Getting at Realistic QCD Events at the LHC

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work with
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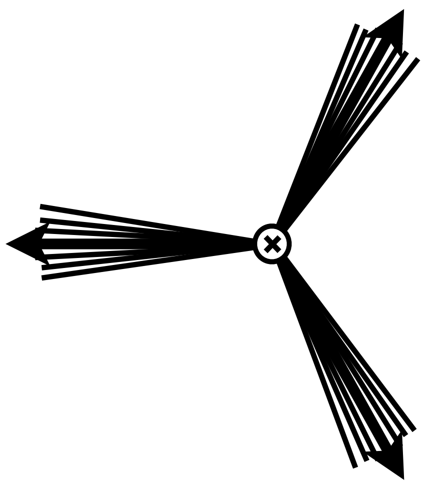
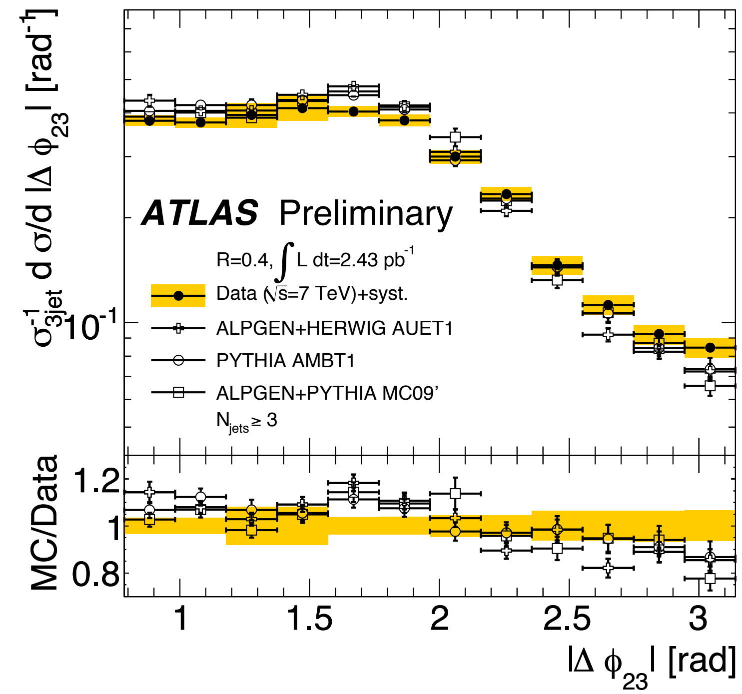
Outline

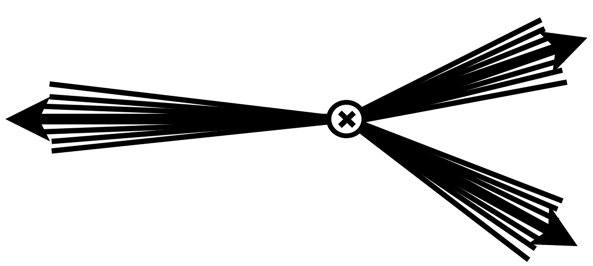
- Realistic multijet events at the LHC
- Nearby jets: kinematics and modes
- $SCET_+$
- Jet substructure and factorization

Multijet Events

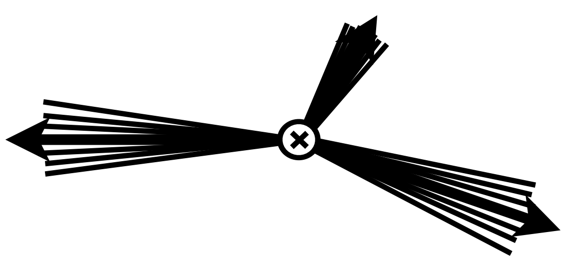


uncommon
well-separated
energetic
all scales ~pT

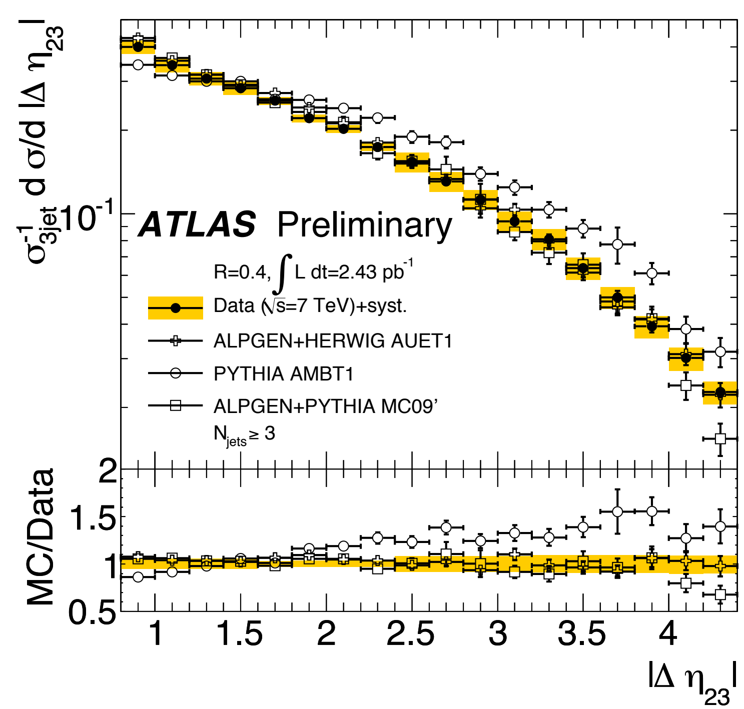




common
nearby jets
energetic
small dijet invariant mass



common
well-separated
hierarchy of jet energies
small dijet invariant masses



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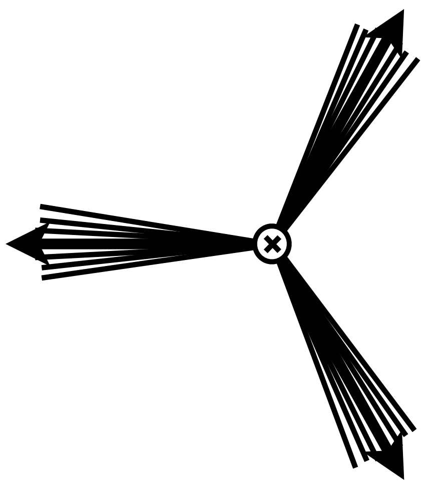
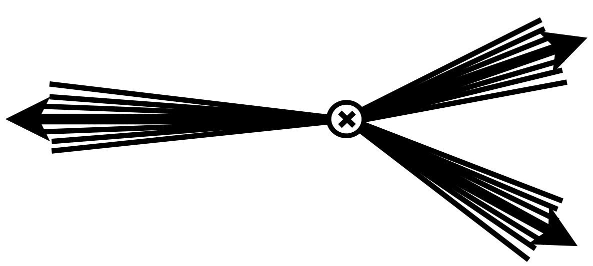
Scales in Multijet Events



uncommon

$$\ln \frac{m_J}{p_{TJ}}$$

factorization theorems exist



common

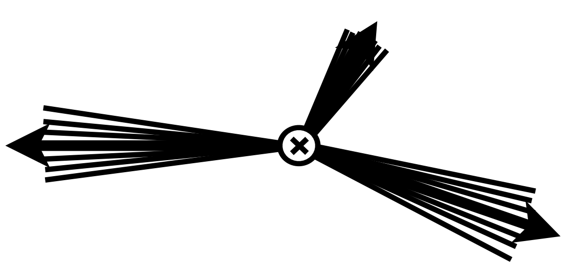
$$\ln \frac{m_{\text{dijet}}}{p_{TJ}}, \quad \ln \frac{m_J}{m_{\text{dijet}}}$$

focus of this talk

all configurations can have non-global logs:

$$\ln \frac{m_{J_1}}{m_{J_2}}, \quad \ln \frac{m}{\Lambda}$$

Dasgupta, Salam

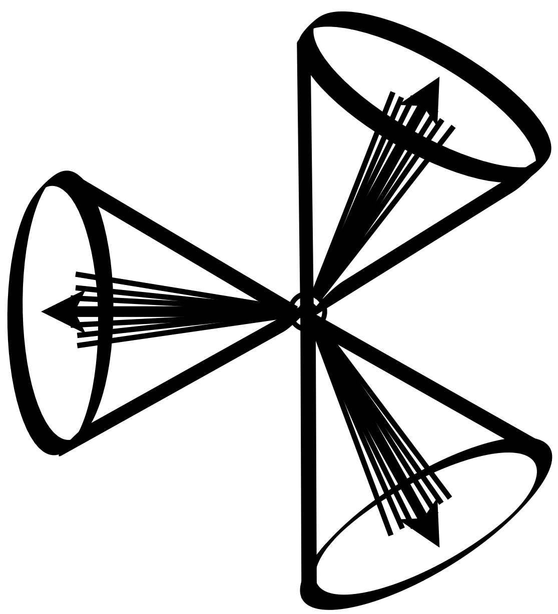


common

$$\ln \frac{p_{T1}}{p_{T2}}$$

not yet explored

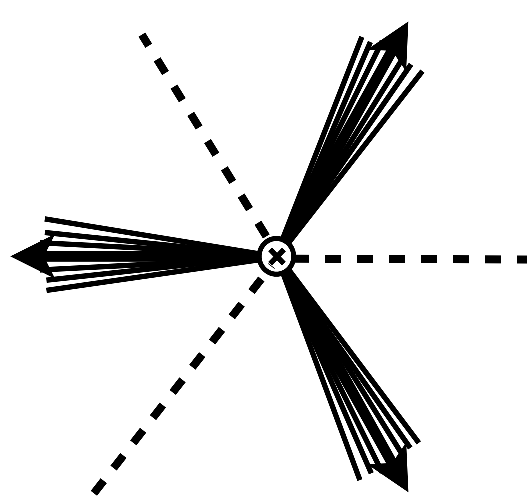
Observables for Multijet Events



Jet algorithms

(largely) fixed jet size
interjet region
experimentally well understood
logs of R difficult to sum

Ellis, Hornig, Lee, Vermilion, JW



N-jettiness

kinematics set jet boundaries
no interjet region
attractive substructure properties
theoretically tractable

Stewart, Tackmann, Waalewijn
Jouttenus, Stewart, Tackmann, Waalewijn

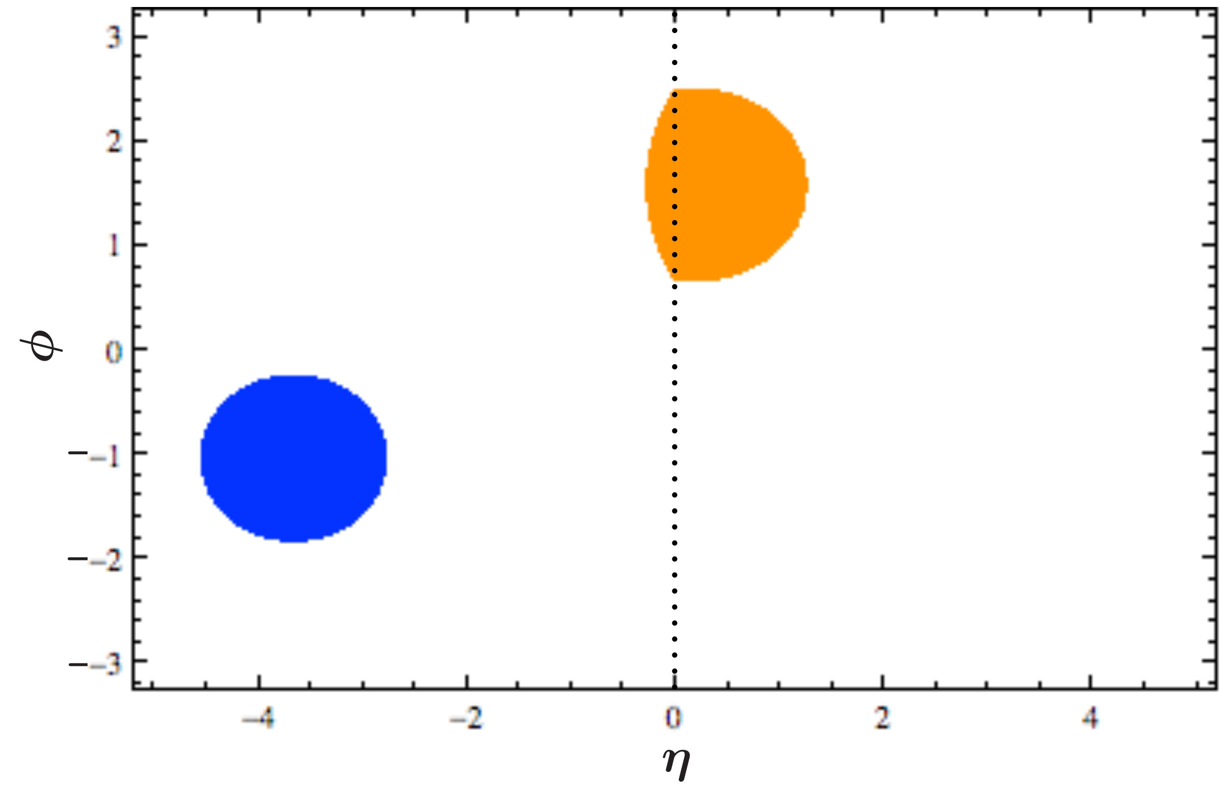
N-jettiness

for each jet:

