathrm{LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5 \times 10^{-11}$
  Values might be too small ! A new detailed analysis is needed along the lines of LMD+V.

Sum of all pseudoscalars (Nyffeler '09; Jegerlehner, Nyffeler '09):

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

where we have assumed a 16% error, as inferred for the pion-exchange contribution.

Pseudoscalar exchanges

Model for $\mathcal{F}_{P^{(*)}\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
modified ENJL (off-shell) [BPP]	59(9)	85(13)
VMD / HLS (off-shell) [HKS,HK]	57(4)	83(6)
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant FF at ext. vertex) [MV]	77(7)	114(10)
nonlocal χQM (off-shell) [DB]	65(2)	—
LMD+V (off-shell) [N]	72(12)	99(16)
AdS/QCD (off-shell ?) [HoK]	69	107
AdS/QCD/DIP (off-shell) [CCD]	65.4(2.5)	—
DSE (off-shell) [FGW]	58(7)	84(13)
[PdRV]	—	114(13)
[JN]	72(12)	99(16)

BPP = Bijnens, Pallante, Prades ’95, ’96, ’02 (ENJL = Extended Nambu-Jona-Lasinio model); HK(S) = Hayakawa, Kinoshita, Sanda ’95, ’96; Hayakawa, Kinoshita ’98, ’02 (HLS = Hidden Local Symmetry model); KN = Knecht, Nyffeler ’02; MV = Melnikov, Vainshtein ’04; DB = Dorokhov, Broniowski ’08 (χQM = Chiral Quark Model); N = Nyffeler ’09; HoK = Hong, Kim ’09; CCD = Cappiello, Catà, D’Ambrosio ’10 (used AdS/QCD to fix parameters in DIP (D’Ambrosio, Isidori, Portolés) ansatz); FGW = Fischer, Goecke, Williams ’10, ’11 (Dyson-Schwinger equation)

Reviews on LbyL: PdRV = Prades, de Rafael, Vainshtein ’09; JN = Jegerlehner, Nyffeler ’09

Pseudoscalar exchanges (cont.)

- BPP use rescaled VMD result for η, η'. Also all LMD+V evaluations use VMD for η, η' !
- Off-shell form factors used in BPP, HKS presumably do not fulfill new short-distance constraint at external vertex and might have too strong damping $\to$ smaller values than off-shell LMD+V.
- Result for pion with off-shell form factors at both vertices in N '09, JN '09 is not too far from value given by MV '04, but this is pure coincidence ! Approaches not comparable ! MV '04 evaluate pion-pole contribution and use on-shell form factors (constant form factor at external vertex).

  Note: Following MV '04 and using $h_2 = -10$ GeV^2 we obtain 79.8×10^{-11} for the pion-pole contribution, close to 79.6×10^{-11} given in Bijnens, Prades '07 and 79.7×10^{-11} in DB '08.
- DB '08: Nonlocal χQM $\to$ strong damping for off-shell pions.
- HoK '09: error estimated in AdS/QCD model to be $< 30\%$. In text it is written that they use off-shell form factor, but only expression for on-shell form factor is given.
- CCA '10: DIP ansatz for off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ does not depend on $(q_1+q_2)^2$ (like for VMD form factor). Therefore, since form factor satisfies short-distance constraint at external vertex with $\chi \neq 0$, it violates the Brodsky-Lepage behavior.
- FGW '10, '11: DSE on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ satisfies OPE and Brodsky-Lepage behavior, quantitatively similar to on-shell LMD+V. New short-distance constraint on the off-shell form factor at external vertex is not discussed.

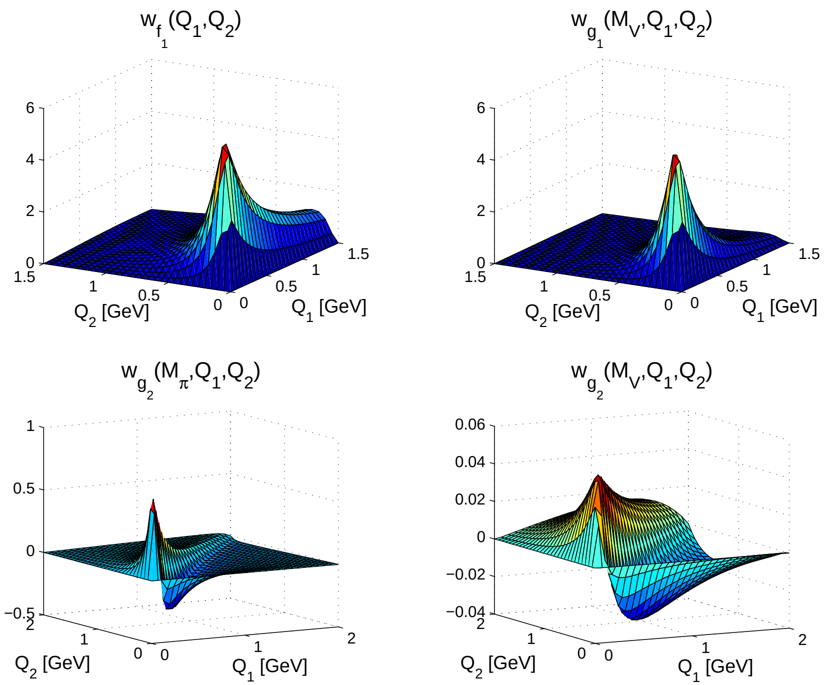
Relevant momentum regions in $a_\mu^{\mathrm{LbyL;PS}}$

In Knecht, Nyffeler '02, a 2-dimensional integral representation for $a_\mu^{\mathrm{LbyL;PS}}$ was derived for a certain class of on-shell form factors (schematically):

$$a_\mu^{\mathrm{LbyL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\, f_i(Q_1, Q_2)$$

with universal weight functions w_i. The dependence on the form factors resides in the f_i.

