Infrared singularities of gauge-theory scattering amplitudes from the anomalous-dimension matrix of n-jet operators in SCET

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INTRODUCTION

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- Understanding the structure of *infrared singularities* in gauge theory amplitude has been a long standing issue.

- Recently, it has been shown that they can be mapped onto UV divergences of n-jet operators in SCET. **(Becher,Neubert, 2009)**

- This means they can be described by means of an anomalous dimension, whose structure is constrained by: **(Becher,Neubert, 2009; Gardi, Magnea 2009)**

  - *soft-collinear factorization,*
  - *color conservation,*
  - *non-abelian exponentiation,*
  - *collinear limit.*

- A conjecture has been formulated, which has an extremely simple form and it should hold to all order in perturbation theory.

MOTIVATION

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- Important phenomenological applications in higher order log resummation for *n*-jet processes.

- Interesting for the understanding of the deeper structure of QCD: the anomalous dimension predicts only pairwise interactions among different partons.

- It implies Casimir scaling of the cusp anomalous dimension of quarks and gluons, in contrast with results obtained using the AdS/CFT correspondence in the strong-coupling behavior.

- This does not tell if and at which order a violation of the Casimir scaling could arise in perturbation theory. A diagrammatic analysis excluded it up to 3 loop, and at 4 loops in terms with higher Casimir invariants.
  (Becher, Neubert 2009; Gardi, Magnea 2009)

- Our aim is to complete the diagrammatic analysis at four loop.

INFRARED DIVERGENCES OF GAUGE THEORY AMPLITUDES

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- Given a UV renormalized, on-shell *n*-parton scattering amplitude with IR divergences regularized in $d = 4 - 2\epsilon$ dimensions, one obtains the finite remainder free from IR divergences from

$$|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0} \mathbf{Z}^{-1}(\epsilon,\{\underline{p}\},\mu)|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle.$$

- The multiplicative renormalization factor **Z** derives from an anomalous dimension $\mathbf{\Gamma}$:

$$\mathbf{Z}(\epsilon,\{\underline{p}\},\mu) = \mathbf{P}\exp\left[\int_\mu^\infty \frac{d\mu'}{\mu'}\mathbf{\Gamma}(\{\underline{p}\},\mu')\right].$$

- The anomalous dimension is conjectured to be very simple:

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\gamma_{\rm cusp}(\alpha_s)\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s).$$

(Becher, Neubert 2009; Gardi, Magnea 2009)

- Semiclassical origin of IR singularities: completely determined by color charges and momenta of external partons; only color dipole correlations.

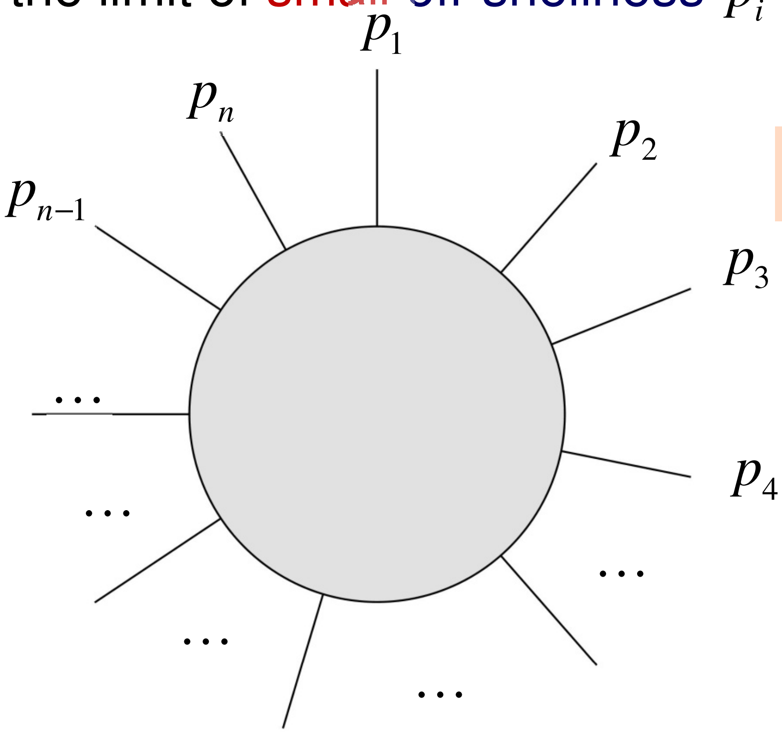
INFRARED DIVERGENCES OF GAUGE THEORY AMPLITUDES

- One easily derives the formal solution for **Z** up to four loops in perturbation theory

$$
\begin{aligned}
\ln \mathbf{Z} = {} & \frac{\alpha_s}{4\pi}\left(\frac{\Gamma_0{}'}{4\epsilon^2}+\frac{\boldsymbol{\Gamma}_0}{2\epsilon}\right)+\left(\frac{\alpha_s}{4\pi}\right)^2\left(-\frac{3\beta_0\Gamma_0{}'}{16\epsilon^3}+\frac{\Gamma_1{}'-4\beta_0\boldsymbol{\Gamma}_0}{16\epsilon^2}+\frac{\boldsymbol{\Gamma}_1}{4\epsilon}\right) \\
& +\left(\frac{\alpha_s}{4\pi}\right)^3\left(\frac{11\beta_0^2\Gamma_0{}'}{72\epsilon^4}-\frac{5\beta_0\Gamma_1{}'+8\beta_1\Gamma_0{}'-12\beta_0^2\boldsymbol{\Gamma}_0}{72\epsilon^3}+\frac{\Gamma_2{}'-6\beta_0\boldsymbol{\Gamma}_1-6\beta_1\boldsymbol{\Gamma}_0}{36\epsilon^2}+\frac{\boldsymbol{\Gamma}_2}{6\epsilon}\right) \\
& +\left(\frac{\alpha_s}{4\pi}\right)^4\left(-\frac{25\beta_0^3\Gamma_0{}'}{192\epsilon^5}+\frac{13\beta_0^2\Gamma_1{}'+40\beta_1\beta_0\Gamma_0{}'-24\beta_0^3\boldsymbol{\Gamma}_0}{192\epsilon^4}\right. \\
& \qquad -\frac{7\beta_0\Gamma_2{}'+9\beta_1\Gamma_1{}'+15\beta_2\Gamma_0{}'-24\beta_0^2\boldsymbol{\Gamma}_1-48\beta_0\beta_1\boldsymbol{\Gamma}_0}{192\epsilon^3} \\
& \qquad \left.+\frac{\Gamma_3{}'-8\beta_0\boldsymbol{\Gamma}_2-8\beta_1\boldsymbol{\Gamma}_1-8\beta_2\boldsymbol{\Gamma}_0}{64\epsilon^2}+\frac{\boldsymbol{\Gamma}_3}{8\epsilon}\right)+O(\alpha_s{}^5)
\end{aligned}
$$