$$\mathcal{T}_j = \sum_i n_j \cdot k_i$$

advantages:
inclusive over phase space
calculable
no boundary parameters

boundary regions for jets, beam



jet assignment depends only on particle direction

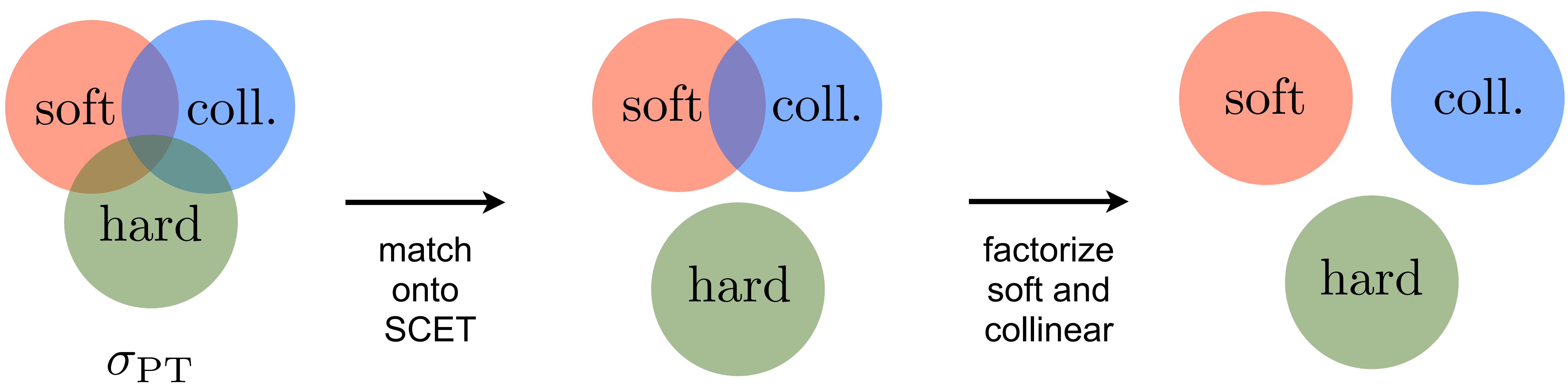
We will study a specific multijet configuration using N-jettiness

But the framework we use applies to other jet definitions and observables

SCET Factorization for Multijet Events

Factorization separates soft and collinear dynamics of jet evolution

Makes cross sections calculable, allows for resummation

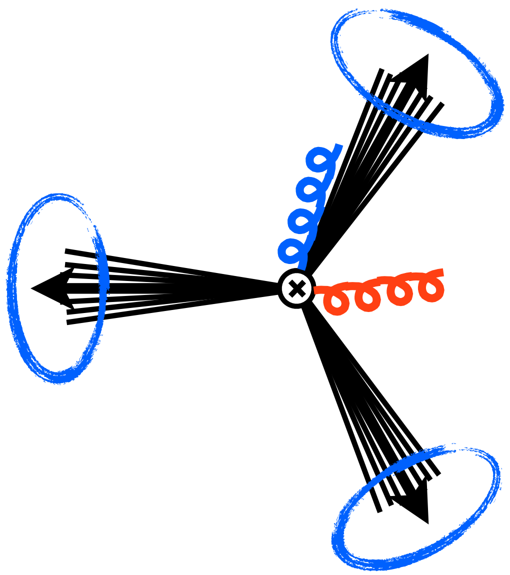


$\sigma_{\rm PT}$

$$\sigma = H_N \otimes |O_N|^2$$

$$\sigma = H_N \otimes \left[\prod_i J_i \right] \otimes S_N$$

Modes for Multijet Events

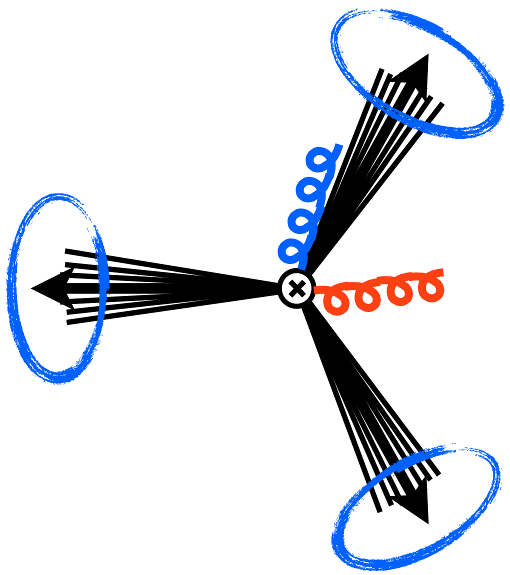


can use the kinematics of the final state to determine the modes that contribute to the observable

correct modes for SCET in this case:

hard:	$p_h \sim \sqrt{s_{ij}}(1,1,1)$	$p_h^2 \sim s_{ij}$
collinear:	$p_c \sim E_J(1,\lambda^2,\lambda)$	$p_c^2 \sim E_J^2\lambda^2 \sim E_J\mathcal{T}$
soft:	$p_s \sim E_J(\lambda^2,\lambda^2,\lambda^2)$	$p_s^2 \sim E_J^2\lambda^4 \sim \mathcal{T}^2$

How Do We Determine the Modes?



collinear modes:

support near the jet axis: $p_c \sim E_c(1, \lambda_c^2, \lambda_c)$

label momentum: $E_c \sim E_J$

contribution to the observable: $n \cdot p_c \sim \mathcal{T}$

$$\Rightarrow \; p_c \sim (E_J, \mathcal{T}, \sqrt{E_J \mathcal{T}})$$

soft modes:

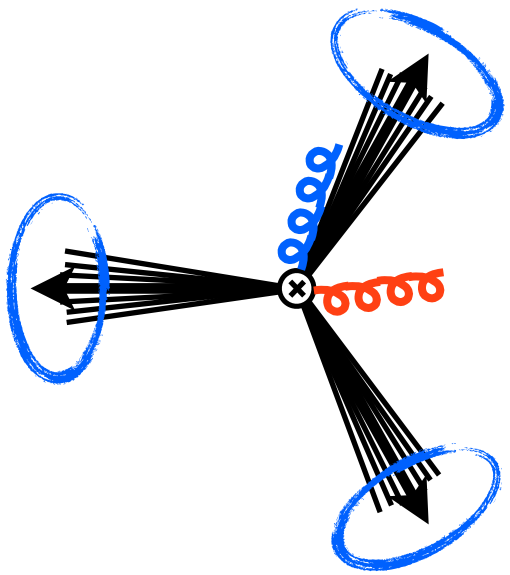
isotropic mode: $p_s \sim E_s(\lambda_s^2, \lambda_s^2, \lambda_s^2)$

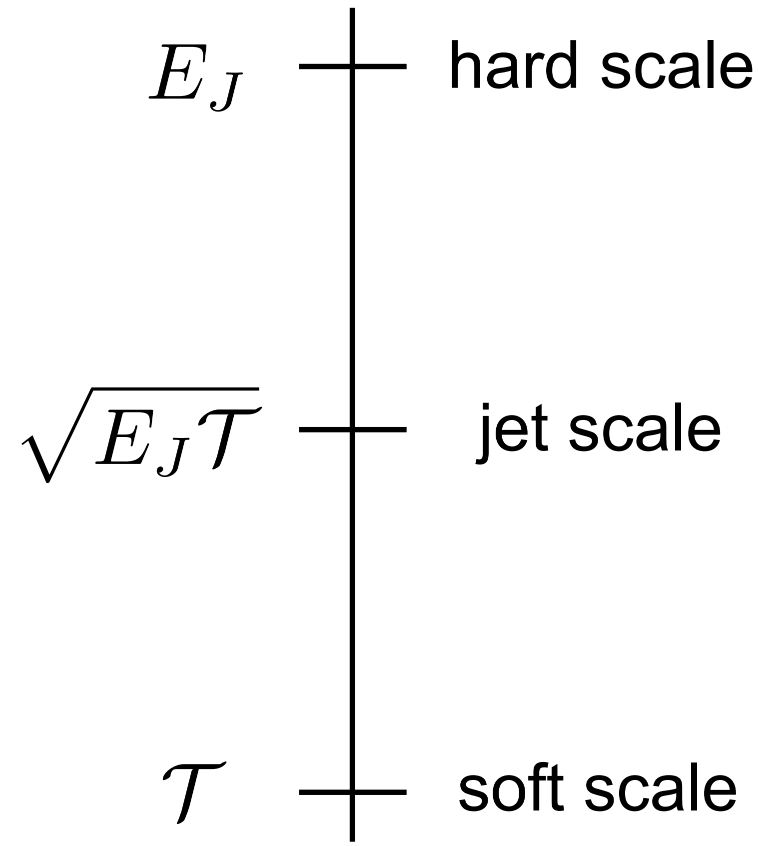
label momentum: $E_s \sim E_J \lambda_s^2$

contribution to the observable: $n \cdot p_s \sim \mathcal{T}$

$$\Rightarrow \; p_s \sim (\mathcal{T}, \mathcal{T}, \mathcal{T})$$

Factorization and Scales in Multijet Events

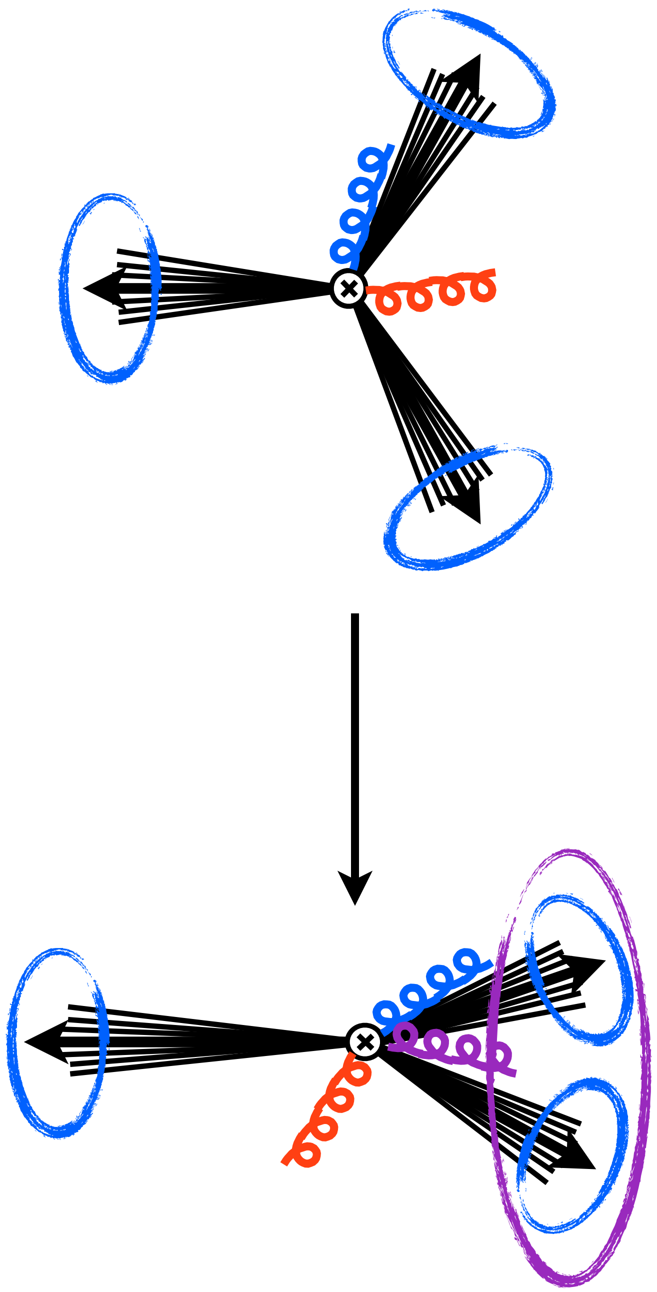




factorization theorem:

$$\frac{d\sigma}{d\mathcal{T}_i} = \frac{d\sigma^0}{d\Phi_3} H_N \left[B_a(\mathcal{T}_a) B_b(\mathcal{T}_b) \prod_i J_i(\mathcal{T}_i) \right] \otimes S_N(\mathcal{T}_a, \ldots, \mathcal{T}_N)$$

The Limit of Nearby Jets



Take two jets to be close in angle
Keep their energies of the same order

$$\gamma_{H_N} = \Gamma_{\rm cusp}[\alpha_s] \sum_{i \neq j} \mathbf{T}_i \cdot \mathbf{T}_j \ln \frac{\mu^2}{s_{ij}} + \gamma_N[\alpha_s]$$