Note: Expressions with on-shell form factors are not valid as they stand ! One needs to set form factor at external vertex to a constant to obtain pion-pole contribution. The expressions are valid, however, for WZW and "off-shell" VMD form factors.



π^0:

- Relevant momentum regions are around $0.25 - 1.25$ GeV. Region below cutoff $\Lambda = 1$ GeV gives bulk of result: 82% for off-shell LMD+V, 92% for VMD.
- No damping from off-shell LMD+V form factor at external vertex, since $\chi \neq 0$. VMD falls off too fast compared to the OPE.
- If form factors in different models lead to damping, we expect comparable results.

η, η':

- Mass of intermediate pseudoscalar is higher than pion mass $\longrightarrow$ expect stronger suppression from propagator.
- Peak of weight functions shifted to higher values of Q_i. Saturation effect and suppression from VMD form factor only fully set in around $\Lambda = 1.5$ GeV: 96% of total for η, 93% for η'.

Hadronic light-by-light scattering in the muon $g-2$: Summary

Some results for the various contributions to $a_{\mu}^{\mathrm{LbyL;had}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN	FGW
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16	84 ± 13
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5	–
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2	–
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13	–
π, K loops $+$subl. N_C	–	–	–	0 ± 10	–	–	–	–
other	–	–	–	–	–	–	–	0 ± 20
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3	107 ± 48
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39	191 ± 81

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09; FGW = Fischer, Goecke, Williams '10, '11 (used values from arXiv:1009.5297v2 [hep-ph], 4 Feb 2011)

- Pseudoscalar-exchange contribution dominates numerically (except in FGW). But other contributions are not negligible. Note cancellation between π, K-loops and quark loops !
- PdRV: Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV '04 on pseudoscalar-pole contribution. Added all errors in quadrature ! Like HK(S). Too optimistic ?
- N, JN: New evaluation of pseudoscalars. Took over most values from BPP, except axial vectors from MV. Added all errors linearly. Like BPP, MV, BP, MdRR. Too pessimistic ?
- FGW: new approach with Dyson-Schwinger equations. Is there some double-counting ? Between their dressed quark loop (largely enhanced !) and the pseudoscalar exchanges.

Hadronic light-by-light scattering in the electron $g-2$

Using the same procedure and models as for the muon we obtain (Nyffeler '09):

$$\begin{aligned} a_e^{\mathrm{LbyL};\pi^0} &= (2.98 \pm 0.34) \times 10^{-14} \\ a_e^{\mathrm{LbyL};\eta} &= 0.49 \times 10^{-14} \\ a_e^{\mathrm{LbyL};\eta'} &= 0.39 \times 10^{-14} \\ a_e^{\mathrm{LbyL;PS}} &= (3.9 \pm 0.5) \times 10^{-14} \end{aligned}$$

Note: naive rescaling yields a too small result:

$$a_e^{\mathrm{LbyL};\pi^0}(\text{rescaled}) = \left(\frac{m_e}{m_\mu}\right)^2 a_\mu^{\mathrm{LbyL};\pi^0} = 1.7 \times 10^{-14}$$

Assuming that pseudoscalars give again dominant contribution to had. LbyL scattering leads to the "guesstimate" (Jegerlehner, Nyffeler '09):

$$a_e^{\mathrm{LbyL;had}} = (3.9 \pm 1.3) \times 10^{-14}$$

Later confirmed by large-log analysis of pseudoscalar pole-contribution by PdRV '09 (published version):

$$a_e^{\mathrm{LbyL;had}} = (3.5 \pm 1.0) \times 10^{-14}$$

6. Conclusions

- Jegerlehner '07, '08: one should use off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ to evaluate pion-exchange contribution. As done in earlier papers by BPP, HKS, HK ! Prescription by Melnikov, Vainshtein '04 to use a constant WZW form factor at the external vertex only yields pion-pole contribution with on-shell form factors $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$.
- New short-distance constraint on off-shell form factor at external vertex (Nyffeler '09):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3}\chi + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

[$\chi =$ chiral condensate magnetic susceptibility]

- New evaluation of pion-exchange contribution within large-N_C approximation using off-shell LMD+V form factor that fulfills all QCD short-distance constraints and model parameters $\chi = (-3.3 \pm 1.1)$ GeV^{-2}, $h_6 = (5 \pm 5)$ GeV^4 (Nyffeler '09):

$$a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

[BPP: 59 ± 9; HKS: 57 ± 4; KN: 58 ± 10; MV: 77 ± 7 in units of 10^{-11}]

- Updated values for η and η' (using simple VMD form factors):

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

[BPP: 85 ± 13; HKS: 83 ± 6; KN: 83 ± 12; MV: 114 ± 10 in units of 10^{-11}]

- Combined with evaluations of other contributions (Nyffeler '09; Jegerlehner, Nyffeler '09):

$$a_\mu^{\mathrm{LbyL;had}} = (116 \pm 40) \times 10^{-11}$$

[PdRV: $(105 \pm 26) \times 10^{-11}$]

- Compare with errors on:
  - Had. vac. pol.: $\pm(40-53) \times 10^{-11}$
  - BNL $g-2$ exp.: $\pm 63 \times 10^{-11}$
  - Future $g-2$ exp.: $\pm 15 \times 10^{-11}$

Outlook on had. LbyL scattering

- Some progress made in recent years for pseudoscalars and axial-vector contributions, implementing many experimental and theoretical constraints. More work needed for η, η' !
- More uncertainty for exchanges of scalars (and heavier resonances) and for (dressed) pion + kaon loop and (dressed) quark loops. Furthermore, there are some cancellations.
- Soon results from Lattice QCD ?

