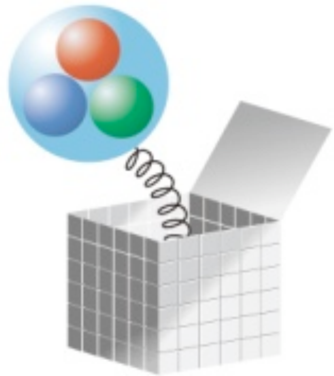


# $\pi^0 \to \gamma^* \gamma^*$

Shoji Hashimoto (KEK)

@ LbL workshop, INT, U of Washington, Mar 1, 2011.

# This work is (being) done by …

- Eigo Shintani (Riken-BNL), talks at Lattice 2009, 2010
- Xu Feng (KEK), a forthcoming talk at Lattice 2011

as a project of the JLQCD collaboration:

- LQCD simulation with the overlap fermion (since 2006). Applications with exact chiral symmetry
  - pion/kaon mass, decay constant, form factors, $B_K$, Dirac operator spectrum (and chiral condensate), topological susceptibility, VV-AA correlator, $\alpha_s$ from vacuum polarization function, nucleon strange quark content.
  - Relatively slow, but a good test bed to apply theoretical ideas.

# Not impossible, but …

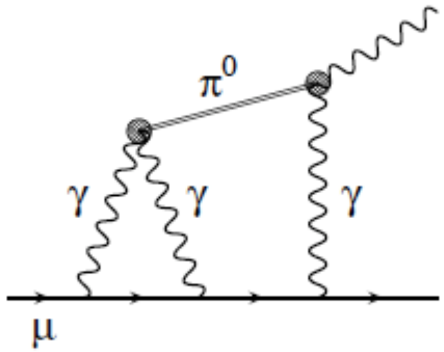
- 4-point function

$$\langle J_\mu(x) J_\nu(y) J_\rho(z) J_\sigma(w) \rangle$$

is calculable on the lattice, in principle. But,

  - Too many form factors (how many?)
  - Fourier transform must be done for three independent momenta.

- Better to start from more tractable target: $\pi^0 \to \gamma^* \gamma^*$
  - Dominant (?) contribution to LbL
  - ⟨PVV⟩ 3-point function to be calculated
    - two independent momenta: $p_1^2, p_2^2$



# How to treat (virtual) photon

- Photon is not a hadronic (ground) state.
  - Conventional method to extract the ground state on the lattice cannot be used. Four-momentum must be fixed from outside.

- Two (related) methods:
  1. 3-point function in the Euclidean momentum space
  2. Wick rotate the relevant amplitude back to Minkowski

# (Euclidean) momentum space

- Lattice is defined in the Euclidean space-time
  - Typically, we (lattice theorists) use the asymptotic behavior of a two-point function

$$C(t,\mathbf{p}) = \int dp_0 e^{ipx} \frac{Z}{p_0^2 + \mathbf{p}^2 + m^2} \sim \frac{Z}{2E} \exp(-Et), \quad E = \sqrt{m^2 + \mathbf{p}^2}$$

to extract the ground state. Not directly looking at the pole.

  - In principle, the same info can be obtained by directly analyzing the correlator