Hierarchy of dijet invariant masses: $s_{ij} = 2E_i E_j \, n_i \cdot n_j$

get large logs of small angles: $\ln n_i \cdot n_j$

Hard scales become widely separated

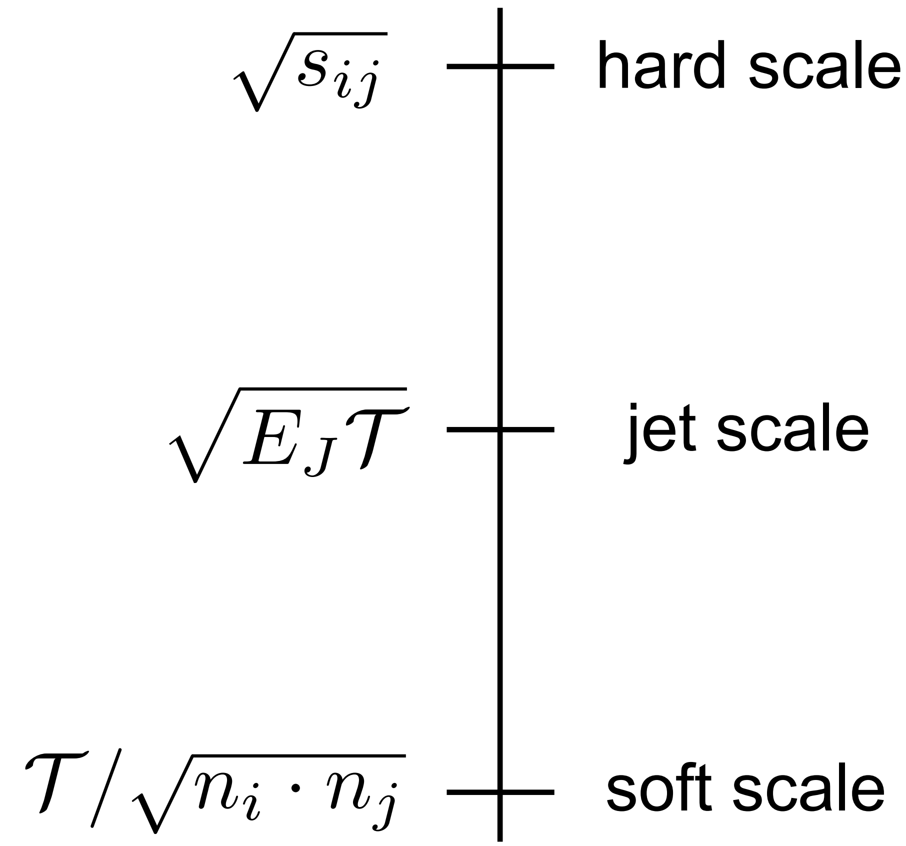
Cannot sum large logarithms in the hard function
- same problem in the soft function

$$\ln n_i \cdot n_j \ : \ \text{ninja}$$

What's the Solution?

The problem is two-fold:

1. Hierarchy of scales in the hard function
2. Hierarchy of scales in the soft function



The two problems are related:

$$\gamma_H + \sum_i \gamma_{J_i} + \gamma_S = 0$$

but the machinery needed to solve them is very different

Hard function: use a tower of EFTs
Soft function: add a new mode (new EFT)

Hard function factorization solved by
Bauer, Schwartz
Baumgart, Marcantonini, Stewart

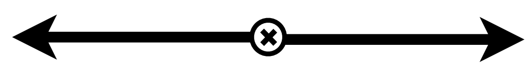
Hard Function Factorization

Bauer, Schwartz
Baumgart, Marcantonini, Stewart

QCD

$\sqrt{s_{ij}}$ ——— $C_2(q_i)$ $\quad$ hard: $\quad p_h \sim \sqrt{s_{ij}}\,(1,1,1)$

O_2
2-jet operator



resolve 2 jets

Hard Function Factorization

Bauer, Schwartz
Baumgart, Marcantonini, Stewart

QCD

$\sqrt{s_{ij}}$ —————— $C_2(q_i)$ hard: $p_h \sim \sqrt{s_{ij}}\,(1,1,1)$

O_2
2-jet operator

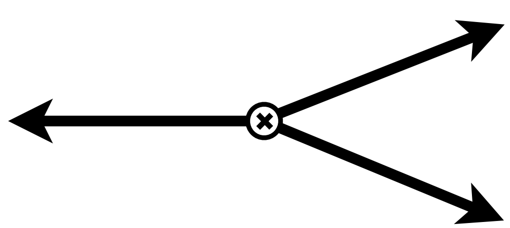
resolve 2 jets

$$\lambda = \frac{m_J^2}{Q^2}$$

$\sqrt{t}$ —————— $C_3(q_i)$ collinear: $p_c \sim E_J(1, \lambda_t^2, \lambda_t)$

$$\lambda_t = \frac{t}{Q^2}$$

O_3
3-jet operator



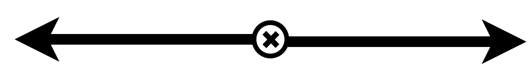
resolve 3 jets

Hard Function Factorization

QCD

$\sqrt{s_{ij}}$ —————— $C_2(q_i)$ hard: $p_h \sim \sqrt{s_{ij}}\,(1,1,1)$

O_2
2-jet operator



resolve 2 jets

$$\lambda = \frac{m_J^2}{Q^2}$$

$\sqrt{t}$ —————— $C_3(q_i)$ collinear: $p_c \sim E_J(1, \lambda_t^2, \lambda_t)$

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O_3
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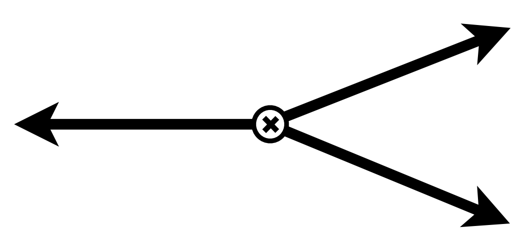
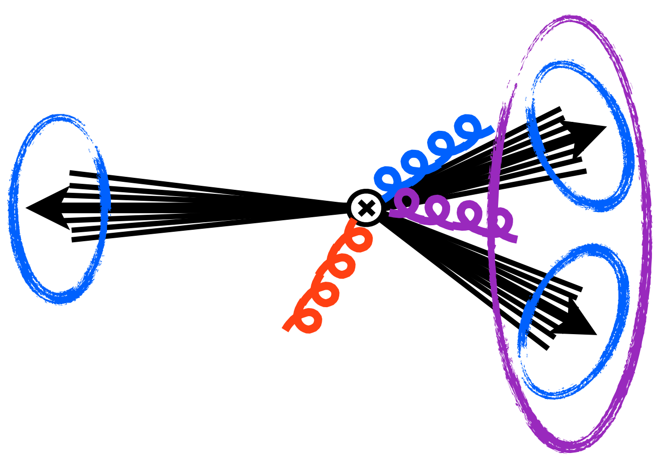


resolve 3 jets

our contribution: proved that the matching coefficient from O_{N-1} onto O_N is universal, depends only on one scale

$$\langle N | \vec{O}^{\dagger}_{N-1}(\mu) | 2 \rangle = \langle N | \vec{O}^{+\dagger}_{N}(\mu) | 2 \rangle \, C_+(t, x, \mu)$$

Soft Function Solution: Add a New Mode

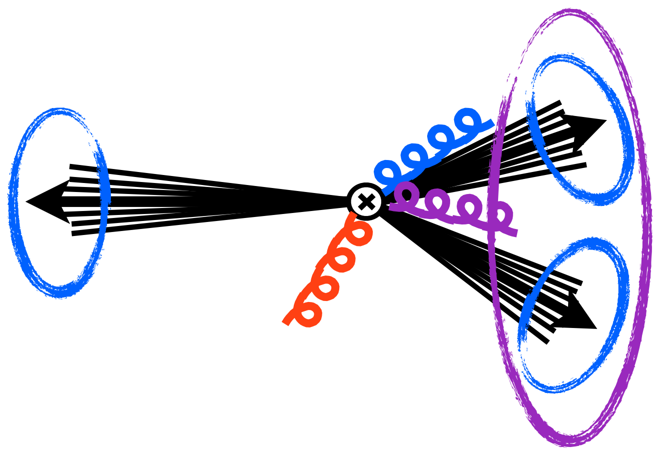


soft radiation between the dijets
lives at a different scale

We will add a new collinear-soft (csoft) mode which
contributes to the dijet system

Build this new mode into a new version of SCET
$SCET_+$: an EFT for multijets with small dijet invariant masses
Also useful for jet substructure: nearby subjets

The csoft mode



collinear modes: $p_c \sim (E_J, \mathcal{T}, \sqrt{E_J \mathcal{T}})$

soft modes: $p_s \sim (\mathcal{T}, \mathcal{T}, \mathcal{T})$

csoft modes:

support near the dijet system: $p_{cs} \sim E_{cs}(1, \lambda_{cs}^2, \lambda_{cs})$

angular support fixed: $\lambda_{cs} \sim \frac{m_{jj}}{\sqrt{\hat{s}}}$

contribution to the observable: $n_{1,2} \cdot p_{cs} \sim \mathcal{T}$

$$\Rightarrow \quad E_{cs} \lambda_{cs}^2 \sim \mathcal{T}$$

$$E_{cs} \sim \sqrt{\hat{s}} \frac{\mathcal{T}}{m_{jj}}$$

csoft modes:

$$p_{cs} \sim \left(\sqrt{\hat{s}} \frac{\mathcal{T}}{m_{jj}}, \mathcal{T}, \mathcal{T} \left(\frac{\sqrt{\hat{s}}}{m_{jj}} \right)^{1/2} \right)$$

SCET$_+$

content of SCET$_+$

collinear modes: $p_c \sim (1, \lambda^2, \lambda)$

soft modes: $p_s \sim (\lambda^2, \lambda^2, \lambda^2)$

csoft modes: $p_{cs} \sim (\eta^2, \lambda^2, \eta\lambda)$

Complete factorization in SCET$_+$

QCD

——— $\sqrt{s_{ij}}$ hard 2 jet

SCET O_2

——— $\sqrt{t}$ hard 3 jet

SCET$_+$ O_3

——— m_J jet

soft$_+$ S_3

——— $\frac{m_J^2}{\sqrt{t}}$ csoft

soft S_2

——— $\frac{m_J^2}{\sqrt{s_{ij}}}$ soft

Constructing SCET$_+$: Go Back to SCET

focus on soft-collinear decoupling:
how do we separate soft and collinear modes
in the leading order Lagrangian?

collinear Lagrangian:

$$\mathcal{L}_n = \bar{\xi}_n \Big[\mathrm{i} n \cdot D_n + g\, n \cdot A_{us} + \mathrm{i} \not{D}_{n\perp} W_n \frac{1}{\bar{n} \cdot \mathcal{P}_n} W_n^\dagger \mathrm{i} \not{D}_{n\perp} \Big] \frac{\bar{\not{n}}}{2} \xi_n$$

collinear Wilson line: $W_n(x) = \Big[\sum_{\text{perms}} \exp\Big(\frac{-g}{\bar{n} \cdot \mathcal{P}_n}\, \bar{n} \cdot A_n(x) \Big) \Big]$

soft Wilson line: $Y_n^\dagger(x) = \mathrm{P} \exp \Big[\mathrm{i} g \int_0^\infty \mathrm{d}s\, n \cdot A_{us}(x + s\, n) \Big]$

Soft-Collinear Decoupling in SCET

BPS field redefinition: separates soft and collinear fields in the Lagrangian at leading power

Bauer, Pirjol, Stewart
Freedman, Luke

recently shown to all orders

BPS field redefintion:

$$\xi_n^{(0)}(x) = Y_n^\dagger(x)\,\xi_n(x)\,,$$
$$A_n^{(0)}(x) = Y_n^\dagger(x)\,A_n(x)\,Y_n(x)$$
$$W_n^{(0)}(x) = Y_n^\dagger(x)\,W_n(x)\,Y_n(x)$$

factorizes the Lagrangian: $\mathcal{L}_{\rm SCET} = \sum_i \mathcal{L}_{n_i}^{(0)} + \mathcal{L}_{us} + \cdots$

Soft-Collinear Decoupling in SCET$_+$

need a new BPS field redefinition to decouple csoft gluons from collinear

also need to decouple soft from csoft!