  There are some (very) preliminary studies by Hayakawa et al., hep-lat/0509016; Rakow et al. (Lattice 2008); Blum and Chowdhury, Nucl. Phys. B (Proc. Suppl.) $\mathbf{189}$, 251 (2009).

  But even result with 50% error (but reliable !) would be very helpful !

Suggested way forward in the meantime:

- Important to have unified consistent framework (model) which deals with all contributions.
- Purely phenomenological approach: resonance Lagrangian where all couplings are fixed from experiment. Non-renormalizable Lagrangian: how to achieve matching with pQCD ?
- Large-N_C framework: matching Green's functions with QCD short-distance constraints. (e.g. using Resonance Chiral Theory $\rightarrow$ many unknown couplings enter).

Outlook on had. LbyL scattering (cont.)

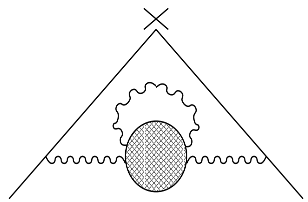
- In both approaches with resonance Lagrangians: experimental information on various hadronic form factors with on-shell and off-shell photons would be very helpful, e.g. on $\mathcal{F}_{P\gamma^*\gamma^*}$, $\mathcal{F}_{S\gamma^*\gamma^*}$, $\mathcal{F}_{A\gamma^*\gamma^*}$ and $\mathcal{F}_{\pi^+\pi^-\gamma^*\gamma^*}$. Experiments at e^+e^- colliders running at $1-3$ GeV, like BES at BEPC, CMD/SND at VEPP, KLOE at DAΦNE, could maybe measure some of these hadronic form factors for $|q_\gamma| < 2$ GeV. Additional information maybe available from BABAR, BELLE or future Super-B-factory.

- Can hopefully get additional informations / cross-checks on form factors from lattice QCD. Lattice studies of $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ were presented by S.D. Cohen et al., arXiv:0810.5550 [hep-lat]; E. Shintani et al., arXiv:0912.0253 [hep-lat].

  Further lattice evaluations of quark condensate magnetic susceptibility $\chi(\mu)$ (in addition to Buividovich et al. '09) would be very useful, not just for had. LbyL.

- Supplement these approaches with model-independent low-energy theorems in some particular limits (e.g. $m_\mu \to 0, m_\pi \to 0$ with m_μ / m_π fixed).

- Test models for had. LbyL scattering by comparison with exp. results for higher order contributions to had. vacuum polarization:



What experiments could do for had. LbyL

- Measurement of decay width $\pi^0 \to \gamma\gamma$: would be useful for normalization of form factor.

- Measurement of transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}(Q^2)$ at low Q^2 (slope at origin): would be useful to check consistency of models fitted to larger (asymptotic) values of Q^2.
Note that in many models there is only one free parameter, once normalization is fixed.
Brodsky-Lepage: no free parameter, once normalization F_π is fixed !

- Note: data on $\mathcal{F}_{\pi^0\gamma^*\gamma}(Q^2)$ do not uniquely fix result for pion-exchange contribution to had. LbyL scattering ! LMD+V and VMD give an equally good fit to the CLEO data, but results differ by about 20%: $a_{\mu;\text{LMD+V}}^{\text{LbyL};\pi^0} = 72 \times 10^{-11}$, $a_{\mu;\text{VMD}}^{\text{LbyL};\pi^0} = 57 \times 10^{-11}$.

- Problem: off-shell form factor (model dependent !) $\to$ no direct experimental information available ! In LMD+V model, the parameters h_3, h_4, h_6, related to off-shellness of pion, are the most uncertain and give the largest contribution to the uncertainty in the final result.

- $\pi^0 \to e^+e^-$ and $\eta \to \mu^+\mu^-, \eta \to e^+e^-$: could fix counterterm χ in ChPT approach (not to be confused with quark condensate magnetic susceptibility !) which enters as coefficient of single log term. Decay $\pi^0 \to e^+e^-$ is also related to parameter h_2 in LMD+V form factor.
But the ChPT approach has its limitations, because of the uncertainty in the $\mathcal{O}(1)$ term without logarithms.

Some questions to be addressed by this Workshop

- What do we really need to calculate ? Resonance-pole contribution with on-shell form factors or resonance-exchange contribution with off-shell form factors ?

- Quark-loop: omit it completely following Prades, de Rafael, Vainshtein ? But why should pion-pole alone match with quark-loop ? In principle, at leading order in N_C, an infinite number of resonances contribute to $\langle VVVV \rangle$ and therefore $g - 2$.

  The contribution from P', A', S' etc. has not been taken into account in any of the calculations.

- Large enhancement of quark-loop according to Fischer, Goecke, Williams. Is it real ? Contradicts all other evaluations, based on constituent quarks, dressed with form factors with $\rho - \gamma$ mixing.

  Is there some double counting when we add it to pseudoscalar exchanges (pole) ?

- How to better control contributions subleading in N_C, like dressed pion/Kaon loops ?

  Question of on-shell vs off-shell form factors also enters there.

Some questions to be addressed by this Workshop (cont.)