N-POINT GREEN'S FUNCTION

- To set up the problem, consider first n-point off-shell Green's function in the limit of small off-shellness p_i^2 and large momentum transfer s_{ij}



$$= G_n(\{p\})$$

$$s_{ij} = (p_i \pm p_j)^2, \quad \text{with}$$

$$\begin{cases} +, \text{ if both incoming or outgoing} \\ -, \text{ otherwise;} \end{cases}$$

$$p_k^2 \ll s_{ij}$$

- Define reference vectors $n_i = (1, \hat{\mathbf{n}}_i)$ and $\bar{n}_i = (1, -\hat{\mathbf{n}}_i)$ with $n_i \cdot \bar{n}_i = 2$

$$\text{with } \lambda = \frac{p_i^2}{s_{ij}}, \quad \rightarrow \quad i\text{-collinear:} \quad (\bar{n}_i \cdot p_i, n_i \cdot p_i, p_\perp) \sim (1, \lambda, \sqrt{\lambda})$$

N-JET PROCESSES IN SCET

- A n-jet process with small off-shelness p_i^2 and large momentum transfer s_{ij} is conveniently described in SCET, introducing a set of collinear fields $\xi_i,\, A_i^\mu$ for each direction, and soft fields $q_s,\, A_s^\mu$

(Bauer,Fleming,Pirjol,Stewart 2000,2001; Beneke,Chapovsky,Diehl,Feldmann,2002 Becher,Hill,Neubert,2002)

$$\chi_i(x) = W_i^\dagger(x)\xi_i(x) = W_i^\dagger(x)\frac{\not{n}_i\bar{\not{n}}_i}{4}\psi_i(x),$$

$$A_\perp^\mu(x) = W_i^\dagger(x)[iD_\perp^\mu, W_i(x)], \qquad \text{with} \qquad W_i(x) = \mathbf{P}\exp\left(ig\int_{-\infty}^0 ds\,\bar{n}_i\cdot A_i(x+s\bar{n}_i)\right)$$

- The Lagrangian reads

$$\mathcal{L}_{\text{SCET}} = \bar{q}_s i\not{D}_s q_s - \frac{1}{4}(F_{\mu\nu}^{s\,a})^2 + \sum_{i=1}^n \left\{ \bar{\xi}_i \frac{\bar{\not{n}}}{2}\left[in_i\cdot D + i\not{D}_{ci\perp}\frac{1}{i\bar{n}_i\cdot D_{ci}} i\not{D}_{ci\perp}\right]\xi_i - \frac{1}{4}(F_{\mu\nu}^{ci\,a})^2 \right\}$$

- And the soft interactions can be decoupled from the collinear Lagrangian by a field redefinition

$$\chi_i(x) = S_i(x_-)\chi_i^{(0)}(x), \qquad \text{with} \qquad S_i(x) = \mathbf{P}\exp\left(ig\int_{-\infty}^0 dt\, n_i\cdot A_s^a(x+tn_i)t^a\right)$$

$$A_{i\perp}^\mu(x) = S_i(x_-)\mathcal{A}_{i\perp}^{(0)\mu}(x)S_i^\dagger(x),$$

LEADING POWER N-JET OPERATORS

- A generic n-jet process is mediated in SCET by n-jet operators, which at leading power contain exactly one field for each collinear sector. Defining a generic collinear field $(\phi_i)^{\alpha_i}_{a_i}(x) = [S_i(x_-)]_{a_i b_i} (\phi_i)^{(0)}(x)$, the operator reads

$$\mathcal{H}_n^{\text{eff}} = \int dt_1 \ldots dt_n \, \tilde{C}^{a_1 \ldots a_n}_{\alpha_1 \ldots \alpha_n}(t_1 \ldots t_n, \mu) (\phi_1)^{\alpha_1}_{a_1}(x + t_1 \bar{n}_1) \ldots (\phi_n)^{\alpha_n}_{a_n}(x + t_n \bar{n}_n)$$

- Using the color space representation, in which an amplitude is represented as a vector in colour space, $| c_1, c_2, \ldots, c_n \rangle$ on which the color generator for parton i acts like a matrix:

$$\mathbf{T}_i^c \, | \ldots, a_i, \ldots \rangle = (\mathbf{T}_i^c)_{b_i a_i} \, | \ldots, b_i, \ldots \rangle$$

- One obtain the usual picture of hard-soft-collinear facorization:

$$\begin{aligned} \mathcal{H}_n^{\text{eff}} &= \int dt_1 \ldots dt_n \, \langle O_n(\{\underline{t}\}, \mu) | \tilde{C}_n(\{\underline{t}\}, \mu) \rangle \\ &= \int dt_1 \ldots dt_n \, \langle O_n^{(0)}(\{\underline{t}\}, \mu) | \tilde{C}_n(\{\underline{t}\}, \mu) \rangle \langle 0 | S_1(0) \ldots S_n(0) | 0 \rangle \\ &= \int dt_1 \ldots dt_n \, \langle O_n^{(0)}(\{\underline{t}\}, \mu) | \tilde{C}_n(\{\underline{t}\}, \mu) \rangle \, \mathcal{S}(\{\underline{n}\}, \mu) \end{aligned}$$