First, factorize the soft modes out - both the collinear and csoft fields appear like collinear fields to the soft modes, so the normal BPS works

$$\xi_n^{(0)}(x) = Y_n^\dagger(x)\, \xi_n(x)\,,$$

$$A_n^{(0)}(x) = Y_n^\dagger(x)\, A_n(x)\, Y_n(x)$$

$$W_n^{(0)}(x) = Y_n^\dagger(x)\, W_n(x)\, Y_n(x)$$

V is the csoft analog to *W* : $$V_n^{(0)}(x) = Y_n^\dagger(x) V_n(x) Y_n(x)$$

Soft-Collinear Decoupling in SCET_+

Now we can use a second field redefinition for csoft modes

$$\xi_{n_1}^{(0,0)}(x) = X_{n_1}^{(0)\dagger}(x)\,\xi_{n_1}^{(0)}(x)\,,$$
$$A_{n_1}^{(0,0)}(x) = X_{n_1}^{(0)\dagger}(x)\,A_{n_1}^{(0)}(x)X_{n_1}^{(0)}(x)$$

X is the csoft analog to *Y*

Just like the soft-collinear decoupling: the csoft mode appears soft to the collinear modes

collinear modes: $p_c \sim (1, \lambda^2, \lambda)$

soft modes: $p_s \sim (\lambda^2, \lambda^2, \lambda^2)$

csoft modes: $p_{cs} \sim (\eta^2, \lambda^2, \eta\lambda)$

all the modes couple through the + momentum

Factorization Theorem

$$e^+e^- \to 3 \text{ jets}$$

$$\frac{d\sigma}{d\mathcal{T}_i} = \frac{d\sigma^0}{d\Phi_3} H_2\, H_3^+ \prod_i J_i \otimes S_c \otimes S_2$$

The csoft function S_c is calculated like a soft function (the amplitude is eikonal), but there is a zero bin from the soft sector

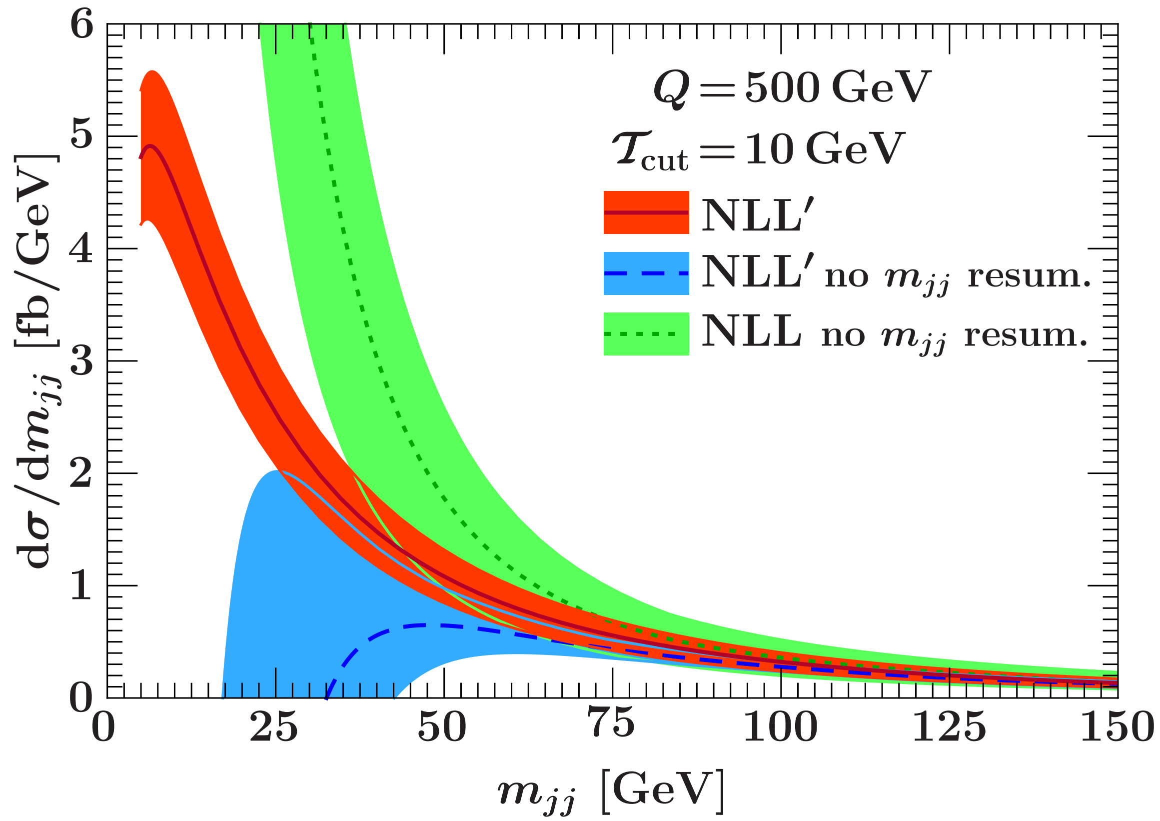
$$pp \to N \text{ jets} + \text{leptons}$$

$$\begin{aligned}
&\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{T}_a\,\mathrm{d}\mathcal{T}_b\,\mathrm{d}\mathcal{T}_1\cdots\mathrm{d}\mathcal{T}_N\,\mathrm{d}t\,\mathrm{d}z} \\
&\quad= \int \mathrm{d}^4q\,\mathrm{d}\Phi_L(q) \int \mathrm{d}\Phi_N(\{q_i\})\, M_N(\Phi_N,\Phi_L)\,(2\pi)^4\delta^4\Big(q_a+q_b-\sum_i q_i - q\Big)\,\delta\big(t-s_{12}\big)\,\delta\Big(z-\frac{E_1}{E_1+E_2}\Big) \\
&\quad\times \sum_\kappa \int \mathrm{d}x_a\mathrm{d}x_b \int \mathrm{d}s_a\mathrm{d}s_b\, B_{\kappa_a}(s_a,x_a,\mu)\, B_{\kappa_b}(s_b,x_b,\mu) \prod_i \int \mathrm{d}s_i\, J_{\kappa_i}(s_i,\mu)\,|C_+^\kappa(t,z,\mu)|^2 \int \mathrm{d}k_1\,\mathrm{d}k_2\, S_+^\kappa(k_1,k_2,\mu) \\
&\quad\times \vec{C}_{N-1}^{\kappa\dagger}(\Phi_N,\Phi_L,\mu)\,\widehat{S}_{N-1}^\kappa\Big(\mathcal{T}_1-\frac{s_1}{Q_1}-k_1,\mathcal{T}_2-\frac{s_2}{Q_2}-k_2,\mathcal{T}_a-\frac{s_a}{Q_a},\ldots,\mathcal{T}_N-\frac{s_N}{Q_N},\mu\Big)\vec{C}_{N-1}^\kappa(\Phi_N,\Phi_L,\mu)
\end{aligned}$$

Resumming Kinematic Logs

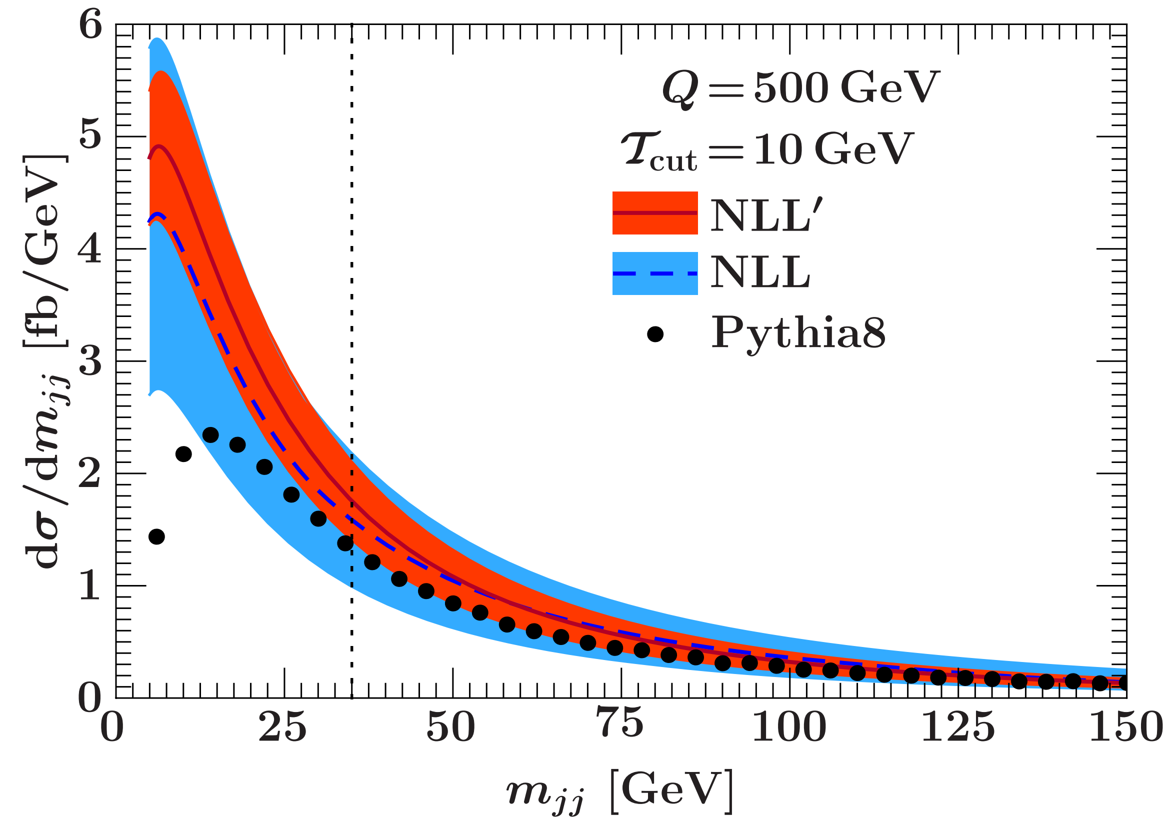
$e^+e^- \to 3$ jets

resummed dijet mass spectrum



gain stability at small m_{jj}

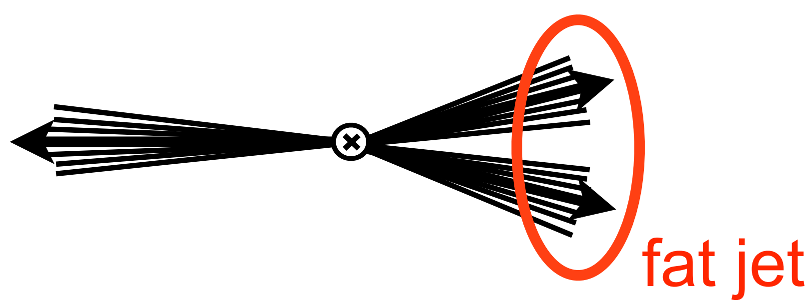
comparison to Pythia



observable breaks down at small m_{jj}
- 3 jet observable for a 2 jet event

Jet Substructure Limit for Ninja

substructure limit:
2 jets merge



Can think about the subjets as their own jets

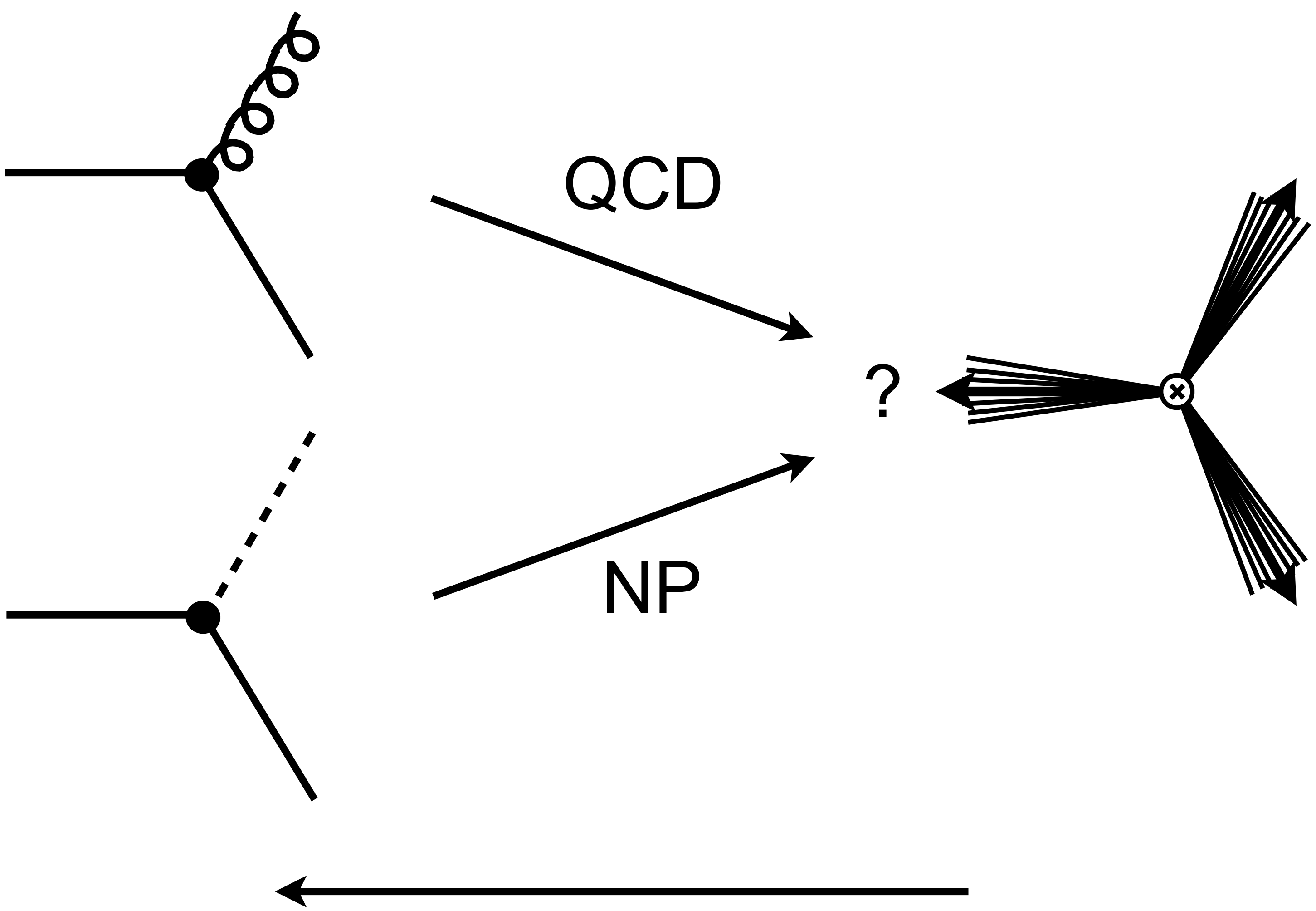
Ask a basic question:

Can jet substructure algorithms be factorized in SCET?