- How to best estimate and quote model errors ?

  Can one be fooled by some precise final result ?

  Example: VMD form factor shows that having only few parameters in a model can give the illusion of a precise determination of had. LbyL scattering which may not be justified.

  If we take the model very seriously, the VMD form factor (even with off-shell pions !) is completely determined by the vector meson mass M_V, once the normalization is fixed by $\Gamma(\pi^0 \to \gamma\gamma)$ or F_π. Fit by the CLEO collaboration of the transition form factor (error includes uncertainty in normalization): $M_V \equiv \Lambda_{\pi^0} = (776 \pm 22)$ MeV

  $\Rightarrow a_\mu^{\mathrm{LbyL};\pi^0} = (57 \pm 2) \times 10^{-11}$.

- How to add errors from different contributions ? Linearly or in quadrature ?

  If we would use one model, then the errors are correlated. Would need to study variation of sum of all contributions, when all model parameters vary.

  Currently we essentially use a different model for each contribution ! The argument that the estimates for each contribution are therefore “independent” and we can add them in quadrature also seems “dangerous”. In particular, if several models give very different results for one contribution.

Backup slides

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$

- 2-loop integral $\to$ 8-dim. integral. Integration over 3 angles can be done easily
- 5 non-trivial integrations: 2 moduli: $|q_1|$, $|q_2|$,   3 angles: $p \cdot q_1$, $p \cdot q_2$, $q_1 \cdot q_2$ (recall $p^2 = m_\mu^2$).
- Observation: $p \cdot q_1$, $p \cdot q_2$ do not appear in the model-dependent form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$
- Can perform those two angular integrations by averaging expression for $a_\mu^{\mathrm{LbyL};\pi^0}$ over the direction of p (Jegerlehner, Nyffeler '09)

Method of Gegenbauer polynomials (hyperspherical approach)

(Baker, Johnson, Willey '64, '67; Rosner '67; Levine, Roskies '74; Levine, Remiddi, Roskies '79)

Denote by $\hat{K}$ unit vector of four-momentum vector K in Euclidean space

Propagators in Euclidean space:

$$\frac{1}{(K-L)^2+M^2} = \frac{Z_{KL}^M}{|K||L|} \sum_{n=0}^{\infty} \left(Z_{KL}^M\right)^n C_n(\hat{K}\cdot\hat{L})$$

$$Z_{KL}^M = \frac{K^2+L^2+M^2-\sqrt{(K^2+L^2+M^2)^2-4K^2L^2}}{2|K||L|}$$

Use orthogonality conditions of Gegenbauer polynomials:

$$\int d\Omega(\hat{K})\, C_n(\hat{Q}_1\cdot\hat{K})\, C_m(\hat{K}\cdot\hat{Q}_2) = 2\pi^2 \frac{\delta_{nm}}{n+1} C_n(\hat{Q}_1\cdot\hat{Q}_2)$$

$$\int d\Omega(\hat{K})\, C_n(\hat{Q}\cdot\hat{K})\, C_m(\hat{K}\cdot\hat{Q}) = 2\pi^2\, \delta_{nm}$$

$\hat{Q}_1 \cdot \hat{K}$ = Cosine of angle between the four-dimensional vectors Q_1 and K

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$ (cont.)

Average over direction $\hat{P}$ (note: $P^2 = -m_\mu^2$):

$$\langle \cdots \rangle = \frac{1}{2\pi^2} \int d\Omega(\hat{P}) \cdots$$

After reducing numerators in the functions T_i in $a_\mu^{\mathrm{LbyL};\pi^0}$ against denominators of propagators, one is left with the following integrals, denoting propagators by $(4) \equiv (P+Q_1)^2 + m_\mu^2, \quad (5) \equiv (P-Q_2)^2 + m_\mu^2$:

$$\begin{aligned}
\langle \frac{1}{(4)} \frac{1}{(5)} \rangle &= \frac{1}{m_\mu^2 R_{12}} \arctan\left(\frac{zx}{1-zt}\right) \\
\langle (P \cdot Q_1) \frac{1}{(5)} \rangle &= -(Q_1 \cdot Q_2) \frac{(1-R_{m2})^2}{8m_\mu^2} \\
\langle (P \cdot Q_2) \frac{1}{(4)} \rangle &= (Q_1 \cdot Q_2) \frac{(1-R_{m1})^2}{8m_\mu^2} \\
\langle \frac{1}{(4)} \rangle &= -\frac{1-R_{m1}}{2m_\mu^2} \\
\langle \frac{1}{(5)} \rangle &= -\frac{1-R_{m2}}{2m_\mu^2}
\end{aligned}$$

$Q_1 \cdot Q_2 = Q_1 Q_2 \cos\theta$, $\quad t = \cos\theta \quad$ (θ = angle between Q_1 and Q_2), $\quad Q_i \equiv |Q_i|$

$R_{mi} = \sqrt{1 + 4m_\mu^2/Q_i^2}$, $\quad x = \sqrt{1-t^2}$, $\quad R_{12} = Q_1 Q_2 x$, $\quad z = \frac{Q_1 Q_2}{4m_\mu^2}(1-R_{m1})(1-R_{m2})$

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$ (cont.)