$$\frac{Z_1}{m_1^2 - q^2} + \frac{Z_2}{m_2^2 - q^2} + \ldots$$

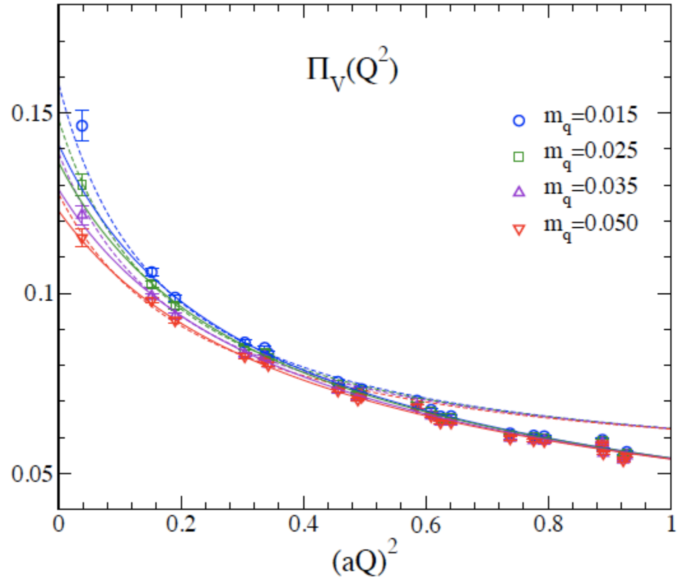
in the space-like ($q^2=-Q^2<0$) region. (In practice, more complicated because of scattering states.)

# Two-point function

- VV correlator as an example

$$\int d^4x e^{iQx} \langle 0 | V_\mu^a(x) V_\nu^{b\dagger}(0) | 0 \rangle = \delta^{ab} (\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu) \Pi_V^{(1)}(Q)$$

  - Only space-like $Q_\mu$ is available, in unit of $2\pi/L$.
  - Lowest vector resonance is well reproduced by a pole fit.
    - Note that the $\pi\pi$ threshold is not open for our quark masses.
  - Higher states may contain $\pi\pi$ continuum, whose functional form is non-trivial.



# Three-point function for $\pi^0 \to \gamma^* \gamma^*$

- Form factor could be extracted from

$$\left\langle 2m_q P(0) V_\mu(x) V_\nu(y) \right\rangle \xrightarrow{F.T.} \frac{m_\pi^2 f_\pi}{m_\pi^2 - q^2} f_{\pi^0\gamma\gamma}(p_1^2, p_2^2) \varepsilon_{\mu\nu\alpha\beta} p_1^\alpha p_2^\beta + \text{higher resonances}$$

  - At small $q^2$, PS channel is dominated by pion, as the pion pole is much closer. The next possible state would be $3\pi$.

  - Momenta $p_1$ and $p_2$ are space-like.
  - Form factor is known in the soft pion limit (ABJ anomaly)

$$f_{\pi^0\gamma\gamma}(0,0) = \frac{1}{4\pi^2 f_0}$$

, with $f_0$ the pion decay constant in the massless limit.

    - guaranteed when chiral symmetry is exact.
    - Away from this limit, chiral expansion is possible. Some LECs appear.

# Lattice result

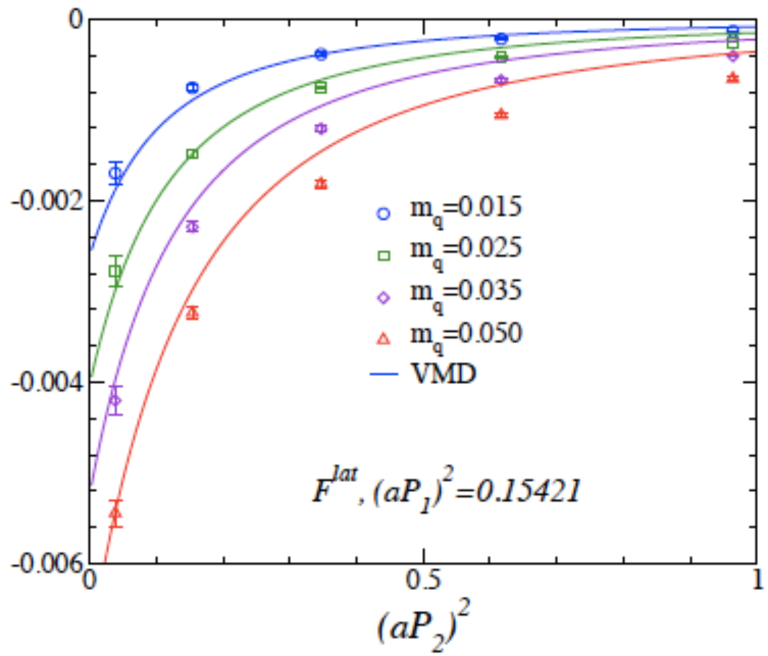
- Sample results: Shintani @ Lattice 2009

$$F(P_1^2,P_2^2) = -\frac{f_\pi m_\pi^2}{Q^2+m_\pi^2} f_{\pi^0\gamma\gamma}(P_1^2,P_2^2)$$