We want jet substructure to be *calculable*

Jet Substructure

Jet substructure helps us
solve an inverse problem:



Understanding jet substructure
lets us go to the left

Main goals:

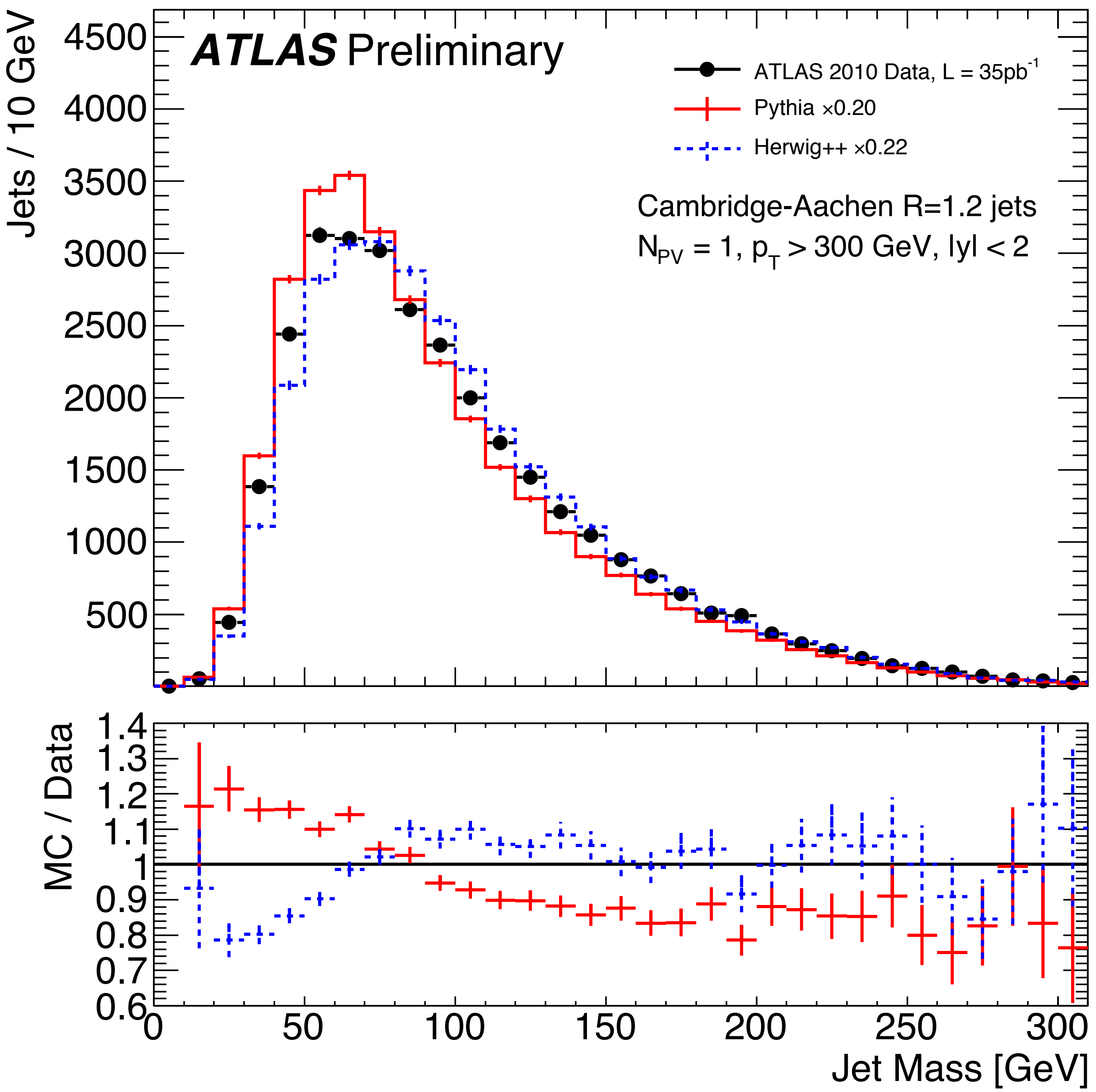
1. Better understand QCD in jets

2. Discriminate between QCD and NP

Jet Substructure

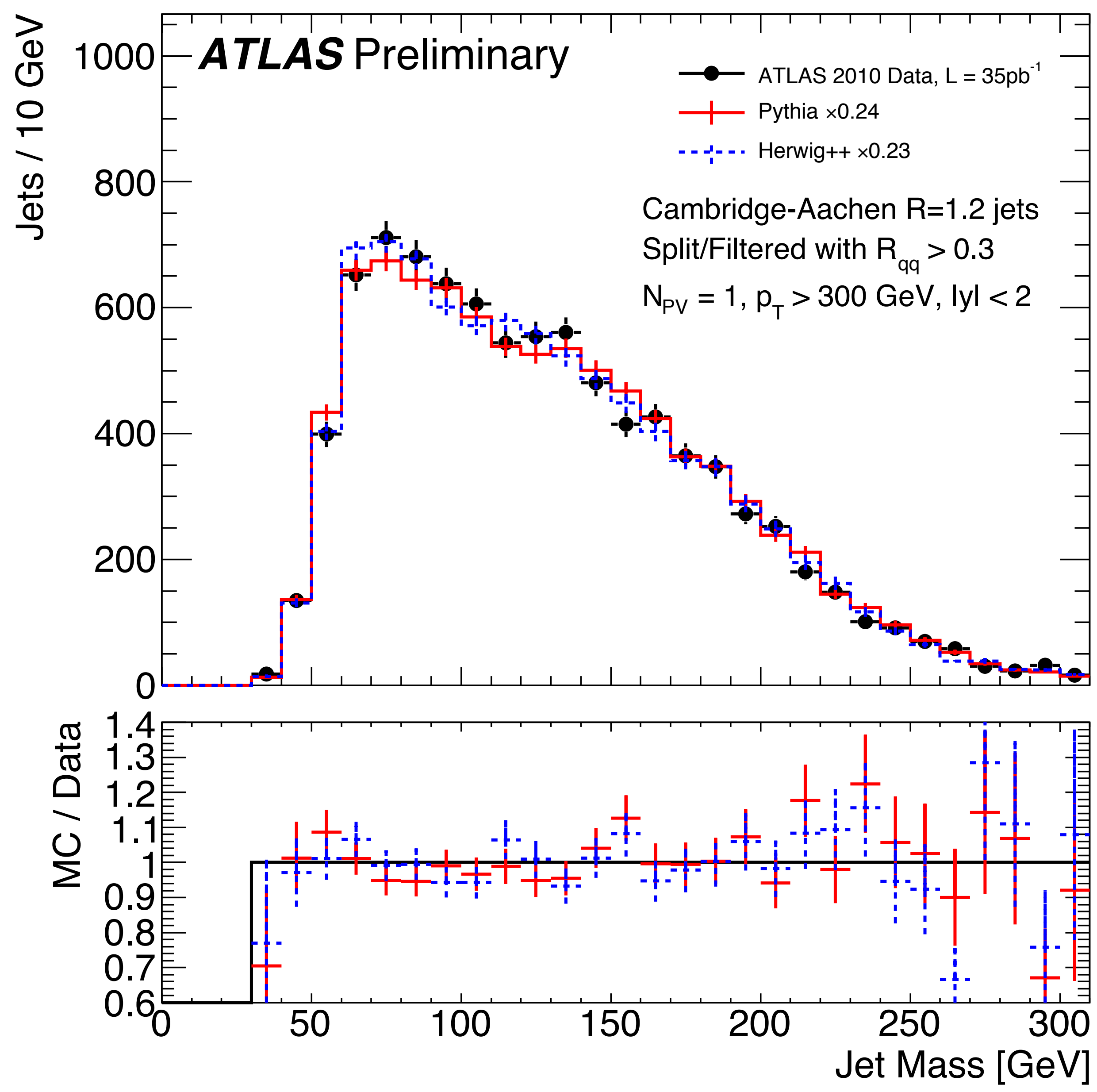
ATLAS-CONF-2011-073

jet mass distribution



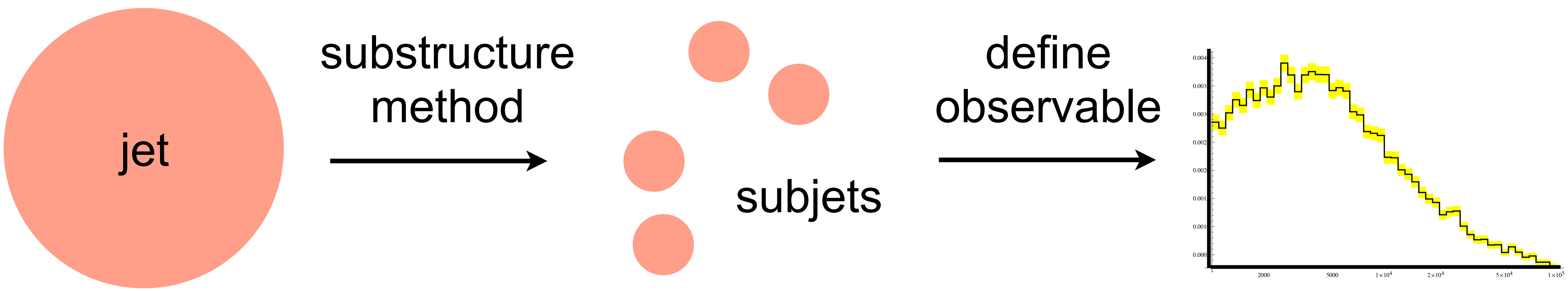
large errors between data/MC

filtered jet mass distribution



reduced data/MC errors

What Jet Substructure Does

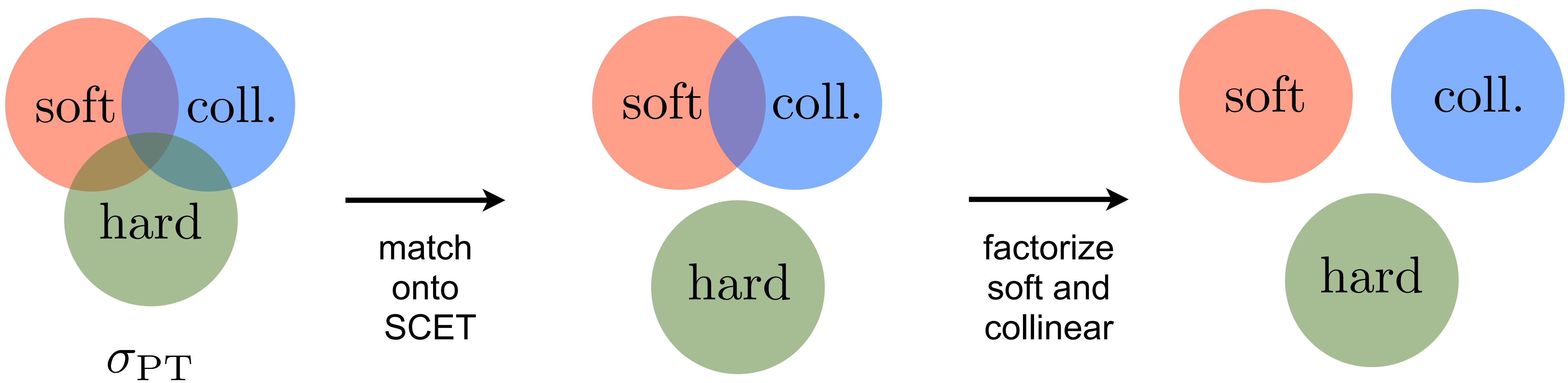


Steps:

1. Define subjets
2. Make kinematic cuts on subjets
3. Define observable

Use SCET power counting to determine if a jet substructure algorithm factorizes

Factorization for Jet Substructure



$$\sigma = H_N \otimes |O_N|^2$$

$$\sigma = H_N \otimes \left[\prod_i J_i\right] \otimes S_N$$

Factorization has two parts:

1. Factorization of the N-jet operators (BPS redefinition)
2. Factorization of the observable

Factorization for Jet Substructure

Start with basic SCET distribution $\frac{d\sigma}{d\tau} = H_N \langle O_N^\dagger \, \hat{\mathcal{R}}(\tau) \, O_N \rangle$