In this way one obtains (Jegerlehner, Nyffeler '09):

$$a_\mu^{\mathrm{LbL};\pi^0} = -\frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 dQ_2 \int_{-1}^{+1} dt\,\sqrt{1-t^2}\,Q_1^3\,Q_2^3 \times \left[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_2^2,-Q_1^2,-Q_3^2)\,\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_2^2,-Q_2^2,0)}{(Q_2^2+m_\pi^2)} I_1(Q_1,Q_2,t) + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(-Q_3^2,-Q_1^2,-Q_2^2)\,\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(-Q_3^2,-Q_3^2,0)}{(Q_3^2+m_\pi^2)} I_2(Q_1,Q_2,t)\right]$$

where $Q_3^2 = (Q_1+Q_2)^2$, $\quad Q_1\cdot Q_2 = Q_1 Q_2\cos\theta$, $\quad t=\cos\theta$

$$\begin{aligned} I_1(Q_1,Q_2,t) = {} & X(Q_1,Q_2,t)\Big(8\,P_1\,P_2\,(Q_1\cdot Q_2) - 2\,P_1\,P_3\,(Q_2^4/m_\mu^2 - 2\,Q_2^2) - 2\,P_1\,(2 - Q_2^2/m_\mu^2 + 2\,(Q_1\cdot Q_2)\,/m_\mu^2) \\ & + 4\,P_2\,P_3\,Q_1^2 - 4\,P_2 - 2\,P_3\,(4 + Q_1^2/m_\mu^2 - 2\,Q_2^2/m_\mu^2) + 2/m_\mu^2\Big) \\ & - 2\,P_1\,P_2\,(1 + (1-R_{m1})\,(Q_1\cdot Q_2)\,/m_\mu^2) + P_1\,P_3\,(2 - (1-R_{m1})\,Q_2^2/m_\mu^2) + P_1\,(1-R_{m1})/m_\mu^2 \\ & + P_2\,P_3\,(2 + (1-R_{m1})^2\,(Q_1\cdot Q_2)\,/m_\mu^2) + 3\,P_3\,(1-R_{m1})/m_\mu^2 \end{aligned}$$

$$\begin{aligned} I_2(Q_1,Q_2,t) = {} & X(Q_1,Q_2,t)\Big(4\,P_1\,P_2\,(Q_1\cdot Q_2) + 2\,P_1\,P_3\,Q_2^2 - 2\,P_1 + 2\,P_2\,P_3\,Q_1^2 - 2\,P_2 - 4P_3 - 4/m_\mu^2\Big) \\ & - 2\,P_1\,P_2 - 3\,P_1\,(1-R_{m2})/(2m_\mu^2) - 3\,P_2\,(1-R_{m1})/(2m_\mu^2) - P_3\,(2 - R_{m1} - R_{m2})/(2m_\mu^2) \\ & + P_1\,P_3\,(2 + 3\,(1-R_{m2})\,Q_2^2/(2m_\mu^2) + (1-R_{m2})^2\,(Q_1\cdot Q_2)\,/(2m_\mu^2)) \\ & + P_2\,P_3\,(2 + 3\,(1-R_{m1})\,Q_1^2/(2m_\mu^2) + (1-R_{m1})^2\,(Q_1\cdot Q_2)\,/(2m_\mu^2)) \end{aligned}$$

where $P_1^2 = 1/Q_1^2$, $P_2^2 = 1/Q_2^2$, $P_3^2 = 1/Q_3^2$, $\quad X(Q_1,Q_2,t) = \frac{1}{Q_1 Q_2\, x}\arctan\left(\frac{zx}{1-zt}\right)$

Constraining the LMD+V model parameter h_6

- Final result for $a_{\mu}^{\mathrm{LbyL};\pi^0}$ is very sensitive to value of h_6. We can get some indirect information on size and sign of h_6 as follows.
- Estimates of low-energy constants in chiral Lagrangians via exchange of resonances work quite well. However, we may get some corrections, if we consider the exchange of heavier resonances as well. Typically, a large-N_C error of 30% can be expected.
- In $\langle VVP \rangle$ appear 2 combinations of low-energy constants from the chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$, denoted by A_{V,p^2} and $A_{V,(p+q)^2}$ in Knecht, Nyffeler, EPJC '01.

$$A_{V,p^2}^{\mathrm{LMD}} = \frac{F_\pi^2}{8M_V^4} - \frac{N_C}{32\pi^2 M_V^2} = -1.11 \, \frac{10^{-4}}{F_\pi^2}$$

$$A_{V,p^2}^{\mathrm{LMD+V}} = \frac{F_\pi^2}{8M_{V_1}^4} \frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2 M_{V_1}^2}\left(1 + \frac{M_{V_1}^2}{M_{V_2}^2}\right) = -1.36 \, \frac{10^{-4}}{F_\pi^2}$$

The relative change is only about 20%, well within expected large-N_C uncertainty !

$$A_{V,(p+q)^2}^{\mathrm{LMD}} = -\frac{F_\pi^2}{8M_V^4} = -0.26 \, \frac{10^{-4}}{F_\pi^2}, \qquad A_{V,(p+q)^2}^{\mathrm{LMD+V}} = -\frac{F_\pi^2}{8M_{V_1}^4 M_{V_2}^4} h_6$$

Note that $A_{V,(p+q)^2}^{\mathrm{LMD}}$ is "small" compared to A_{V,p^2}^{LMD}. About same size as absolute value of the shift in A_{V,p^2} when going from LMD to LMD+V !