SOFT-COLLINEAR FACTORIZATION FOR THE N-JETS AMPLITUDE

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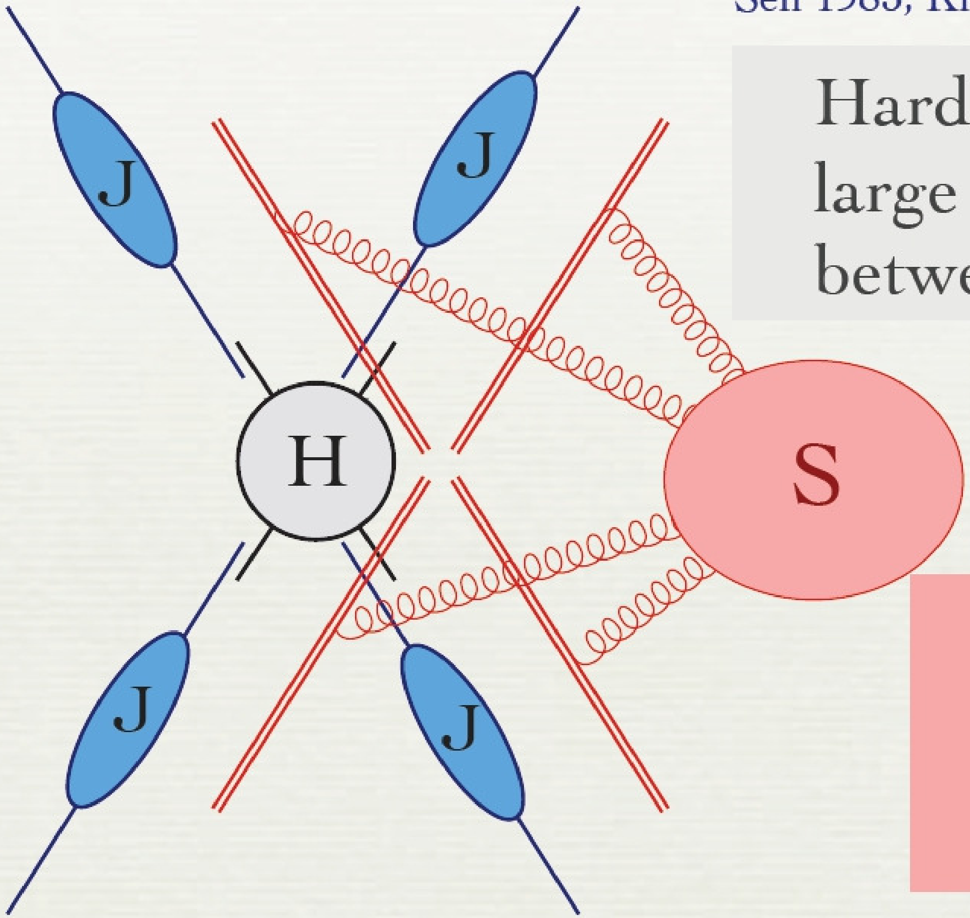
Sen 1983; Kidonakis, Oderda, Sterman 1998



Hard function H depends on large momentum transfers s_{ij} between jets

Soft function S depends on scales $\Lambda_{ij}^2 = \dfrac{M_i^2 M_j^2}{s_{ij}}$

Jet functions $J_i = J_i(M_i^2)$

(Courtesy of M. Neubert)

N-POINT GREEN'S FUNCTION AND ON-SHELL MATCHING

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- The off-shell n-points green function $G_n(\{\underline{p}\})$ is given by the matrix element of the effective Hamiltonian:

$$\underbrace{G_n(\{\underline{p}\})}_{\text{"finite"}} = \mathcal{C}_n(\{\underline{p}\},\mu)\langle O_n^{\text{ren}}(\epsilon,\mu)\rangle = \lim_{\epsilon\to 0}\mathcal{C}_n(\{\underline{p}\},\mu) \underbrace{\mathbf{Z}(\epsilon,\mu)}_{\text{renormalize UV}, \propto\frac{1}{\epsilon}} \underbrace{\langle O_n^{bare}(\epsilon)\rangle}_{\text{UV}, \propto\frac{1}{\epsilon}}$$

- In order to have on-shell n-parton scattering amplitude, take the limit $p_i^2 \to 0$ This introduces IR divergences, regulated evaluating the matrix element in $d = 4-2\epsilon$ dimension. The matrix elements of the operators becomes scaleless and reduces to trivial Dirac and color structures:

$$p_i^2 \to 0 \quad \Rightarrow \quad \underbrace{G_n(\epsilon,\{\underline{p}\})}_{\text{IR}, \propto\frac{1}{\epsilon}} = \mathcal{C}_n(\{\underline{p}\},\mu) \underbrace{\mathbf{Z}(\epsilon,\mu)}_{\text{renormalize UV}, \propto\frac{1}{\epsilon}} \underbrace{\langle O_n^{bare}(\epsilon)\rangle}_{\text{scaleless, UV and IR}, \propto\frac{1}{\epsilon}-\frac{1}{\epsilon}}$$

- One recover the identity

$$|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle \propto |\mathcal{C}_n(\{\underline{p}\},\mu)\rangle = \lim_{\epsilon\to 0}\mathbf{Z}^{-1}(\epsilon,\{p\},\mu)\,|G_n(\epsilon,\{\underline{p}\})\rangle$$

EXAMPLE: ONE LOOP DERIVATION OF THE ANOMALOUS DIMENSION

- An explicit calculation of the divergent part of the jet and soft function gives:

$$\mathcal{J}_q(p^2,\mu) = 1 + \frac{\alpha_s}{4\pi} C_F \left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-p^2} + \frac{3}{2\epsilon} \right) + \mathcal{O}(\epsilon^0),$$

$$\mathcal{J}_g(p^2,\mu) = 1 + \frac{\alpha_s}{4\pi} \left[C_A \left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-p^2} \right) + \frac{\beta_0}{2\epsilon} \right] + \mathcal{O}(\epsilon^0),$$

$$\mathcal{S}_g(\{\underline{p}\},\mu) = 1 + \frac{\alpha_s}{4\pi} \sum_{(i,j)} \frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2} \left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{-\sigma_{ij} n_i\cdot n_j \bar{n}_i\cdot p_i \bar{n}_j\cdot p_j \mu^2}{2(-p_i^2)(-p_j^2)} \right) + \mathcal{O}(\epsilon^0),$$

- The one loop divergences of the complete effective theory n-particle matrix elements are thus

$$\mathcal{S}_g(\{\underline{p}\},\mu)\prod_i \mathcal{J}_i(p_i^2,\mu) = 1 - \frac{\alpha_s}{4\pi}\left[\sum_{(i,j)} \frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2} \left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-s_{ij}} \right) + \sum_i \frac{\gamma_0^i}{2\epsilon} + \mathcal{O}(\epsilon^0) \right],$$

- Comparing with the explicit result for Z, one can derive the anomalous dimension Γ at one loop.