  - Vector Meson Dominance (VMD) suggests

$$F^{VMD}(P_1^2,P_2^2) = -\frac{f_\pi m_\pi^2}{Q^2+m_\pi^2} \times X_a \frac{m_v^2}{P_1^2+m_v^2}\frac{m_v^2}{P_2^2+m_v^2}$$

    - Agreement at the lowest momentum with $X_a$=0.0260(6), c.f. $1/4\pi^2$=0.02533
    - Becomes worse at higher momenta. 2nd lowest is already 600~700 MeV.



# Lessons

- Good:
  - The method works. Computational cost is not particularly higher than that for other 3pt functions.
  - ABJ anomaly is well reproduced. (Was not trivial to me.)
  - Finite quark mass correction can be extracted, in principle = (precise) calculation of $\pi^0$ decay width.
- Marginal
  - Momentum resolution is not sufficient to determine the form factor in detail, especially in the low momentum region... Matter of the computational power.

# Wick rotate back to Minkowski

- The other method: Ji, Jung, Phys. Rev. Lett. 86, 208 (2001); Phys. Rev. D64, 034506 (2001). Dudek, Edwards, Phys. Rev. Lett. 97, 172002 (2006).
  - After doing the LSZ reduction, what we want to calculate (in the Minkowski space-time) is

$$\int d^4y\, e^{ip_2(x-y)} \left\langle 0 \middle| T\left[V^\mu(x)V^\nu(y)\right] \middle| M(q) \right\rangle, \quad x \text{ can be set to } 0$$

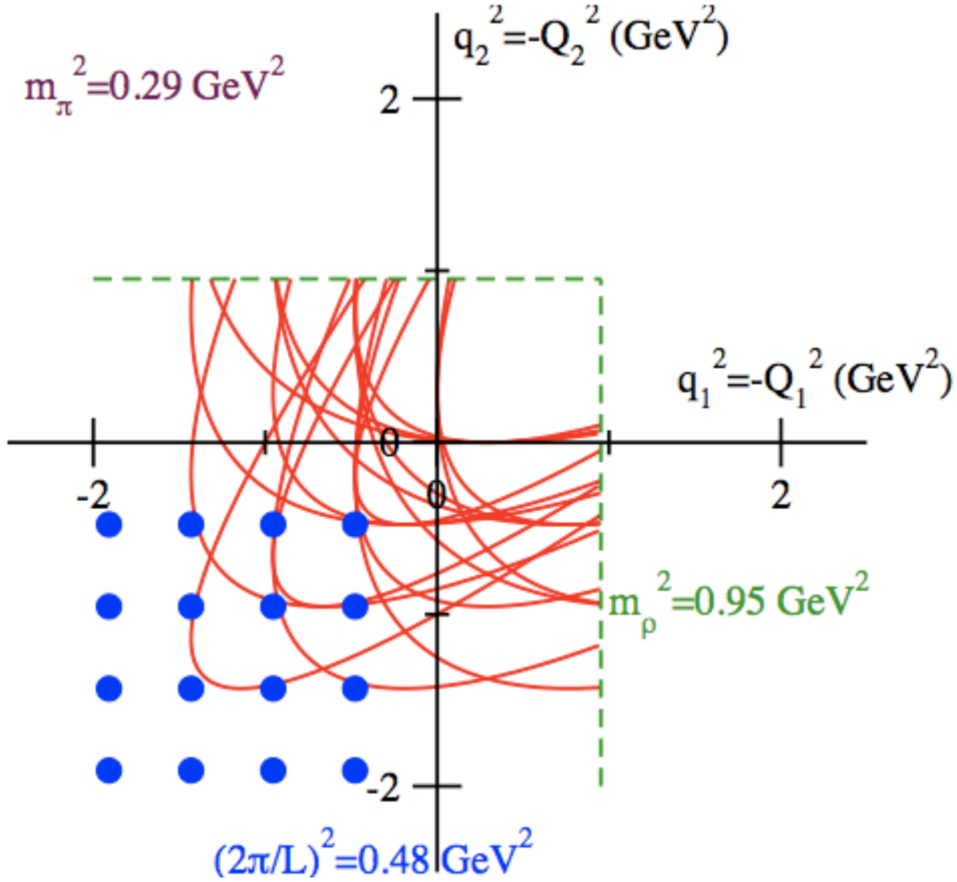
    - $p_1$ is fixed by a condition $p_1+p_2=q$.
  - Wick rotate; possible when $p_{1,2}^2<m_V^2$