The restriction operator specifies the phase space cuts and measurement of the observable

Bauer, Fleming, Lee, Sterman

O_N factorizes into jet and soft operators: $O_N = O_J^N O_{S_N}$

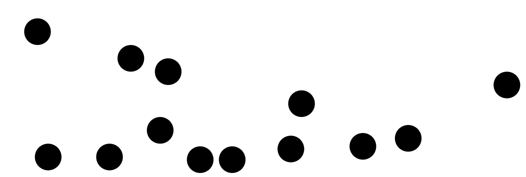
Bauer, Pirjol, Stewart

Need to show the restriction operator factorizes: $\hat{\mathcal{R}} = \hat{\mathcal{R}}_c + \hat{\mathcal{R}}_s$

- A necessary condition for factorization

Soft-Collinear Factorization

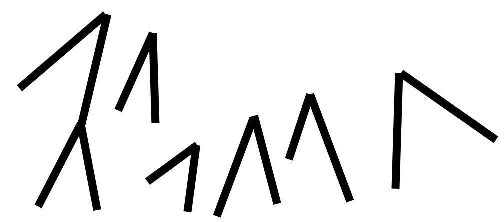
QCD: build the jet from
successive recombinations



final state particles

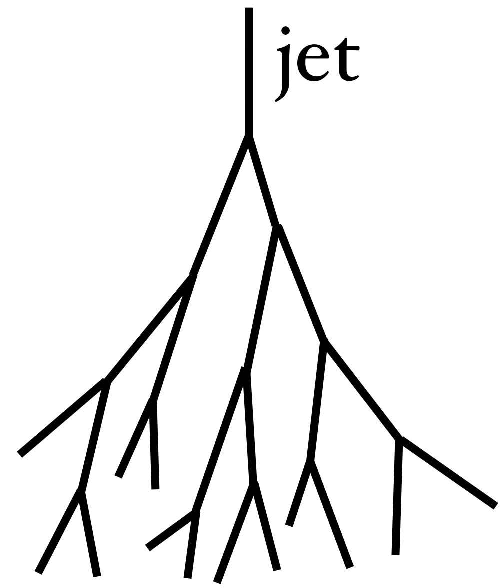
Soft-Collinear Factorization

QCD: build the jet from successive recombinations



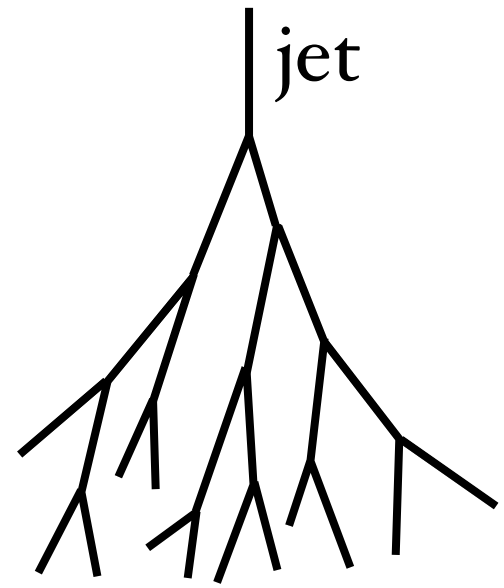
Soft-Collinear Factorization

QCD: build the jet from
successive recombinations

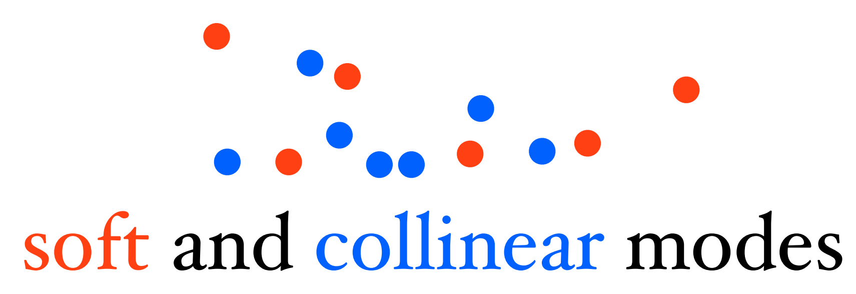


Soft-Collinear Factorization

QCD: build the jet from successive recombinations

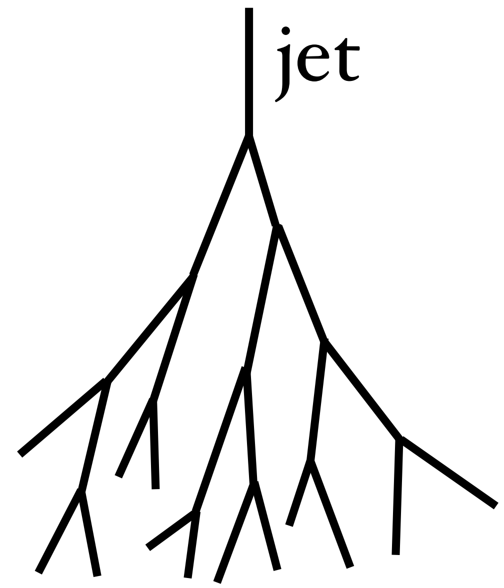


SCET: phase space cuts on collinear and soft particles must separate



Soft-Collinear Factorization

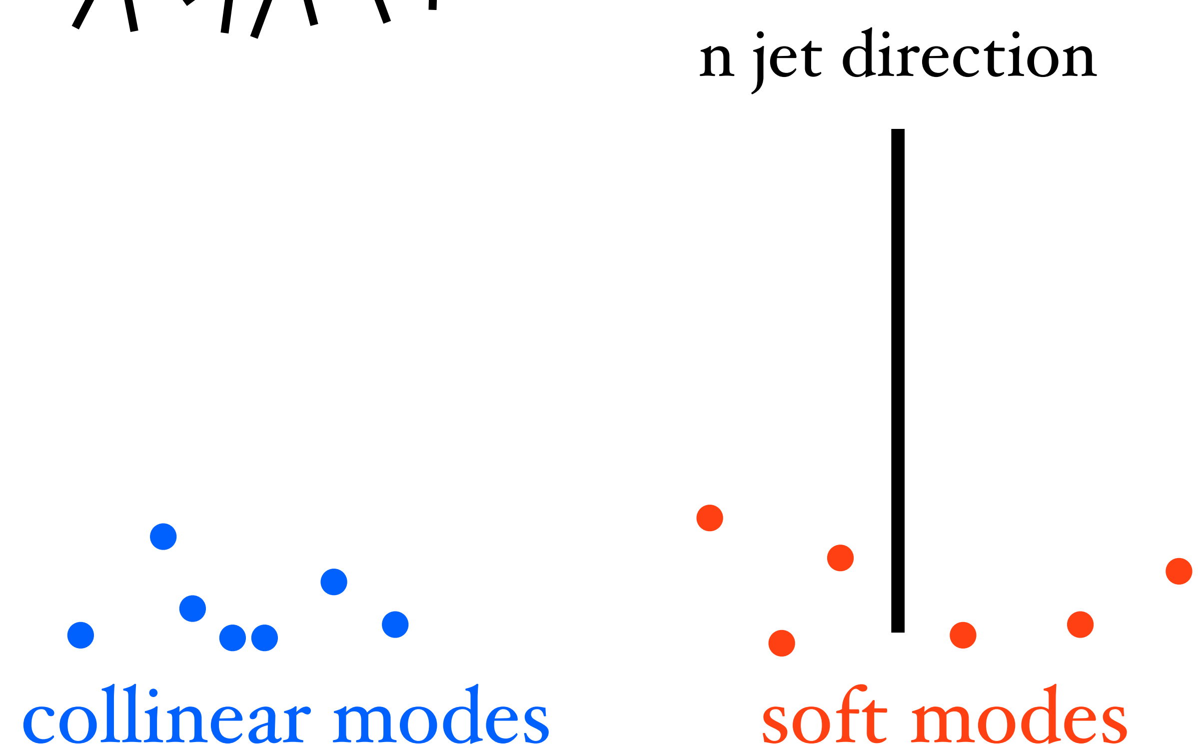
QCD: build the jet from successive recombinations



SCET: phase space cuts on collinear and soft particles must separate

↓

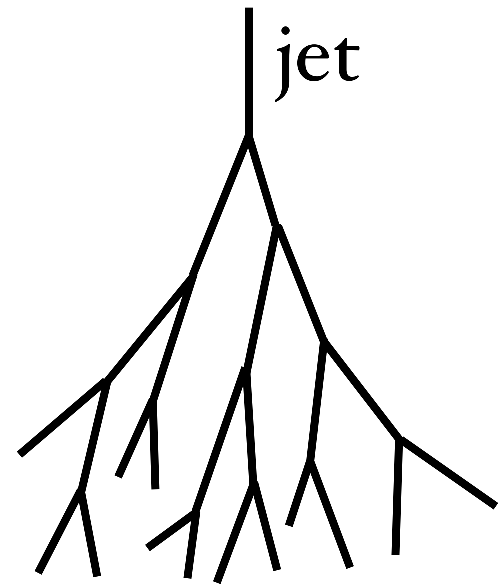
build up jet independently in the soft and jet functions



Cheung, Luke, Zuberi
Ellis, Hornig, Lee, Vermilion, JW

Soft-Collinear Factorization

QCD: build the jet from successive recombinations



SCET: phase space cuts on collinear and soft particles must separate

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build up jet independently in the soft and jet functions

Cheung, Luke, Zuberi
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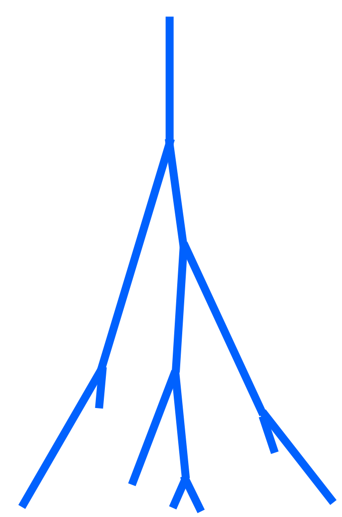
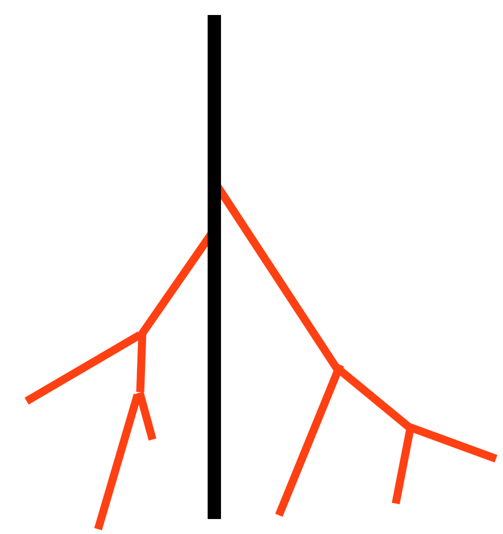


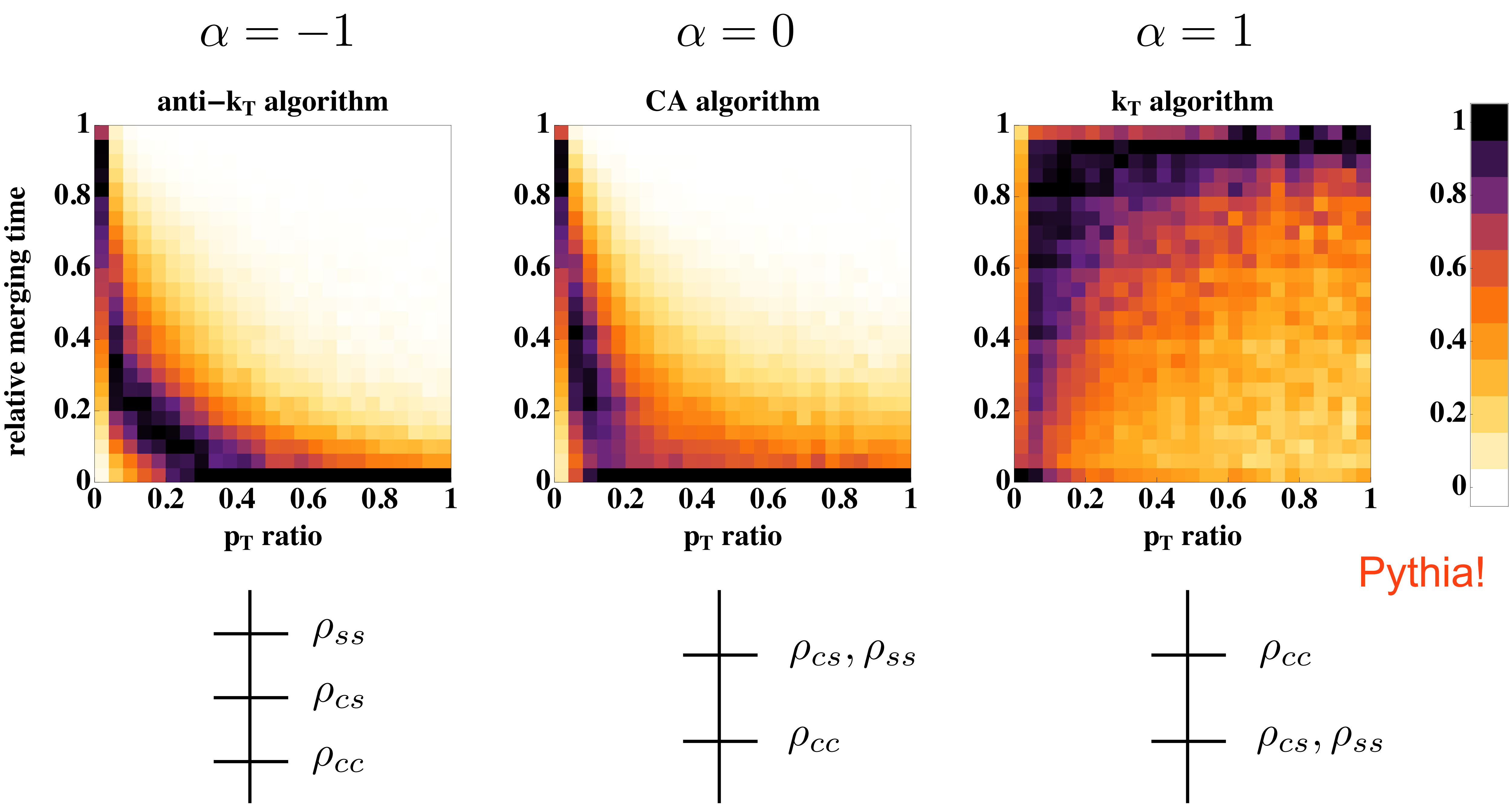
collinear modes



soft modes

Jet Algorithm Ordering

pairwise metric: $\rho_{ij}^{\alpha} = \min\left(p_{Ti}^{\alpha}, p_{Tj}^{\alpha}\right) \Delta R_{ij}$