- Assuming that LMD/LMD+V framework is self-consistent, but allowing for a 100% uncertainty of $A_{V,(p+q)^2}^{\mathrm{LMD}}$, we get the range $h_6 = (5 \pm 5)$ GeV^4

Detailed results for the pion-exchange contribution (Nyffeler ’09)

$a_{\mu}^{\mathrm{LbyL};\pi^0} \times 10^{11}$ with the off-shell LMD+V form factor:

	$h_6 = 0$ GeV^4	$h_6 = 5$ GeV^4	$h_6 = 10$ GeV^4
$h_3 = -10$ GeV^2	68.4	74.1	80.2
$h_3 = 0$ GeV^2	66.4	71.9	77.8
$h_3 = 10$ GeV^2	64.4	69.7	75.4
$h_4 = -10$ GeV^2	65.3	70.7	76.4
$h_4 = 0$ GeV^2	67.3	72.8	78.8
$h_4 = 10$ GeV^2	69.2	75.0	81.2

$\chi = -3.3$ GeV^{-2}, $h_1 = 0$ GeV^2, $h_2 = -10.63$ GeV^2 and $h_5 = 6.93$ GeV$^4 - h_3 m_\pi^2$

When varying h_3 (upper half of table), h_4 is fixed by constraint $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$.
In the lower half the procedure is reversed.

Within scanned region:
Minimal value: 63.2×10^{-11} [$\chi = -2.2$ GeV^{-2}, $h_3 = 10$ GeV^2, $h_6 = 0$ GeV^4]
Maximum value: 83.3×10^{-11} [$\chi = -4.4$ GeV^{-2}, $h_4 = 10$ GeV^2, $h_6 = 10$ GeV^4]

Take average of results for $h_6 = 5$ GeV^4 for $h_3 = 0$ GeV^2 and $h_4 = 0$ GeV^2 as estimate:

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

Added errors from χ, h_3 (or h_4) and h_6 linearly. Do not follow Gaussian distribution !

Parametrization of $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ for arbitrary model parameters h_i

The h_i enter the form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}$ linearly in the numerator, therefore (Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\sum_{i=1}^{7} c_i\, \tilde{h}_i + \sum_{i=1}^{7}\sum_{j=i}^{7} c_{ij}\, \tilde{h}_i\, \tilde{h}_j\right]$$

with dimensionless coefficients c_i, c_{ij}, if we measure the h_i in appropriate units of GeV: $\tilde{h}_i = h_i/\mathrm{GeV}^2$ for $i = 1,2,3,4$, $\tilde{h}_i = h_i/\mathrm{GeV}^4$ for $i = 5,6$ and $\tilde{h}_7 = h_7/\mathrm{GeV}^6$.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$c_i \times 10^4$	−1.4530	0	−1.4530	−1.4530	0.4547	0.4547	−1.2048

$c_{ij} \times 10^4$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
$i=1$	0.4447	0.0729	0.7428	0.7120	−0.3620	−0.3332	0.9916
$i=2$	...	0	0.0730	0.0729	−0.0557	−0.0557	0.2221
$i=3$	...	...	0.2980	0.5653	−0.1967	−0.1679	0.1796
$i=4$	...	...	...	0.2672	−0.1679	−0.1391	0.1162
$i=5$	...	...	...	...	0.1215	0.1796	−0.8072
$i=6$	...	...	...	...	...	0.0581	−0.3052
$i=7$	...	...	...	...	...	...	1.6122

There is no term without the constants h_i, because of the form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_3^2, q_3^2, 0)$ with a soft photon at the external vertex. This also leads to $c_2 = 0$ and $c_{22} = 0$.

h_1, h_3, h_4 are not independent, but must obey the relation $h_1 + h_3 + h_4 = M_{V_1}^2\, M_{V_2}^2\, \chi$, because of the new short-distance constraint on the form factor at the external vertex.

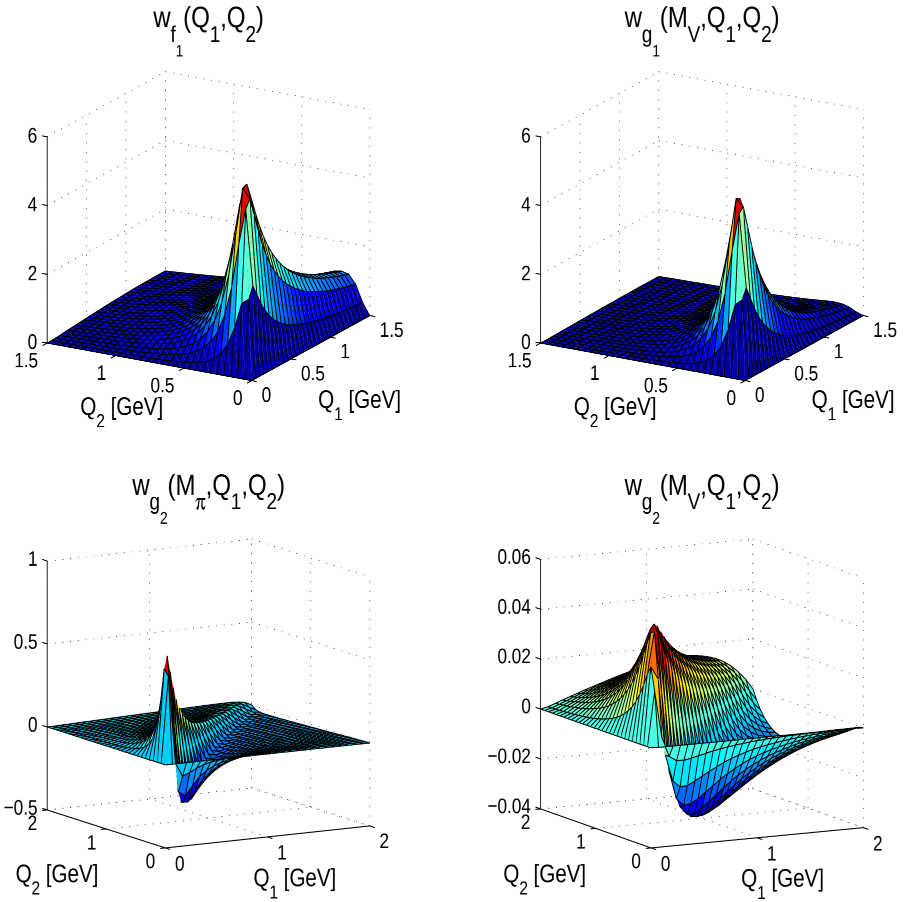
Relevant momentum regions in $a_\mu^{\mathrm{LbyL};\pi^0}$