$$\frac{\int dt_2\; e^{-\omega_2(t_1-t_2)} \left\langle V^\mu(\mathbf{0},t_1)\left[\int d^3\mathbf{y}\, e^{i\mathbf{p}_2\cdot\mathbf{y}} V^\nu(\mathbf{y},t_2)\right]\left[\int d^3\mathbf{z}\, e^{-i\mathbf{q}\cdot\mathbf{z}} P(\mathbf{z},t_0)\right]\right\rangle}{\dfrac{Z_M(q)}{2E_M(q)} e^{-E_M(q)(t_1-t_0)}}$$

    - Available momenta: $p_2^2=\omega_2^2-|\mathbf{p}_2|^2$ and $p_1^2$ extend over wider range.

# Momentum region covered

- Limitation due to the analyticity
  - $p_{1,2}^2 < m_V^2$
- With a typical (not very aggressive) set up:
  - L=1.8 fm
  - $m_\pi$ = 540 MeV, $m_\rho$ = 970 MeV
  - Measurement extended to the time-like region.



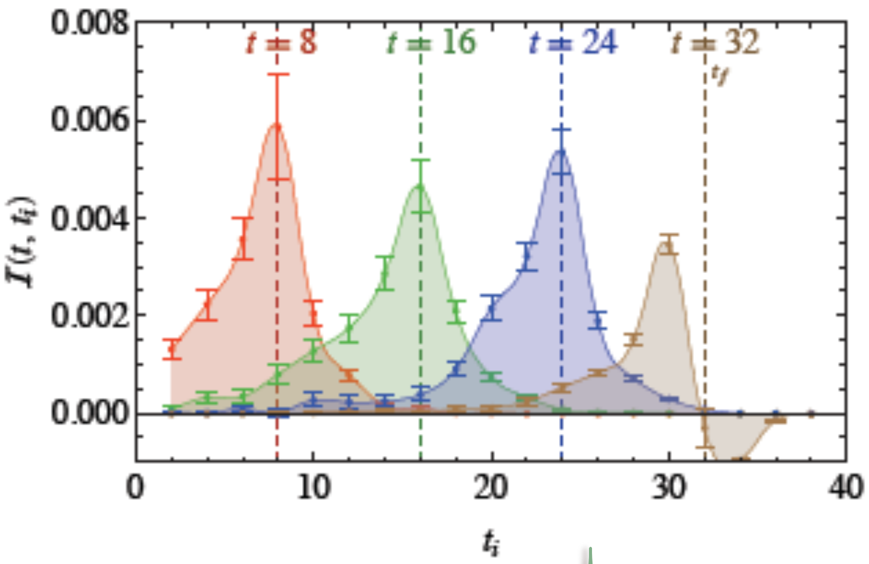

c.f. VMD

$$\mathscr{F}(Q_1^2, Q_2^2) = \frac{F}{\left(1 + Q_1^2/M_{\text{pole}}^2\right)\left(1 + Q_2^2/M_{\text{pole}}^2\right)}$$