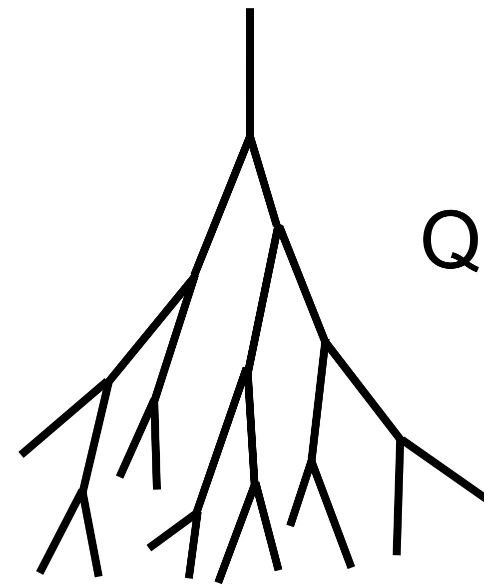


Ambiguity for Jet Substructure

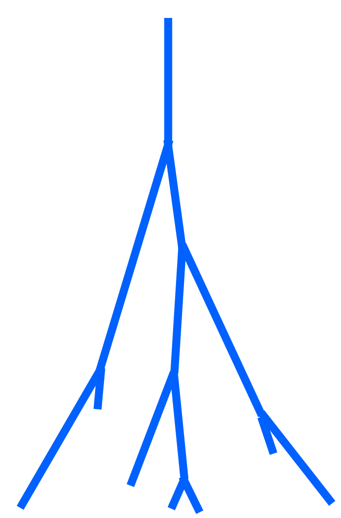
constraints on the order of merging of jets:

Decoupling of soft and collinear phase space constraints introduces ambiguities into merging order

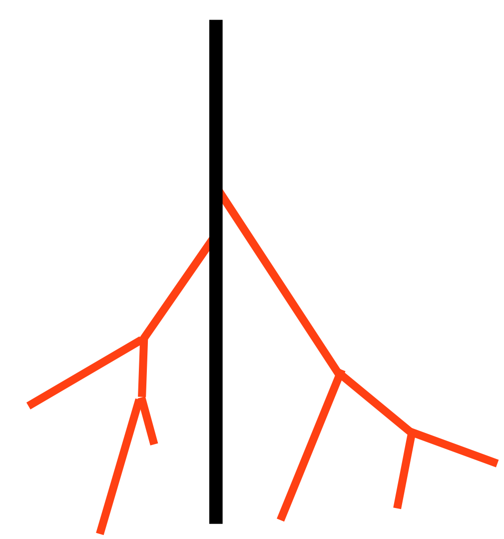
Many observables depending on merging order do not factorize



QCD

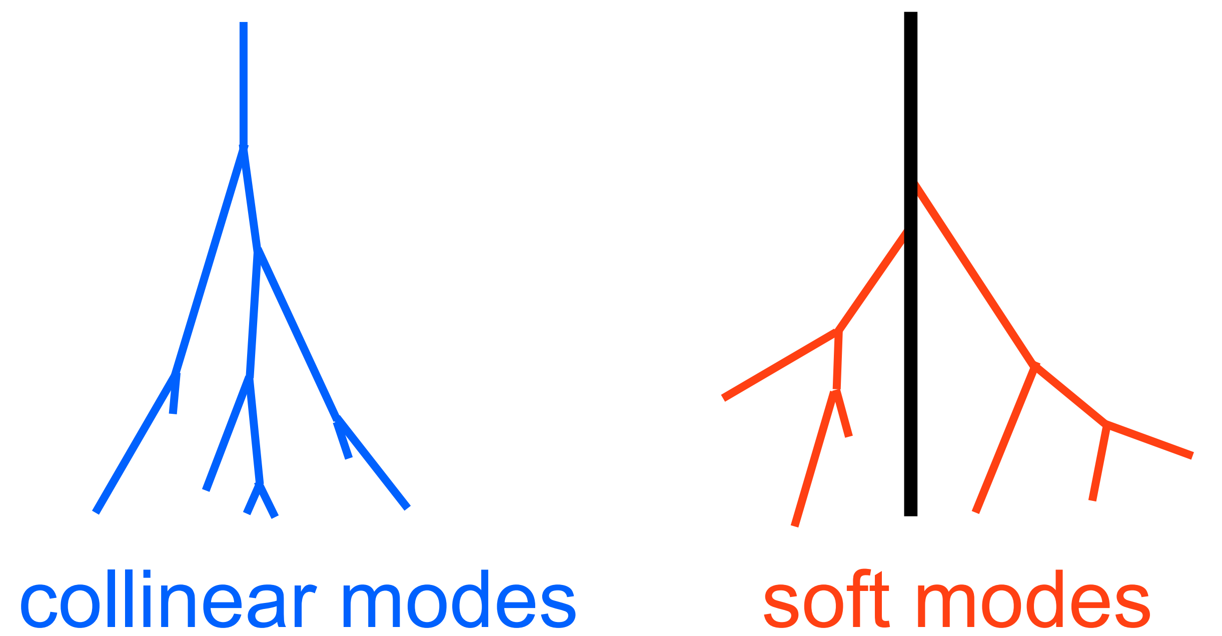


collinear modes



soft modes

Ambiguity for Jet Substructure



- Cannot determine merging order between soft and collinear sectors
- Cannot determine which softs were merged with a specific collinear particle

soft - collinear merging for k_T

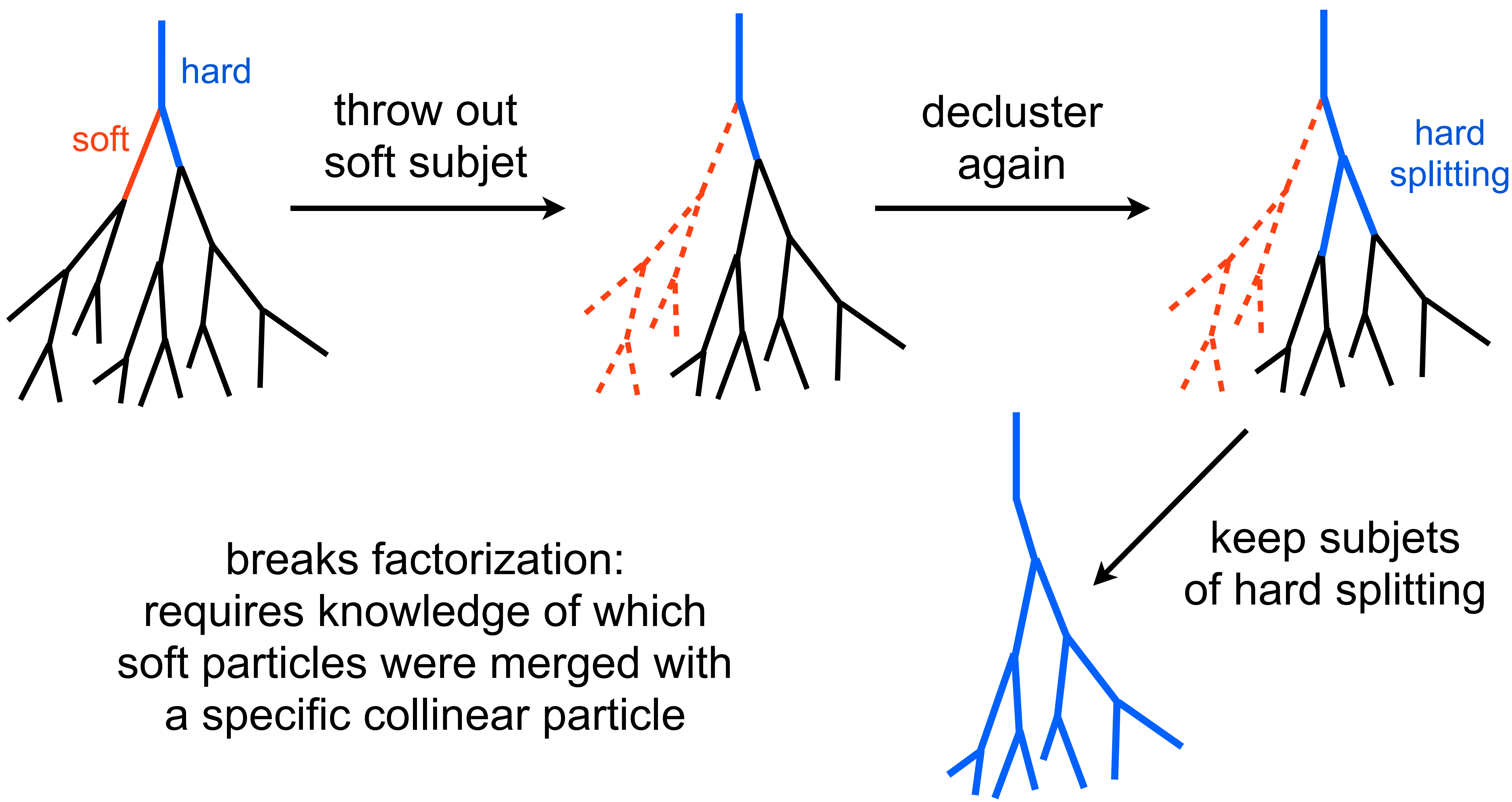
$$\rho_{cs} = E_s \frac{\theta_{ns}}{R}$$

$$\downarrow$$

$$\rho_{c_i s} = \rho_{c_j s}$$

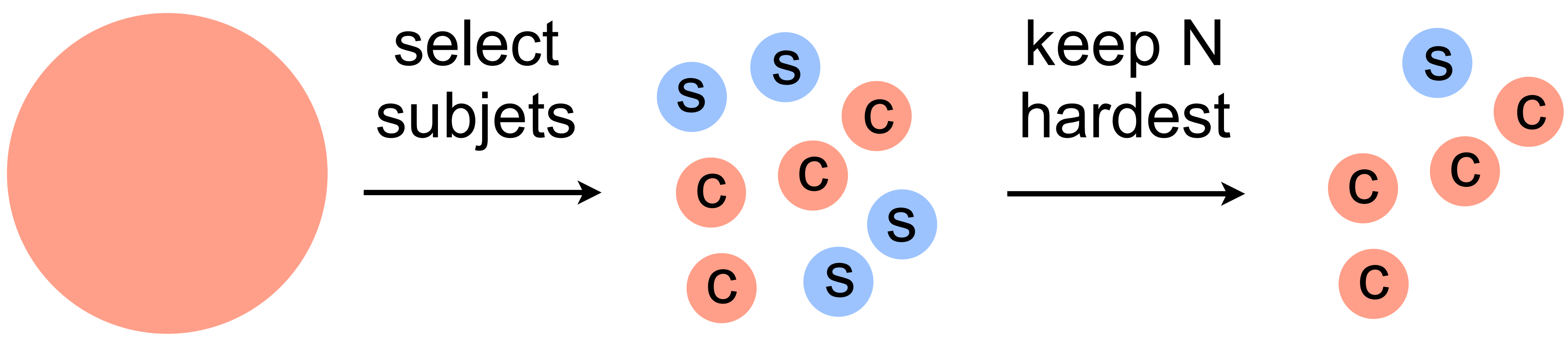
Common Jet Substructure Steps

- Declustering: step back through the recombinations until one step passes a kinematic cut