CONSTRAINT ON Γ:

SOFT-COLLINEAR FACTORIZATION

- The identification $|\mathcal{M}_n\rangle = |\mathcal{C}_n\rangle$ allows to use properties of the soft-collinear factorization to constrain $\boldsymbol{\Gamma}$. First

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{\mathbf{h}}$$

- Then, invariance under the renormalization group assure that

$$\boldsymbol{\Gamma}_{\mathbf{h}} = \boldsymbol{\Gamma}_{\mathbf{c+s}}$$

- Soft-collinear factorization gives then

$$\boldsymbol{\Gamma}_{\mathbf{c+s}}(s_{ij}) = \boldsymbol{\Gamma}_{\mathbf{s}}(\Lambda_{ij}) + \underbrace{\sum_{\mathbf{i}} \Gamma_c^i(p_i^2)\mathbf{1}}_{p_i^2 \text{ dependence must cancel}}$$

- This identity implies some consequences:

  - *nontrivial rewriting of the hard scale in term of the collinear and soft scale,*
  - *the collinear sectors are color diagonal:* $\boldsymbol{\Gamma}$ *and* $\boldsymbol{\Gamma}_s$ *must have the same color dependence*

CONSTRAINT ON Γ: NON-ABELIAN EXPONENTIATION

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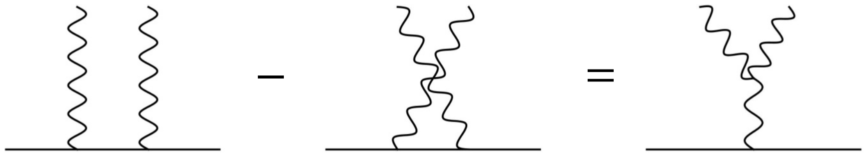
- The soft function is a matrix element of Wilson lines:

$$\mathcal{S}(\{\underline{n}\},\mu) = \langle 0 | \mathbf{S}_1(0)\ldots\mathbf{S}_n(0) | 0 \rangle = \exp(\widetilde{\mathcal{S}}(\{\underline{n}\},\mu))$$

- The exponent $\widetilde{\mathcal{S}}$ receives contributions only from Feynman diagrams whose color weights are color-connected ("maximally non-abelian")

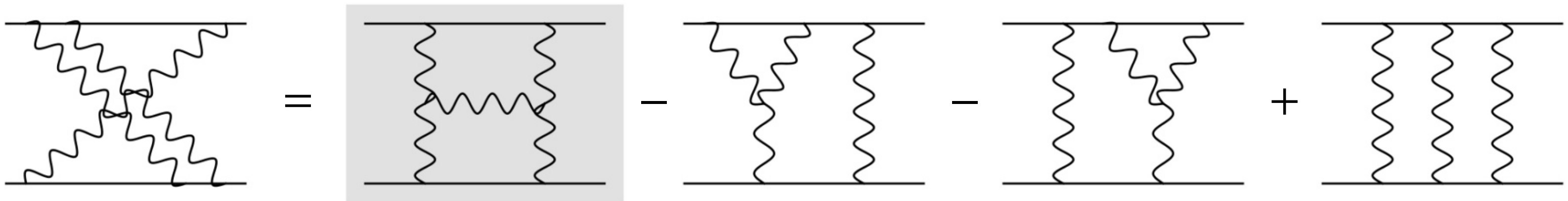
(Gatheral 1983; Frenkel and Taylor 1984)

- Color structures can be simplified using the Lie commutation relation:



$$\mathbf{T}^a\mathbf{T}^b - \mathbf{T}^b\mathbf{T}^a = if^{abc}\mathbf{T}^c$$

- Use this to decompose color structures into a sum over products of connected webs



- Only single connected webs contribute to the exponent $\widetilde{\mathcal{S}}$.

CONSTRAINT ON Γ:
SOFT FUNCTION AND WILSON LOOPS

- Wilson lines require UV renormalization beyond the renormalization of the coupling constant, when the integration path is not smooth: The simplest case is a Wilson loop with a single cusp.

- If the cusp is formed by two light-like segments with tangent vector n_1 and n_2, these UV divergences can be removed by a factor $Z(\beta_{12})$, which is a function of the hyperbolic cusp angle β_{12},

$$\cosh \beta_{12} = \frac{n_1 \cdot n_2}{\sqrt{n_1^2 n_2^2}}$$

(Polyakov1980,
Korchemsky, Radyushin, 1987
Korchemskya, Korchemsky, 1992)

- The corresponding anomalous dimension reads

$$\Gamma(\beta_{12}) \xrightarrow{n_{1,2}^2 \to 0} \Gamma^i_{\rm cusp}(\alpha_s) \ln \frac{\mu^2}{\Lambda_s} + \ldots$$

- This suggest a Sudakov-type log, which are well explained in effective field theory. The form of Λ_s in the anomalous dimension can be obtained from SCET.