# Work by the JLab group

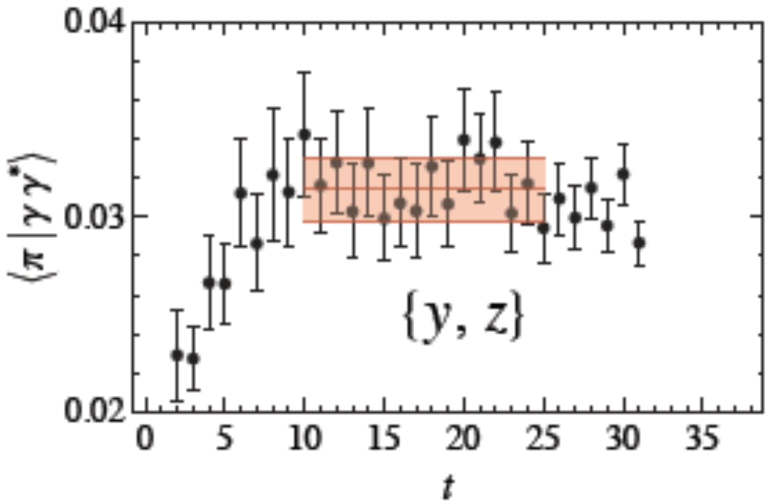
S. Cohen @ Lattice 2008; arXiv:0810.5550 [hep-lat].

- Integrand




$$\frac{\int dt_2\, e^{-\omega_2(t_1-t_2)} \left\langle V^\mu(\mathbf{0},t_1)\left[\int d^3\mathbf{y}\, e^{i\mathbf{p}_2\cdot\mathbf{y}} V^\nu(\mathbf{y},t_2)\right]\left[\int d^3\mathbf{z}\, e^{-i\mathbf{q}\cdot\mathbf{z}} P(\mathbf{z},t_0)\right]\right\rangle}{\frac{Z_M(q)}{2E_M(q)} e^{-E_M(q)(t_1-t_0)}}$$

- Plateau




- Exraction of the ground state pion is achieved.