Common Jet Substructure Steps

- Filtering: decluster down to a fixed level, keep the hardest *N* subjets



breaks factorization:
If there are “soft subjets”, whether or not they pass the cut depends on the number of collinear subjets

Power Counting for Pruning

Pruning

- Recluster found jet with an algorithm
- Remove wide angle soft particles by making a cut at each merging step:

$$z_{ij} = \frac{\min(p_{Ti}, p_{Tj})}{p_{Ti+j}} < z_{\rm cut} \quad \text{and} \quad \Delta R_{ij} > D_{\rm cut}$$

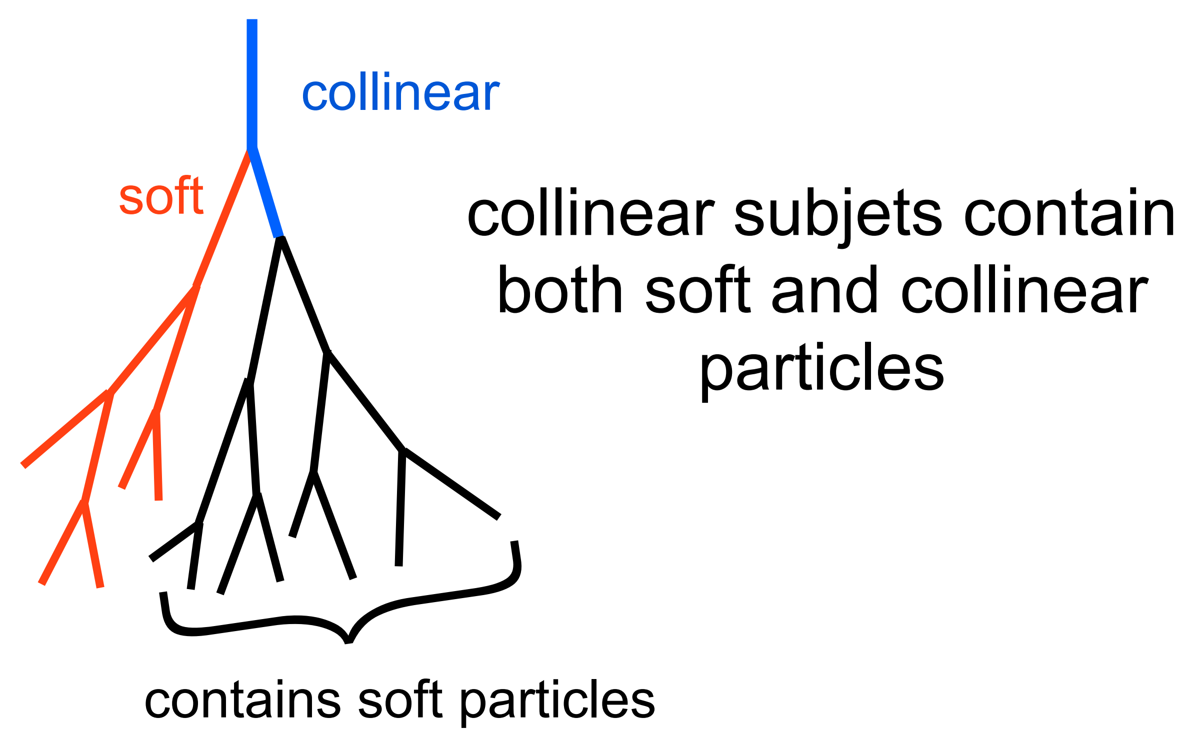
- For recombinations passing these cuts, *prune* the softer of particles i and j
- Surviving (unpruned) particles form the new jet

Power Counting for Pruning

Factorization requirements:

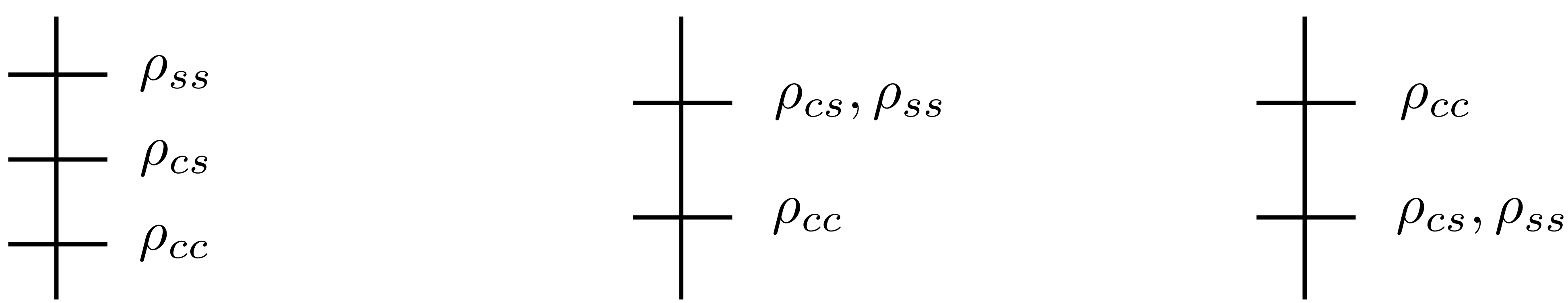
- c-c merging not pruned
- require $z_{cut} \sim \lambda$
    - ensures that any soft particle farther away than D_{cut} from the jet axis will be pruned
- Can look at different reclustering algorithms to see the behavior of pruning

merging	z	ΔR
$c_i\,, c_j \to c_{ij}$	$\frac{\min(p_{Ti}, p_{Tj})}{p_{Tij}} \sim \lambda^0$	$\Delta R_{ij} \sim \lambda$
$c\,, s \to c$	$\frac{p_{Ts}}{p_{Tc}} \sim \lambda^2$	$\Delta R_{ns} \sim \lambda^0$
$s_i\,, s_j \to s_{ij}$	$\frac{\min(p_{Ti}, p_{Tj})}{p_{Tij}} \sim \lambda^0$	$\Delta R_{ij} \sim \lambda^0$



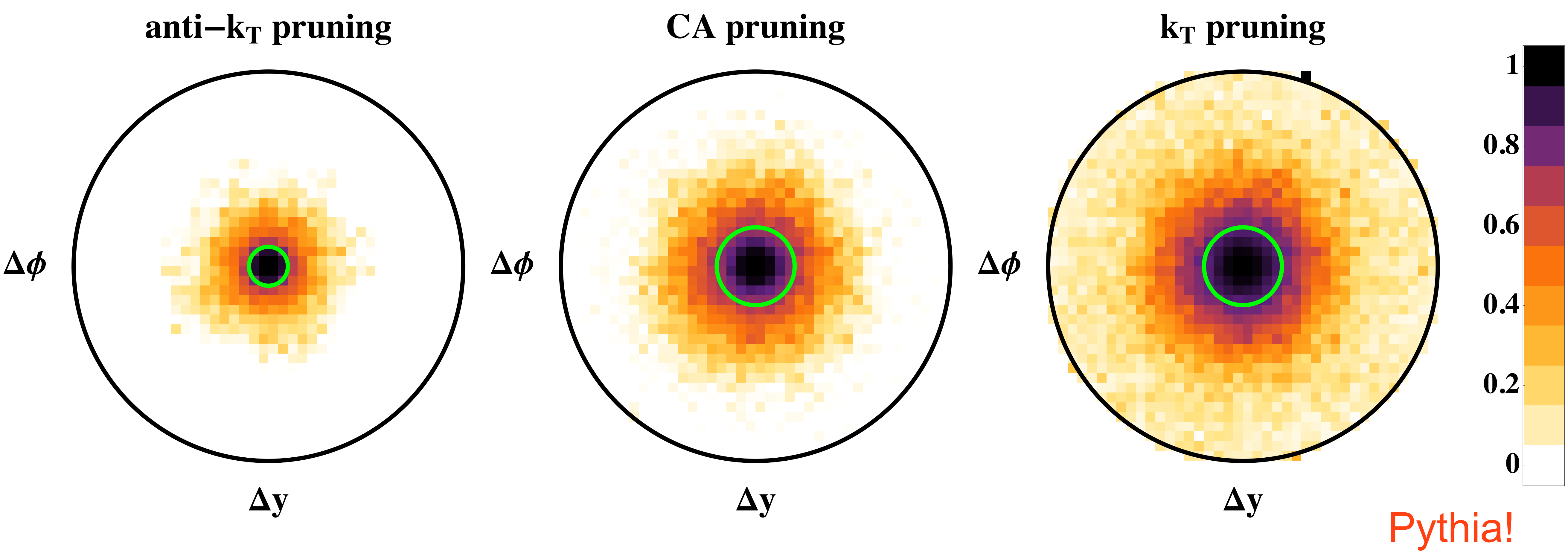
Power Counting for Pruning

- anti-kT: soft PS is just a circle around the jet axis - expect the soft PS to be a circle of radius D_{cut}
- CA: c-s and s-s merging simultaneous, so soft particles at larger angles can be merged near the axis and not pruned - expect unpruned soft PS to be a circle of radius $2D_{cut}$
- kT: metric prefers soft recombinations earlier, so can merge more soft PS into the jet - expect unpruned soft PS to be a circle of radius $2D_{cut}$, with more support outside the circle



Power Counting for Pruning

fraction of remaining pT after pruning
as a function of the location in the jet



green circles: power counting prediction for the region will little pruning

Conclusions

- Many realistic multijet configurations contain large logs
- Can use the final state kinematics to determine the required modes for SCET
  - Built $SCET_+$ to describe nearby jets
- This limit also applies to jet substructure
  - Few theoretical constraints imposed on jet substructure, factorization is a basic but essential test

Extra Slides

Kinematics from SCET

- Energies: $E_c \sim \lambda^0,\, E_s \sim \lambda^2$
- Angles:
- collinear - collinear: $\theta_{cc} \sim \lambda$
- soft - soft: $\theta_{ss} \sim \lambda^0$
- collinear - soft: $p_c \cdot p_s = 2E_c E_s (1 - \cos\theta_{cs})$

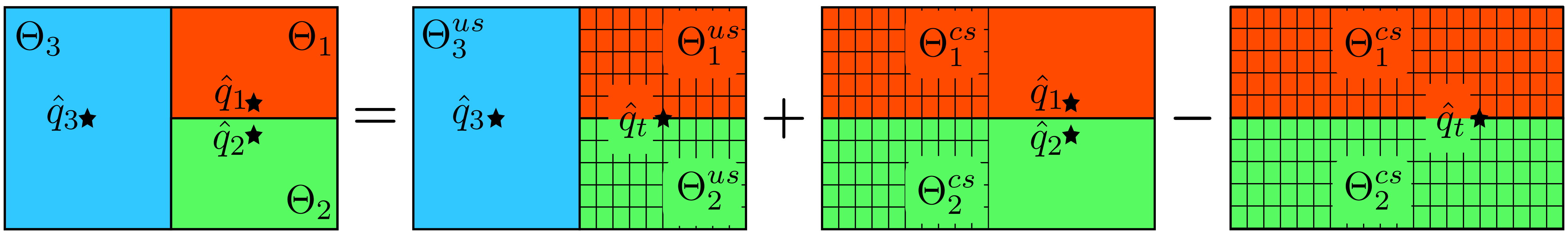
independent of p_c

$$\frac{p_c \cdot p_s}{E_c E_s} = 2\frac{p_c^- p_s^+}{p_c^- (p_s^+ + p_s^-)} + \mathcal{O}(\lambda) = \boxed{\frac{2p_s^+}{p_s^+ + p_s^-}} + \mathcal{O}(\lambda)$$

soft Wilson line depends only on label direction

⟶ will write θ_{cs} as θ_{ns}

soft + csoft calculation



(a) $\mathcal{T}_i$ (b) $\mathcal{T}_i^{us}$ (c) $\mathcal{T}_i^{cs}$ (d) $\mathcal{T}_i^{cs\,0}$

$$S_2(\mathcal{T}_1^{us}, \mathcal{T}_2^{us}, \mathcal{T}_3^{us}, \mu) = \frac{1}{N_C} \langle 0 | \bar{T} \left[Y_{n_3}^\dagger Y_{n_t} \right]_{ji} \mathcal{M}_3^{us}(\mathcal{T}_1^{us}, \mathcal{T}_2^{us}, \mathcal{T}_3^{us}) \, T \left[Y_{n_t}^\dagger Y_{n_3} \right]_{ij} | 0 \rangle \,,$$

$$S_+^{\{q,g,\bar{q}\}}(\mathcal{T}_1^{cs}, \mathcal{T}_2^{cs}, \mu) = \frac{1}{N_C\, C_F} \langle 0 | \bar{T} \left[V_{n_t}^\dagger X_{n_g} T^A X_{n_g}^\dagger X_{n_q} \right]_{ji} \mathcal{M}^{cs}(\mathcal{T}_1^{cs}, \mathcal{T}_2^{cs}) \, T \left[X_{n_q}^\dagger X_{n_g} T^A X_{n_g}^\dagger V_{n_t} \right]_{ij} | 0 \rangle \,.$$