In Knecht, Nyffeler '02, a 2-dimensional integral representation was derived for a certain class of on-shell form factors (schematically):

$$a_\mu^{\mathrm{LbyL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_i w_i(Q_1, Q_2)\, f_i(Q_1, Q_2)$$

with universal weight functions w_i. The dependence on the form factors resides in the f_i.

Note: Expressions with on-shell form factors are not valid as they stand ! One needs to set form factor at external vertex to a constant to obtain pion-pole contribution. The expressions are valid, however, for WZW and "off-shell" VMD form factors.



- $w_{f_1}(Q_1, Q_2)$ enters for WZW form factor. Tail leads to $\ln^2 \Lambda$ divergence for momentum cutoff Λ.
- $w_{g_1}(M_V, Q_1, Q_2)$ enters for VMD form factor.
- Relevant momentum regions are therefore around $0.25 - 1.25$ GeV. As long as form factors in different models lead to damping, we expect comparable results for $a_\mu^{\mathrm{LbyL};\pi^0}$, at the level of 20%. Similarly for η, η'.
- Weight functions $w_{g_2}(M, Q_1, Q_2)$ enter in integral originating from T_2. Because of cancellation between positive and negative parts, finite even with constant WZW form factors ! Only very small contribution to final result.

Relevant momentum regions in $a_\mu^{\mathrm{LbyL;PS}}$

Result for pseudoscalar exchange contribution $a_\mu^{\mathrm{LbyL;PS}} \times 10^{11}$ for off-shell LMD+V and VMD form factors obtained with momentum cutoff Λ in 3-dimensional integral representation of JN '09 (integration over square). In brackets, relative contribution of the total obtained with $\Lambda = 20$ GeV.

Λ [GeV]	π^0 LMD+V ($h_3 = 0$)	π^0 LMD+V ($h_4 = 0$)	π^0 VMD	η VMD	η' VMD
0.25	14.8 (20.6%)	14.8 (20.3%)	14.4 (25.2%)	1.76 (12.1%)	0.99 (7.9%)
0.5	38.6 (53.8%)	38.8 (53.2%)	36.6 (64.2%)	6.90 (47.5%)	4.52 (36.1%)
0.75	51.9 (72.2%)	52.2 (71.7%)	47.7 (83.8%)	10.7 (73.4%)	7.83 (62.5%)
1.0	58.7 (81.7%)	59.2 (81.4%)	52.6 (92.3%)	12.6 (86.6%)	9.90 (79.1%)
1.5	64.9 (90.2%)	65.6 (90.1%)	55.8 (97.8%)	14.0 (96.1%)	11.7 (93.2%)
2.0	67.5 (93.9%)	68.3 (93.8%)	56.5 (99.2%)	14.3 (98.6%)	12.2 (97.4%)
5.0	71.0 (98.8%)	71.9 (98.8%)	56.9 (99.9%)	14.5 (99.9%)	12.5 (99.9%)
20.0	71.9 (100%)	72.8 (100%)	57.0 (100%)	14.5 (100%)	12.5 (100%)

π^0:

- Although weight functions plotted earlier are not applicable to off-shell LMD+V form factor, region below $\Lambda = 1$ GeV gives the bulk of the result: 82% for LMD+V, 92% for VMD.
- No damping from off-shell LMD+V form factor at external vertex since $\chi \neq 0$ (new short-distance constraint). Note that VMD falls off too fast, compared to the OPE.

η, η':

- Mass of intermediate pseudoscalar is higher than pion mass $\rightarrow$ expect a stronger suppression from propagator.
- Peak of relevant weight functions shifted to higher values of Q_i. For η', vector meson mass is also higher $M_V = 859$ MeV. Saturation effect and the suppression from the VMD form factor only fully set in around $\Lambda = 1.5$ GeV: 96% of total for η, 93% for η'.

Axial-vector exchanges

Model for $\mathcal{F}_{A^*\gamma^*\gamma^*}$	$a_\mu(a_1) \times 10^{11}$	$a_\mu(a_1, f_1, f_1') \times 10^{11}$
ENJL-VMD [BPP] (nonet symmetry)	2.5(1.0)	—
ENJL-like [HKS,HK] (nonet symmetry)	1.7(1.7)	—
LMD [MV] (f_1 pure octet, f_1' pure singlet)	5.7	17
LMD [MV] (ideal mixing)	5.7	22(5)
[PdRV]	—	15(10)
[JN]	—	22(5)

- MV '04: derived QCD short-distance constraint for axial-vector pole contribution with on-shell form factor $\mathcal{F}_{A\gamma^*\gamma^*}$ at both vertices
- Simple VMD ansatz: short-distance constraints forbids form factor at external vertex. Assuming all axial-vectors in the nonet have same mass M leads to

$$a_\mu^{\mathrm{AV}} = \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{M^2} N_C \mathrm{Tr}\left[\hat{Q}^4\right] \left(\frac{71}{192} + \frac{81}{16} S_2 - \frac{7\pi^2}{144}\right) + \ldots \approx 1010 \frac{m_\mu^2}{M^2} \times 10^{-11}$$