CONSTRAINT ON Γ: SOFT-COLLINEAR FACTORIZATION

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- The nontrivial interplay among the hard, collinear and soft scale is suggested by SCET: it is of the form

$$\ln\frac{\mu^2}{\mu_h^2} = 2\ln\frac{\mu^2}{\mu_c^2} - \ln\frac{\mu^2}{\mu_s^2}$$

- Namely:

$$\beta_{ij} \equiv L_i + L_j - \ln\frac{\mu^2}{-s_{ij}}$$

hard log

$$\beta_{ij} = \ln\frac{-s_{ij}\mu^2}{(-p_i^2)(-p_j^2)}$$

soft log

$$L_i = \ln\frac{\mu^2}{-p_i^2}$$

collinear log

CONSTRAINT ON Γ: SOFT-COLLINEAR FACTORIZATION

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$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \mathbf{\Gamma}_{\mathbf{s}}(\{\underline{\beta}\},\mu) + \sum_{\mathbf{i}} \Gamma_c^i(L_i,\mu)\mathbf{1}$$

- Given that the form of the collinear anomalous dimension is known,

$$\Gamma_c^i(L_i) = -\Gamma_{\text{cusp}}^i(\alpha_s)L_i + \gamma_c^i(\alpha_s)$$

- One obtains a strong constraint from the requirement of no dependence on the collinear momentum, when one combine the soft and the collinear anomalous dimension:

$$\frac{\partial \mathbf{\Gamma}_s(\{\mathbf{L}\})}{\partial L_i} = \Gamma_{\text{cusp}}^i(\alpha_s)$$

(Becher, Neubert 2009; Gardi,Magnea 2009)

- The conjecture on $\mathbf{\Gamma}$ becomes a conjecture on $\mathbf{\Gamma}_{\mathbf{s}}$:

$$\mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) \stackrel{?}{=} -\sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \gamma_{\text{cusp}}(\alpha_s)\beta_{ij} + \sum_i \gamma_s^i(\alpha_s).$$

- Only exception could be a more complicated dependence on β_{ij} , such that the dependence on the collinear log cancels: e.g. the conformal cross ratio:

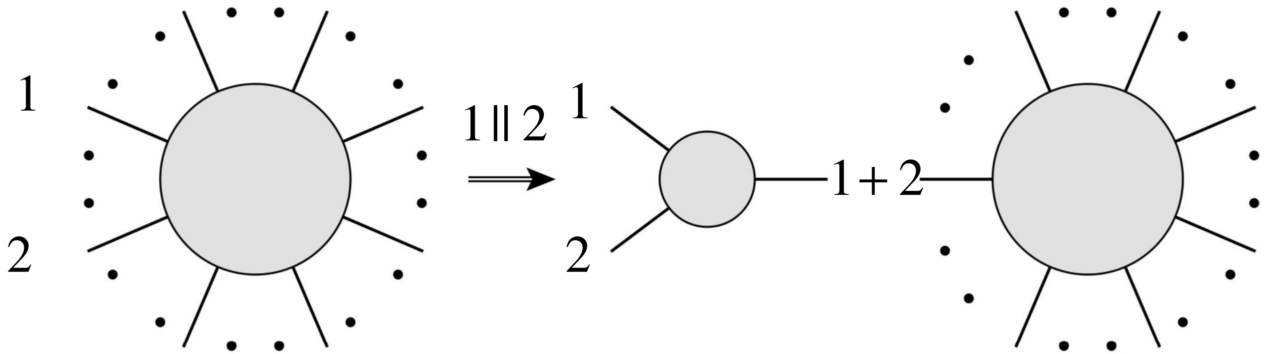
$$\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl}$$

CONSTRAINT ON Γ: CONSISTENCY WITH THE COLLINEAR LIMIT

- When two partons become collinear, an *n*-point amplitudes reduces to a (*n*-1)-parton amplitude times a splitting function:

(Berends, Giele 1989; Mangano, Parke 1991; Kosower 1999; Catani,De Florian, Rodrigo 2003)

$$|\mathcal{M}_n(\{p_1, p_2, p_3, \ldots, p_n\})\rangle = \mathbf{Sp}(\{p_1, p_2\})\,|\mathcal{M}_{n-1}(\{P, p_3, \ldots, p_n\})\rangle + \ldots$$



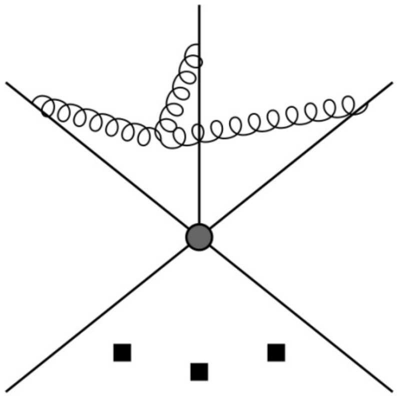
$$\mathbf{\Gamma}_{\mathrm{Sp}}(\{p_1, p_2\}, \mu) = \mathbf{\Gamma}(\{p_1, \ldots, p_n\}, \mu) - \mathbf{\Gamma}(\{P, p_3, \ldots, p_n\}, \mu)\big|_{\mathbf{T}_P \to \mathbf{T}_1 + \mathbf{T}_2}$$

(Becher, Neubert 2009)

- $\mathbf{\Gamma}_{\mathrm{Sp}}$ must be independent of momenta and colors of partons 3,…*n.*

Diagrammatic Analysis: One and Two Loops

- Recipe: attach single connected gluon web to the Wilson lines of the soft ("Mercedes star") operator



- And study color and momentum dependence: symmetries in the color structure must match symmetries in the momentum dependence. Use

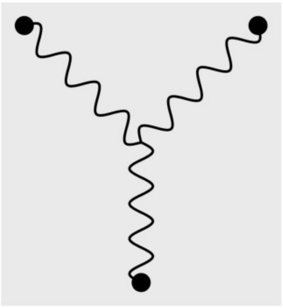
$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c, \quad f^{abc} f^{abd} = C_A \delta^{cd},$$

$$\mathrm{tr}_{\text{adj.}} (\mathbf{T}^a \mathbf{T}^b \mathbf{T}^c) = i f^{ade} f^{beg} f^{cgd} = \frac{i C_A}{2} f^{abc}.$$

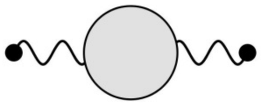
- One loop

  - one leg: $\mathbf{T}_i^2 = C_i$
  - two legs: $\mathbf{T}_i \cdot \mathbf{T}_j$