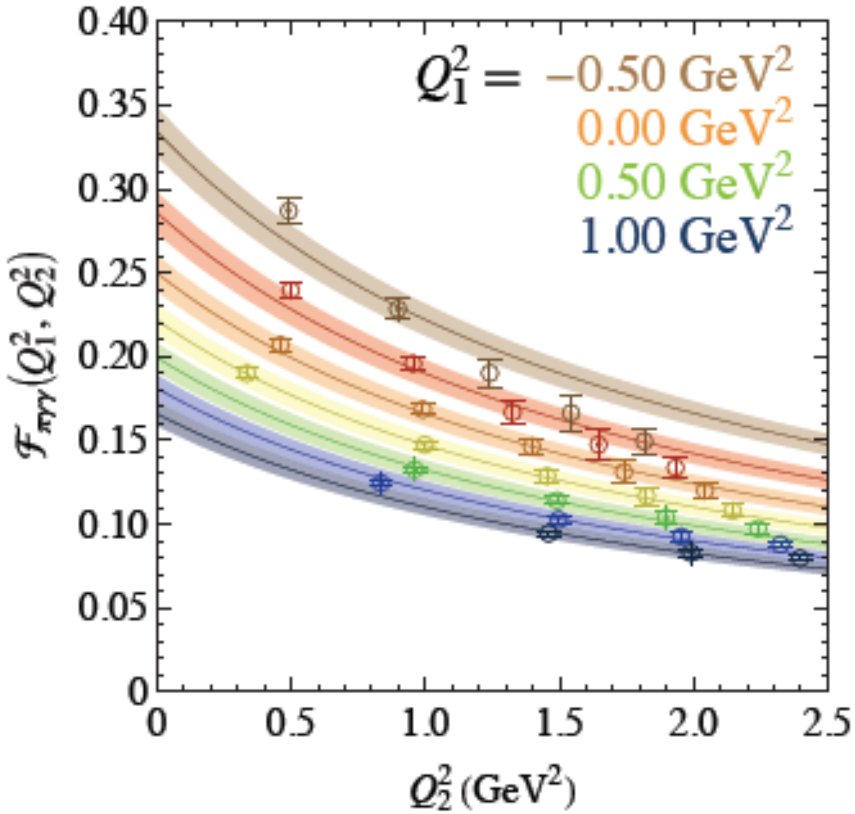
# Work by the JLab group

S. Cohen @ Lattice 2008; arXiv:0810.5550 [hep-lat].

- Form factor
  - obtained for several combinations of $p_1^2$ and $p_2^2$.
  - Fit is attempted with the VMD form

$$\mathscr{F}(Q_1^2, Q_2^2) = \frac{F}{\left(1 + Q_1^2/M_{\mathrm{pole}}^2\right)\left(1 + Q_2^2/M_{\mathrm{pole}}^2\right)}$$

    - Still at a heavy pion mass.
  - Limitation due to computational costs to repeat calc for different values of $t_1$, $t_2$.

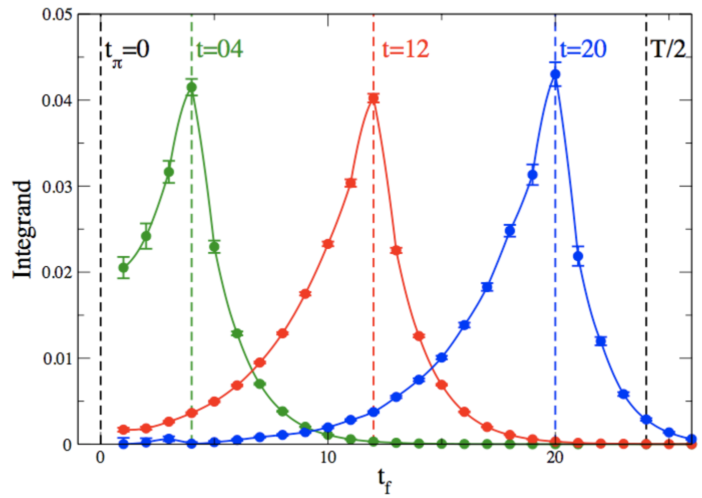


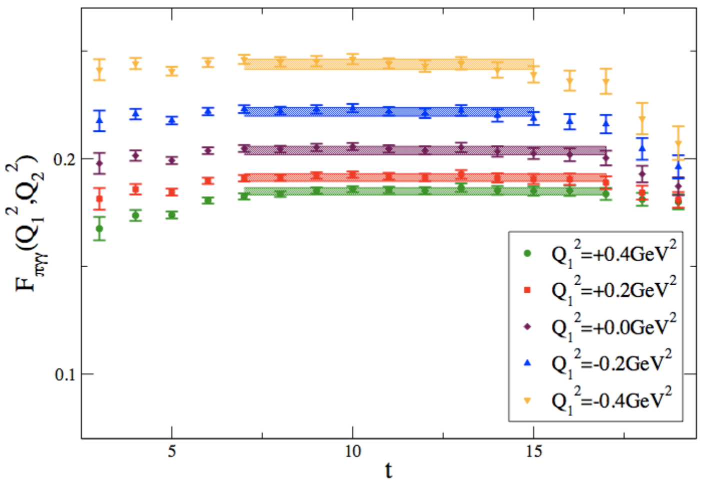
# JLQCD work (very preliminary)

Xu Feng et al. (2011)

- overlap fermion
  - exact chiral symmetry
- all-to-all
  - JLQCD stores
    - low-lying eigenvalues/eigenvectors
    - quark propagators from random noises

  all-to-all propagator can be constructed.
  - No (little) extra cost for various vector, momentum configs.