($\hat{Q} = \mathrm{diag}(2/3, -1/3, -1/3)$, $S_2 = 0.26043$)

Strong dependence on mass M:
$M = 1300$ MeV: $a_\mu^{\mathrm{AV}} = 7 \times 10^{-11}$, $M = M_\rho$: $a_\mu^{\mathrm{AV}} = 28 \times 10^{-11}$ (with $+\ldots$)

- More sophisticated LMD ansatz (Czarnecki, Marciano, Vainshtein '03): see Table. Now there is form factor at external vertex. Dressing leads to lower effective mass M. Furthermore f_1, f_1' have large coupling to photons $\longrightarrow$ huge enhancement compared to BPP, HKS !

Scalar exchanges

Model for $\mathcal{F}_{S^*\gamma^*\gamma^*}$	a_μ(scalars) $\times 10^{11}$
Point coupling	$-\infty$
ENJL [BPP]	$-7(2)$
[PdRV]	$-7(7)$
[JN]	$-7(2)$

- Within ENJL model: scalar exchange contribution related by Ward identities to (constituent) quark loop $\rightarrow$ HK argued that effect of (broad) scalar resonances below several hundred MeV might already be included in sum of (dressed) quark loops and (dressed) $\pi + K$ loops !
- Potential double-counting is definitely an issue for the broad sigma meson $f_0(600)$ ($\leftrightarrow \pi^+\pi^-; \pi^0\pi^0$). Ongoing debate whether the scalar resonances $f_0(980), a_0(980)$ are two-quark or four-quark states.
- It is not clear which scalar resonances are described by ENJL model. Model parameters fixed by fitting various low-energy observables and resonance parameters, among them $M_S = 980$ MeV. However, model then yields $M_S^{\mathrm{ENJL}} = 620$ MeV.
- Can the usually broad scalar resonances be described by a simple resonance Lagrangian which works best in large-N_C limit, i.e. for very narrow states ?

Charged pion and kaon loops

Model $\pi^+\pi^-\gamma^*(\gamma^*)$	$a_\mu(\pi^\pm) \times 10^{11}$	$a_\mu(\pi^\pm, K^\pm) \times 10^{11}$
Point coupling (scalar QED)	-45.3	-49.8
VMD [KNO, HKS]	-16	$-$
full VMD [BPP]	$-18(13)$	$-19(13)$
HLS [HKS,HK]	-4.45	$-4.5(8.1)$
[MV] (all N_C^0 terms !)	$-$	$0(10)$
[PdRV]	$-$	$-19(19)$
[JN]	$-$	$-19(13)$

- Dressing leads to a rather huge suppression compared to scalar QED ! Very model dependent.
- MV '04 studied HLS model via expansion in $(m_\pi/M_\rho)^2$ and $(m_\mu - m_\pi)/m_\pi$:

$$a_{\mu;\mathrm{HLS}}^{\mathrm{LbL};\pi^\pm} = \left(\frac{\alpha}{\pi}\right)^3 \sum_{i=0}^{\infty} f_i\left[\frac{m_\mu - m_\pi}{m_\pi}, \ln\left(\frac{M_\rho}{m_\pi}\right)\right]\left(\frac{m_\pi^2}{M_\rho^2}\right)^i = \left(\frac{\alpha}{\pi}\right)^3 (-0.0058)$$

$$= (-46.37 + 35.46 + 10.98 - 4.70 - 0.3 + \ldots) \times 10^{-11} = -4.9(3) \times 10^{-11}$$

- Large cancellation between first three terms in series. Expansion converges only very slowly. Main reason: typical momenta in the loop integral are of order $\mu = 4m_\pi \approx 550$ MeV and the effective expansion parameter is μ/M_ρ, not m_π/M_ρ.
- MV '04: Final result is very likely suppressed, but also very model dependent $\rightarrow$ chiral expansion looses predictive power $\rightarrow$ lumped together all terms subleading in N_C.

Dressed quark loops

Model	$a_\mu(\text{quarks}) \times 10^{11}$
Point coupling	$62(3)$
ENJL + bare heavy quark [BPP]	$21(3)$
VMD [HKS, HK]	$9.7(11.1)$
DSE [FGW]	$107(48)$
[PdRV] (Bare c-quark only !)	2.3
[JN]	$21(3)$

- de Rafael '94: dressed quark loops can be interpreted as irreducible contribution to the 4-point function $\langle VVVV \rangle$. Also appear as short-distance complement of low-energy hadronic models (absorb as counterterms the remaining cutoff dependences).
- Quark-hadron duality: the quark loops also model contributions from exchanges and loops of heavier hadronic states, like $\pi', a_0', f_0', p, n, \ldots$.
- Again very large model-dependent effect of the dressing (form factors).
- BPP employ the ENJL model up to some cutoff μ and then add the bare quark loop with constituent quark mass $M_Q = \mu$. The latter contribution simulates the high-momentum component of the quark loop which is non-negligible.
- PdRV '09 argued that the dressed light-quark loops should not be included as separate contribution. They assume them to be already covered by using the short-distance constraint from MV '04 for the pseudoscalar-pole contribution.
- FGW '10, '11: in contrast to other models, dressing using Schwinger-Dyson equations leads to large enhancement. Potential double counting with pseudoscalar exchanges ?