⟹ *No new structures*



- Two loops



  - one leg: $-i f^{abc} \mathbf{T}_i^a \mathbf{T}_i^b \mathbf{T}_i^c = \frac{C_A C_i}{2}$
  - two legs: $-i f^{abc} \mathbf{T}_i^a \mathbf{T}_i^b \mathbf{T}_j^c = \frac{C_A}{2} \mathbf{T}_i \cdot \mathbf{T}_j$

⟹ *No new structures*

  - three legs: $-i f^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c$

⟹ *Incompatible with soft-collinear factorization*

DIAGRAMMATIC ANALYSIS: THREE LOOPS

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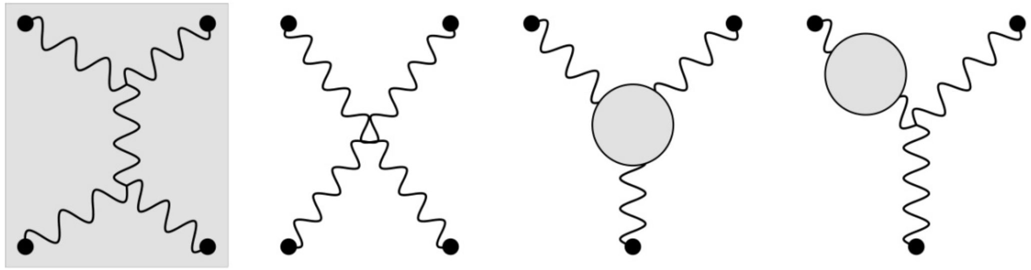
- The color structure of the first two diagrams is



$$\mathcal{T}_{ijkl} = -\mathcal{T}_{ikjl} = -\mathcal{T}_{ljki} = \mathcal{T}_{jilk} = \mathcal{T}_{klij} = f^{adx} f^{bcx} (\mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d)_+$$

- One finds three new structures compatible with soft-collinear factorization:

$$\mathbf{\Delta\Gamma_3} = -\frac{\bar{f}_1(\alpha_s)}{4} \sum_{(i,j,k,l)} \mathcal{T}_{ijkl} \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})} - \bar{f}_2(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\, F(\beta_{ijkl}, \beta_{iklj} - \beta_{iljk})$$

(Becher, Neubert 2009; Dixon,Gardi,Magnea,2009)

- $\bar{f}_1$ and $\bar{f}_2$ are not compatible with collinear limit: the splitting function depends on colors and momenta of additional partons.

$$\Delta\Gamma_{\text{Sp}}(\{p_1, p_2\}, \mu)|_{\bar{f}_1(\alpha_s)} = 2 \sum_{(i,j)\neq 1,2} \mathcal{T}_{12ij} \left[\ln \frac{(-s_{Pi})(-s_{Pj})}{(-s_{12})(-s_{ij})} + \ln z(1-z) \right],$$

$$\Delta\Gamma_{\text{Sp}}(\{p_1, p_2\}, \mu)|_{\bar{f}_2(\alpha_s)} = 2\mathcal{T}_{1122} - 4 \sum_{i\neq 1,2} \mathcal{T}_{12ii}.$$

DIAGRAMMATIC ANALYSIS: THREE LOOPS

- The function $F(\beta_{ijkl}, \beta_{iklj} - \beta_{iljk}) = F(x,y) = -F(-x,y)$ is also incompatible with the two-parton collinear limit, unless it vanishes in all collinear limits. Write

$$p_1^\mu = zEn^\mu + p_\perp^\mu - \frac{p_\perp^2}{4zE}\bar{n}^\mu, \qquad p_2^\mu = (1-z)En^\mu - p_\perp^\mu - \frac{p_\perp^2}{4(1-z)E}\bar{n}^\mu,$$

- Then

$$\Delta\Gamma_{\mathrm{Sp}}(\{p_1,p_2\},\mu)|_F = \sum_{(i,j)\neq 1,2}\left[8\mathcal{T}_{12ij}F(\omega_{ij},\omega_{ij}) + 4\mathcal{T}_{1ij2}F(\epsilon_{ij},-2\omega_{ij})\right],$$

- With

$$\epsilon_{ij} = \beta_{1ij2} = \frac{1}{z(1-z)E}\left(\frac{p_\perp\cdot p_i}{n\cdot p_i} - \frac{p_\perp\cdot p_j}{n\cdot p_j}\right) \to 0,$$

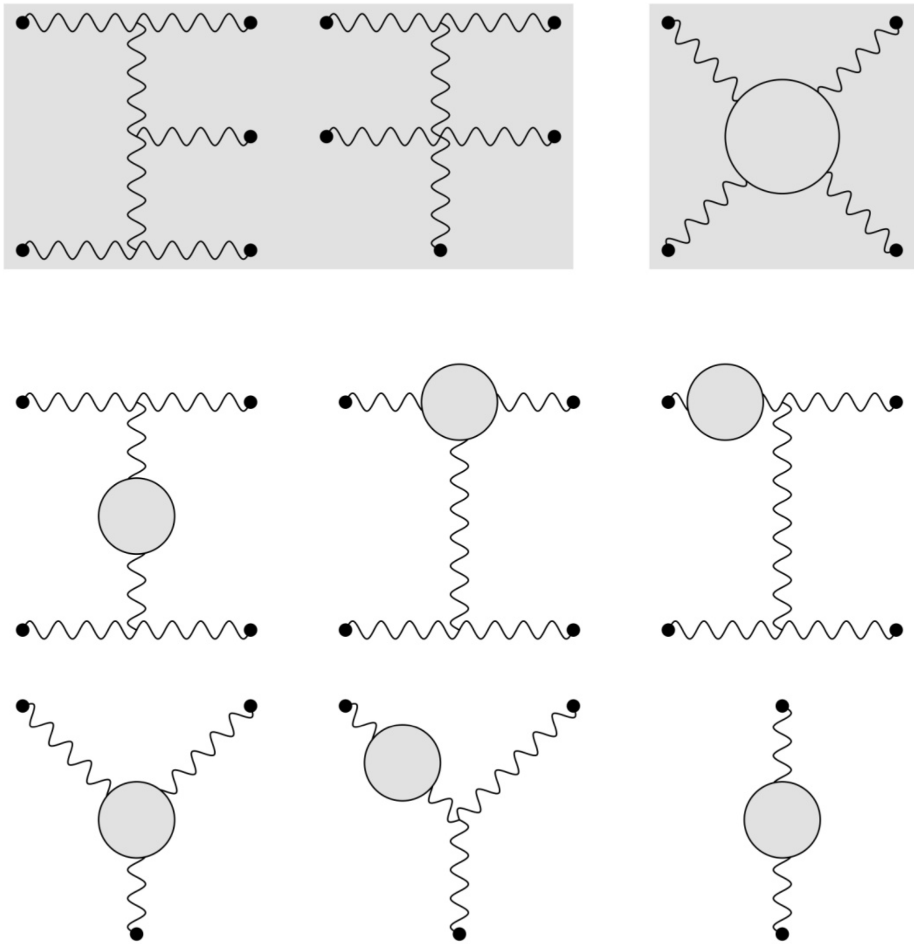
$$\omega_{ij} = \beta_{12ij} = \ln\frac{p_\perp}{4z^2(1-z)^2E^2} + \ln\frac{-s_{ij}}{(-n\cdot p_i)(-n\cdot p_j)} \to -\infty$$