# All-to-all

## Crucial tool for $\pi^0 \to \gamma^*\gamma^*$ or LbL

- Usually, the quark propagator is calculated with a fixed initial point, color and spin (one-to-all);
- All-to-all stands for a propagator from any lattice point to any other lattice point. Stochastic estimate is possible, much improved by treating low-mode contrib exactly.

$$D^{-1}(x,y) = \sum_{k=1}^{N_{ev}} \frac{1}{\lambda^{(k)}} u^{(k)}(x) u^{(k)\dagger}(y) + \sum_{d=1}^{N_d} \left[ D_{high}^{-1} \eta^{(d)} \right](x) \eta^{(d)}(y)$$

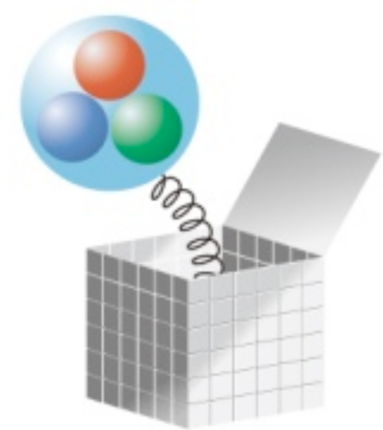
Low-mode contribution

High-mode propagation from random-noise vectors

Random noise

# Status and beyond

- Lattice calculation of the $\pi^0 \to \gamma^* \gamma^*$ form factor
  - Feasible with current resources.
  - With the overlap fermion, the ABJ anomaly is precisely reproduced (as it should be).
  - Many combinations of vector current location and component; all-to-all works fine.
  - Seems to work in the region $|p^2| < 1$ GeV$^2$.
- Going beyond the VMD model
  - Precise calculation near $|p^2| \sim 0$: test of NLO ChPT
  - Higher momentum: connection to the perturbative region?
  - Which region is (most) relevant to g-2??



Thank you for your attention.

# Backup slides

# Overlap fermion

Neuberger, Narayanan (1998)

$$D = \frac{1}{a}\left[1 + \frac{X}{\sqrt{X^\dagger X}}\right],\ X = aD_W - 1$$
$$= \frac{1}{a}[1 + \gamma_5 \,\mathrm{sgn}(aH_W)],\ aH_W = \gamma_5(aD_W - 1)$$

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D$$

- Exact chiral symmetry via the Ginsparg-Wilson relation.

$$\delta\bar{\psi} = i\alpha\bar{\psi}\left(1 - \frac{a}{2}D\right)\gamma_5,\ \delta\psi = i\alpha\gamma_5\left(1 - \frac{a}{2}D\right)\psi$$

  - Continuum-like Ward-Takahashi identities hold.
  - Index theorem (relation to topology) satisfied.
  - Topology change is costly; large-scale simulation is feasible only at fixed topology
    - induces O(1/V) effects in general; can be accounted for in the spectral function analysis

# Parameters

**$N_f$ = 2 runs**

- $\beta$=2.30 (Iwasaki), $a$=0.12 fm, $16^3$x32
- 6 sea quark masses covering $m_s/6$~$m_s$
- Q=0 sector only, except for Q=-2, -4 runs at $m_q$=0.050
- $\varepsilon$-regime run at m=0.002 ($m_q$~ 3 MeV), $\beta$=2.30

**$N_f$ = 2+1 runs**

- $\beta$=2.30 (Iwasaki), $a$=0.11 fm, $16^3$x48
- 5 ud quark masses, covering $m_s$/ 6~$m_s$
  - x 2 s quark masses
- Q=0 sector only, except for Q=1 at $m_{ud}$=0.015
- Larger volume lattice $24^3$x48 running at $m_{ud}$=0.015, 0.025.
- $\varepsilon$-regime run at m=0.002 ($m_q$ ~ 3 MeV)