- It is not clear whether such a function appears in loop calculation. An example for this function has recently been given:

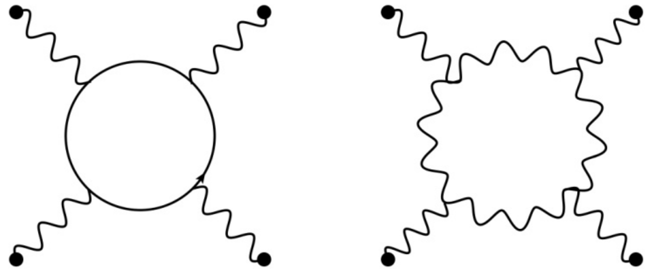
$$F(x,y) = x^3(x^2 - y^2)$$

(Dixon,Gardi,Magnea,2009)

DIAGRAMMATIC ANALYSIS: FOUR LOOPS



DIAGRAMMATIC ANALYSIS: FOUR LOOPS



- At four loops structures involving higher Casimir invariants appears:

$$\mathcal{D}_{ijkl} = d_F^{abcd}\left(\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\right)_+, \quad d_R^{a_1\ldots a_n} = \mathrm{tr}[(\mathbf{T}_R^{a_1}\ldots\mathbf{T}_R^{a_n})_+]$$

- There are possible new structures compatible with soft-collinear factorization:

$$\Delta\Gamma_{s4,1} \propto \sum_{(i,j)} \beta_{ij}\left[D_{iijj}\, g_1(\alpha_s) + D_{iiij}\, g_2(\alpha_s)\right] + \sum_{(i,j,k)} \beta_{ij}\, D_{ijkk}\, g_3(\alpha_s),$$

(Becher, Neubert 2009)

$$\Delta\Gamma_{s4,2} = \sum_{(i,j)}\left[D_{iijj}\, g_4(\alpha_s) + D_{iiii}\, g_5(\alpha_s)\right] + \sum_{(i,j,k,l)} D_{ijkl}\, G_1(\beta_{ijkl}, \beta_{iklj} - \beta_{iljk}).$$

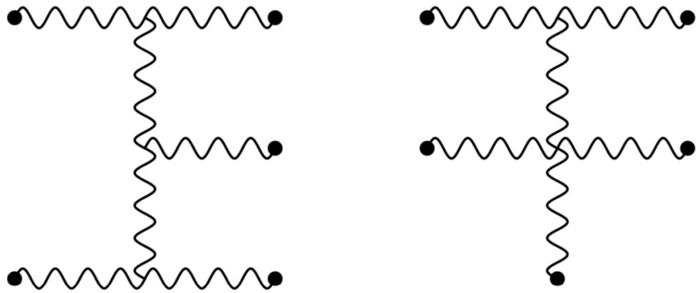
New

- Again, they are not compatible with the collinear limit, except $G_1(\beta_{ijkl}, \beta_{iklj} - \beta_{iljk}) = G_1(x,y) = G_1(-x,y)$ if it vanishes in all collinear limits. A possible example reads

$$G_1(x,y) = x^2(x^2 - y^2)^2$$

(Dixon,Gardi,Magnea,2009)

DIAGRAMMATIC ANALYSIS: FOUR LOOPS



- The two webs have color structure

$$\mathcal{T}_{ijklm} \equiv f^{adx} f^{bcy} f^{exy} (\mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d \mathbf{T}_m^e)_+ .$$

- Try to find all possible new contribution to the anomalous dimension compatible with the symmetries of $\mathcal{T}_{ijklm}$:

$$\mathcal{T}_{ijklm} = -\mathcal{T}_{ikjlm} = -\mathcal{T}_{ljkim} = -\mathcal{T}_{klijm} = -\mathcal{T}_{jilkm}$$

- Examples are e.g.

$$\Delta\Gamma_{s,4} \propto \sum_{(i,j,k)} \mathcal{T}_{iijki}\, g_6(\alpha_s)\beta_{ij} + \sum_{(i,j,k,l)} \mathcal{T}_{iijki} (g_{15}(\alpha_s)\beta_{ij} + g_{16}(\alpha_s)\beta_{il})$$
$$+ \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm} (g_{17}(\alpha_s)\beta_{ij} + g_{18}(\alpha_s)\beta_{im}) + \ldots,$$

- Simplification occurs summing over indices not involved in the β_{ij} factors, using

$$\sum_i \mathbf{T}_i^a \,|\mathcal{M}_n(\epsilon,\{\underline{p}\})\rangle = 0, \quad \text{eg.} \quad \sum_{(i,j,k,l)} \mathcal{T}_{iijkl}\beta_{ij} = -\sum_{(i,j,k)} \left(\mathcal{T}_{iijki} + \mathcal{T}_{iijkj} + \mathcal{T}_{iijkk}\right)\beta_{ij}$$

DIAGRAMMATIC ANALYSIS: FOUR LOOPS

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- There are two structures compatible with soft-collinear factorization:

$$\Delta\Gamma_{s,4} = \sum_{(i,j,k)} \mathcal{T}_{iijkk}\, \bar{g}(\alpha_s)\beta_{ij} + \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, G_2(\beta_{ijkm}, \beta_{ikmj} - \beta_{imjk}, \beta_{ijml}, \beta_{imlj} - \beta_{iljm}),$$

New

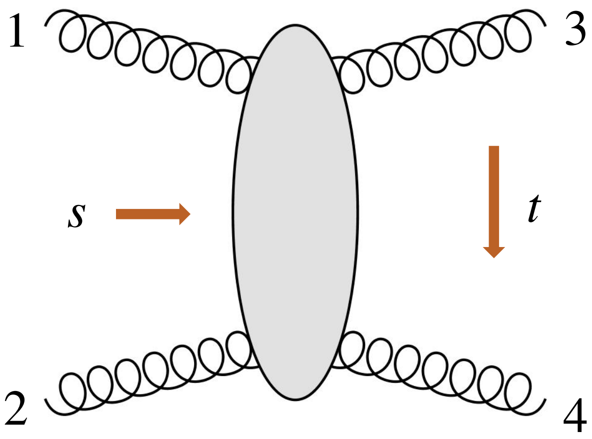
- The first function is incompatible with the collinear limit, the second function cannot be excluded, if it vanishes in all collinear limits.

- Applied to the two-jet case, it means that the Casimir scaling of the cusp anomalous dimension is still preserved:

$$\frac{\Gamma^q_{\text{cusp}}(\alpha_s)}{C_F} = \frac{\Gamma^g_{\text{cusp}}(\alpha_s)}{C_A} = \gamma_{\text{cusp}}(\alpha_s)$$

THE HIGH ENERGY LIMIT

- Recently, *Del-Duca, Duhr, Gardi, Magnea and White (2011)* have shown that the dipole formula can be used in the high energy limit to study Reggeization properties of gauge theories.



- In the $t/s \to 0$ limit particles exchaged in the t-channel may "Reggeize":
- Large logs of t/s are generated by a simple replacement of the t-channel propagator:

$$\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

- The Regge trajectory has a perturbative expansion with IR divergent coefficients:

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}(\alpha_s^3)$$

- The gluon has been shown to Reggeize at NLL, and the two-loop trajectory is known:

$$\mathcal{M}^{gg\to gg}_{a_1a_2a_3a_4}(s,t) = 2g_s^2\frac{s}{t}\left[(T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1,k_3)\right]\left(\frac{s}{-t}\right)^{\alpha(t)}\left[(T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1,k_3)\right]$$

THE HIGH ENERGY LIMIT

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- What can we learn from the dipole formula at high energy? Introduce the Mandelstam color operator

$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 = -(\mathbf{T}_3 + \mathbf{T}_4),$$
$$\mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_3 = -(\mathbf{T}_2 + \mathbf{T}_4),$$
$$\mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_4 = -(\mathbf{T}_2 + \mathbf{T}_3)$$

$$s + t + u = 0,$$
$$\mathbf{T}_s^2 + \mathbf{T}_s^2 + \mathbf{T}_s^2 = \sum_{i=1}^{4} C_i$$

(Del-Duca, Duhr, Gardi, Magnea and White 2011)

- At high energy the dipole formula factorizes

$$Z\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \tilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_1\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

- The operator Z_1 is s-independent and proportional to the unit matrix in color space;
- Color and s-dependence are collected into the factor

$$\tilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K(\alpha_s(\mu^2), \epsilon)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + i\pi\mathbf{T}_s^2\right]\right\}$$

- This result governs Reggeization and its breaking: at LL accuracy, the s-channel contribution can be dropped, and one has

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K(\alpha_s(\mu^2), \epsilon)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2\right]\right\} Z_1 \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

- If at LO and at leading t/s the amplitude is dominated by t-channel exchange, the hard function is an eigenstate of the color operator $\mathbf{T}_t^2$
- It is possible to prove that Reggeization holds at NLL for the real part of the amplitudes, while it breaks down at NNLL; the result can be generalized to multiparticle scattering.

THE HIGH ENERGY LIMIT

- The result can be used in the opposite direction, i.e. use reggeization as an additional constraint on the dipole formula:
- Consider the high energy limit of the additional terms found at three and four loops: consider a $2 \to 2$ scattering process: The conformal ratios in the high energy limit read:

$$\beta_{1234} = \ln\frac{(-s_{12})(-s_{34})}{(-s_{13})(-s_{24})} = 2\ln\left(\frac{s}{-t}\right) - 2i\pi$$

$$\beta_{1342} = \ln\frac{(-s_{13})(-s_{24})}{(-s_{14})(-s_{23})} = 2\ln\left(\frac{-t}{s+t}\right) \simeq -2\ln\left(\frac{s}{-t}\right)$$

$$\beta_{1423} = \ln\frac{(-s_{14})(-s_{23})}{(-s_{12})(-s_{34})} = 2\ln\left(\frac{s+t}{s}\right) + 2i\pi \simeq 2i\pi$$

(Del-Duca, Duhr, Gardi, Magnea and White 2011)

- And the functions found at three and four loops become

$$F(x,y) = x^3(x^2-y^2) \to \propto \ln\left(\frac{s}{-t}\right)^4, \qquad G_1(x,y) = x^2(x^2-y^2)^2 \to \propto \ln\left(\frac{s}{-t}\right)^4,$$

- F contains a superleading log and must be ruled out; This is not the case for G_1, but consistency with Regge limit requires cancellation of the $\ln^3$ as well.
- More complicated functions of F and G_1 in which these logs cancel are still possible.

CONCLUSION

- *Infrared singularities* in gauge theory amplitude can be mapped onto UV divergences of n-jet operators in SCET.
- They can be described by means of an anomalous dimension, whose structure is constrained by soft-collinear factorization, non-abelian exponentiation, and two-parton collinear limit.
- The anomalous dimension is expected to have a very simple structure. It should hold to all order in perturbation theory.
- We perform a diagrammatic analysis up to four loop, showing that only new structures proportional to functions vanishing in all collinear limits can appear.
- No violation of Casimir scaling of the cusp anomalous dimension arise.